This is a preprint of: "Analytic non-integrability of the Suslov problem", Adam Mahdi, Clàudia Valls, *J. Math. Phys.*, num. 53, 122901, 2012.
DOI: [10.1063/1.4763464]

ANALYTIC NON-INTEGRABILITY OF THE SUSLOV PROBLEM

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Abstract. In this work we consider the Suslov problem, which consists of a rotation motion of a rigid body, whose center of mass is located at one axis of inertia, around a fixed point O in a constant gravity field restricted to a nonholonomic constraint. The integrability and non-integrability has been established by a number of authors for the nongeneric values of $\mathbf{b} = (b_1, b_2, b_3)$ which is the unit vector along the line connecting the point O with the center of mass of the body. Here we prove the analytic non-integrability for the remaining (generic) values of $\mathbf{b}$.

1. Introduction

The Suslov problem is one of the most famous problems in nonholonomic dynamics with no shape space and was formulated in [6]. It is a generalized rigid body with some of its body angular velocity components set equal to zero, i.e., it consists of a rotational motion of a rigid body around a fixed point O in a constant gravity field when restricted by a non–holonomic constraint

$$\langle \mathbf{n}, \boldsymbol{\omega} \rangle = 0,$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the body angular velocity, $\mathbf{n}$ is a vector fixed in the body and $\langle \cdot, \cdot \rangle$ denotes the standard metric in $\mathbb{R}^3$. To be more precise, the equations of motion of the Suslov problem are

$$\mathbf{I}\,\dot{\boldsymbol{\omega}} = \mathbf{I}\,\boldsymbol{\omega} \times \boldsymbol{\omega} + \varepsilon \boldsymbol{\gamma} \times \mathbf{b} + \lambda \mathbf{n}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \quad \langle \mathbf{n}, \boldsymbol{\omega} \rangle = 0, \tag{1}$$

where λ is the Langrange multiplier; the diagonal matrix $\mathbf{I} = \mathrm{diag}(I_1, I_2, I_3)$ represents the inertia of the body; $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ is the unit vertical vector and $\mathbf{b} = (b_1, b_2, b_3)$ is the unit vector along the line connecting the point O with the center of mass of the body; ε is the product of the mass of the body and the gravity constant. We assume that $\mathbf{n} = (0, 0, 1)$, i.e., we assume that the center of mass is located at the third axis of inertia, and thus the equation of the constraint is $\omega_3 = 0$. The Suslov equations can be written as

$$\begin{gathered} \dot{\omega}_1 = \frac{\varepsilon}{I_1}(\gamma_2 b_3 - \gamma_3 b_2), \qquad \dot{\omega}_2 = \frac{\varepsilon}{I_2}(\gamma_3 b_1 - \gamma_1 b_3), \\ \dot{\gamma}_1 = -\omega_2 \gamma_3, \qquad \dot{\gamma}_2 = \omega_1 \gamma_3, \qquad \dot{\gamma}_3 = \omega_2 \gamma_1 - \omega_1 \gamma_2, \end{gathered} \tag{2}$$

where $I_1, I_2 > 0$. System (2) has two polynomial first integrals

$$\mathcal{F}_1 = \frac{1}{2}(I_1 \omega_1^2 + I_2 \omega_2^2) + \varepsilon(b_1 \gamma_1 + b_2 \gamma_2 + b_3 \gamma_3) \quad \text{and} \quad \mathcal{F}_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2. \tag{3}$$

The integrability of the Suslov problem and its generalization have been studied by a number of authors (see [3], [4], [5], [1], [8], [7] and the references therein). There are three known integrable cases of system (2), which are:

(i) The Suslov case (see [6]), i.e., when $\varepsilon = 0$. Then $\mathcal{F}_3 = \omega_1$. This case was studied by Suslov in [6] without the assumption on the location of the center of mass, i.e., with $\mathbf{n}$ being arbitrary.

(ii) The Kharlamova-Zabelina case (see [2]), i.e., when $\langle \mathbf{b}, \mathbf{n} \rangle = 0$. Then $\mathcal{F}_3 = I_1 \omega_1 b_1 + I_2 \omega_2 b_2$.

2010 *Mathematics Subject Classification.* 70F25, 70E40, 37J30.
Key words and phrases. Suslov problem, analytic integrability.
The second author is supported by the grants AGAUR PIV-DGR-2010 and by FCT through CAMGDS, Lisbon.

(iii) The Kozlov case, i.e., when $\mathbf{b}$ parallel to $\mathbf{n}$, i.e., $b_1 = b_2 = 0$, and $I_1 = I_2$. Then $\mathcal{F}_3 = \omega_1\gamma_1 + \omega_2\gamma_2$ (see [3]).

There are also a number of non-integrability results. Namely, using different type of techniques such as the Moralis-Ramis theory, or the study of the connection between the properties of solutions on the complex time plane and the existence of first integrals it was proved in [8, 7] and [4] that when $\mathbf{b}$ is parallel to $\mathbf{n}$ only the case $I_1 = I_2$ admits an additional complex or real meromorphic first integral. When $\mathbf{b}$ is not parallel to $\mathbf{n}$ and $b_2 = 0$ it was proved in [5] and [4] that system (2) does not possess any third complex or real meromorphic first integral which is functionally independent of $\mathcal{F}_1$ and $\mathcal{F}_2$. The main purpose of this paper is to study the integrability of the remaining case, i.e., when $\varepsilon b_2 b_3 \neq 0$. We note that if $\varepsilon \neq 0$, then with a rescaling of time we can always assume that $\varepsilon = 1$. Moreover, we note that system (2) is invariant with respect to the change of variables

$$\gamma_1 \to -\gamma_2, \quad \gamma_2 \to -\gamma_1, \quad \omega_1 \to \omega_2, \quad \omega_2 \to \omega_1,$$
$$I_2 \to I_1, \quad I_1 \to I_2, \quad b_2 \to -b_1, \quad b_1 \to -b_2.$$

Therefore we can also assume that $b_1 \neq 0$ otherwise with the above change we are in the case $b_2 = 0$, which was studied in [8, 7] and [4]. In short, we will consider the polynomial integrability of system (2) when $\varepsilon b_1 b_2 b_3 \neq 0$. The following theorem is the main result of the paper.

Theorem 1. *If $\varepsilon b_1 b_2 b_3 \neq 0$, then system (2) does not admit any third analytic first integral which is functionally independent of $\mathcal{F}_1$ and $\mathcal{F}_2$.*

This theorem completes the characterization of the integrability of the Suslov problem in the analytic category. The article is organised as follows. In Section 2 we shortly review some of the definitions and results used in the paper. The proof of our main result is given in Section 3

2. Preliminaries

We consider polynomial differential systems of the form

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{P}(\mathbf{x}), \quad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n, \tag{4}$$

with $\mathbf{P}(\mathbf{x}) = (P_1(\mathbf{x}), \ldots, P_n(\mathbf{x}))$ and $P_i \in \mathbb{C}[x_1, \ldots, x_n]$ for $i = 1, \ldots, n$. As usual $\mathbb{C}[x_1, \ldots, x_n]$ denotes the polynomial right over $\mathbb{C}$ in the variables $x_1, \ldots, x_n$. Here, t can be real or complex. We say that $\mathcal{F}$ is a *first integral* of (4) if and only if $\mathcal{F}$ is constant on the solution curves of (4). We associate to (4) a vector field $\mathcal{X}$ as follows

$$\mathcal{X} = P_1(\mathbf{x})\frac{\partial}{\partial x_1} + \ldots + P_n(\mathbf{x})\frac{\partial}{\partial x_n}.$$

Then if $\mathcal{F}$ is differentiable, then it is a first integral of (4) if and only if $\mathcal{X}\mathcal{F} = 0$. We say that $\mathcal{F}$ is a *polynomial*, *analytic* or *meromorphic first integral* depending whether $\mathcal{F}$ belongs to the corresponding category of functions. Also, we say that system (4) is *weight-homogeneous* if there exists $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{N}^n$ and $d \in \mathbb{N}$ such that for an arbitrary $\alpha \in \mathbb{R}^+$, where $\mathbb{R}^+$ denote the set of positive real numbers, we have

$$P_i(\alpha^{s_1}x_1, \ldots, \alpha^{s_n}x_n) = \alpha^{s_i - 1 + d}P_i(x_1, \ldots, x_n)$$

for $i = 1, \ldots, n$. We call $\mathbf{s} = (s_1, \ldots, s_n)$ the *weight-exponent* of system (4) and d the *weight degree* with respect to the weight exponent $\mathbf{s}$. In the particular case that $\mathbf{s} = (1, \ldots, 1)$ system (4) is called a *homogeneous polynomial differential system of degree d*. We say that a polynomial $F(x_1, \ldots, x_n)$ is a *weight-homogeneous polynomial with weight exponent* $\mathbf{s} = (s_1, \ldots, s_5)$ and *weight-degree d* if

$$F(\alpha^{s_1}x_1, \ldots, \alpha^{s_n}x_n) = \alpha^d F(x_1, \ldots, x_n).$$

If $\mathbf{s} = (1, \ldots, 1)$ then we say that F is a *homogeneous polynomial of degree d*. The following well-known proposition reduces the study of the existence of an analytic first integral of a weight-homogeneous polynomial differential system (4) to the study of the existence of a weight-homogeneous polynomial first integrals.

Proposition 1. *Let H be an analytic function and let $H = \sum_k H_k$ be its decomposition into weight-homogeneous polynomials of weight degree k with respect to the weight exponent* $\mathbf{s}$. *Then H is an analytic first integral of the weight-homogeneous polynomial differential system* (4) *with weight exponent* $\mathbf{s}$ *if and only if each weight-homogeneous part H_k is a first integral of system* (4) *for all k.*

Introducing the change of variables

$$\Omega_1 = \ell_1\omega_1 + i\ell_2\omega_2, \qquad \Omega_2 = \ell_1\omega_1 - i\ell_2\omega_2, \qquad \xi_1 = \gamma_1 + i\gamma_3, \qquad \xi_3 = \gamma_1 - i\gamma_3,$$

where $i = \sqrt{-1}$ and $\ell_1 = \sqrt{I_1}$ and $\ell_2 = \sqrt{I_2}$, system (2) writes

$$\begin{aligned}
\dot{\Omega}_1 &= -\frac{i}{2\ell_2}(b_3(\xi_1+\xi_3) + ib_1(\xi_1-\xi_3)) + \frac{1}{2\ell_1}(2b_3\gamma_2 + ib_2(\xi_1-\xi_3)),\\
\dot{\Omega}_2 &= \frac{i}{2\ell_2}(b_3(\xi_1+\xi_3) + ib_1(\xi_1-\xi_3)) + \frac{1}{2\ell_1}(2b_3\gamma_2 + ib_2(\xi_1-\xi_3)),\\
\dot{\xi}_1 &= \frac{\xi_1}{2\ell_2}(\Omega_1-\Omega_2) - \frac{i\gamma_2}{2\ell_1}(\Omega_1+\Omega_2),\\
\dot{\gamma}_2 &= -\frac{i}{4\ell_1}(\xi_1-\xi_3)(\Omega_1+\Omega_2),\\
\dot{\xi}_3 &= -\frac{\xi_3}{2\ell_1}(\Omega_1-\Omega_2) + \frac{i\gamma_2}{2\ell_2}(\Omega_1+\Omega_2).
\end{aligned} \tag{5}$$

The first integral $\mathcal{F}_1$ in these new variables becomes

$$\overline{\mathcal{F}}_1 = \mathcal{F}_1(\gamma_1,\gamma_2,\gamma_3,\Omega_1,\Omega_2) = \Omega_1\Omega_2 + b_1(\xi_1+\xi_3) + 2b_2\gamma_2 - ib_3(\xi_1-\xi_3).$$

The second integral $\mathcal{F}_2$ in these new variables becomes

$$\overline{\mathcal{F}}_2 = \mathcal{F}_2(\gamma_1,\gamma_2,\gamma_3,\Omega_1,\Omega_2) = \xi_1\xi_3 + \gamma_2^2.$$

System (5) is weight-homogeneous of weight exponent $(1,1,2,2,2)$ and weight degree $d = 2$. By Proposition 1 the characterization of the polynomial first integral of weight exponent $(1,1,2,2,2)$ and weight degree $d = 2$ of system (5) is sufficient to know all the analytic first integrals of system (5). Thus for each $n \geq 1$, we can express any analytic first integrals G as

$$G = \sum_{l_1+l_2+2j_1+2j_2+2j_3=n} G_{l_1,l_2,j_1,j_2,j_3}\Omega_1^{l_1}\Omega_2^{l_2}\xi_1^{j_1}\gamma_2^{j_2}\xi_3^{j_3}, \tag{6}$$

where $G_{l_1,l_2,j_1,j_2,j_3} \in \mathbb{C}$. In the rest of this section we shall use that fact that the five dimensional system (5) possesses the two functionally independent polynomial first integrals $\overline{\mathcal{F}}_1$ and $\overline{\mathcal{F}}_2$ in order to reduce it to the three dimensional polynomial system. We shall restrict system (5) to the zero level of the first integral $\overline{\mathcal{F}}_2$. Hence, setting $\overline{\mathcal{F}}_2 = 0$ and solving for ξ_3 we get

$$\xi_3 = -\frac{\gamma_2^2}{\xi_1}. \tag{7}$$

Now introducing $\xi_3 = -\gamma_2^2/\xi_1$ into $\overline{\mathcal{F}}_1 = 0$ sand solving for Ω_1 we get

$$\Omega_1 = -\frac{1}{\Omega_2\xi_1}(b_1(\xi_1^2-\gamma_2^2) + 2b_2\gamma_2\xi_1 - ib_3(\xi_1^2+\gamma_2^2). \tag{8}$$

Let

$$\begin{aligned}
\mathcal{S} &= ib_1(\xi_1^2-\gamma_2^2) + b_3(\xi_1^2+\gamma_2^2) + 2ib_2\xi_1\gamma_2,\\
\mathcal{S}_1 &= (ib_2\ell_2 - b_1\ell_1)(\xi_1^2+\gamma_2^2) + b_3(2\ell_2\xi_1\gamma_2 + i\ell_1(\xi_1^2-\gamma_2^2)).
\end{aligned} \tag{9}$$

Then system (5) with the restriction $\overline{\mathcal{F}}_1 = \overline{\mathcal{F}}_2 = 0$ and with a rescaling of time by setting $dt = \ell_1\ell_2\Omega_2\xi_1\, d\tau$ becomes

$$\begin{aligned} \dot{\Omega}_2 &= 2\Omega_2\xi_1\mathcal{S}_1, \\ \dot{\xi}_1 &= 2\xi_1\big(i(\ell_1\xi_1 - i\ell_2\gamma_2)\mathcal{S} - \xi_1(\ell_1\xi_1 + i\ell_2\gamma_2)\Omega_2^2\big), \\ \dot{\gamma}_2 &= \ell_2(\xi_1^2+\gamma_2^2)\big(\mathcal{S} - i\xi_1\Omega_2^2\big). \end{aligned} \tag{10}$$

where the dot denotes derivative with respect to τ. Then if we denote by $\overline{G} = \overline{G}(\Omega_2, \xi_1, \gamma_2)$ the restriction of G given in (6) to ξ_3 and Ω_1 as in (7) and (8) we get that $\overline{G}$ is a first integral of system (10) and is of the form

$$\overline{G} = \sum_{l_1+l_2+2j_1+2j_2+2j_3=n} \overline{G}_{1,2,3}\Omega_2^{l_2-l_1}(b_1(\xi_1^2-\gamma_2^2)+2b_2\gamma_2\xi_1 - ib_3(\xi_1^2+\gamma_2^2))^{l_1}\xi_1^{j_1-j_3-l_1}\gamma_2^{j_2+2j_3}, \tag{11}$$

where $\overline{G}_{1,2,3} = (-1)^{l_1+j_3}G_{l_1,l_2,j_1,j_2,j_3} \in \mathbb{C}$.

3. Proof of the main result

It follows from Section 2 that to prove Theorem 1 it is sufficient to show that system (10) has no first integral of the form (11). For each $n \geq 1$, we can express it as

$$G = \sum_{k=-[n/2]}^{[n/2]} G_k(\xi_1, \gamma_2)\Omega_2^k,$$

where as usual $[\cdot]$ denotes the integer part function. Without loss of generality we can assume that G has no constant term. We note that G_k is a polynomial in the variable γ_2 and a rational function with respect to the variable ξ_1. Clearly, G is a polynomial in the variable γ_2 and a rational function in the variables ξ_1 and Ω_2. To prove Theorem 1 it is enough to show that $G_k = 0$ for each $k = -[n/2], \ldots, [n/2]$. If G is a first integral system (10) it must satisfy

$$\frac{\partial G}{\partial \Omega_2}\dot{\Omega}_2 + \frac{\partial G}{\partial \xi_1}\dot{\xi}_1 + \frac{\partial G}{\partial \gamma_2}\dot{\gamma}_2 = 0. \tag{12}$$

Setting

$$\mathcal{A}[f_j] = \mathcal{S}\left[\ell_2(\xi_1^2+\gamma_2^2)\frac{\partial f_j}{\partial \gamma_2} + 2i\xi_1(\ell_1\xi_1 - i\ell_2\gamma_2)\frac{\partial f_j}{\partial \xi_1}\right] + 2j\xi_1\mathcal{S}_1 f_j, \tag{13}$$

and

$$\mathcal{B}[f_j] = -i\ell_2(\xi_1^2+\gamma_2^2)\xi_1\frac{\partial f_j}{\partial \gamma_2} - 2\xi_1^2(\ell_1\xi_1 + i\ell_2\gamma_2)\frac{\partial f_j}{\partial \xi_1},$$

computing the different coefficients of Ω_2^j in (12), we obtain

$$\begin{aligned} \mathcal{A}[G_j] &= 0, \quad j = -n, -n+1, \\ \mathcal{A}[G_j] + \mathcal{B}[G_{j-2}] &= 0, \quad j = -n+2, \ldots, n, \\ \mathcal{B}[G_j] &= 0, \quad j = n-1, n. \end{aligned} \tag{14}$$

We consider a preliminary result.

Lemma 2. *Let $\mathcal{S}$ and $\mathcal{S}_1$ be as in (9), then $\mathcal{S}$ does not divide $\mathcal{S}_1$.*

Proof. Setting $\mathcal{S} = 0$ and solving with respect to ξ_1 we get

$$\xi_1 = \frac{\gamma_2}{b_1 - ib_3}\big(-b_2 \pm \sqrt{b_1^2+b_2^2+b_3^3}\big).$$

Note that since $b_1, b_3 \in \mathbb{R}$, ξ_1 is well defined. Now introducing ξ_1 in $\mathcal{S}_1$ we obtain

$$\begin{aligned} &\frac{2\gamma_2^2}{(b_1-ib_3)^2}\sqrt{b_1^2+b_2^2+b_3^2}\big(b_2\ell_1(b_3+ib_1) + \ell_2(b_2^2+ib_1b_3+b_3^2) \\ &\quad + \sqrt{b_1^2+b_2^2+b_3^2}(ib_2\ell_2 - (b_1-ib_3)\ell_1)\big). \end{aligned}$$

Solving it with respect to b_1 we get

$$b_1 = \pm i\sqrt{b_2^2 + b_3^2} \quad \text{and} \quad b_1 = \frac{1}{I_1}[ib_2\sqrt{I_1 I_2} \pm b_3\sqrt{I_1(I_2 - I_1)}],$$

which is not possible since $b_j \in \mathbb{R}$ for $j = 1, 2, 3$ and $I_1, I_2 > 0$. □

Lemma 3. *Let $l_1 \neq l_2$ and f_j be a polynomial in the variable γ_2 and a rational function with respect to the variable ξ_1. Then condition $\mathcal{A}[f_j] = 0$ implies*

$$f_j = \begin{cases} 0, & \textit{for} \quad j \in \mathbb{Z} \setminus \{0\}, \\ \textit{constant}, & \textit{for} \quad j = 0. \end{cases}$$

Proof. Let $l_1 \neq l_2$ and $j \in \mathbb{Z} \setminus \{0\}$. From $\mathcal{A}[f_j] = 0$ we have that

$$\mathcal{S}\left[\ell_2(\xi_1^2 + \gamma_2^2)\frac{\partial f_j}{\partial \gamma_2} + 2i\xi_1(\ell_1\xi_1 - i\ell_2\gamma_2)\frac{\partial f_j}{\partial \xi_1}\right] + 2j\xi_1\mathcal{S}_1 f_j = 0. \tag{15}$$

Since by Lemma 2 $\mathcal{S}$ does not divide $\mathcal{S}_1$, in view of (15) we get that $\mathcal{S}$ must divide f_j. Therefore we can write $f_j = \mathcal{S}^m g_j$ for some $m \geq 1$ and g_j is not divisible by $\mathcal{S}$. Then g_j satisfies

$$m\mathcal{S}\left[\ell_2(\xi_1^2 + \gamma_2^2)\frac{\partial g_j}{\partial \gamma_2} + 2i\xi_1(\ell_1\xi_1 - i\ell_2\gamma_2)\frac{\partial g_j}{\partial \xi_1}\right] + G_j g_j = 0, \tag{16}$$

where

$$\begin{aligned} G_j = -2\big(&b_1 j l_1 \xi_1^3 + b_1 j l_1 \xi_1 \gamma_2^2 - 2b_1 m l_1 \xi_1^2 \gamma_2 + ib_1 m l_2 \xi_1^2 \gamma_2 + 2ib_1 m l_2 \xi_1 \gamma_2^2 + ib_1 m l_2 \gamma_2^3 - ib_2 j l_2 \xi_1^3 \\ &- ib_2 j m_2 \xi_1 \gamma_2^2 + 2b_2 m l_1 \xi_1^3 - ib_2 m l_2 \xi_1^3 - 2ib_2 l l_2 \xi_1^2 \gamma_2 - ib_2 m l_2 \xi_1 \gamma_2^2 - ib_3 j l_1 \xi_1^3 + ib_3 j l_1 \xi_1 \gamma_2^2 \\ &- 2b_3 j l_2 \xi_1^2 \gamma_2 - 2ib_3 m l_1 \xi_1^2 \gamma_2 - b_3 m l_2 \xi_1^2 \gamma_2 - 2b_3 m l_2 \xi_1 \gamma_2^2 - b_3 m l_2 \gamma_2^3\big). \end{aligned}$$

Since g_j is not divisible by $\mathcal{S}$, then G_j must be divisible by $\mathcal{S}$, i.e.

$$G_j = \mathcal{S}(\alpha_1\xi_1 + \alpha_2\gamma_2 + \alpha_3),$$

for some $\alpha_i \in \mathbb{C}$, $i = 1, 2, 3$. Then introducing G_j in (16), and after simplifying by $\mathcal{S}$, we get

$$m\left[\ell_2(\xi_1^2 + \gamma_2^2)\frac{\partial g_j}{\partial \gamma_2} + 2i\xi_1(\ell_1\xi_1 - i\ell_2\gamma_2)\frac{\partial g_j}{\partial \xi_1}\right] + (\alpha_1\xi_1 + \alpha_2\gamma_2 + \alpha_3)g_j = 0.$$

Solving it we obtain

$$g_j = K_j\Big(\frac{l_2^2\xi_1^2 - 2il_1l_2\xi_1\gamma_2 - l_2^2\gamma_2^2}{\xi_1}\Big)e^{-\frac{\xi_1 h(\xi_1,\gamma_2)}{2m\sqrt{l_1^2 - l_2^2}(l_2^2\xi_1^2 - 2il_1l_2\xi_1\gamma_2 - l_2^2\gamma_2^2)}},$$

where K_j is any smooth function in the variable $(l_2^2\xi_1^2 - 2il_1l_2\xi_1\gamma_2 - l_2^2\gamma_2^2)/\xi_1$ and

$$\begin{aligned} h_1 = &-2\alpha_3\sqrt{l_1^2 - l_2^2}(l_1\xi_1 - il_2\gamma_2) - \big(l_2(\xi_1^2 - \gamma_2^2) - 2il_1\xi_1\gamma_2\big)\Bigg(i\alpha_2\sqrt{l_1^2 - l_2^2}\log\alpha_1 \\ &- 2i(\alpha_2 l_1 + i\alpha_1 l_2)\log\Big(\frac{-2i\sqrt{l_1^2 - l_2^2}\xi_1 - 2i(l_1\xi_1 - il_2\gamma_2)}{\sqrt{x_1}}\Big)\Bigg). \end{aligned}$$

Since $m \geq 1$, $l_1 \neq l_2$, $l_1 l_2 \neq 0$ and g_j must be a polynomial in the variable γ_2 and a rational function with respect to the variable ξ_1, we must have $\alpha_1 = \alpha_2 = \alpha_3 = 0$. But then $G_j = 0$, which is not possible. Hence, $g_j = 0$ which yields $f_j = 0$. Now assume $j = 0$. From $\mathcal{A}[f_0] = 0$ after simplifying by $\mathcal{S}$, we have that

$$\ell_2(\xi_1^2 + \gamma_2^2)\frac{\partial f_0}{\partial \gamma_2} + 2i\xi_1(\ell_1\xi_1 - i\ell_2\gamma_2)\frac{\partial f_0}{\partial \xi_1} = 0. \tag{17}$$

Solving it we obtain

$$f_0 = K_0\Big(\frac{l_2^2\xi_1^2 - 2il_1l_2\xi_1\gamma_2 - l_2^2\gamma_2^2}{\xi_1}\Big), \tag{18}$$

where K_j is any smooth function. Since f_0 must be a polynomial in the variable γ_2 and a Laurent polynomial with respect to the variable ξ we must have that K_0 is a polynomial. In view of equation (11), we get that f_0 is of the form

$$f_0 = \sum_{2l_2+2j_1+2j_2+2j_3=n} \overline{G}_{1,2,3}(b_1(\xi_1^2-\gamma_2^2)+2b_2\gamma_2\xi_1-ib_3(\xi_1^2+\gamma_2^2))^{l_2}\xi_1^{j_1-j_3-l_2}\gamma_2^{j_2+2j_3}. \tag{19}$$

Then n must even, and comparing (18) and (19) if we set $j_3+l_2-j_1=p$, we must have $2l_2+j_2+2j_3=2p$. This implies, $2j_3+2l_2-2j_1=2l_2+j_2+2j_3$ which yields $-2j_1=j_2$, i.e. $j_1=j_2=0$. Hence, if we denote $G_{l_2,l_2,0,0,j_3}=\tilde{G}_{l_2,,j_3}$ we get

$$\begin{aligned} f_0 &= \sum_{2l_2+2j_3=n}(-1)^{j_3+l_2}\tilde{G}_{l_2,j_3}(b_1(\xi_1^2-\gamma_2^2)+2b_2\gamma_2\xi_1-ib_3(\xi_1^2+\gamma_2^2))^{l_2}\xi_1^{-j_3-l_2}\gamma_2^{2j_3}\\ &= \frac{(-1)^{n/2}}{\xi_1^{n/2}}\sum_{l_2=0}^{n/2}\tilde{G}_{l_2,\frac{n-2l_2}{2}}(b_1(\xi_1^2-\gamma_2^2)+2b_2\gamma_2\xi_1-ib_3(\xi_1^2+\gamma_2^2))^{l_2}\gamma_2^{n-2l_2}. \end{aligned}$$

Comparing with (18) we obtain that if f_0 is not a constant then

$$(l_2^2\xi_1^2-2il_1l_2\xi_1\gamma_2-l_2^2\gamma_2^2)^{n/2}=\sum_{l_2=0}^{n/2}\bar{G}_{l_2,l_2,\frac{n-2l_2}{2}}(b_1(\xi_1^2-\gamma_2^2)+2b_2\gamma_2\xi_1-ib_3(\xi_1^2+\gamma_2^2))^{l_2}\gamma_2^{n-2l_2},$$

which is not possible (note that $b_1b_2b_3\neq 0$). This concludes the proof of the lemma. □

Lemma 4. *Let $l_1=l_2$ and f_j be a polynomial in the variable γ_2 and a rational function with respect to the variable ξ_1. Then condition $\mathcal{A}[f_j]=0$ implies*

$$f_j=\begin{cases}0, & \text{for } j\in\mathbb{Z}\setminus\{0\},\\ \text{constant}, & \text{for } j=0.\end{cases}$$

Proof. Let $l_1=l_2$ and $j\in\mathbb{Z}\setminus\{0\}$. From $\mathcal{A}[f_j]=0$ and after simplifying by $l_2(\xi_1-i\gamma_2)$ we have that

$$\mathcal{S}\Big[(\xi_1+i\gamma_2)\frac{\partial f_j}{\partial\gamma_2}+2i\xi_1\frac{\partial f_j}{\partial\xi_1}\Big]+2j\xi_1\big((ib_2-b_1)(\xi_1+i\gamma_2)+ib_3(\xi_1-i\gamma_2)\big)f_j=0.$$

Solving it we get

$$f_j=K_j\Big(\frac{x_2+ix_1}{\sqrt{x_1}}\Big)e^{-ij\tan^{-1}\left(\frac{-2b_2\xi_1^2+b_3(\xi_1-i\gamma_2)^2+2b_1\xi_1(\gamma_2+i\xi_1)}{2(b_2-b_3)\xi_1((i\xi_1+\gamma_2)+b_1(3\xi_1^2-2i\xi_1\gamma_2-\gamma_2^2))}\right)}$$
$$\frac{(5b_1+4ib_2-3ib_3)\xi_1^2+2(-2ib_1+b_2-2b_3)\xi_1\gamma_2-(b_1-ib_3)\gamma_2^2)}{\xi_1^2((b_1-ib_3)\xi_1^2+2b_2\xi_1\gamma_2-(b_1+ib_3)\gamma_2^2)^{j/2}}.$$

Since f_j must be a polynomial in the variable γ_2 and a rational function with respect to the variable ξ_1 and $b_1b_2b_3\neq 0$ we must have $f_j=0$. Now assume $j=0$. From $\mathcal{A}[f_0]=0$ after simplifying by $l_2\mathcal{S}(\xi_1-i\gamma_2)$, we have that

$$(\xi_1+i\gamma_2)\frac{\partial f_0}{\partial\gamma_2}+2i\xi_1\frac{\partial f_0}{\partial\xi_1}=0. \tag{20}$$

Solving it we get

$$f_0=K_0\Big(\frac{(\gamma_2+i\xi_1)^2}{\xi_1}\Big),$$

where K_j is any smooth function. Since f_0 must be a polynomial in the variable γ_2 and a rational function with respect to the variable ξ_1 we must have that K_0 is a polynomial. Proceeding as in the proof of Lemma 3 we get that either f_0 is constant or

$$(\gamma_2+i\xi_1)^n=\sum_{l_2=0}^{n/2}\bar{G}_{l_2,l_2,\frac{n-2l_2}{2}}(b_1(\xi_1^2-\gamma_2^2)+2b_2\gamma_2\xi_1-ib_3(\xi_1^2+\gamma_2^2))^{l_2}\gamma_2^{n-2l_2}.$$

But this last case is not possible and thus f_0 is constant. This concludes the proof of the lemma. ☐

Proof of Theorem 1. We first show by backwards induction that $G_j = 0$ for $j = -n, \ldots, -1$. For $j = -n$ we need to solve $\mathcal{A}[G_{-n}] = 0$, and it follows from Lemmas 3 and 4 that $G_{-n} = 0$. Then using that $\mathcal{B}[0] = 0$ by induction we obtain that for any $j = -n, \ldots, -1$, the coefficient of Ω_2^j in (12) satisfies that $\mathcal{A}[G_j] = 0$. Then the same arguments that we used for G_{-n} imply that G_j=0 for $j = -n, \ldots, -1$. Moreover, G_0 satisfies that $\mathcal{A}[G_0] = 0$ and again by Lemmas 3 and 4 we get that G_0 is a constant. Since we can always assume that G has no constant term, then $G_0 = 0$. Note that $\mathcal{B}[0] = 0$ and by induction we obtain that for any $j = 1, \ldots, n$, the coefficient of Ω_2^j in (12) satisfies $\mathcal{A}[G_j] = 0$. Again by Lemmas 3 and 4, we get $G_j = 0$ for $j = 1, \ldots, n$. This completes the proof of the theorem. ☐

4. Acknowledgment

The authors would like to thank Prof. Andrzej Maciejewski for pointing out the remaining case in the integrability of the Suslov problem.

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