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DARBOUX INTEGRABILITY OF A SIMPLIFIED FRIEDMAN-ROBERTSON-WALKER HAMILTONIAN SYSTEM

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ABSTRACT. We characterize the Darboux first integrals of a simplified Friedman–Robertson–Walker Hamiltonian system depending on one parameter.

1. Introduction and statement of the main result

Given a system of ordinary differential equations depending on parameters in general is very difficult to recognize for which values of the parameters the equations have first integrals because there are no satisfactory methods to answer this question.

In this paper we study the first integrals of the simplified Friedman–Robertson–Walker Hamiltonian differential system in $\mathbb{R}^4$

$$
\begin{aligned}
\dot{x} &= \quad -p_x, \\
\dot{y} &= \quad p_y, \\
\dot{p}_x &= \quad x - b\,xy^2, \\
\dot{p}_y &= \quad -y - b\,x^2 y,
\end{aligned} \tag{1}
$$

where $b \in \mathbb{R}$ is a parameter and the dot denotes derivative with respect to time t. The Hamiltonian of this system is

$$H_0 = \frac{1}{2}(p_y^2 - p_x^2) + \frac{1}{2}(y^2 - x^2) + \frac{b}{2}x^2y^2.$$

System (1) models a universe, filled by a conformally coupled but massive real scalar field, see [2] for more details where the authors present analytical and numerical evidence of the existence of chaotic motion. For a more general point of view on the Friedman-Robertson-Walker system see for instance the works [5, 7] and the references quoted therein.

The vector field associated to system (1) is

$$\mathcal{X} = -p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + x(1 - by^2)\frac{\partial}{\partial p_x} - y(1 + bx^2)\frac{\partial}{\partial p_y}.$$

Let $U \subset \mathbb{C}^4$ be an open set. We say that the non–constant function $H\colon U \to \mathbb{C}$ is a *first integral* of the polynomial vector field $\mathcal{X}$ on U if

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$H(x(t), y(t), p_x(t), p_y(t))$ is constant for all values of t for which the solution $(x(t), y(t), p_x(t), p_y(t))$ of $\mathcal{X}$ is defined on U. Clearly H is a first integral of $\mathcal{X}$ on U if and only if

$$\mathcal{X}H = -p_x\frac{\partial H}{\partial x} + p_y\frac{\partial H}{\partial y} + x(1-by^2)\frac{\partial H}{\partial p_x} - y(1+bx^2)\frac{\partial H}{\partial p_y} = 0$$

on U.

In this paper we want to study the so-called Darboux first integrals of the Friedman–Robertson–Walker polynomial differential systems (1), using the Darboux theory of integrability (originated in the papers [4]). For a present state of this theory see the Chapter 8 of [6], the paper [8], and the references quoted in them.

We emphasize that the study of the existence of first integrals is a classical problem in the theory of differential systems, because the knowledge of first integrals of a differential system can be very useful in order to understand and simplify the topological structure of their orbits. Thus, their existence or not can also be viewed as a measure of the complexity of a differential system.

We recall that a first integral is of *Darboux* type if it is of the form

$$f_1^{\lambda_1}\cdots f_p^{\lambda_p}F_1^{\mu_1}\cdots F_q^{\mu_q}, \tag{2}$$

where $f_1,\ldots,f_p$ are Darboux polynomials (see section 2 for a definition), $F_1,\ldots,F_q$ are exponential factors (see section 2 for a definition), and λ_j, $\mu_k\in\mathbb{C}$ for all j and k.

The functions of the form (2) are called *Darboux* functions, and they are the base of the Darboux theory of integrability, which looks when these functions are first integrals or integrating factors. In this last case, the first integrals associated to integrating factors given by Darboux functions are the Liouvillian first integrals, see for more details [6, 8].

The Darboux theory of integrability is essentially an algebraic theory of integrability based in the invariant algebraic hypersurfaces that a polynomial differential system has. In fact to every Darboux polynomial there is associated some invariant algebraic hypersurface (see again section 2), and the exponential factors appear when an invariant algebraic surface has multiplicity larger than 1, for more details see [3, 6, 8]. As far as we know is the unique theory of integrability which is developed for studying the first integrals of polynomial differential systems. In general the other theories of integrability do not need that the differential system be polynomial.

The main result of this paper is the following.

Theorem 1. *The unique first integrals of Darboux type of the Friedman–Robertson–Walker Hamiltonian system* (1) *with* $b\neq 0$ *are functions of Darboux type in the variable* H_0.

We prove Theorem 1 in section 3.

Since the Darboux theory of integrability of a polynomial differential system is based on the existence of Darboux polynomials and their multiplicity, the study of the existence or not of Darboux first integrals needs to look for the Darboux polynomials. These will be done in the next Theorems 7 and 8, whose results are the main steps for proving Theorem 1.

We must mention that any comment on the existence or non–existence of first integrals additional to the Hamiltonian itself appears in the reference [2]. In that paper the authors are mainly concerned with the chaotic behavior of the Friedman–Robertson–Walker polynomial Hamiltonian system. Of course, roughly speaking the chaotic motion is against the existence of first integrals.

The Hamiltonian system (1) is *completely integrable* if it has two independent first integrals H_1 and H_2 in involution. That is, H_1 and H_2 are *independent* if their gradients are linearly independent over a set of full Lebesgue measure in $\mathbb{C}^4$. Moreover, H_1 and H_2 are in *involution* if the Poisson bracket $\{H_i, H_0\} = 0$ for $i \in \{1, 2\}$.

Corollary 2. *The Friedman–Robertson–Walker Hamiltonian system* (1) *is completely integrable with first integrals of Darboux type if and only if* $b = 0$.

Corollary 2 is proved at the end of section 3.

2. Basic results

Let $h = h(x, y, p_x, p_y) \in \mathbb{C}[x, y, p_x, p_y] \setminus \mathbb{C}$. As usual $\mathbb{C}[x, y, p_x, p_y]$ denotes the ring of all complex polynomials in the variables x, y, p_x, p_y. We say that h is a *Darboux polynomial* of system (1) if it satisfies

$$\mathcal{X}h = Kh,$$

the polynomial $K = K(x, y, p_x, p_y) \in \mathbb{C}[x, y, p_x, p_y]$ is called *the cofactor* of h and has degree at most two. Every Darboux polynomial h defines an *invariant algebraic hypersurface* $h = 0$, i.e. if a trajectory of system (1) has a point in $h = 0$, then the whole trajectory is contained in $h = 0$, see for more details [6]. When $K = 0$ the Darboux polynomial h is a polynomial first integral.

An *exponential factor* E of system (1) is a function of the form $E = \exp(g/h) \notin \mathbb{C}$ with $g, h \in \mathbb{C}[x, y, p_X, p_y]$ satisfying $(g, h) = 1$ and

$$\mathcal{X}E = LE,$$

for some polynomial $L = L(x, y, p_x, p_y)$ of degree at most 2, called the *cofactor* of E.

A geometrical meaning of the notion of exponential factor is given by the next result.

Proposition 3. *If* $E = \exp(g/h)$ *is an exponential factor for the polynomial differential system* (1) *and* h *is not a constant polynomial, then* $h = 0$ *is* *an invariant algebraic hypersurface, and eventually* e^g *can be exponential factors, coming from the multiplicity of the infinite invariant hyperplane.*

The proof of Proposition 3 can be found in [3, 9]. We explain a little the last part of the statement of Proposition 3. If we extend to the projective space $\mathbb{PR}^4$ the polynomial differential system (1) defined in the affine space $\mathbb{R}^4$, then the hyperplane at infinity always is invariant by the flow of the extended differential system. Moreover, if this invariant hyperplane has multiplicity higher than 1, then it creates exponential factors of the form e^g, see for more details [9].

Theorem 4. *Suppose that the polynomial vector field* $\mathcal{X}$ *of degree* m *defined in* $\mathbb{C}^4$ *admits* p *invariant algebraic hypersurfaces* $f_i = 0$ *with cofactors* K_i, *for* $i = 1, \ldots, p$ *and* q *exponential factors* $E_j = \exp(g_j/h_j)$ *with cofactors* L_j, *for* $j = 1, \ldots, q$. *Then there exists* $\lambda_i, \mu_j \in \mathbb{C}$ *not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$$

if and only if the function of Darboux type

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} E_1^{\mu_1} \cdots E_q^{\mu_q}$$

is a first integral of $\mathcal{X}$.

Theorem 4 is proved in [6].

From Theorem 4 it follows easily the next well–known result.

Corollary 5. *The existence of a rational first integral for a polynomial differential system implies either the existence of a polynomial first integral or the existence of two Darboux polynomials with the same non–zero cofactor.*

The following result is well–known.

Lemma 6. *Assume that* $\exp(g_1/h_1), \ldots, \exp(g_r/h_r)$ *are exponential factors of some polynomial differential system*

$$x' = P(x, y, p_x, p_y),\ y' = Q(x, y, p_x, p_y),\ p_x' = R(x, y, p_x, p_y),\ p_y' = U(x, y, p_x, p_y) \tag{3}$$

with $P, Q, R, U \in \mathbb{C}[x, y, p_x, p_y]$ *with cofactors* L_j *for* $j = 1, \ldots, r$. *Then*

$$\exp(G) = \exp(g_1/h_1 + \cdots + g_r/h_r)$$

is also an exponential factor of system (3) *with cofactor* $L = \sum_{j=1}^{r} L_j$.

3. Proof of Theorem 1

According with section 2 for proving Theorem 1 we need to characterize the Darboux polynomials of system (1).

The following result characterizes the polynomial first integrals of system (1) with $b \neq 0$, i.e. it characterizes the Darboux polynomial with zero cofactor.

Theorem 7. *The unique polynomial first integrals of system* (1) *with* $b \neq 0$ *are polynomials in the variable* H_0.

Proof. Doing the change of variables

$$z_1 = p_x - p_y, \quad z_2 = p_x + p_y, \tag{4}$$

system (1) becomes

$$\begin{aligned} \dot{x} &= -\frac{1}{2}(z_1 + z_2), \\ \dot{y} &= -\frac{1}{2}(z_1 - z_2), \\ \dot{z}_1 &= x + y + bxy(x - y), \\ \dot{z}_2 &= x - y - bx(x + y), \end{aligned} \tag{5}$$

and H_0 writes as

$$H_0 = \frac{1}{2}(y^2 - x^2 - z_1 z_2 + bx^2y^2).$$

Now we restrict to $H_0 = 0$. We get that

$$z_2 = \frac{bx^2y^2 + y^2 - x^2}{z_1}. \tag{6}$$

Then system (5) on $H_0 = 0$ becomes, after the rescaling by $d\tau = z_1 dt$:

$$\begin{aligned} x' &= -\frac{1}{2}z_1^2 - \frac{1}{2}(bx^2y^2 + y^2 - x^2), \\ y' &= -\frac{1}{2}z_1^2 + \frac{1}{2}(bx^2y^2 + y^2 - x^2), \\ z_1' &= z_1(x + y + bxy(x - y)), \end{aligned} \tag{7}$$

where the prime denotes the derivative with respect to the variable τ. If $f = f(x, y, p_x, p_y)$ is a polynomial first integral of system (1) then, if we denote by $g = g(x, y, z_1, z_2)$ the polynomial first integral f written in the new variables (x, y, z_1, z_2), we have that g is also a polynomial first integral of the differential system (5). Furthermore, if we denote by $h = h(x, y, z_1)$ the polynomial first integral g restricted to the invariant hypersurface (6), then h is a rational first integral of the differential system (7) in the sense that it is the quotient of a polynomial in the variables (x, y, z_1) and a power of z_1. Furthermore h satisfies

$$\begin{aligned} &\Big(-\frac{1}{2}z_1^2 - \frac{1}{2}(bx^2y^2 + y^2 - x^2)\Big)\frac{\partial h}{\partial x} + \\ &\Big(-\frac{1}{2}z_1^2 + \frac{1}{2}(bx^2y^2 + y^2 - x^2)\Big)\frac{\partial h}{\partial y} + z_1(x + y + bxy(x - y))\frac{\partial h}{\partial z_1} = 0. \end{aligned} \tag{8}$$

We write

$$h = \sum_{j=0}^{n} h_j(x, y, z_1) \tag{9}$$

where each h_j is the quotient of a polynomial in the variable (x, y, z_1) with degree j and the denominator a power in the variable z_1. Here we define the degree of h_j as j. Hence, computing the terms of degree $n+4$ we have

$$\frac{1}{2}bx^2y^2\Big(\frac{\partial h_n}{\partial y}-\frac{\partial h_n}{\partial x}\Big)+z_1bxy(x-y)\frac{\partial h_n}{\partial z_1}=0.$$

We write it as

$$L[h_n]=0, \quad \text{where} \quad L=\frac{1}{2}bx^2y^2\Big(\frac{\partial}{\partial y}-\frac{\partial}{\partial x}\Big)+z_1bxy(x-y)\frac{\partial}{\partial z_1}. \tag{10}$$

The characteristic equations associated to the linear partial differential operator L are

$$\frac{dx}{dy}=-1, \quad \frac{dy}{dz_1}=\frac{xy}{2z_1(x-y)}.$$

The system has the general solution

$$x+y=d_1, \quad \frac{y^{2x}}{e^{2y}z_1^x}=d_2,$$

where d_1 and d_2 are constants of integration. According to this, we make the change of variables

$$u=x+y, \quad v=\frac{y^{2x}}{e^{2y}z_1^x}, \quad w=y. \tag{11}$$

Its inverse transformation is

$$x=u-w, \quad z=y^2\left(e^{2w}v\right)^{1/(w-u)}, \quad y=w. \tag{12}$$

Under the change of variables (11) and (12) equation (10) becomes the following ordinary differential equation (for fixed u and v):

$$\frac{b}{2}w^2(u-w)^2\frac{\partial \bar{h}_n}{\partial w}=0$$

where $\bar{h}_n$ is h_n written in the variables u, v and w. Solving it we have

$$\bar{h}_n=\bar{h}_n(u,v)=h_n\left(x+y,\frac{y^{2x}}{e^{2y}z_1^x}\right).$$

Since the numerator of h_n is a polynomial of degree n we must have

$$h_n=\sum_{k=0}^{n}\alpha_k(x+y)^k, \quad \alpha_k\in\mathbb{R}. \tag{13}$$

Now using the transformations (11) and (12) and working in a similar manner to solve $\bar{h}_n$, computing the terms of degree $n+3$, we get

$$h_{n-1}=\sum_{k=0}^{n-1}\beta_k(x+y)^k, \quad \beta_k\in\mathbb{R}. \tag{14}$$

Now, using the transformations (11) and (12) and working in a similar manner to solve $\bar{h}_{n-2}$, computing the terms of degree $n+2$, we get

$$\frac{b}{2}w^2(u-w)^2\frac{\partial \bar{h}_{n-2}}{\partial w} = w^4\left(e^{2w}v\right)^{2/(w-u)}\sum_{k=0}^{n} k\alpha_k u^{k-1}.$$

Integrating this equation we obtain that $\bar{h}_{n-2} = \bar{h}_{n-2}(u,v,w)$ is

$$\frac{1}{b}\left(\sum_{k=0}^{n} k\alpha_k u^{k-1}\right)\left(\frac{\left(e^{2w}v\right)^{2/(w-u)}\left(5u^2-8wu+4w^2+2(u-w)\log\left(e^{2w}v\right)\right)}{-2u+2w-\log\left(e^{2w}v\right)}\right.$$
$$\left.-4e^4\mathrm{Ei}\left(\frac{2\log\left(e^{2w}v\right)}{w-u}-4\right)\left(3u-2w+\log\left(e^{2w}v\right)\right)\right)+k_{n-2}(u,v),$$

where $Ei(z)$ is the exponential integral function, see for instance [1]. Going back to the variables x, y and z_1 we obtain that $h_{n-2}(x,y,z_1)$ is

$$\frac{1}{b}\left(\sum_{k=0}^{n} k\alpha_k (x+y)^{k-1}\right)\left(-\frac{\left(y^{2x}z_1^{-x}\right)^{-2/x}\left(5x^2+2yx+2\log\left(y^{2x}z_1^{-x}\right)x+y^2\right)}{2x+\log\left(y^{2x}z_1^{-x}\right)}-\right.$$
$$\left.4e^4\mathrm{Ei}\left(-\frac{2\log\left(y^{2x}z_1^{-x}\right)}{x}-4\right)\left(3x+y+\log\left(y^{2x}z_1^{-x}\right)\right)\right)+k_{n-2}\left(x+y,\frac{y^{2x}}{e^{2y}z_1^x}\right).$$

Since the numerator of $h_{n-2}(x,y,z_1)$ is a polynomial of degree $n-2$ we must have

$$\sum_{k=0}^{n} k\alpha_k (x+y)^{k-1} = 0.$$

So $\alpha_k = 0$ for $k = 1,\ldots,n$. Then $h_n = \alpha_0$, which implies that $h = \alpha_0/z_1^m$ for some nonnegative integer m. Then, from (8) we get

$$-\frac{m\alpha_0}{z_1^m}(x+y+bxy(x-y)) = 0.$$

Therefore $m\alpha_0 = 0$, and consequently $h =$ constant, in contradiction with the fact that h is a first integral. □

Now we characterize the existence of Darboux polynomials with non–zero cofactor when $b \neq 0$.

Theorem 8. *System* (1) *with* $b \neq 0$ *has no Darboux polynomial with non–zero cofactor.*

Proof. Let f be a Darboux polynomial with a non–zero cofactor K. Since K is a polynomial of degree at most 2, we write it as

$$K = a_0 + a_1x + a_2y + a_3p_x + a_4p_y + a_5x^2 + a_6xy + a_7xp_x + a_8xp_y + a_9y^2$$
$$+ a_{10}yp_x + a_{11}yp_y + a_{12}p_x^2 + a_{13}p_xp_y + a_{14}p_y^2.$$

Note that system (1) is invariant under the involution $\sigma\colon \mathbb{C}[x,y,p_x,p_y] \to \mathbb{C}[x,y,p_x,p_y]$ defined by

$$\sigma(x) = -x,\ \sigma(y) = -y,\ \sigma(p_x) = -p_x,\ \sigma(p_y) = -p_y.$$

If f is a Darboux polynomial of system (1) then $g = f \cdot \sigma(f)$ is a Darboux polynomial of system (1) invariant by σ and with cofactor

$$K_1 = K + \sigma(K) = 2(a_0 + a_5x^2 + a_6xy + a_7xp_x + a_8xp_y + a_9y^2 + a_{10}yp_x + a_{11}yp_y + a_{12}p_x^2 + a_{13}p_xp_y + a_{14}p_y^2).$$

Let f be a Darboux polynomial of system (1) with non-zero cofactor K. If f is invariant by σ then $K = K_1$ and we take $g = f$. If f is not invariant by σ then we consider $g = f \cdot \sigma(f)$ a new Darboux polynomial of system (1) invariant by σ and with cofactor K_1. We have

$$-p_x\frac{\partial g}{\partial x} + p_y\frac{\partial g}{\partial y} + x(1-by^2)\frac{\partial g}{\partial p_x} - y(1+bx^2)\frac{\partial g}{\partial p_y} = K_1g. \tag{15}$$

We write g as a polynomial in the variable p_x as

$$g = \sum_{i=0}^{k} g_i(x,y,p_y)p_x^j,$$

where each g_i is a polynomial in the variables x, y and p_y. Without loss of generality we can assume that $g_k(x,y,p_y) \neq 0$. Then computing the coefficient of p_x^{k+2} in (15) we get

$$2a_{12}g_k = 0 \quad \text{which yields} \quad a_{12} = 0.$$

Now computing the coefficient of p_x^{k+1} in (15) we obtain

$$-\frac{\partial g_k}{\partial x} = 2(a_7x + a_{10}y + a_{13}p_y)g_k.$$

Solving it we get

$$g_k = K_k(y,p_y)e^{a_7x^2+2a_{10}yx+2a_{13}p_yx}.$$

Since g_k must be a polynomial we get $a_7 = a_{10} = a_{13} = 0$.

Now we write g as a polynomial in the variable p_y as

$$g = \sum_{i=0}^{m} \bar{g}_i(x,y,p_x)p_y^j,$$

where each $\bar{g}_i$ is a polynomial in the variables x, y and p_x. Without loss of generality we can assume that $\bar{g}_m(x,y,p_x) \neq 0$. Then computing the coefficient of p_y^{m+2} in (15) we get

$$2a_{14}\bar{g}_m = 0 \quad \text{which yields} \quad a_{14} = 0.$$

Now computing the coefficient of p_y^{m+1} in (15) we obtain

$$\frac{\partial \bar{g}_m}{\partial y} = 2(a_8x + a_{11}y)\bar{g}_m.$$

Solving it we get

$$\bar{g}_m = K_m(x, p_x)e^{2a_8xy+a_{11}y^2}.$$

Since $\bar{g}_m$ must be a polynomial we get $a_8 = a_{11} = 0$. Hence $K_1 = 2(a_0 + a_5x^2 + a_6xy + a_9y^2)$. Then

$$-p_x\frac{\partial g}{\partial x}+p_y\frac{\partial g}{\partial y}+x(1-by^2)\frac{\partial g}{\partial p_x}-y(1+bx^2)\frac{\partial g}{\partial p_y} = 2(a_0+a_5x^2+a_6xy+a_9y^2)g.$$

Now we write $g = \sum_{j=0}^{\ell} g_j(x, y, p_x, p_y)$ where each g_j is a homogeneous polynomial of degree j. Then computing the terms of degree $\ell + 2$ we get

$$-bxy\left(y\frac{\partial g_\ell}{\partial p_x} + x\frac{\partial g_\ell}{\partial p_y}\right) = 2(a_5x^2 + a_6xy + a_9y^2)g_\ell.$$

Solving it we get

$$g_\ell = K\left(x, y, \frac{-xp_x + yp_y}{y}\right)e^{-\frac{2p_x(a_5x^2+a_6xy+a_9y^2)}{bxy^2}}.$$

Since it must be a polynomial we get $a_5 = a_6 = a_9 = 0$. Then $K_1 = 2a_0$.

Now proceeding as in the proof of Theorem 7 introducing the change of variables (4) and the restriction $H_0 = 0$ (see (7)) we get that $g(x, y, p_x, p_y) = h(x, y, z_1)$ with

(16)
$$\Big(-\frac{1}{2}z_1^2 - \frac{1}{2}(bx^2y^2 + y^2 - x^2)\Big)\frac{\partial h}{\partial x} + \Big(-\frac{1}{2}z_1^2 + \frac{1}{2}(bx^2y^2 + y^2 - x^2)\Big)\frac{\partial h}{\partial y}$$
$$+ z_1(x + y + bxy(x - y))\frac{\partial h}{\partial z_1} = 2a_0z_1h.$$

We write h as in (9) and we get h_n and h_{n-1} as in (13) and (14). Now, using the transformations (11) and (12) and working in a similar manner to solve $\bar{h}_{n-2}$, computing the terms of degree $n + 2$ in (16), we get

$$\begin{aligned}\frac{b}{2}w^2(u-w)^2\frac{\partial \bar{h}_{n-2}}{\partial w} = \;& w^4\left(e^{2w}v\right)^{2/(w-u)}\sum_{k=0}^{n} k\alpha_k u^{k-1} \\ & +2a_0w^2\left(e^{2w}v\right)^{1/(w-u)}\sum_{k=0}^{n}\alpha_k u^k.\end{aligned}$$

Integrating this equation and going back to the variables x, y and z_1 we get that $h_{n-2}(x,y,z_1)$ is

$$\frac{1}{b\left(-2y+2(x+y)+\log\left(y^{2x}z_1^{-x}\right)\right)}\Bigg(-4a_0\left(y^{2x}z_1^{-x}\right)^{-1/x}\sum_{k=0}^{n}\alpha_k(x+y)^k-$$
$$\Big(\left(4y^2-8(x+y)y+5(x+y)^2+2x\log\left(y^{2x}z_1^{-x}\right)\right)\left(y^{2x}z_1^{-x}\right)^{-2/x}+$$
$$4e^4\mathrm{Ei}\left(-\frac{2\log\left(y^{2x}z_1^{-x}\right)}{x}-4\right)\left(-2y+2(x+y)+\log\left(y^{2x}z_1^{-x}\right)\right)$$
$$\left(-2y+3(x+y)+\log\left(y^{2x}z_1^{-x}\right)\right)\Big)\sum_{k=0}^{n}k\alpha_k(x+y)^{k-1}\Bigg)+k_{n-2}\left(x+y,\frac{y^{2x}}{e^{2y}z_1^x}\right).$$

Since the numerator of h_{n-2} must be a polynomial of degree $n-2$ we have

$$\sum_{k=0}^{n}k\alpha_k(x+y)^{k-1}=0,\quad \text{and}\quad a_0\sum_{k=0}^{n}\alpha_k(x+y)^k=0.$$

This implies that $\alpha_k=0$ for $k=1,\ldots,n$ and $a_0\alpha_0=0$.

If $\alpha_0=0$ then $h=0$, in contradiction that h is a Darboux polynomial. Therefore $a_0=0$ and $K_1=0$. Thus $g=f\cdot\sigma(f)$ is a polynomial first integral. By Theorem 7 the unique polynomial first integrals are polynomials in the variable H_0, we must have that f is a polynomial in the variable H_0 which is not possible because f is a Darboux polynomial of system (1) with non–zero cofactor. This concludes the proof of the theorem. □

Theorem 9. *The unique rational first integrals of system* (1) *with* $b\neq 0$ *are rational functions in the variable* H_0.

Proof. It follows directly from Corollary 5 and Theorems 7 and 8. □

Now we proceed as in the proof of Theorem 8.

Proof of Theorem 1. It follows from Theorems 4, 7 and 8 and Proposition 3 that in order to have a first integral of Darboux type we must have q exponential factors $E_j=\exp(g_j/h_j(H_0))$ with cofactors L_j such that $\sum_{j=1}^q\mu_jL_j=0$. Let $G=\sum_{j=1}^q\mu_jg_j/h_j(H_0)$, then $E=\exp(G)$ is an exponential factor of system (1) with cofactor $L=\sum_{j=1}^q\mu_jL_j$ (see Lemma 6) and G satisfies that $\mathcal{X}G=0$, that is, G must be a rational first integral of system (1). By Theorem 9 it must be a rational function in the variable H_0. □

Proof of Corollary 2. When $b=0$ it easy to check that $x^2+p_x^2$ and $y^2+p_y^2$ are two independent first integrals in involution for the Hamiltonian system (1). Note that both first integrals are given by functions of Darboux type.

When $b\neq 0$ Theorem 1 shows that there exists a unique independent first integral of Darboux type. So, the corollary is proved. □

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