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Solitary Traveling Water Waves of Moderate Amplitude

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Abstract

We prove the existence of solitary traveling wave solutions for an equation describing the evolution of the free surface for waves of moderate amplitude in the shallow water regime. This non-linear third order partial differential equation arises as an approximation of the Euler equations, modeling the unidirectional propagation of surface water waves. We give a description of the solitary wave profiles by performing a phase plane analysis and study some qualitative features of the solutions.

1 Introduction

Ever since Scott Russell's first recorded observation of "the great wave of translation" [33], there has been growing interest in the study of solitary wave solutions of the equations for water waves. The existence theory for irrotational waves of small amplitude dates back to works of Krasovski, Lavrentiev and Ter-Krikorov [26, 27, 34], and was later improved by Friedrichs and Hyers [17], Beale [6] and Amick and Toland [4]. Although at the time no existence results for waves with arbitrary amplitude were available, Keady and Pritchard [24] proved that symmetric and monotone solitary wave solutions are necessarily waves of elevation which propagate at supercritical Froude number. It was shown by Amick and Toland [3] that such waves of elevation actually exist for all amplitudes from zero up to the solitary wave of greatest height and that they decay exponentially at infinity, under the assumption that the wave profile is symmetric and monotone from crest to trough. Craig and Sternberg [16] proved that any supercritical solitary wave solution is symmetric and decays monotonically to a constant on either side of the crest. More recently, results on existence, symmetry and regularity were obtained in the rotational case, cf. [20, 29], and the flow beneath an irrotational solitary wave was investigated in [10] and [11]. In parallel with the aforementioned research on the exact water wave problem, the past fifty years have seen a resurgence in interest in approximate model equations. With the appearence of integrable equations like the KdV [25] or Camassa–Holm [15], whose solitary wave solutions are in fact solitons (cf. [31, 18, 1, 7, 22]), approximations to the full governing equations received renewed attention from the mathematical community. These same equations have long been a staple of the applied ocean sciences, with myriad applications in coastal engineering and tsunami modelling, cf. [12]. Many results have been obtained for waves of small amplitude, but it is also interesting and important to look at larger amplitude waves. Departing from an equation first derived by Johnson in [22], which at a certain depth below the fluid surface is a Camassa–Holm equation, one can derive a corresponding equation for the free surface valid for waves of moderate amplitude in the shallow water regime. Constantin and Lannes discuss large-time well-posedness of this equation in [13] and prove existence and uniqueness of solutions on some maximal time interval, also showing that that singularites can develop only in the form of wave breaking. To our knowledge, not a great deal is known so far about global solutions. In the present paper we prove existence of solitary traveling wave solutions for this equation and provide some qualitative features of the wave profile including symmetry, exponential decay at infinity and the fact that the profile has a unique crest point.

2 Preliminaries

Our mathematical model is based on a number of simplifying assumptions regarding the fluid and the physical quantities that play a role in the equations of motion. We assume that the water is inviscid, incompressible and that it has constant density. Furthermore we restrict our attention to gravity water waves, meaning that the only external force relevant to the propagation of the waves is due to the gravitational acceleration g. Concerning the fluid domain Ω, our analysis is valid for fluid flows over a flat bed at depth $y = -h_0$ that extends to infinity in both horizontal directions. The fluid domain is bounded from above by the one dimensional free surface which describes the elevation of the wave above the bed by means of $\eta(x, t)$, a function of space x and time t. We denote the fluid velocity field by (u, v) and impose the additional assumption that the flow is irrotational. The equations governing the motion of the fluid are taken

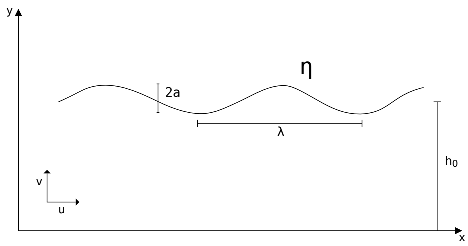

Figure 1: The fluid domain Ω for one dimensional surface waves.

to be Euler's equations (1), which arise from Newton's second law of motion. Furthermore, based on the assumption that mass is neither generated nor destroyed anywhere in the fluid, we employ the equation of mass conservation which reduces to (2) below, because the density is constant. Due to the fact that we consider irrotational flows the curl of the velocity field (i.e. the vorticity) must be zero, which leads to relation (3). In addition to these equations of motion we impose suitable conditions on the surface and the bottom of the fluid. The kinematic boundary conditions express the fact that the boundaries of the fluid domain behave like surfaces which move with the fluid. This ensures that there is no flow through the bottom and that fluid particles do not leave the fluid domain through the surface. The dynamic boundary condition decouples the motion of air from that of water by setting the atmospheric pressure equal to a constant, which is reasonable since the density of air is very small compared to that of water (see [8] and [28] for a justification of the assumptions on inviscid homogeneous fluid flows for gravity water waves). In what follows we will be interested mostly in approximations of the full governing equations. To this end, the variables are nondimensionalized and scaled using appropriate reference quantities (cf. [21] for a detailed discussion). The resulting system of equations valid in the fluid domain $\Omega = \{(x,y) | -h_0 < y < \eta(x,t)\}$ reads in nondimensionalized and scaled form

$$\left.\begin{aligned} u_t + \varepsilon(uu_x + vu_y) &= -P_x, \\ \delta^2\Big(v_t + \varepsilon(uv_x + vv_y)\Big) &= -P_y, \end{aligned}\right\} \quad \text{in } \Omega, \tag{1}$$

$$u_x + v_y = 0 \quad \text{in } \Omega, \tag{2}$$

$$u_y - \delta^2 v_x = 0 \quad \text{in } \Omega, \tag{3}$$

with boundary conditions

$$\begin{aligned} P = \eta, \quad v = \eta_t + \varepsilon u \eta_x \quad &\text{on } y = \varepsilon\eta, \\ v = 0 \quad &\text{on } y = -1, \end{aligned}$$

where P is the nondimensional pressure relative to the hydrostatic pressure distribution. The so called amplitude and shallowness parameters

$$\varepsilon = \frac{a}{h_0}, \qquad \delta = \frac{h_0}{\lambda},$$

appearing in this formulation arise naturally in the process of nondimensionalization and relate the average wave length λ, amplitude a and water depth h_0. They characterize various physical regimes in which simplified equations can be derived by means of asymptotic expansions in terms of ε and δ. The resulting model equations serve as a basis to construct approximate solutions to the full governing equations which, under certain circumstances, still model accurately the type of waves of interest. In the following we shall focus on the long-wave limit, or shallow water regime, where $\delta \ll 1$, and are concerned with waves of moderate amplitude, characterized by $\varepsilon = O(\delta)$.

3 An equation for waves of moderate amplitude in shallow water

In the shallow water regime, one can derive the Green–Naghdi equations (cf. [19]),

$$\begin{cases} \eta_t + [(1+\varepsilon\eta)u]_x = 0, \\ u_t + \eta_x + \varepsilon u u_x = \dfrac{\delta^2}{3}\dfrac{1}{1+\varepsilon\eta}\Big[(1+\varepsilon\eta)^3\left(u_{xt} + \varepsilon u u_{xx} - \varepsilon u_x^2\right)\Big]_x, \end{cases}$$

which couple the free surface η to the vertically averaged horizontal velocity component

$$u(x,t) = \frac{1}{1+\varepsilon\eta(x,t)} \int_{-1}^{\varepsilon\eta(x,t)} u(x,y,t)\,dy.$$

For one dimensional surface waves propagating over a flat bed, this set of equations provides an approximation to the Euler equations up to terms of order δ^2, cf. [2]. Under the additional assumption that $\varepsilon = O(\delta)$, one can study an equation for the velocity, the Johnson equation

$$u_t + u_x + \frac{3}{2}\varepsilon u u_x + \frac{\delta^2}{12}u_{xxx} - \frac{\delta^2}{12}u_{xxt} + \frac{7\varepsilon\delta^2}{24}u u_{xxx} + \frac{4\varepsilon\delta^2}{3}u_x u_{xx} = 0, \tag{4}$$

which was first derived in [22] by means of asymptotic expansions in terms of ε and δ. Constantin and Lannes showed in [13] that, defining η in terms of u by an expression which arises in the asymptotic derivation of (4),

$$\eta = u + \frac{\varepsilon}{4}u^2 - \frac{\delta^2}{6}u_{xx} - \frac{5\varepsilon\delta^2}{12}u u_{xx} - \frac{17\varepsilon\delta^2}{48}u_x^2, \tag{5}$$

a solution u of the Johnson equation satisfies the Green–Naghdi equations up to terms of order δ^4, providing thus a good approximation to the governing equations for water waves. The Johnson equation actually belongs to a wider family of equations of this type, none of which is integrable unless the averaged horizontal velocity component is replaced by the horizontal velocity evaluated at a specific depth in the fluid domain,

$$u_\rho(x,t) = \partial_x \Phi(x,y,t)\Big|_{y=(1+\varepsilon\eta)\rho-1},$$

where Φ is the velocity potential associated with the irrotational velocity field. Precisely at the level line $\rho = \frac{1}{\sqrt{2}}$ the Johnson equation turns out to be a Camassa–Holm equation and is therefore integrable, cf. [22, 13, 15]. However, all of these equations describe the evolution of the velocity at a certain depth below the water surface and, unlike the case of model equations like the Korteweg–de Vries equation [25], they are not identical to the equation for the free surface. Using the expression (5) in Johnson's equation (4), one can derive a corresponding evolution equation for the free surface of waves of moderate amplitude in the shallow water regime,

$$\eta_t + \eta_x + \frac{3}{2}\varepsilon\eta\eta_x - \frac{3}{8}\varepsilon^2\eta^2\eta_x + \frac{3}{16}\varepsilon^3\eta^3\eta_x + \frac{\delta^2}{12}\eta_{xxx} - \frac{\delta^2}{12}\eta_{xxt} \tag{6}$$

$$+\frac{7\varepsilon\delta^2}{24}\Big(\eta\eta_{xxx} + 2\eta_x\eta_{xx}\Big) = 0. \tag{7}$$

Large-time well-posedness results were obtained for this equation in [13] using a semi-group approach due to Kato, cf. [23]. It is shown that for any initial data $\eta_0 \in H^3(\mathbb{R})$ and a maximal existence time $t_m > 0$ there exists a unique solution $\eta \in \mathcal{C}(H^3(\mathbb{R}); [0,t_m)) \cap \mathcal{C}^1(H^2(\mathbb{R}); [0,t_m))$ which depends continuously on initial data. Furthermore it is proved that if the maximal existence time is finite blow-up occurs in the form of wave breaking, e.g. the wave profile remains bounded but its slope becomes unbounded as t approaches t_m (cf. [9, 35] for discussions of this phenomenon). Proving results on global solutions seems to be quite an intricate task as the third order partial differential equation (6) contains non linear terms of high order. The aim of this paper is to prove existence of solitary traveling wave solutions of (6) by performing a phase plane analysis of the corresponding system of ordinary differential equations and providing a qualitative description of the wave profile.

4 Existence of solitary traveling wave solutions

Traveling wave solutions have the property that wave profiles propagate at constant speed $c > 0$ without changing their shape. Defining characteristic variables and scaling out the amplitude and shallowness parameters by means of the transformation

$$\tau = \frac{1}{\delta}\,(x - ct), \quad \eta(\tau) = \frac{\varepsilon}{2}\,\eta(x,t), \tag{8}$$

we transform (6) into the ordinary differential equation

$$\eta'(1-c) + 3\eta\eta' - \frac{3}{2}\eta^2\eta' + \frac{3}{2}\eta^3\eta' + \frac{1+c}{12}\eta''' + \frac{7}{12}\Big(\eta\eta''' + 2\eta'\eta''\Big) = 0,$$

which, upon integration with respect to τ, yields

$$12(1-c)\eta + 18\eta^2 - 6\eta^3 + \frac{9}{2}\eta^4 + (1+c)\eta'' + \frac{7}{2}\Big((\eta')^2 + 2\eta\eta''\Big) + C = 0, \tag{9}$$

where C is a constant of integration. Among all the traveling wave solutions of (6) we shall focus on solutions which have the additional property that the waves are localized and that η and its derivatives decay at infinity, that is,

$$\eta^{(n)}(\tau) \to 0 \quad \text{as } |\tau| \to \infty, \text{ for } n \in \mathbb{N}. \tag{10}$$

Under this decay assumption the constant of integration in (9) vanishes and we can conveniently rewrite it as the planar autonomous system

$$\begin{cases} \eta' = \zeta, \\ \zeta' = \dfrac{12(c-1)\eta - 18\eta^2 + 6\eta^3 - 9/2\eta^4 - 7/2\zeta^2}{1 + c + 7\eta}. \end{cases} \tag{11}$$

Our goal is to determine a homoclinic orbit in the phase plane starting and ending in $(0,0)$ which corresponds to a solitary traveling wave solution of (6). The existence of such an orbit depends on the parameter c, the wave speed. In accordance with the results in [24, 3, 16, 30], it appears only for $c > 1$ which reflects the fact that solitary waves travel at supercritical speed with Froude number greater than one. We start our analysis by determining the critical points of (11), that is, points where $(\eta', \zeta') = (0,0)$. After linearizing the system in the vicinity of those points to determine the local behaviour, we prove existence of a homoclinic orbit by analyzing the phase plane.

System (11) has at most two critical points: one at the origin, $P_0 = (0,0)$, and one given by $P_c = (\eta_c, 0)$, where η_c is the unique real root of the polynomial

$$p(\eta) = 12(c-1) - 18\eta + 6\eta^2 - 9/2\eta^3. \tag{12}$$

Indeed, one can show that the discriminant of $p(\eta)$ is always negative, so there are no multiple roots. Furthermore, its derivative $p'(\eta) = -27/2\eta^2 + 12\eta - 18$ has no real roots, so $p(\eta)$ has no (local) extrema and therefore has exactly one real root. Since the highest coefficient is negative and the constant term is a multiple of $(c-1)$, the real root η_c is positive for $c > 1$ and negative for $c < 1$.

When $c = 1$, the root is zero in which case the two fixed points coincide at the origin. Hence, only for $c > 1$ both fixed points lie in the right half-plane where $\eta > 0$ and we expect physically relevant solitary waves of elevation. To explicitly determine η_c (cf. also Figure 3 in Section 5) one can use Cardano's formula for third order polynomials to find that

$$\eta_c = \frac{1}{9}\Big(f(c)^{1/3} - 92 f(c)^{-1/3} + 4\Big), \tag{13}$$

where

$$f(c) = -1556 + 972c + 36\sqrt{2469 - 2334c + 729c^2}.$$

To linearize the system near its critical points we compute the Jacobian Matrix J of (11) and evaluate it at P_0 and P_c. All fixed points lie on the horizontal axis of the phase plane, so the Jacobian takes the form

$$J = \begin{pmatrix} 0 & 1 \\ J_c & 0 \end{pmatrix}, \text{ where } J_c = \partial_\eta \zeta'.$$

Since the trace of J is zero, all eigenvalues at the critical points are of the form

$$\lambda^\pm = \pm\sqrt{J_c},$$

and the behaviour of the system in the vicinity of the fixed points depends on the sign of J_c. At P_0 we find that

$$J_c\Big|_{(0,0)} = \frac{12(c-1)}{c+1}\begin{cases} < 0 & \text{if } c < 1, \\ = 0 & \text{if } c = 1, \\ > 0 & \text{if } c > 1, \end{cases}$$

so the eigenvalues of J at the origin are $\lambda_0^\pm = \pm\sqrt{\frac{12(c-1)}{c+1}}$. When $c > 1$, we get two distinct real eigenvalues of opposing sign and hence P_0 is a saddle point for the linearized system. For $c < 1$, the number J_c is negative, so the eigenvalues are purely imaginary and hence P_0 is a center for the linearized system. Evaluating J_c at the other critical point $P_c = (\eta_c, 0)$ we find that

$$J_c\Big|_{(\eta_c,0)} = \frac{p'(\eta_c)\,\eta_c}{1 + c + 7\eta_c}\begin{cases} > 0 & \text{if } c < 1, \\ = 0 & \text{if } c = 1, \\ < 0 & \text{if } c > 1, \end{cases}$$

where $p(\eta)$ was defined in (12) and has η_c as its unique real root. Indeed, since p' has no real roots and is negative in zero it is always negative, hence $J_c(\eta_c, 0) < 0$ if and only if $\eta_c > 0$ which holds true whenever $c > 1$. The other case follows by the same argument. Hence, the two fixed points P_0 and P_c exchange stability as c passes from less than 1 to greater than 1. Important for our analysis is the fact that only when $c > 1$, the fixed point P_c lies in the right half-plane where $\eta > 0$. In this case, we can hope to find a homoclinic orbit emerging and returning to the origin since P_c is a center whereas P_0 is a saddle point for the linearized system. Observe that, since J_c is non zero whenever $c \neq 1$, both fixed points are hyperbolic which means that a (topological) saddle point for the linearized system remains a saddle also for the non-linear system (cf. [32], p.140). Since (11) is symmetric with respect to the horizontal axis, i.e.

invariant under the transformation $(t, \zeta) \mapsto (-t, -\zeta)$, a linear center remains a center for the non-linear system (cf. [32], p.144). When $c = 1$ the two fixed points coincide at the origin and the Jacobian evaluated at P_0 reduces to

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{14}$$

which is a nilpotent matrix, so $P_0 = P_c$ is a non-hyperbolic fixed point. In particular, J has two zero eigenvalues in which case one can show, using an approach described in [5], that the origin is a degenerate equilibrium state, i.e. a cusp[1].

To prove existence of a homoclinic orbit we look for a solution of (11) which leaves the saddle point P_0 in the direction of the unstable Eigenspace spanned by the eigenvector $(1, \lambda_0^+)$, encircles the center fixed point P_c and returns to the fixed point at the origin. Such a solution exists for all times, because the right hand side of (11) is analytic for $\eta > -(1+c)/7$. Since $\eta' > 0$ in the upper half-plane, a solution which leaves the origin in the direction $(1, \lambda_0^+)$ goes to the right and eventually has to cross the vertical line where $\eta = \eta_c$, because ζ' is bounded from above. Indeed,

$$\zeta' = \frac{\eta\, p(\eta) - 7/2\zeta^2}{1 + c + 7\eta} \leq \eta\, p(\eta) - 7/2\zeta^2 \leq P_m,$$

where P_m is the unique maximum of $\eta\, p(\eta)$, with $p(\eta)$ defined in (12). Then the solution goes down and to the right, since $\eta' = \zeta > 0$ and $\zeta' < 0$ whenever $\eta > \eta_c$. Consequently, it has to cross the horizontal axis since otherwise, assuming that ζ tends monotonically to a constant this would imply $\zeta' \to -\infty$ in view of (11), which is a contradiction. Once the solution has crossed the horizontal axis, it returns to the origin in the same way in the lower half-plane by symmetry. The solution cannot return to the origin without encircling the fixed point P_c since in this case, P_0 would have an elliptic sector but we already showed that it is a saddle point (i.e. there are four hyperbolic sectors plus separatrices in the neighbourhood of P_c, see [5]). This concludes the proof of existence of a homoclinic orbit starting and ending in the origin which corresponds to a solitary traveling wave solution of (6), cf. Figure 2.

Remark: For $c < 1$ and as long as $\eta_c > -(1+c)/7$, the fixed point at the origin is a center and $P_c = (\eta_c, 0)$ a saddle point which now lies in the left half-plane of the phase space, so we could still hope for the existence of a homoclinic orbit

[1]It is shown in Theorem 67 on p.362 of [5] that if a system can be put in the form

$$\begin{cases} \eta' = \zeta, \\ \zeta' = a_k \eta^k [1 + h(\eta)] + b_n \eta^n \zeta [1 + g(\eta)] + \zeta^2 f(\eta, \zeta). \end{cases}$$

where $h(\eta), g(\eta)$ and $f(\eta, \zeta)$ are analytic in a neighbourhood of the origin, $h(0) = g(0) = 0, k \geq 2, a_k \neq 0$ and $n \geq 1$, then, if $k = 2m$, $m \geq 1$ and $b_n = 0$, the equilibrium state is degenerate. In our case when $c = 1$, we can write the system (11) as

$$\begin{cases} \eta' = \zeta, \\ \zeta' = -9\eta^2 [1 + h(\eta)] - \frac{7}{2(2+7\eta)} \zeta^2, \end{cases}$$

where $h(\eta) = -23/6\eta + O(\eta^2)$, which satisfies the assumptions of the theorem. Hence we infer that the origin is degenerate and that the neighbourhood of zero consists of the union of two hyperbolic sectors and two separatrices.

emerging from $(\eta_c, 0)$, which would correspond to a solitary wave solution of (6) with negative water level far out. This contradicts, however, the fact that we assumed η to decay to zero at infinity.

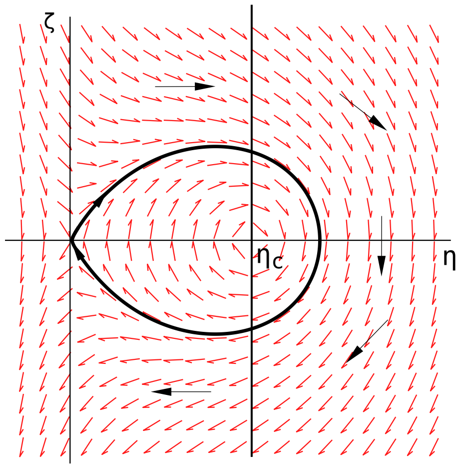

Figure 2: Phase portrait of system (11) for $c = 2$ with a homoclinic orbit emerging from the origin which corresponds to a solitary wave solution of the equation for surface waves of moderate amplitude in the shallow water regime.

5 Qualitative study of solitary traveling wave solutions

Despite the fact that we are not able to explicitly solve (11) we can work out certain features of its solitary traveling wave solutions for $c > 1$ along the lines of ideas in [14], to qualitatively describe the wave profile. Let η be a solitary traveling wave solution of (11). We claim that η has a single maximum. To this end, multiply equation (9) by η' and integrate, using the decay assumption (10). This gives

$$6(1-c)\eta^2 + 6\eta^3 - \frac{3}{2}\eta^4 + \frac{9}{10}\eta^5 + \frac{1+c}{2}\,[\eta']^2 + \frac{7}{2}\eta\,[\eta']^2 = 0,$$

which we rewrite as

$$[\eta']^2 = \eta^2 \frac{m(\eta)}{1+c+7\eta}, \tag{15}$$

where

$$m(\eta) = 12(c-1) - 12\eta + 3\eta^2 - 9/5\eta^3. \tag{16}$$

The discriminant of this third order polynomial is always negative, so there are no multiple roots. Furthermore, its derivative $m'(\eta)$ has no real roots, so $m(\eta)$ has no local extremum. Hence, $m(\eta)$ has only one real root which is positive since the highest coefficient is negative and the constant term is positive for $c > 1$. We conclude that η' vanishes precisely at the unique real root η_m of $m(\eta)$, so there exists a unique maximal wave height. Furthermore, this value

η_m is attained at a single value of τ. To see this, assume to the contrary that there exists an interval (τ_1, τ_2) with $\eta(\tau) = \eta_m$ for all $\tau \in (\tau_1, \tau_2)$. Hence, η_m would have to satisfy equation (9) in that interval, which reduces to

$$12(1-c) + 18\eta_m - 6\eta_m^2 + 9/2\eta_m^3 = 0.$$

On the other hand, since η_m is a root of $m(\eta)$, it must also satisfy the equation

$$12(1-c) + 12\eta_m - 3\eta_m^2 + 9/5\eta_m^3 = 0,$$

but this is not possible. Also, assuming that there are two distinct and isolated values τ_1, τ_2 with $\eta(\tau_1) = \eta(\tau_2) = \eta_m$, there must be another maximum or minimum between these points since η decays for $|\tau| \to \infty$. Hence, there exists $\tau_0 \in (\tau_1, \tau_2)$ where $\eta'(\tau_0) = 0$ but $\eta(\tau_0) \neq \eta_m$ which contradicts the fact that $m(\eta)$ has a unique real root. To determine η_m, one can use again Cardano's formula to find that

$$\eta_m = \frac{5}{9} - \frac{155}{81} r(c)^{-1/3} + r(c)^{1/3}, \tag{17}$$

where

$$r(c) = 5/729(-731 + 486c + 18\sqrt{3(703 - 731c + 243c^2)}).$$

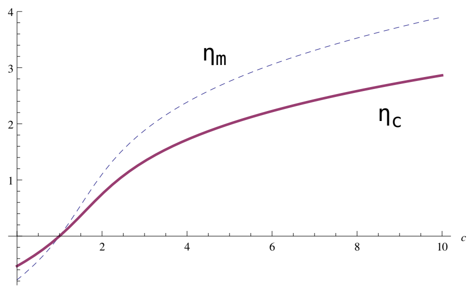

Figure 3: The x-coordinate η_c of the critical point $P_c = (\eta_c, 0)$ and the maximum height η_m of the wave profile as a function of the wave speed c.

It is important to notice that the maximal height of the wave η_m is an increasing function of the wave speed, cf. Figure 3, which means that higher waves travel faster. To see this, recall that η_m is the positive real root of the polynomial $m(\eta)$ which displays a dependence on c only in the constant term. Hence, since for $c_1 < c_2$ we have $m(\eta, c_1) - m(\eta, c_2) = 12(c_1 - c_2) < 0$, the graph of $m(\eta)$ is shifted upwards if we increase c and the zero of $m(\eta)$ is shifted to the right. We will use the fact that η_m grows with c in the comparison of solitary wave profiles of different speeds below.

We claim that the wave profile is symmetric with respect to the vertical axis, that is, we have to show that $\eta(\tau)$ is an even function of τ. To get a heuristic idea of this statement, recall (15) and regard η' as a function of η. This relation ensures that for every height of the profile η we get two values for the steepness of the wave at that point which only differ by sign. Hence the wave cannot be steeper on one side of the crest than on the other at the same height above the bed. To make this more precise, fix $c > 0$ and let η be a solution of (9) with crestpoint at $\tau = 0$. Define the function

$$\tilde{\eta}(\tau) = \begin{cases} \eta(\tau) & \text{for } \tau \in (-\infty, 0], \\ \eta(-\tau) & \text{for } \tau \in [0, \infty), \end{cases} \tag{18}$$

which is even by construction. On $[0, \infty)$, both η and $\tilde{\eta}$ satisfy (9) and since $\eta, \tilde{\eta} \in \mathcal{C}^1$ in $\tau = 0$, there exists a unique solution to the right of zero. Hence, $\eta = \tilde{\eta}$ on $[0, \infty)$ and in particular $\eta(\tau) = \tilde{\eta}(\tau) = \eta(-\tau)$ on $[0, \infty)$ which proves the claim.

The solitary wave profile decays exponentially at infinity, which can be seen by performing a Taylor expansion of the right hand side of (15) in η around zero. This yields $(\eta')^2 = 12\frac{c-1}{c+1}\eta^2 + O(\eta^3)$ for $|\tau| \to \infty$ and we conclude that

$$\eta(\tau) = O\left(\exp\left(-\sqrt{12\frac{c-1}{c+1}}|\tau|\right)\right) \text{ as } |\tau| \to \infty. \tag{19}$$

Note that the decay rate at infinity is given by the eigenvalue λ_0^- of the Jacobian matrix at the fixed point P_0, which determines the angle at which the homoclinic orbit corresponding to the solitary wave solution leaves the origin.

It is also interesting to investigate variations of the wave profile η upon changing the wave speed c. We show that two wave profiles with different speeds intersect precisely in two points. Indeed, let η be a solitary solution of (11) with crest point at $\tau = 0$. Since our system displays an analytic dependence on the parameter c, so does its solution and we can define the even function $f(\tau) = \partial_c \eta(\tau)$ for which we claim that it has precisely two zeros on $\mathbb{R}$. At $\tau = 0$, f is positive since the maximal height $\eta(0) = \eta_m$ is an increasing function of the wave speed c. Furthermore, notice that the decay rate of η at infinity is faster for larger c, since differentiating (19) with respect to c yields

$$\partial_c \eta \approx -\frac{12|\tau|}{\sqrt{12\frac{c-1}{c+1}}} \frac{1}{(1+c)^2}\eta < 0 \text{ for } |\tau| \to \infty,$$

so f is negative for large $|\tau|$. Moreover we show that the graph of f is decreasing whenever it crosses the horizontal axis, i.e. if there exists $\tau_0 > 0$ such that $f(\tau_0) = 0$, then $f'(\tau_0) < 0$. Indeed, differentiating (15) with respect to c gives

$$(1+7f)(\eta')^2 + (2+2c+14\eta)\,\eta'\,f' = 2\eta\,f\,m(\eta) + \eta^2 f\,(-27/5\eta^2 + 6\eta - 12) + 12\eta^2,$$

where $'$ denotes the derivative with respect to τ and we used that $\partial_c \eta' = f'$. Evaluating this equation at τ_0, so that all the terms involving f disappear, we find that

$$(2 + 2c + 14\eta)\,\eta'\,f' = 12\eta^2 - (\eta')^2 > 0.$$

The fact that the right hand side is positive follows from (15), noting that $\frac{m(\eta)}{1+c+7\eta} < 12$[2]. Since the wave profile is decreasing to the right of the crest

[2] This inequality follows by showing that $-9/5\eta^3 + 3\eta^2 - 96\eta - 24 < 0$, which amounts to proving that the polynomial has only one negative real root and then, since the highest order coefficient is negative, the left hand side is negative for all $\eta > 0$.

point, $\eta'(\tau_0) < 0$ and it follows from the above inequality that also $f'(\tau_0) < 0$ which proves the claim. For the solitary wave solutions of (11) this means that, if we fix a wave speed c_1 and find for the corresponding wave profile $\eta(\tau, c_1)$ the unique value τ_0 at which f vanishes, then a wave profile $\eta(\tau, c_2)$ corresponding to a higher speed $c_2 > c_1$ lies above the original wave profile $\eta(\tau, c_1)$ to the left of τ_0 where f is positive, and below $\eta(\tau, c_1)$ to the right of τ_0 where f is negative. Since the wave profiles are symmetric with respect to the vertical axis, the same is true on the other side of the crest point, cf. Figure 4.

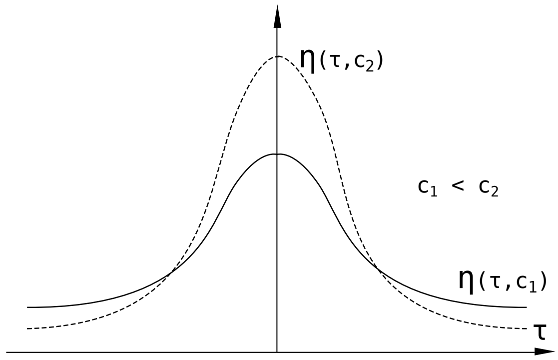

Figure 4: Solitary wave profiles with two different speeds.

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