

A Note on Uniqueness and Compact Support of Solutions in a recent Model for Tsunami Background Flows

Anna Geyer

Abstract

We present an elementary proof of uniqueness for solutions of an initial value problem which is not Lipschitz continuous, generalizing a technique employed in [20]. This approach can be applied for a wide class of vorticity functions in the context of [6], where, departing from a recent model for the evolution of tsunami waves developed in [10], the possibility of modelling background flows with isolated regions of vorticity is rigorously established.

1 Introduction

Tsunami waves are gravity water waves mostly generated by undersea earthquakes, cf. [2], which cause a vertical displacement of the entire column of water above the fault region, thus giving the tsunami its initial wave profile. The waves then propagate over large distances without essentially changing their shape, a characteristic feature observable for example in the May 1960 tsunami that set out off the coast of Chile and travelled almost $17000km$ across the Pacific Ocean until it hit Japan (cf. [8, 21, 22]). Tsunami waves travel at very high, almost constant speed and their wave length is typically hundreds of kilometers long whereas their amplitude is relatively small (about $0.5m$, cf. [21]). While out in the open ocean, where the water depth is relatively uniform (eg. the Central Pacific Basin is approximately $4.3km$ deep, cf. [8]), the evolution of a tsunami is essentially governed by linear theory, the typical wave speed being \sqrt{gh} , where g is the gravitational acceleration and h the average water depth, cf. [11]. When a tsunami approaches the shoreline, the front of the wave slows down as the depth decreases, causing the water to pile up vertically near the coast since the back of the wave is still out in the open ocean travelling at very high speed. The resulting damage by surging water and inundation is often far more devastating than the effects of the preceding earthquake itself. In the case of the tsunami that hit Japan on March 11, 2011, an undersea megathrust earthquake of magnitude 9 occurred in the region where the pacific plate is subducting under the plate beneath the Japanese island of Honshu, as reported by the U.S. Geological Survey. When the stresses that had been building up in this process were finally released, the break caused the sea floor to rise by several meters in a rupture zone $300km$ long and $150km$ wide, with the epicenter about $70km$ off the coast of the island of Honshu. The earthquake resulted in a major tsunami which devastated entire towns along the pacific coast of northern Japan, inundating a

total area of approximately $500km^2$ and causing the loss of thousands of lives. The tsunami not only hit the coast of Japan but also travelled across the Pacific Ocean reaching as far as the coast of Chile, about $17,000km$ from the epicenter of the earthquake.

Although it is beyond the scope of mathematical analysis to predict such catastrophes, this recent tsunami has reminded the world once again of how dangerous the destructive forces of nature can be to human lives. To obtain a better understanding of such phenomena, several models have been developed which describe the time evolution of tsunamis after the wave has obtained its initial shape. The equations governing the behaviour of these very long waves can be derived from Euler's equations and the equation of mass conservation under some general assumptions on the water, the oceans bathymetry and the type of water wave under consideration. The resulting system of equation with suitable boundary conditions provides a general model for tsunami waves which applies not only in the situation of the recent Japan tsunami, but also for the 2004 boxing day tsunami and the Chilean tsunami of 1960, cf. [14, 9, 4, 8, 21]. Most investigations of tsunami waves do not take into account the various states the ocean might exhibit prior to the arrival of waves near the shore, that is, the models are restricted to irrotational flows, which model background states of still water, cf. [17, 18, 19]. However, underlying currents might have significant effects on the evolution of tsunami waves, cf. the discussion in [10]. The tsunami model analysed in this paper allows for slow bottom variations and includes the possibility of having a background flow with vorticity which might enhance or repress the evolution of tsunami waves. Departing from the analysis of [10], the possibility of incorporating a non-trivial background flow which models isolated regions of vorticity surrounded by still water near the shore is rigorously first established in [6] for the governing equations without passing to shallow-water approximations, and later generalized in [15] for a wider class of vorticity functions.

In the present paper, we improve the result obtained in [15] by simplifying the existence and uniqueness proof therein, thus allowing for even more general vorticity distributions in the model. Furthermore, we present an alternative proof of the fact that solutions have compact support under the additional assumption that far out in the ocean and close to the surface and bottom, the water is still.

2 Preliminaries

We can reasonably model the evolution of tsunami waves in a two-dimensional setting, since the direction of propagation is essentially perpendicular to the fault line. This simplifying assumption is justified for the prominent examples of tsunamis mentioned above, where the motion was almost uniform along the fault line, the length of the rupture zone exceeded the wave length and the ocean depth over which the tsunami travelled was relatively uniform, cf.[5, 21]. We assume the water to be inviscid, incompressible and to have constant density. Furthermore, we neglect the effect of surface tension which plays a minor role in the modelling of gravity water waves. We are interested in the motion of water

near the coast beneath a water surface which is flat in the absence of waves. Hence, we want the model to admit a shoreline at the intersection of the flat surface and the sea bed. In Cartesian coordinates (x, y) , let the shoreline be centered at the origin and assume that the water extends to $-\infty$ in the negative horizontal direction, with constant depth h_0 out in the open ocean. Denote the two-dimensional fluid domain by $D = \{(x, y) \in \mathbb{R}^2 : x < 0, b(x) < y < 0\}$, where $b(x)$ describes the fixed impermeable bed. In our two-dimensional setting we can introduce a stream function ψ such that the fluid's velocity field is given by $(\psi_y, -\psi_x)$. We assume that the vorticity ω can be written as a function $\gamma(\psi)$ called vorticity function and let $\omega = \gamma(\psi)$. This specifies a vorticity distribution throughout the entire fluid domain and it can be proven that in the absence of stagnation points a vorticity distribution may be given by means of a vorticity function (cf. the discussion in [12, 7]). The equations of motion, which can be derived from Euler's equation and the equation of mass conservation (cf. [10, 12]), and boundary conditions governing the background state of the water may be formulated in terms of the stream function ψ as

$$\begin{cases} \Delta\psi = -\gamma(\psi) & \text{in } D, \\ \psi = \psi_y = 0 & \text{on } y = 0, \\ \psi = 0 & \text{on } y = b(x), \end{cases} \quad (1)$$

for a given seabed profile $b(x)$ and a vorticity function $\gamma(\psi)$. The goal is to model a background state of the ocean near the shore which contains isolated regions of vorticity surrounded by still water, assuming the surface is flat prior to the arrival of waves. That is, we look for radially symmetric solutions of (1) that have compact support in the fluid domain.

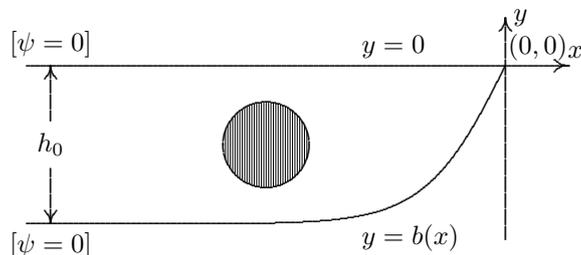


Figure 1: Fluid domain D with an isolated region of non-zero vorticity.

Using the Ansatz

$$\psi(x, y) = \psi(r), \quad \text{where } r = \sqrt{(x - x_0)^2 + (y - y_0)^2},$$

for some $(x_0, y_0) \in D$, the problem is reduced to an initial value problem for the second order ordinary differential equation

$$\begin{cases} \psi''(r) + \frac{1}{r} \psi'(r) = -\gamma(\psi(r)), & r > 0, \\ \psi(0) = a, \quad \psi'(0) = 0, \end{cases} \quad (2)$$

for some initial value $a > 0$. Since (1) is an over determined boundary value problem, we expect non-trivial solutions to exist only for certain classes of functions γ . One can show using maximum principles, cf. [10], that for linear

vorticity functions, the system (1) has only trivial solutions due to the fact that the model admits a shoreline, the free surface is flat in the absence of waves and the water is still outside the region of vorticity. It turns out that these requirements also impose restrictions on the regularity of γ : taking $\gamma \in \mathcal{C}^1$ precludes radially symmetric solutions with compact support in the fluid domain, as we could find $T > 0$ such that $\psi(T) = \psi'(T) = 0$ and it follows from the backward uniqueness property, cf. [13], that $\psi \equiv 0$. For the vorticity function

$$\gamma(\psi) = \begin{cases} \psi - \psi|\psi|^{-\alpha} & \text{for } \psi \neq 0, \\ 0 & \text{for } \psi = 0, \end{cases} \quad \alpha \in (0, 1), \quad (3)$$

which is non-linear and continuous but not \mathcal{C}^1 , non-trivial solutions of system (2) with compact support in $[0, \infty)$ were obtained for $\alpha = 1/2$ in [6], and arbitrary $\alpha \in (0, 1)$ in [15]. These solutions model background states of the ocean prior to the arrival of waves with isolated regions of vorticity under a flat free surface outside of which the water is at rest, cf. Figure (1). The proof of this result is based on a dynamical systems approach, but relies only on basic tools from the theory of ordinary differential equations and consists essentially of two parts: In the first part, it is shown that for any initial value a greater than some constant $a_\alpha > 0$ there exists a unique \mathcal{C}^2 -solution ψ which depends continuously on the initial data a . The second part shows that for certain initial data a big enough, the corresponding unique solution has compact support. There are essentially two difficulties to overcome: the first is due to the fact that (2) is not a classical initial value problem as the equation displays a discontinuity at $r = 0$. This can be remedied by performing a change of variables in the vicinity of the discontinuity and solving the resulting system using an integral Ansatz and a version of Banach's fixed point theorem. Furthermore, since γ is not locally Lipschitz in $\psi = 0$, one cannot merely rely on the classical theory to obtain existence and uniqueness of solutions. A complex chain of arguments involving the reparametrization of the system recast in polar coordinates and an application of the inverse function theorem is carried out to establish uniqueness of solutions also at points where $\psi = 0$. However, this approach might lead to serious difficulties if one wishes to work with more complicated vorticity functions, as pointed out in [20].

The aim of the present note is to avoid these problems by simplifying the uniqueness proof in [15], using elementary arguments to show that if a (vorticity) function γ satisfies the set of hypotheses given below, uniqueness of solutions of (2) is guaranteed in the neighborhood of points where the right hand side fails to be Lipschitz continuous, cf. Section (3). Furthermore, in Section (4) we present an alternative way of proving compact support of solutions of the initial value problem (2) under the additional assumption that for certain initial data a the corresponding solution ψ_a tends asymptotically to zero.

3 Uniqueness

Consider the initial value problem

$$\begin{cases} \psi'' + \frac{1}{r} \psi' = -\gamma(\psi), & r \geq r_0 \geq 1, \\ \psi(r_0) = 0, \quad \psi'(r_0) = \psi_1, \end{cases} \quad (4)$$

where $\psi_1 \neq 0$ and γ is given by (3). We are going to show that a unique solution to (4) exists and that it depends continuously on initial data. This is not straightforward, since the right hand side of the differential equation is not Lipschitz continuous for $\psi = 0$, hence we cannot apply classical existence and uniqueness theorems right away. To obtain the desired result we rely upon the following

Theorem 3.1 *Assume that a continuous function $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the hypotheses*

$$\begin{cases} (i) & \gamma(0) = 0, \\ (ii) & \psi \cdot \gamma(\psi) < 0, \\ (iii) & |\gamma(\psi_1) - \gamma(\psi_2)| \leq \frac{C}{(\min\{|\psi_1|, |\psi_2|\})^\alpha} |\psi_1 - \psi_2|, \quad \alpha \in (0, 1), \end{cases} \quad (5)$$

for any $\psi, \psi_1, \psi_2 \in [-\delta, 0) \cup (0, \delta]$ with $\psi_1 \cdot \psi_2 > 0$ and $C, \delta > 0$. Then, given $\psi_1 \neq 0$, the initial value problem (4) has a unique solution to the right of r_0 .

This is a variation of Theorem 2.1 in [20] and can be proven in almost the same way. The vorticity function $\gamma(\psi)$ defined in (3) is continuous and satisfies the hypotheses (5) in Theorem (3.1) as long as $\delta < 1$. Indeed, by definition $\gamma(0) = 0$, whereas $\psi \cdot \gamma(\psi) < 0$ if and only if $\psi^2(1 - |\psi|^{-\alpha}) < 0$, which is true for $|\psi| \leq \delta < 1$. To show that the third hypothesis is fulfilled assume without loss of generality that $\psi_1 < \psi_2$. Under the assumption that $\psi_1 \cdot \psi_2 > 0$ it suffices to consider the case where $\psi_1, \psi_2 > 0$, since γ is an odd function. Then, since $-\psi_2(1 - \psi_2^{-\alpha}) < -\psi_2(1 - \psi_1^{-\alpha})$, we have

$$\begin{aligned} |\gamma(\psi_1) - \gamma(\psi_2)| &= |\psi_1 - \psi_1|\psi_1|^{-\alpha} - \psi_2 + \psi_2|\psi_2|^{-\alpha}| \\ &< |\psi_1(1 - |\psi_1|^{-\alpha}) - \psi_2(1 - |\psi_1|^{-\alpha})| \\ &< |\psi_1 - \psi_2| |1 - |\psi_1|^{-\alpha}|, \end{aligned}$$

and there exists a constant $C > 0$ such that $|1 - |\psi_1|^{-\alpha}| < C|\psi_1|^{-\alpha}$, since $|\psi_1| \leq \delta$. Therefore,

$$|\gamma(\psi_1) - \gamma(\psi_2)| < |\psi_1 - \psi_2| C |\psi_1|^{-\alpha} = \frac{C}{(\min\{|\psi_1|, |\psi_2|\})^\alpha} |\psi_1 - \psi_2|.$$

From continuity of the function γ we infer that a solution to (4) exists and that it is continuous for all $r \geq 1$. Hence there is a time interval centered at r_0 where $|\psi(r)| \leq \delta < 1$ and Theorem (3.1) applies. We conclude that solutions to (4) are uniquely determined by their initial values, at least in a small interval to the right of r_0 . Away from the zeros of ψ , i.e. in any interval I where $|\psi(r)| > 0$ for $r \in I$, we can use the fact that γ is (locally) \mathcal{C}^1 to infer uniqueness from the theorem of Picard–Lindelöf. Once uniqueness is established, continuous dependence of the solution on initial conditions follows immediately (cf. Theorem 3.4 in [16]). Hence we can prove existence and uniqueness of solutions to the original initial value problem (2) by applying Theorem (3.1) in neighborhoods of values of r where $\psi(r) = 0$ and by employing standard results away from the zeros of ψ , where γ is locally \mathcal{C}^1 .

4 Compact Support

To obtain isolated regions of vorticity for the background state in the model for tsunami waves presented above, one has to prove that the solutions of the

initial value problem (2) have compact support. In [15] this is achieved by an involved argument using a coercive functional which decreases along solutions and performing a detailed analysis of the dynamics of the system in the (ψ, ψ') -plane. We present here a simpler approach which relies on the additional assumption that ψ tends asymptotically to zero, that is, close to the boundaries of the fluid domain, the water is at rest. More precisely, for a solution ψ_a of (2) corresponding to some initial value $a > 0$ and under the assumption that $\lim_{r \rightarrow \infty} \psi_a(r) = 0$, we give an elementary proof of the fact that $\psi_a(r)$ has compact support in $[0, \infty)$.

Consider the decreasing \mathcal{C}^2 -function $\psi_+ : [0, \infty) \rightarrow [0, \infty)$ defined implicitly by

$$r = \int_{\psi_+(r)}^{\psi_+(0)} \frac{ds}{\sqrt{\frac{2}{2-\alpha}|s|^{2-\alpha} - s^2}}, \quad r \in [0, I], \quad (6)$$

where

$$I = \int_0^{(1-\alpha)^{\frac{1}{\alpha}}} \frac{ds}{\sqrt{\frac{2}{2-\alpha}|s|^{2-\alpha} - s^2}}.$$

Let $\psi_+ \equiv 0$ for $r > I$. Then ψ_+ satisfies the second order ordinary differential equation

$$\psi_+'' + \psi_+ - \psi_+|\psi_+|^{-\alpha} = 0, \quad \text{for } r > 0, \quad (7)$$

where the values at the boundary of $[0, I]$ are given by

$$\begin{aligned} \psi_+(0) &= (1-\alpha)^{\frac{1}{\alpha}}, & \psi_+'(0) &= -(1-\alpha)^{\frac{1}{\alpha}} \sqrt{\frac{\alpha(3-\alpha)}{(2-\alpha)(1-\alpha)}}, \\ \psi_+(I) &= \psi_+'(I) = 0. \end{aligned}$$

This can be easily checked, as

$$r = \int_{\psi_+(r)}^{\psi_+(0)} \frac{ds}{\sqrt{\frac{2}{2-\alpha}|s|^{2-\alpha} - s^2}} = \int_r^0 \frac{\psi_+'(s) ds}{\sqrt{\frac{2}{2-\alpha}|\psi_+(s)|^{2-\alpha} - \psi_+^2(s)}}$$

is equivalent to

$$(\psi_+'(r))^2 = \frac{2}{2-\alpha}|\psi_+(r)|^{2-\alpha} - \psi_+^2(r).$$

Differentiating with respect to r yields

$$2\psi_+' \psi_+'' = 2|\psi_+|^{1-\alpha} \psi_+' - 2\psi_+ \psi_+',$$

which in view of the fact that $\psi_+' \neq 0$ and $\psi_+ \geq 0$ gives (7). Furthermore,

$$\begin{aligned} \psi_+'(0) &= -\sqrt{\frac{2}{2-\alpha}|\psi_+(0)|^{2-\alpha} - \psi_+^2(0)} = -\sqrt{\frac{2}{2-\alpha}(1-\alpha)^{\frac{2-\alpha}{\alpha}} - (1-\alpha)^{\frac{2}{\alpha}}} \\ &= -\sqrt{(1-\alpha)^{\frac{2}{\alpha}} \left(\frac{2}{(2-\alpha)(1-\alpha)} - 1 \right)} = -(1-\alpha)^{\frac{1}{\alpha}} \sqrt{\frac{\alpha(3-\alpha)}{(2-\alpha)(1-\alpha)}}. \end{aligned}$$

Notice that $(1-\alpha)^{\frac{1}{\alpha}} > 0$ is the minimum of the function $s \mapsto s - s|s|^{-\alpha}$ for $s > 0$, and since $\lim_{r \rightarrow \infty} \psi_a(r) = 0$ there exists $r_0 > 0$ such that $|\psi_a(r)| < (1-\alpha)^{\frac{1}{\alpha}}$ for all $r \geq r_0$. We claim that

$$|\psi_a(r)| \leq \psi_+(r - r_0) \quad \text{for } r \geq r_0. \quad (8)$$

If equation (8) holds then $\psi_a(r)$ vanishes for $r \geq r_0 + I$ since $\psi_+(r) = 0$ for $r > I$, and we have proved via another approach that a solution ψ_a to (2) has compact support in $[0, \infty)$.

To prove the claim, let us assume that the upper bound were false. By construction, $\lim_{r \rightarrow \infty} \psi_a(r) - \psi_+(r - r_0) = 0$ and $r \mapsto \psi_a(r) - \psi_+(r - r_0)$ is negative at $r = r_0$, since $\psi_a(r_0) < (1-\alpha)^{\frac{1}{\alpha}} = \psi_+(0)$. By assumption, there exists $R > r_0$ such that $\psi_a(R) > \psi_+(R - r_0)$. Therefore, the function $r \mapsto \psi_a(r) - \psi_+(r - r_0)$ has a positive maximum in $[r_0, \infty)$ at some point $r_1 > r_0$ with $\psi'_a(r_1) - \psi'_+(r_1 - r_0) = 0$ and $\psi''_a(r_1) - \psi''_+(r_1 - r_0) \leq 0$. Recalling that ψ_a is a solution of system (2) and that ψ_+ satisfies (7) leads to a contradiction, since

$$\begin{aligned} 0 &\geq \psi''_a(r_1) - \psi''_+(r_1 - r_0) \\ &= -\frac{1}{r} \psi'_a(r_1) - [\psi_a(r_1) - \psi_a(r_1)|\psi_a(r_1)|^{-\alpha}] \\ &\quad + \psi_+(r_1 - r_0) - \psi_+(r_1 - r_0)|\psi_+(r_1 - r_0)|^{-\alpha} \\ &> -\frac{1}{r} \psi'_+(r_1 - r_0) \geq 0. \end{aligned}$$

The second to last inequality is due to the fact that $s \mapsto s - s|s|^{-\alpha}$ is strictly decreasing on $(0, (1-\alpha)^{\frac{1}{\alpha}})$ and $\psi_+(r_1 - r_0) < \psi_a(r_1) < (1-\alpha)^{\frac{1}{\alpha}}$. Analogously, we can show that the lower bound of inequality (8) holds. This proves the claim.

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