ON SOME BACKGROUND FLOWS FOR TSUNAMI WAVES

ANNA GEYER

ABSTRACT. With the aim to describe the state of the sea in a coastal region prior to the arrival of a tsunami, we show the existence of background flow fields with a flat free surface which model isolated regions of vorticity outside of which the water is at rest.

1. INTRODUCTION

Tsunami waves are generated by a sudden vertical displacement of a body of water on a massive scale, caused by landslides, volcanic eruptions or, most commonly, by undersea earthquakes [2]. Tectonic collisions in the form of thrust (or normal) faults sometimes make the ocean floor rise (or drop) by a few meters, causing the column of water directly above to rise (or fall) as well and thereby creating an initial wave profile of elevation (or of depression), as it was the case with the December 2004 tsunami cf. [16, 23, 9, 10, 4, 5]. Tsunami waves are a special type of gravity water waves, with typical wavelength of hundreds of kilometers, which can travel over thousands of kilometers at very high speed with little loss of energy, a spectacular example being the May 1960 tsunami that originated near the Chilean coast (due to the largest earthquake ever recorded) and propagated across the Pacific Ocean devastating coastal areas in Hawaii and Japan, $10000\,km$ respectively $17000\,km$ far from the Chilean coast [8, 24]. Away from the shore, where the ocean can be assumed to have uniform depth over large distances (e.g. the ocean floor of the Central Pacific Basin is relatively uniform, with a mean water depth of about $4300\,m$ cf. [8]), the evolution of the wave is governed essentially by linear water wave theory, the typical wave speed being $\sqrt{gh}$ with g the gravitational constant of acceleration and h the average depth of the sea [23, 11]. The amplitude of a tsunami wave out in the open sea is typically very small (roughly about $0.5\,m$ cf. [23]), but when it approaches a gently sloping beach the front of the wave slows down causing the water to pile up vertically, since the back of the wave is still hundreds of kilometers out in the sea, travelling at much higher speed. The enormous amounts of water involved in this process, account for much of the devastating effects tsunami waves have in coastal areas.

Before the arrival of the tsunami waves at the shore, the water in that region is unlikely to be still: even in the presence of surface waves of small amplitude or for a flat free surface, beneath the surface there could be considerable motion due to the presence of currents (already for irrotational flows with a free surface, an underlying uniform current complicates considerably the dynamics of the flow since without a current all particle paths describe a non-closed loop [3] whereas certain currents can produce closed particle paths [14]). Taking into account currents, it seems essential in a reasonable model for tsunami waves to allow for some kind of background flow field, which models the motion of water in the absence of waves. While most investigations are restricted to irrotational flows which model background states of still water, the possibility of incorporating pre-existing vorticity has only recently been studied in [10]. Various vorticity distributions were obtained in the shallow water regime and it was found that the requirement of a flat free surface is too restrictive, as it invalidates even the simple choice of constant non-zero vorticity throughout the flow field. As opposed to passing to the long wave limit and studying approximations for the shallow water regime, background flows that are governed by the full Euler equations and model isolated regions of vorticity outside of which the water is still have been only recently studied in [6], where a rigorous proof of the existence of a non-trivial solution to the equations governing such background flows which allow for a flat surface is given for a particular choice of vorticity distribution. The aim of the present work is to discuss a generalization of this result. While in [6] a special type of vorticity distribution was provided, we present a whole family of

vorticity distributions (which includes that considered in [6]) admitting a vorticity region surrounded by still water.

2. PHYSICAL ASSUMPTIONS AND THE FORMULATION OF THE PROBLEM

We can reasonably model the evolution of tsunami waves in a two dimensional setting, a simplifying assumption which is justified for the December 2004 tsunami off the coast of Indonesia [23] and the 1960 Chile tsunami [4]. The direction of propagation of tsunami waves was mainly perpendicular to the fault line, with the length of the rupture zone exceeding the wavelength, and the ocean depth over which the tsunami waves travelled was relatively uniform. Furthermore we assume the water to be inviscid and consider its density to be constant. As we are concerned with gravity water waves, we neglect surface tension. We want the model to admit a shoreline and assume that at the bottom we have a fixed impermeable bed. In Cartesian coordinates (x, y), let the origin be the intersection of the flat free surface and the seabed at the shoreline $x = 0$. Let the horizontal x-axis be in the direction of the incoming right-running waves and the vertical y-axis pointing upwards. We assume the fluid to extend to $-\infty$ in the negative horizontal direction and let the bed's topography for a gently sloping beach be given by the function $b(x)$ where $b(0) = 0$, $b(x) < 0$ for $x < 0$ and $b'(0) > 0$. In the open sea we assume uniform depth h_0 such that $b(x) = h_0$ for x far away from the shoreline $x = 0$. We will denote the fluid domain by $D = \{(x, y) \in \mathbb{R}^2 : x < 0, \, b(x) < y < 0\}$.
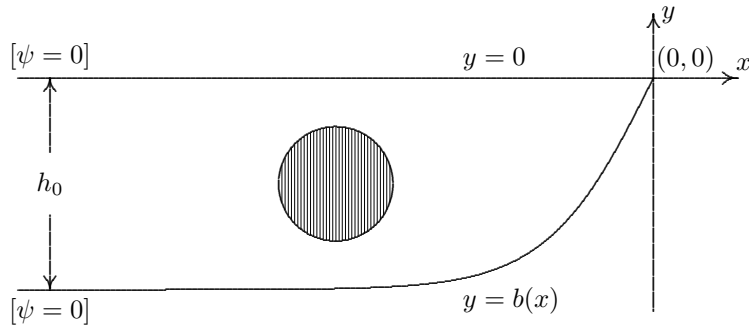

FIGURE 1. Fluid domain D with an isolated region of non-zero vorticity

In the two-dimensional setting we can introduce a stream function ψ, such that the fluid's velocity field is given by $(\psi_y, -\psi_x)$. We consider the vorticity ω to be a function of ψ, $\omega = \gamma(\psi)$, where γ is called vorticity function. Clearly $\omega = \gamma(\psi)$ specifies a vorticity distribution throughout the flow and notice that in the absence of stagnation points (that is, points where $\nabla\psi = (0, 0)$), one can prove that the vorticity distribution is specified by means of a vorticity function cf. the discussion in [13, 7]. The equations governing a background state with flat free surface can be reformulated in terms of ψ as

$$
(2.1) \qquad \begin{cases} \Delta\psi = -\gamma(\psi) & \text{in } D, \\ \psi = \psi_y = 0 & \text{on } y = 0, \\ \psi = 0 & \text{on } y = b(x), \end{cases}
$$

given a vorticity distribution γ and the bottom profile b of the fluid domain D. For a detailed discussion of how these equations governing the fluid motion can be derived from the principle of mass conservation and the Euler equations we refer to [13] and [10].

Our aim is to show existence of an isolated region of non-zero vorticity in the fluid domain, outside of which the water is at rest (see Figure 1). That is, we have to find a suitable vorticity distribution γ and prove that (2.1) has a non-trivial radially symmetric solution with compact support in D.

Radial solutions are obtained via the Ansatz

$$
\psi(x, z) = \psi(r) \text{ with } r = \sqrt{(x - x_0)^2 + (y - y_0)^2} \text{ for } (x_0, y_0) \in D,
$$

turning the system (2.1) into the semi-linear second order differential equation

$$
(2.2) \qquad \psi'' + \frac{1}{r}\psi' = -\gamma(\psi), \qquad r > 0,
$$

where $'$ denotes the derivative with respect to r. Note that for solutions with compact support the boundary conditions in (2.1) will be trivially satisfied, as $\psi \equiv 0$ outside some compact region. To be able to uniquely determine a solution to (2.2), we have to specify initial values for ψ and ψ' at $r = 0$, say

$$(2.3) \qquad\qquad (\psi(0), \psi'(0)) = (a, 0).$$

We require $\psi'(0) = 0$ to produce classical solutions.

The boundary value problem (2.1) is over-determined and it is expected that a non-trivial solution will only exist for certain classes of functions γ. The fact that our model admits a shoreline and we require the water to be still outside the region of vorticity imposes restrictions on the regularity of γ. For linear vorticity functions $\gamma(\psi) = a\psi + b$ it can be shown (see [10]) that system (2.1) admits only trivial solutions. The argument relies mainly upon maximum principles and the fact that the streamlines of the flat free surface and the seabed intersect at the shoreline and are equal to zero.

The argument uses essentially the restrictive boundary conditions on the flat free surface and the seabed and follows from maximum principles.

We can therefore not hope to find non-trivial solutions with compact support inside a circular boundary for a linear vorticity distribution as suggested for example in [1], since these arise in the context of an unbounded fluid which is at rest at infinity. As we are interested in classical solutions, γ has to be at least continuous. However, requiring $\gamma \in \mathcal{C}^1$ precludes radially symmetric solutions with compact support in the fluid domain, since we could find a value $T > 0$ sufficiently large, such that $\psi(T) = \psi'(T) = 0$. Then, by the backward uniqueness property [15] for (2.2) with $\gamma \in \mathcal{C}^1$, we would have $\psi(r) \equiv 0$ for all values of $r > 0$. The considerations made in [12] and [6] lead us to consider the vorticity function

$$(2.4) \qquad\qquad \gamma(\psi) = \begin{cases} \psi - \psi|\psi|^{-\alpha} & \text{for } \psi \neq 0, \\ 0 & \text{for } \psi = 0, \end{cases} \qquad \alpha \in (0, 1).$$
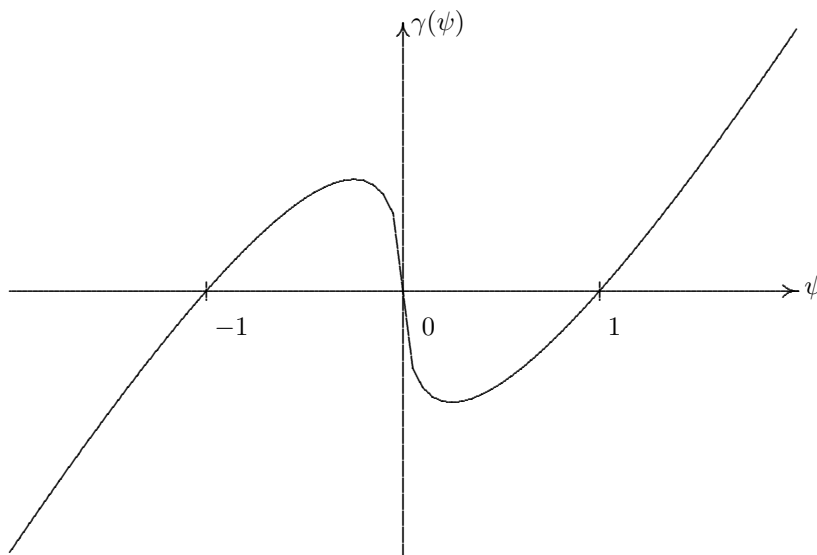

FIGURE 2. The vorticity function γ for the value of $\alpha = \frac{1}{2}$

We now state the main result of this paper.

Theorem 2.1. *For vorticity functions of type (2.4) there exists $a > 0$ such that (2.2)-(2.3) has a non-trivial, $\mathcal{C}^2$-solution ψ with compact support on $[0, \infty)$. This models a background state in the fluid domain D with an isolated region of non-zero vorticity outside of which the water is still.*

3. Proof of the main result

Instead of solving the second order initial value problem (2.2)-(2.3), consider the equivalent planar system of first order ordinary differential equations

$$
(3.1) \qquad \begin{cases} \psi' = \beta, \\ \beta' = -\frac{1}{r}\beta - \psi + \psi|\psi|^{-\alpha}, \end{cases} \qquad r > 0,
$$

with initial values

$$
(3.2) \qquad \psi(0) = a, \qquad \beta(0) = 0.
$$

Define

$$
(3.3) \qquad a_\alpha := \left(\frac{2}{2-\alpha}\right)^{\frac{1}{\alpha}} > 1 \quad \text{and} \quad M_\alpha := \begin{cases} a_\alpha^{\frac{2}{\alpha^4}} & \text{for } 0 < \alpha \le \frac{1}{2}, \\ a_\alpha^{\frac{8}{(1-\alpha)\alpha^2}} & \text{for } \frac{1}{2} \le \alpha < 1. \end{cases}
$$

The proof of Theorem 2.1, using a dynamical system approach which relies upon basic theory of ordinary differential equations, follows essentially from the results of the following two propositions, which we will prove in Sections 3.1 and 3.2, respectively.

Proposition 3.1. *For all $a > a_\alpha$ there exists a unique $\mathcal{C}^2$-solution (ψ, β) to (3.1)-(3.2) which depends continuously on the initial data $(a, 0)$ on any compact interval on which $\psi^2(r) + \beta^2(r) > 0$. Furthermore the solution satisfies $\psi > 1$ for $r \in [0, 1]$.*

Proposition 3.2. *There exists $a > M_\alpha > a_\alpha$ such that for the corresponding solution (ψ, β) of (3.1)-(3.2) there is a finite value $T > 0$ with $\psi(T) = \beta(T) = 0$.*

Proof of Theorem 2.1 By virtue of Propositions 3.1 and 3.2 there exists a value of $a > M_\alpha > a_\alpha$ such that for the corresponding uniquely defined $\mathcal{C}^2$-solution to (3.1)-(3.2), we can find $T > 0$ such that $\psi(T) = \beta(T) = 0$. Then by setting $\psi(r) = 0$ for $r \ge T$ we obtain a compactly supported solution of (2.2) defined for all $r \ge 0$. Furthermore, recall from Proposition 3.1 that $\psi(r) > 1$ for $r \in [0, 1]$, which in view of (2.4) yields $\omega = \gamma(\psi) > 0$. Since ψ has compact support, we obtain an isolated region of non-zero vorticity ω which contains a ball of unit radius where $\omega > 0$, outside of which the water is at rest.

3.1. **Proof of Proposition 3.1.** We claim that for any $a > a_\alpha$ there exists a unique $\mathcal{C}^2$-solution (ψ, β) to (3.1)-(3.2) which depends continuously on the initial data $(a, 0)$ on any compact interval on which $\psi^2(r) + \beta^2(r) > 0$ and for which $\psi > 1$ for $r \in [0, 1]$.

This is not immediately clear for two reasons:
- the right hand side of (3.1) displays a discontinuity at $r = 0$, so the system is not a classical initial value problem.
- since the vorticity function $\gamma(\psi)$ fails to be locally Lipschitz when $\psi = 0$ the right hand side of (3.1) is not locally Lipschitz and we cannot apriori expect uniqueness of solutions or continuous dependence on initial data from the standard theory of ordinary differential equations.

In the first part of the proof, summed up in Lemma 3.1, we consider the system in the vicinity of the discontinuity, for $r \in [0, 1]$. By a simple change of variables (3.6) we overcome the problem of the discontinuity and solve the equivalent system (3.4) using an integral Ansatz and Banach's fixed point theorem. We ensure continuous dependence of solutions on the initial data $(a, 0)$ and find that the solutions of the integral equation (3.7) are always greater than one. In Lemma 3.2 we introduce an important functional (3.14) which decreases along solutions and will be helpful in deriving results throughout the proofs of both Proposition 3.1 and 3.2 as it ensures global existence of solutions. In Lemma 3.3 we tackle the second part of the proof by analyzing the system away from the discontinuity. The difficulty in this case lies in the fact that the right hand side of (3.1) fails to be locally Lipschitz continuous whenever $\psi = 0$. By rewriting the system in polar coordinates we obtain another equivalent formulation (3.16), for which existence and uniqueness of solutions as well as continuous dependence on initial data follows from standard results whenever the right hand side is $\mathcal{C}^1$. In the vicinity of points where $\psi = 0$ an application of the inverse

function theorem yields yet another local reformulation (3.20), shifting the lack of Lipschitz continuity in the dependent variable for (3.16) to the independent variable for the new system and thereby gaining $\mathcal{C}^1$-regularity of the dependent variable for (3.20). We thus obtain local uniqueness and continuous dependence also at points where the right hand side of (3.16) fails to be Lipschitz.

Lemma 3.1. *For $r \in [0,1]$ system (3.1) can be equivalently written as*

$$(3.4) \qquad v'' + e^{-2s}(v - v|v|^{-\alpha}) = 0, \qquad s \geq 0,$$

where the initial values (3.2) are described by the limits

$$(3.5) \qquad v(s) \to a \quad and \quad v'(s)e^s \to 0 \quad for \ s \to \infty.$$

Equation (3.4) has a unique $\mathcal{C}^2$-solution which depends continuously on the parameter a and with $v(s) > 1$ for $s \geq 0$. In particular, this means that for $r \leq 1$, (3.1)-(3.2) has a unique $\mathcal{C}^2$-solution (ψ, β) which depends continuously on a and is such that $\psi(r) > 1$ for $r \in [0,1]$.

Proof. We perform the change of variables

$$(3.6) \qquad s = -\ln r, \qquad \psi(r) = v(s),$$

and find that (3.1) is equivalent to

$$v'' + e^{-2s}(v - v|v|^{-\alpha}) = 0, \qquad s \in \mathbb{R}.$$

This follows from the fact that (3.1) is equivalent to (2.2) which in view of

$$\psi'' + \frac{1}{r}\psi' + \psi - \psi|\psi|^{-\alpha} = v''\frac{1}{r^2} + v'\frac{1}{r^2} - \frac{1}{r^2}v' + v - v|v|^\alpha = 0$$

yields

$$v'' + r^2(v - v|v|^\alpha) = 0.$$

The restriction $0 \leq r \leq 1$ is equivalent to $s \geq 0$ in the new variable.

Let an arbitrary $a > a_\alpha$ be fixed. We can deal with local existence and uniqueness issues of a solution to (3.1)-(3.2) by considering the integral equation

$$(3.7) \qquad v(s) = a - \int_s^\infty (\tau - s)e^{-2\tau}\left(v(\tau) - v(\tau)|v(\tau)|^{-\alpha}\right) d\tau, \qquad s \geq 0.$$

The corresponding asymptotic behavior (3.5) is ensured by

$$(3.8) \qquad v'(s) = \int_s^\infty e^{-2\tau}\gamma(v(\tau)) \, d\tau, \qquad s \geq 0,$$

since

$$\lim_{s \to \infty} v(s) = \lim_{s \to \infty} a - \int_s^\infty (\tau - s)e^{-2\tau}\gamma(v(\tau))d\tau = a$$

and

$$\lim_{s \to \infty} v'(s)e^s = \lim_{s \to \infty} \frac{\int_s^\infty e^{-2\tau}\gamma(v(\tau))d\tau}{e^{-s}} = \lim_{s \to \infty} \frac{e^{-2s}\gamma(v(s))}{e^{-s}} = \lim_{s \to \infty} e^{-s}\gamma(v(s)) = 0,$$

where we used the rule of de l'Hospital in the second equality.

As long as $v(s) \geq 1$ we have that v is non-decreasing, since $\gamma(v) \geq 0$ which in view of (3.8) gives $v'(s) \geq 0$. We even have

$$(3.9) \qquad v(s) > a^{1-\alpha} > 1 \quad for \ s \geq 0.$$

Indeed, if this were not so, define $s_1 := \sup\{s \geq 0 : v(s) = a^{1-\alpha}\}$. Then for all $s \geq s_1$ we have $1 < a^{1-\alpha} \leq v(s) \leq a$, which in view of (3.7) yields a contradiction as

$$0 < a - a^{1-\alpha} = a - v(s_1) = \int_{s_1}^\infty (\tau - s_1)e^{-2\tau}\left(v(\tau) - v(\tau)^{1-\alpha}\right)d\tau$$

$$\leq (a - a^{1-\alpha})\int_{s_1}^\infty (\tau - s_1)e^{-2\tau}d\tau = (a - a^{1-\alpha})\frac{e^{-2s_1}}{4} \leq \frac{(a - a^{1-\alpha})}{4},$$

in view of the fact that $\gamma(v)$ is strictly increasing for $v \in [a^{1-\alpha}, a]$. So for $s \geq 0$ we have that $v(s) > a^{1-\alpha} > 1$ is non-decreasing.

These considerations allow us to view the solution of the integral equation (3.7) as the unique fixed point of the contraction T_a defined by

$$(3.10) \qquad T_a(v)(s) := a - \int_s^\infty (\tau - s)e^{-2\tau}(v(\tau) - v(\tau)|v(\tau)|^{-\alpha})d\tau, \qquad s \geq 0,$$

on the closed subspace $X_a := \{v \in X : a^{1-\alpha} \leq v(s) \leq a, \ s \geq 0\}$ of the Banach space X of bounded continuous functions on $[0,\infty)$ endowed with the supremum norm $\|v\| = \sup_{s \geq 0}\{|v(s)|\}$. To be able to apply Banach's contraction principle (in the following form: *For $\bar{F} \subset X$ a closed subspace of a Banach space X, any contraction $T : F \to F$ has a unique fixed point*) to (3.10) and subsequently to the integral equation (3.7), we have to check the hypotheses.

Notice that for $v \in X_a$ we have $v \geq 1$, since $a > a_\alpha$ and thus $a^{1-\alpha} > (\frac{2}{2-\alpha})^{\frac{1-\alpha}{\alpha}} > 1$ for $0 < \alpha < 1$. Let us check that $T_a(v) \in X_a$, i.e.

$$a^{1-\alpha} \leq a - \int_s^\infty (\tau - s)e^{-2\tau}\gamma(v(\tau))d\tau \leq a, \qquad s \geq 0.$$

The upper bound follows from the fact that the integral is positive, since for $v \in X_a$, $v \geq 1$ and thus $\gamma(v) \geq 0$. For the lower bound, we use the same reasoning as in the proof of (3.9). Now we show that T_a as defined above is a contraction. Since the vorticity function γ defined in (2.4) is $\mathcal{C}^1$ on $[1,\infty)$, by the mean value theorem (cf. [?]) there exists $\xi \in (v,w)$ for $v, w \geq 1$ such that $\gamma(v) - \gamma(w) = \gamma'(\xi)(v - w)$. This yields

$$(3.11) \qquad |\gamma(v) - \gamma(w)| \leq |v - w| \qquad \text{for } v, w, \geq 1,$$

since $\gamma'(\xi) \leq 1$ for $\xi \geq 1$. Then for $s \geq 0$ we have

$$\left| \int_s^\infty (\tau-s)e^{-2\tau}[\gamma(v(\tau)) - \gamma(w(\tau))]d\tau \right| \leq \int_s^\infty (\tau-s)e^{-2\tau}|v(\tau) - w(\tau)|d\tau$$

$$\leq \|v-w\| \int_s^\infty (\tau-s)e^{-2\tau}d\tau = \frac{1}{4}\|v-w\|,$$

whenever $v, w \in X_a$. Thus

$$\|T_a(v) - T_a(w)\| \leq \left| \int_s^\infty (\tau - s)e^{-2\tau}\left(\gamma(v(\tau)) - \gamma(w(\tau))\right)d\tau \right|$$

$$\leq \frac{1}{4}\|v - w\| \qquad \text{for } v, w \in X_a, s \geq 0,$$

which shows that T_a is a contraction on X_a with contraction constant $K \leq \frac{1}{4}$. Therefore, according to Banach's contraction principle, T_a has a unique fixed point, i.e. the integral equation (3.7) has a unique solution $v \in X_a$ which is of class $\mathcal{C}^2$ since $v''(s) = -e^{-2s}(v(s) - v(s)^{1-\alpha})$ is continuous for $s \geq 0$.

To show continuous dependence of the solution on the parameter a, let $v_1 \in X_{a_1}, v_2 \in X_{a_2}$. Then the integral equation (3.7) yields, in view of (3.11), that for $s \geq 0$

$$(3.12) \qquad |v_1(s) - v_2(s)| \leq |a_1 - a_2| + \int_s^\infty (\tau - s)e^{-2\tau}|v_1(\tau) - v_2(\tau)|d\tau$$

$$\leq |a_1 - a_2| + \frac{1}{4}\|v_1 - v_2\|,$$

and therefore

$$(3.13) \qquad \|v_1 - v_2\| \leq \frac{4}{3}|a_1 - a_2|,$$

so we actually even obtain that the solution is stable, cf. [15]. $\square$

Before we proceed to the case where $r \geq 1$, we prove the following useful

Lemma 3.2. *The function*

$$(3.14) \qquad E(r) = E(\psi, \beta) = \frac{1}{2}\beta^2 + \frac{1}{2}\psi^2 - \frac{1}{2-\alpha}|\psi|^{2-\alpha}$$

satisfies

$$(3.15) \qquad E'(r) = -\frac{1}{r}\beta^2, \qquad r > 0,$$

as long as solutions to (3.1)-(3.2) exist and remains bounded for all $r > 0$. Furthermore, $E(r)$ is strictly decreasing whenever $(\psi(r), \beta(r)) \notin \{(0,0), (\pm 1, 0)\}$. We conclude that solutions to (3.1)-(3.2) are defined for all $r \geq 0$ and that ψ and β are bounded functions of r.

Proof. As long as a solution to (3.1)-(3.2) exists, we have $E'(r) = -\frac{1}{r}\beta^2$, since the derivative with respect to r of the function $E(r)$ given by (3.14) in view of (3.1) can be computed as

$$E'(r) = \psi\beta - \beta|\psi|^{-\alpha}\psi + \beta(-\frac{1}{r}\beta - \psi + \psi|\psi|^{-\alpha}) = -\frac{1}{r}\beta^2.$$

Notice that E attains its minimum $E_{min} = \frac{\alpha}{2(\alpha-2)} < 0$ at $(\psi, \beta) = (\pm 1, 0)$, so that (3.15) ensures that E remains bounded. Furthermore,

$$\inf_{\psi \in \mathbb{R}} \left\{ \frac{\psi^2}{2} - \frac{1}{2-\alpha}|\psi|^{2-\alpha} \right\} = \frac{\alpha}{2(\alpha-2)} \quad \text{and} \quad \lim_{|\psi| \to \infty} \left\{ \frac{\psi^2}{2} - \frac{1}{2-\alpha}|\psi|^{2-\alpha} \right\} = \infty.$$

Therefore ψ and β remain bounded as long as solutions exist, since otherwise E would become unbounded. We conclude that the solutions to (3.1)-(3.2) are defined for all $r \geq 0$.

Let us now prove that $E(r)$ is strictly decreasing whenever $(\psi(r), \beta(r)) \notin \{(0,0), (\pm 1, 0)\}$. Otherwise, for $r_2 > r_1 > 0$ with $E(r_2) = E(r_1)$, we would have

$$0 = E(r_2) - E(r_1) = \int_{r_1}^{r_2} E'(r)dr = -\int_{r_1}^{r_2} \frac{\beta^2(r)}{r}dr.$$

This implies $\beta(r) = 0$ on $[r_1, r_2]$ and consequently from (3.1) we have that $\psi'(r) = \beta'(r) = 0$ for all $r \in [r_1, r_2]$. Thus, $\psi(r) = \psi(r_1) = \psi(r_1)|\psi(r_1)|^{-\alpha}$ is constant in $[r_1, r_2]$, so that $\psi(r) \in \{0, \pm 1\}$ in view of (3.1), a contradiction. $\qquad\square$

Now we consider the system away from the discontinuity at $r = 0$ and prove existence, uniqueness and continuous dependence of solutions on the parameter a as long as $\psi^2 + \beta^2 > 0$.

Lemma 3.3. *For $r \geq 1$ system (3.1) can be equivalently reformulated as*

$$(3.16) \qquad \begin{cases} \theta'(r) = -\dfrac{1}{2r}\sin(2\theta) - 1 + R^{-\alpha}|\cos(\theta)|^{2-\alpha}, \\ \\ R'(r) = -\dfrac{1}{r}R\sin^2(\theta) + R^{1-\alpha}\sin(\theta)\dfrac{\cos(\theta)}{|\cos(\theta)|^\alpha}, \end{cases} \qquad r \geq 1.$$

As long as $R > 0$ this system of first order differential equations has a unique C^2-solution which depends continuously on the initial data $(\theta(1), R(1))$, which in turn depends continuously on the parameter a.

Proof. We introduce polar coordinates

$$(3.17) \qquad \psi = R\cos(\theta), \qquad \beta = R\sin(\theta),$$

to show that (3.16) is yet another equivalent formulation of (3.1):

$$\theta'(r) = \frac{d}{dr}\arctan\left(\frac{\beta(r)}{\psi(r)}\right) = \frac{\beta'\psi - \beta\psi'}{\psi^2 + \beta^2} \overset{(3.1)}{=} \frac{-\frac{1}{r}\beta\psi - \psi^2 + |\psi|^{2-\alpha} - \beta^2}{\psi^2 + \beta^2}$$

$$\overset{(3.17)}{=} \frac{-\frac{1}{r}R^2\sin(\theta)\cos(\theta) - R^2\cos^2(\theta) + R^{2-\alpha}|\cos(\theta)|^{2-\alpha} - R^2\sin^2(\theta)}{R^2}$$

$$= -\frac{1}{2r}\sin(2\theta) + R^{-\alpha}|\cos(\theta)|^{2-\alpha} - 1,$$

and

$$R'(r) = \frac{\psi' \cos(\theta) + \psi \sin(\theta)\theta'}{\cos^2(\theta)}$$

$$= \frac{R \sin(\theta) \cos(\theta) + R \cos(\theta) \sin(\theta)(-\frac{1}{r} \sin(\theta) \cos(\theta) + R^{-\alpha} |\cos(\theta)|^{2-\alpha} - 1)}{\cos^2(\theta)}$$

$$= -\frac{1}{r} \sin^2(\theta) R + R^{1-\alpha} \sin(\theta) \cos(\theta) |\cos(\theta)|^{-\alpha}.$$

The initial data $(\theta(1), R(1))$ is specified after solving the integral equation (3.7) on $[0, \infty)$. To show continuous dependence of $(\theta(1), R(1))$ on a, notice that (3.8) in view of (3.11) and (3.13) yields for $s \geq 0$

$$|v_1'(s) - v_2'(s)| \leq \int_s^\infty e^{-2\tau} |\gamma(v_1(\tau)) - \gamma(v_2(\tau))| d\tau$$

$$\leq \int_s^\infty e^{-2\tau} |v_1(\tau) - v_2(\tau)| d\tau$$

(3.18) $$\leq \frac{1}{4} \|v_1 - v_2\| \leq \frac{1}{3} |a_1 - a_2|.$$

Evaluating inequalities (3.12) and (3.18) at $s = 0$ together with (3.13) yields

(3.19) $$|v_1(0) - v_2(0)| + |v_1'(0) - v_2'(0)| \leq \frac{5}{3} |a_1 - a_2|.$$

In view of the formulation (3.16) of the initial value problem (3.1)-(3.2) this means that ψ and $\psi' = \beta$ vary little at $r = 1$. Thus, $\theta(1) = \arctan\left(\frac{\beta(1)}{\psi(1)}\right)$ and $R(1) = \sqrt{\psi^2(1) + \beta^2(1)}$ depend continuously on a. The considerations we made in Lemma 3.1 show that $\psi(1) > a^{1-\alpha} > 0$ and $R(1) > 0$, so $\cos(\theta(1)) = \frac{\psi(1)}{R(1)} > 0$. As long as $R > 0$ and $\cos(\theta) > 0$ the right hand side of (3.16) is $\mathcal{C}^1$. Thus we get local existence and uniqueness as well as continuous dependence on initial data $(\theta(1), R(1))$ for a solution to (3.16) by standard results.

We show that, as long as $R > 0$, this holds true even if $\cos(\theta(r)) = 0$, that is, at points where a solution intersects the vertical axis in the (ψ, β)-phase plane. At such points, the right hand side of (3.16) is still continuous and bounded, but fails to be locally Lipschitz. Thus, while we can still rely for the local existence of solutions on the Cauchy–Peano theorem [15], uniqueness and continuous dependence on initial data on the other hand are no longer guaranteed. We overcome this problem by transforming the system in a neighborhood of such values of r, taking advantage of its local structure.

Denote by r_0 the smallest value of $r > 1$ where $\cos(\theta(r_0)) = 0$, say $\theta(r_0) = -\frac{\pi}{2}$. Since for $r \in (1, r_0)$ the right hand side of (3.16) is $\mathcal{C}^1$, the solution is unique and depends continuously on the initial data $(\theta(1), R(1))$ up to r_0. We then select one of the possible continuations of the solution across $r = r_0$ and show that this selection is unique and depends continuously on $(\theta(1), R(1))$ close to $r = r_0$. Since $\cos(\theta(r_0)) = 0$ and $\theta'(r_0) = -1$, the inverse function theorem (cf. [17]) guarantees the existence of neighborhoods $(r_0 - \varepsilon, r_0 + \varepsilon)$ of r_0 and $(-\delta, \delta^+)$ of 0 for sufficiently small $\varepsilon > 0$ and $\delta, \delta^+ > 0$, as well as a uniquely determined $\mathcal{C}^1$-function

$$\varphi(\tau) = r$$

such that $\varphi(0) = r_0$, $\varphi(-\delta) = r_0 - \varepsilon$ and $\varphi(\delta^+) = r_0 + \varepsilon$ which allows us to locally set

$$\cos(\theta(r)) = -\tau.$$

Notice that this transformation preserves the monotonicity of the respective independent variables r and τ, since $\theta' < 0$ and $\cos(\theta)$ is increasing in a neighborhood of $-\frac{\pi}{2}$. Thus $r_0 - \varepsilon < r_0 < r_0 + \varepsilon$ implies $\cos(\theta(r_0 - \varepsilon)) > \cos(\theta(r_0)) > \cos(\theta(r_0 + \varepsilon))$, or, equivalently, $\cos(\varphi(-\delta)) > 0 > \cos(\varphi(\delta^+))$, which in view of $\cos(\theta(\varphi(\tau))) = -\tau$ implies $-\delta < 0 < \delta^+$. Differentiating the equation $\varphi(\tau) = \varphi(-\cos(\theta(r))) = r$ with respect to $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$ yields

$$\varphi'(\tau) = \frac{1}{\theta'(r) \sin(\theta(r))}, \qquad \tau \in (-\delta, \delta^+).$$

Setting

$$\rho(\tau) = R(r)$$

yields
$$\rho'(\tau) = R'(r)\,\varphi'(\tau), \qquad \text{for } r \in (r_0 - \varepsilon, r_0 + \varepsilon),\ \tau \in (-\delta, \delta^+).$$
Now we transfer (3.16) for $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$ and $\tau \in (-\delta, \delta^+)$ into the system

(3.20)
$$\begin{cases} \varphi'(\tau) = \dfrac{1}{\sqrt{1-\tau^2} + \frac{\tau(1-\tau^2)}{\varphi(\tau)} - \rho(\tau)^{-\alpha}|\tau|^{2-\alpha}\sqrt{1-\tau^2}}, \\[4mm] \rho'(\tau) = -\dfrac{\frac{1}{\varphi(\tau)}\rho(\tau)\sqrt{1-\tau^2} - \rho(\tau)^{1-\alpha}\frac{\tau}{|\tau|^\alpha}}{-1 - \frac{\tau\sqrt{1-\tau^2}}{\varphi(\tau)} + \rho^{-\alpha}(\tau)|\tau|^{2-\alpha}}. \end{cases}$$

A straightforward calculation and the fact that
$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2(-\sqrt{1-\cos^2(\theta)})\cos(\theta) = 2\tau\sqrt{1-\tau^2}$$
for $\varepsilon > 0$ small enough shows that (3.20) and (3.16) are equivalent.

The advantage of the system (3.20) with respect to (3.16) is that the lack of $\mathcal{C}^1$-regularity in θ was shifted into a lack of $\mathcal{C}^1$-regularity in τ. Consequently, the new system is $\mathcal{C}^1$ in the unknown variables $(\varphi, \rho) \in (1, \infty) \times (0, \infty)$ and continuous in the independent variable τ. This is enough to ensure uniqueness and continuous dependence on initial data $(\varphi(-\delta), \rho(-\delta))$ of the solutions to (3.20). Furthermore, $(\varphi(-\delta), \rho(-\delta))$ depends continuously on $(\theta(r_0 - \varepsilon), R(r_0 - \varepsilon))$ via the $\mathcal{C}^1$-function $\varphi(\tau) = r$ and we already mentioned the continuous dependence of solutions $(\theta(r), R(r))$ on $(\theta(1), R(1))$ and thus on the parameter a for $r \in (1, r_0)$. We can therefore deduce that uniqueness and continuous dependence on a of the solution to (3.16) holds also in a neighborhood of r_0.

This procedure can be repeated in almost the same way for the next value of $r > r_0$ where $\cos(\theta(r)) = 0$, i.e. where the solution intersects the vertical axis in the upper half plane. Then again an application of the inverse function theorem (cf. [17]) guarantees the existence of a $\mathcal{C}^1$-function $\varphi(\tau) = r$ such that we can locally set
$$\cos(\theta(r)) = \tau.$$
Notice that this time we choose $\cos(\theta(r)) = \tau$ instead of $-\tau$ to preserves monotonicity of the respective independent variables. As before, we transfer (3.16) into a system which differs from (3.20) only by a change of sign in the second equation. Thus by the same reasoning as above we deduce that uniqueness and continuous dependence on a of the solution to (3.16) also holds in neighborhoods of points where the solution intersects the vertical axis in the upper half plane. This procedure can be repeated for all values of r where the right hand side of (3.16) fails to be locally Lipschitz, as long as $R > 0$.
Summing up, we can say that for values of r where $\cos(\theta(r)) = 0$, that is, where $\theta(r) = -\frac{\pi}{2} + 2k\pi$ or $\theta(r) = \frac{\pi}{2} + 2k\pi$ for $k \in \mathbb{Z}$, the above local transformations guarantee uniqueness and continuous dependence on a of the solution to (3.16) also in neighborhoods of such values as long as $R > 0$. In between these values of r, the right hand side of (3.16) is $\mathcal{C}^1$ and everything follows from standard results. $\square$

This concludes the proof of Proposition 3.1, as we have seen that for any $a > a_\alpha$ there exists a unique $\mathcal{C}^2$-solution (ψ, β) to (3.1)-(3.2) for which $\psi > 1$ on $[0, 1]$ by virtue of (3.9) and which depends continuously on the initial data $(a, 0)$ on any compact interval on which $\psi^2(r) + \beta^2(r) > 0$.

3.2. **Proof of Proposition 3.2.** We show that there exists a value of $a > a_\alpha$ such that for the corresponding solution to (3.1)-(3.2) we can find some $0 < T < \infty$ with $\psi(T) = \beta(T) = 0$. The idea is to perform a detailed qualitative analysis for the system (3.1)-(3.2), similar to the phase-plane analysis of autonomous systems. We introduce two sets $\Omega_\pm$ defined by the solution sets of the equation $E(\psi, \beta) = 0$, where E is the functional defined in Section 3.1. In Lemma 3.4 we show that for initial data a large enough the solution can enter the region $\Omega_\pm$ only for values of $r > \frac{2}{\alpha}$. We find that the value of r at which the solution can enter $\Omega_\pm$ tends to infinity as $a \to \infty$. After that, Lemma 3.5 ensures that there exists an initial value a_+ such that the corresponding solution stays outside $\Omega_- \cup \Omega_+$ for all $r \geq 0$. Finally,

in Lemma 3.6, we prove that for solutions corresponding to such initial data a_+ there exists a finite value $T > 0$ such that $E(T) = 0$, and therefore also $\psi(T) = \beta(T) = 0$.

Let us start with defining the sets $\Omega_\pm$. From (3.14) in Lemma 3.2 we have that

$$E(\psi, \beta) = 0 \quad \text{if and only if} \quad \beta^2 = \frac{2}{2-\alpha}|\psi|^{2-\alpha} - \psi^2.$$

In the plane (ψ, β) the set where $E < 0$ consists of the interiors $\Omega_\pm$ of the closed curves rep-
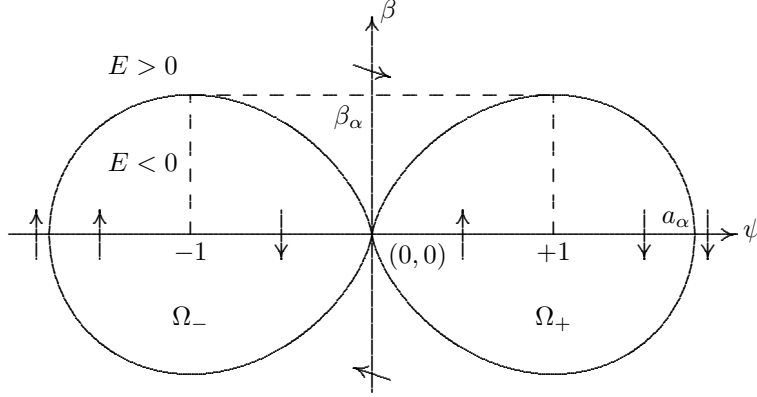

FIGURE 3. The solution set of $E = 0$ in the phase plane (ψ, β) with arrows indicating the dynamics of the system

resenting the solution set of the above equation. These curves are symmetrical with respect to the vertical and the horizontal axis and are tangential to one another and to the vertical axis at the origin. Note from (3.3) that the curves reach their maximum $\beta_\alpha = (\frac{\alpha}{2-\alpha})^{\frac{1}{2}}$ at $\psi = \pm 1$ and they intersect the horizontal axis at the points $\psi = 0$ and $\psi = \pm(\frac{2}{2-\alpha})^{\frac{1}{\alpha}} = \pm a_\alpha$. To get a better understanding of the dynamics of the system (3.1), consider the right half plane, where $\psi > 0$. At $\beta = 0$ we have $\psi' = \beta = 0$ and $\beta' = -\frac{1}{r}\beta - \psi + \psi|\psi|^{-\alpha} > 0$ when $\psi|\psi|^{-\alpha} > \psi$ which is true for $0 < \psi < 1$, whereas $\beta' < 0$ for $\psi > 1$. In the left half plane, we have exactly the opposite situation. Therefore, solutions intersect the horizontal axis perpendicularly from the upper to the lower half plane for $\psi > 1$ and for $-1 < \psi < 0$. On the complement of these sets, they intersect the axis in the opposite direction. For $\psi = 0$ and $\beta > 0$ we have that $\psi' > 0$ and $\beta' < 0$, which means that solutions intersect the vertical axis from left to right in the upper half plane. In the lower half plane, the opposite is true (see Figure 3).

By Lemma 3.2, E is strictly decreasing as long as $(\psi, \beta) \notin \{(0,0), (\pm 1, 0)\}$. Therefore, once a solution reaches the boundary of $\Omega_\pm$ at a point other than $(0,0)$ it will enter $\Omega_\pm$. Once inside, a solution will stay in either Ω_+ or Ω_- for all subsequent times, as E is strictly decreasing.

Recall from (3.3) that we defined

$$M_\alpha := \begin{cases} a_\alpha^{\frac{2}{\alpha^4}} & \text{for } 0 < \alpha \le \frac{1}{2}, \\ a_\alpha^{\frac{8}{(1-\alpha)\alpha^2}} & \text{for } \frac{1}{2} \le \alpha < 1. \end{cases}$$

For certain initial data, solutions stay outside of $\Omega_\pm$ for some time:

Lemma 3.4. *For $a > M_\alpha$ we have that $E(r) > 0$ as long as $r \in [0, \frac{2}{\alpha}]$. This means that a solution to (3.1)-(3.2) with $a > M_\alpha$ can enter $\Omega_\pm$ only for values of $r > \frac{2}{\alpha}$. Additionally, we find that the value of r such that a solution can enter $\Omega_\pm$ tends to infinity as $a \to \infty$.*

Proof. Let $a > a_\alpha^{\frac{2}{1-\alpha}}$. We know from the results in Lemma 3.1 that for $r \in [0, 1]$, $R(r) = \frac{\psi(r)}{\cos(\theta(r))} > a^{1-\alpha} > a_\alpha$. If a solution with initial data a enters the region $\Omega_+ \cup \Omega_-$, then for

some value of $r^* > 1$ we will have $R(r^*) = a_\alpha < a^{\frac{1-\alpha}{2}}$. We can therefore define

$$r_0 = \inf\{r > 0 : R(r) = a_\alpha\}$$

(3.21) $$r_1 = \sup\{0 < r < r_0 : R(r) = a^{1-\alpha}\} > 1$$

$$r_2 = \sup\{r_1 < r < r_0 : R(r) = a^{\frac{1-\alpha}{2}}\}$$

so that

$$a^{1-\alpha} = R(r_1) \geq R(r) \geq R(r_2) = a^{\frac{1-\alpha}{2}} \quad \text{for } r \in [r_1, r_2].$$

The argument leading to the desired result requires us to consider separately the case where $\alpha \in (0, \frac{1}{2}]$ and the case where $\alpha \in [\frac{1}{2}, 1)$. Some inequalities involving functions of α in the exponent will be denoted by (a)-(c) and will be shown at the end of the proof of this Lemma.

Let $\alpha \in (0, \frac{1}{2}]$. We claim that

(3.22) $$r_2 \geq a^{\alpha^3}.$$

Indeed, assume to the contrary that $r_2 < a^{\alpha^3}$, then from (3.16) we infer that

$$R'(r) = -\frac{1}{r} R \sin^2(\theta) + R^{1-\alpha} \sin(\theta) \frac{\cos(\theta)}{|\cos(\theta)|^\alpha} \geq -\frac{1}{r} R - R^{1-\alpha} > -\frac{2}{r} R.$$

In the last inequality we have used that $R(r) \geq a^{\frac{1-\alpha}{2}}$, thus for $\alpha \in (0, \frac{1}{2}]$ we have that $R^\alpha \geq a^{\frac{(1-\alpha)\alpha}{2}} \overset{(a)}{\geq} a^{\alpha^3} > r_2 \geq r$ for $r \in [r_1, r_2]$, which yields $-R^{1-\alpha} > -\frac{1}{r} R$. Integrating the differential inequality

$$\frac{R'(r)}{R(r)} > -\frac{2}{r}, \qquad r \in [r_1, r_2]$$

with respect to r on $[r_1, r_2]$ yields

$$\ln r_2 > \ln r_1 + \frac{1}{2} \ln\left(\frac{R(r_1)}{R(r_2)}\right) = \ln r_1 + \frac{1}{2} \ln\left(\frac{a^{1-\alpha}}{a^{\frac{1-\alpha}{2}}}\right) = \ln r_1 + \ln(a^{\frac{1-\alpha}{4}}),$$

which in turn gives

$$r_2 > r_1 \, a^{\frac{1-\alpha}{4}} > a^{\frac{1-\alpha}{4}} \overset{(a)}{\geq} a^{\alpha^3}, \qquad \text{for } \alpha \in (0, \frac{1}{2}].$$

This last argument yields a contradiction and we are done proving the claim that $r_2 \geq a^{\alpha^3}$. Note that this also means that $r_0 > r_2 \geq a^{\alpha^3}$ and the smallest value of r such that a solution can enter Ω_- or Ω_+ is r_0. We can therefore deduce that solutions corresponding to initial data $a > a_{\alpha}^{\frac{2}{\alpha^4}}$ will stay outside $\Omega_- \cup \Omega_+$ at least for values of $r \in [0, \frac{2}{\alpha}]$, since $a > a_{\alpha}^{\frac{2}{\alpha^4}}$ implies

$$a^{\alpha^3} > a_{\alpha}^{\frac{2}{\alpha}} = \left(\frac{2}{2-\alpha}\right)^{\frac{2}{\alpha^2}} \overset{(c)}{>} \frac{2}{\alpha}$$

and thus $r_0 > a^{\alpha^3} > \frac{2}{\alpha}$.

Now let $\alpha \in [\frac{1}{2}, 1)$. We claim that in this case

(3.23) $$r_2 \geq a^{\frac{1-\alpha}{4}}.$$

If we assume to the contrary that $r_2 < a^{\frac{1-\alpha}{4}}$, then from (3.16) we infer again that

$$R'(r) \geq -\frac{1}{r} R - R^{1-\alpha} > -\frac{2}{r} R,$$

since $R(r) \geq a^{\frac{1-\alpha}{2}}$ and thus

$$R^\alpha \geq a^{\frac{(1-\alpha)\alpha}{2}} \overset{(b)}{\geq} a^{\frac{1-\alpha}{4}} > r_2 \geq r$$

for $r \in [r_1, r_2]$ and $\alpha \in [\frac{1}{2}, 1)$, which yields $-R^{1-\alpha} > -\frac{1}{r} R$. Integrating the differential inequality

$$\frac{R'(r)}{R(r)} > -\frac{2}{r}, \qquad r \in [r_1, r_2],$$

with respect to r on $[r_1, r_2]$ gives

$$r_2 > r_1\, a^{\frac{1-\alpha}{4}} > a^{\frac{1-\alpha}{4}}, \qquad \alpha \in [\tfrac{1}{2}, 1).$$

This last argument again yields a contradiction and we are done proving the claim that $r_2 \geq a^{\frac{1-\alpha}{4}}$. Note that this also means that $r_0 > r_2 \geq a^{\frac{1-\alpha}{4}}$ and as before the smallest value of r such that a solution can enter Ω_- or Ω_+ is r_0. We can therefore deduce that solutions corresponding to initial data $a > a_\alpha^{\frac{8}{(1-\alpha)\alpha^2}}$ will stay outside $\Omega_- \cup \Omega_+$ at least for values of $r \in [0, \frac{2}{\alpha}]$, since

$$a^{\frac{(1-\alpha)\alpha}{4}} > a_\alpha^{\frac{2}{\alpha}} = \left(\frac{2}{2-\alpha}\right)^{\frac{2}{\alpha^2}} \overset{(c)}{>} \frac{2}{\alpha}$$

and thus

$$r_0 > a^{\frac{1-\alpha}{4}} > a_\alpha^{\frac{2}{\alpha^2}} \geq a_\alpha^{\frac{2}{\alpha}} > \frac{2}{\alpha}.$$

Summing up, we can state that for $a > M_\alpha$ and $r \leq \frac{2}{\alpha}$ we have $R(r) > a_\alpha$ and it follows that

$$(3.24) \qquad\qquad E(r) > 0 \quad \text{for } r \in [0, \frac{2}{\alpha}], \quad \text{if } a > M_\alpha,$$

since

$$(3.25) \qquad E(r) = \frac{1}{2}(\beta^2 + \psi^2) - \frac{1}{2-\alpha}|\psi|^{2-\alpha} = \frac{1}{2}R^2 - \frac{1}{2-\alpha}R^{2-\alpha}|\cos\theta|^{2-\alpha}$$

$$(3.26) \qquad\qquad \geq R^{2-\alpha}\left(\frac{1}{2}R^\alpha - \frac{1}{2-\alpha}\right) > a_\alpha^{2-\alpha}\left(\frac{1}{2}a_\alpha^\alpha - \frac{1}{2-\alpha}\right) = 0.$$

This means that a solution with $a > M_\alpha$ can enter Ω_- or Ω_+ only for a value of $r > \frac{2}{\alpha}$. Moreover, (3.22) and (3.23) show that as $a \to \infty$ the value of $r > 0$ at which a solution enters the region $\Omega_- \cup \Omega_+$ approaches infinity.

To finish the proof of this Lemma we show that the following inequalities hold:
(a) $a^{\frac{1-\alpha}{4}} > a^{\frac{(1-\alpha)\alpha}{2}} > a^{\alpha^3}$ for $\alpha \in (0, \frac{1}{2})$,
(b) $a^{\frac{(1-\alpha)\alpha}{2}} > a^{\frac{1-\alpha}{4}}$ for $\alpha \in (\frac{1}{2}, 1)$,
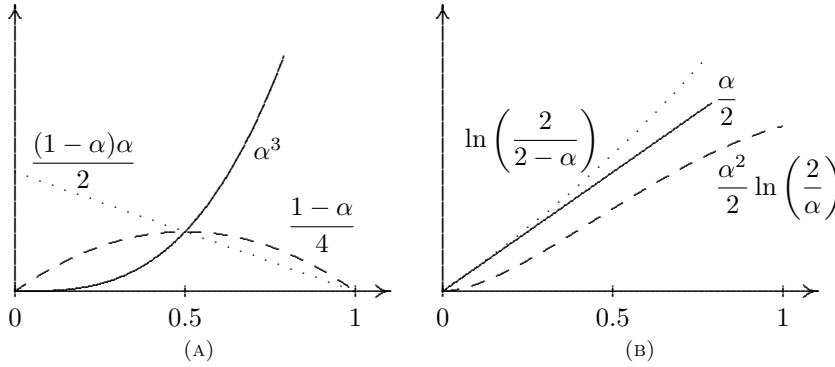(c) $\left(\frac{2}{2-\alpha}\right)^{\frac{2}{\alpha^2}} > \frac{2}{\alpha}$ for $\alpha \in (0, 1)$.

FIGURE 4

Since $a > 1$, in (a) and (b) it suffices to consider the exponent functions of α in and compare them in size (cf. Figure 4a). Inequality (c) can be shown by noting (cf. Figure 4b) that the statement is equivalent to

$$\ln\frac{2}{2-\alpha} > \frac{\alpha^2}{2}\ln\frac{2}{\alpha} \qquad \text{for } \alpha \in (0, 1).$$

Then, define $f(\alpha) = \frac{\alpha}{2}$, $g(\alpha) = \ln\frac{2}{2-\alpha}$ for $\alpha \in (0, 1)$. Both functions are monotone increasing and equal to 0 at $\alpha = 0$. Since $g'(\alpha) = \frac{1}{2-\alpha} > \frac{1}{2} = f'(\alpha)$, $f(\alpha) < g(\alpha)$ for all $\alpha \in (0, 1)$.

Furthermore, let $h(\alpha) = \alpha \ln \frac{2}{\alpha}$, then $h(\alpha) \to 0$ as $\alpha \to 0$ and $h(2) = 0$. It is easy to check that $h(\alpha)$ has its only maximum at $\alpha = \frac{2}{e}$ with $h(\frac{2}{e}) = \frac{2}{e} < 1$ which shows that $h(\alpha) = \alpha \ln \frac{2}{\alpha} < 1$. Thus

$$\frac{\alpha^2}{2} \ln \frac{2}{\alpha} = \frac{\alpha}{2} h(\alpha) < \frac{\alpha}{2} = f(\alpha) < g(\alpha) = \ln \frac{2}{2 - \alpha}.$$

$\square$

Now we know that the solution to certain initial conditions $a > M_\alpha$ can only enter the region $\Omega_\pm$ for a value of $r > \frac{2}{\alpha}$. Furthermore, this value of r increases if we let the starting point a tend to infinity. The question that arises is: will all solutions with initial data $a > M_\alpha$ enter $\Omega_\pm$? The answer is no, which represents our next result.

Lemma 3.5. *There exists initial data $a_+ > M_\alpha$ such that the corresponding solution to (3.1)-(3.2) stays outside of $\Omega_+ \cup \Omega_-$ for all $r \geq 0$.*

Proof. Outside of $\Omega_- \cup \Omega_+$ we have $E > 0$ so that $\psi^2 + \beta^2 > \frac{2}{2-\alpha}|\psi|^{2-\alpha}$. Passing to polar coordinates (3.17) we find

(3.27)
$$-1 - \frac{1}{2r} \leq \theta'(r) \leq -\frac{\alpha}{2} + \frac{1}{2r},$$

for the values of $r > 0$ where $E(r) > 0$. This is easy to see, as the derivative with respect to r of $\theta(r)$, in view of (3.1), is given by

$$\theta'(r) = \frac{\beta'\psi - \beta\psi'}{\psi^2 + \beta^2} \overset{(3.1)}{=} -1 - \frac{\beta\psi}{r(\psi^2 + \beta^2)} + \frac{|\psi|^{2-\alpha}}{\psi^2 + \beta^2} = -1 - A + B$$

where

$$A = \frac{\beta\psi}{r(\psi^2 + \beta^2)}, \quad B = \frac{|\psi|^{2-\alpha}}{\psi^2 + \beta^2}.$$

Since $r > 0$ we have $|A| = \frac{|\beta\psi|}{r(\psi^2+\beta^2)} \leq \frac{1}{2r}$, and since $E > 0$ we have $B < \frac{2-\alpha}{2}$. Thus

$$\theta'(r) = -1 - A + B \geq -1 - A \geq -1 - \frac{1}{2r},$$

and

$$\theta'(r) = -1 - A + B \leq -1 + \frac{1}{2r} + \frac{2-\alpha}{2} = -\frac{\alpha}{2} + \frac{1}{2r}.$$

These estimates on $\theta'(r)$ gives an upper and lower bound on the angular velocity of the solution for all values of $r > 0$ where $E(r) > 0$. In the previous lemma we showed that $E(r) > 0$ at least for $r < \frac{2}{\alpha}$ if $a > M_\alpha$. Now, let us consider values of $r > \frac{2}{\alpha}$ for which $E(r)$ is still positive. For such values, (3.27) reads

(3.28)
$$-1 - \frac{\alpha}{4} < \theta'(r) < -\frac{\alpha}{4}.$$

Denote by D_+, D_- as the sets of points $\{(a, 0) : a > M_\alpha\}$ such that a solution (ψ, β) to (3.1)-(3.2) with initial data $(a, 0)$ will enter Ω_+, Ω_-, respectively, for some finite value of r. Both sets D_+ and D_- are open by continuous dependence of the solution on initial data, since for a point $(a^*, 0) \in D_+$, whose corresponding solution enters the region Ω_+ at time $r = r^*$, a solution whose starting point $(a, 0)$ lies sufficiently close to $(a^*, 0)$ will also enter Ω_+ at some time close to r^*. In the beginning of the discussion of the proof of Proposition 3.2 we analyzed the dynamics of the system and found that in the plane (ψ, β), outside of $\Omega_- \cup \Omega_+$ a solution to (3.1)-(3.2) intersects the horizontal axis from the upper to the lower half-plane on the right of the origin, and in the other direction on the left of the origin. Notice also that a solution intersects the horizontal axis a finite number of times as it winds around the region $\Omega_- \cup \Omega_+$ before entering it. Denote by $D_N \subseteq D_- \cup D_+$ the set of initial data such that corresponding solutions intersect the positive horizontal axis exactly N times prior to entering Ω_- or Ω_+. Since these intersections are transversal, they are stable under small perturbations (cf. [18]). Thus, again by continuous dependence on initial data, for any N these sets D_N are open. In view of the fact that solutions are unique once we specify the initial condition $(a, 0)$, they are disjoint. We can therefore write $D_- \cup D_+ = \dot{\bigcup}_N D_N$. Assume for a moment that all solutions will at one point enter Ω_+ or Ω_-. (3.22) and (3.23) in Lemma 3.4 show that as $a \to \infty$ the value of $r > 0$ at which a solution can enter the

region where $E < 0$ approaches infinity. In view of the above inequality (3.28) and since by virtue of (3.15), E is strictly decreasing whenever $(\psi, \beta) \notin \{(0,0), (\pm 1, 0)\}$, this means that for $r > \frac{2}{\alpha}$ a solution to (3.1)-(3.2) with $a > M_\alpha$ keeps winding around the region $\Omega_- \cup \Omega_+$ before entering it as $a \to \infty$. We deduce that there exist infinitely many open, non-empty sets D_N with $N \to \infty$ as a tends to infinity. By assumption, $D_- \cup D_+ = (a_\alpha, \infty)$. But this is an open interval in $\mathbb{R}$ which cannot be written as the disjoint union of open non-empty sets D_N. Hence, there exists $a_+ > M_\alpha$ such that $(a_+, 0) \notin D_+ \cup D_-$. Solutions to such initial data will therefore not enter the region $\Omega_- \cup \Omega_+$. $\qquad\square$

Lemma 3.6. *For solutions to* (3.1)-(3.2) *corresponding to initial data $a > M_\alpha$ such that they stay outside of $\Omega_- \cup \Omega_+$ for all $r \geq 0$, there exist $0 < T < \infty$ such that $E(T) = 0$ and $\psi(T) = \beta(T) = 0$.*

Proof. Let us assume that $E(r) > 0$ for all $r \geq 0$ and show that this leads us to a contradiction. Recall (3.28) from the previous lemma. Under the assumption that $E > 0$ for all $r \geq 0$, this bound on θ' holds in particular for all $r > \frac{2}{\alpha}$. Consequently, a solution to (3.1)-(3.2) with $a > M_\alpha$ and $r > \frac{2}{\alpha}$ would surround the region $\Omega_- \cup \Omega_+$ with angular velocity between $1 + \frac{\alpha}{4}$ and $\frac{\alpha}{4}$ in clockwise direction. Thus we can construct an increasing sequence $\{r_n\}_{n \geq 1}$ with $r_1 > \frac{2}{\alpha}$ such that $\theta(r_n) = \frac{\pi}{6} + 2(n-k)\pi$ where $k \in \mathbb{N}$ is fixed. We infer that

$$(3.29) \qquad \frac{8\pi}{\alpha + 4} < r_{n+1} - r_n < \frac{8\pi}{\alpha} \qquad \text{for } r_1 > \frac{2}{\alpha},\ n \geq 1,$$

since in view of (3.28) we have $2\pi = \theta(r_{n+1}) - \theta(r_n) < (1 + \frac{\alpha}{4})(r_{n+1} - r_n)$ and $2\pi = \theta(r_{n+1}) - \theta(r_n) > \frac{\alpha}{4}(r_{n+1} - r_n)$. This shows that independent of the number of cycles n the solution has completed, the "time" it takes the solution to return to the ray $\theta(r_n) = \frac{\pi}{6}$ is bounded from above and below by constants. Now consider the region

$$(3.30) \qquad A := \left\{ (\psi, \beta) : E > 0, \frac{\pi}{6}\left(1 - \frac{1}{3}\alpha\right) < \theta < \frac{\pi}{6} \right\}.$$

From the dynamics of the system (3.1) we infer that the solutions enter the region A crossing the ray $\theta = \frac{\pi}{6}$ at some time $r = r_n$ and leave it crossing the ray $\theta = \frac{\pi}{6}(1 - \frac{1}{3}\alpha)$ at some bigger value of $r = r_n^+$. This value r_n^+ satisfies

$$(3.31) \qquad r_n + \frac{2\pi}{9}\frac{\alpha}{\alpha + 4} < r_n^+ < r_n + \frac{2\pi}{9},$$

since we infer from (3.28) that

$$\left(-1 - \frac{\alpha}{4}\right)(r_n^+ - r_n) < \theta(r_n^+) - \theta(r_n) = \frac{\pi}{6}\left(1 - \frac{1}{3}\alpha\right) - \frac{\pi}{6} = -\alpha\frac{\pi}{18}$$

$$\text{and } -\frac{\alpha}{4}(r_n^+ - r_n) > \theta(r_n^+) - \theta(r_n) = -\alpha\frac{\pi}{18}.$$

Passing to polar coordinates (3.17), we find that

$$(3.32) \qquad R > \left(\frac{\sqrt{3}}{2}\right)^{\frac{2-\alpha}{\alpha}}\left(\frac{2}{2-\alpha}\right)^{\frac{1}{\alpha}} \quad \text{in A.}$$

To see this, note that for $\theta < \frac{\pi}{6}$ we have $\cos\theta > \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and thus $E > 0$ whenever $\psi^2 + \beta^2 > \frac{2}{2-\alpha}|\psi|^{2-\alpha}$ or, equivalently,

$$R^2 > \frac{2}{2-\alpha}R^{2-\alpha}|\cos\theta|^{2-\alpha} \geq \frac{2}{2-\alpha}R^{2-\alpha}\left(\frac{\sqrt{3}}{2}\right)^{2-\alpha}.$$

Consequently, in view of (3.17),

$$(3.33) \qquad \beta^2(r) > \sin^2\left(\frac{\pi}{9}\right)\left(\frac{\sqrt{3}}{2}\right)^{\frac{2(2-\alpha)}{\alpha}} =: K_\alpha \qquad \text{for } r \in (r_n, r_n^+),$$

since $\frac{\pi}{6}(1 - \frac{1}{3}\alpha) > \frac{\pi}{9}$ for $\alpha \in (0,1)$ and $\sin(\theta) > \sin\left(\frac{\pi}{6}(1 - \frac{1}{3}\alpha)\right)$ in A. From (3.32) we infer that

$$\beta^2(r) = R^2 \sin^2(\theta) > \left(\frac{\sqrt{3}}{2}\right)^{\frac{2(2-\alpha)}{\alpha}} \left(\frac{2}{2-\alpha}\right)^{\frac{2}{\alpha}} \sin^2(\frac{\pi}{9})$$

$$> \sin^2(\frac{\pi}{9})\left(\frac{\sqrt{3}}{2}\right)^{\frac{2(2-\alpha)}{\alpha}} \qquad \text{in } A.$$

Furthermore, by virtue of (3.29) and (3.31),

$$(3.34) \qquad\qquad r_n + \frac{2\pi}{9}\frac{\alpha}{\alpha+4} < r_n^+ < r_{n+1},$$

since $r_n^+ < r_n + \frac{2\pi}{9} < r_n + \frac{8\pi}{\alpha+4} < r_{n+1}$ for all $\alpha \in (0,1)$. But now from (3.15), together with (3.33), (3.31) and (3.34), we get a contradiction:

$$E\left(\frac{2}{\alpha}\right) - E(\infty) = -\int_{2/\alpha}^{\infty} E'(r)dr = \int_{\frac{2}{\alpha}}^{\infty} \frac{\beta^2(r)}{r}dr$$

$$\geq K_\alpha \sum_{n\geq 1} \int_{r_n}^{r_n^+} \frac{1}{r}dr \geq K_\alpha \sum_{n\geq 1}\frac{1}{r_n^+}(r_n^+ - r_n)$$

$$> K_\alpha \frac{2\pi}{9}\frac{\alpha}{\alpha+4}\sum_{n\geq 1}\frac{1}{r_n + \frac{2\pi}{9}} = \infty.$$

The series is divergent since $r_n < \frac{8\pi}{\alpha}(n-1) + r_1$ in view of (3.29), and $E(\infty)$ is some finite number, as E is bounded by virtue of Lemma 3.2. Recall that we assumed $E(r)$ for all $r \geq 0$, which lead to the above contradiction. Thus there exists a finite value of $T > 0$ such that $E(T) = 0$. Notice that for such values of T we also have $\psi(T) = \beta(T) = 0$. If this were not the case, the dynamics of the system would force the solution to enter Ω_+ or Ω_- at this point, which is contradictory to the assumption of this Lemma. $\qquad\qquad\square$

3.3. **Limiting cases of the parameter α.** In the case where $\alpha = 1$, the vorticity function γ simplifies to

$$(3.35) \qquad\qquad \gamma(\psi) = \psi - \frac{\psi}{|\psi|}$$

which has a point of discontinuity at $\psi = 0$. As we are only interested in classical solutions, we will not consider this case.

In the case where $\alpha = 1$ we simply have

$$(3.36) \qquad\qquad \gamma(\psi) \equiv 0.$$

Thus, system (2.3)-(2.2), for which we seek compactly supported $\mathcal{C}^2$-solutions, reads

$$(3.37) \qquad\qquad \begin{cases} \psi'' + \frac{1}{r}\psi' = 0, & r > 0, \\ \psi(0) = a, \ \psi'(0) = 0, \end{cases}$$

which we can solve easily, obtaining

$$\psi(r) = C_1 \ln(r) + C_2,$$

for some constants $C_1, C_2 \in \mathbb{R}$. In view of the boundary conditions,

$$C_1 = 0 \text{ and } C_2 = a,$$

and we conclude that $\psi(r) \equiv a$ is constant for all $r \geq 0$. In the setting of ψ being the stream function on the fluid domain D, this means that $\psi \equiv a$ is constant throughout the flow field. The boundary conditions $\psi = \psi_y = 0$ on the flat free surface require this constant to be zero. So $\psi \equiv 0$ and the water is still throughout the fluid domain, which is why we do not consider this case.

References

[1] G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, 1967.

[2] E. Bryant, *Tsunami: The Underrated Hazard*, Springer Praxis Books, Springer Berlin Heidelberg, 2008.

[3] A. Constantin, *The trajectories of particles in Stokes waves*, Invent. Math. **166** (2006), 523–535.

[4] A. Constantin, *On the relevance of soliton theory to tsunami modelling*, Wave Motion **46** (2009), 420–426.

[5] A. Constantin, *On the propagation of tsunami waves, with emphasis on the tsunami of 2004*, Discrete Contin. Dyn. Syst. Ser. B **12** (2009), 525–537.

[6] A. Constantin, *A dynamical systems approach towards isolated vorticity regions for tsunami background states*, Arch. Ration. Mech. Anal. DOI 10.1007/S00205-010-0347-1.

[7] A. Constantin and J. Escher, *Analyticity of periodic traveling free surface water waves with vorticity*, Ann. of Math. **172** (2010).

[8] A. Constantin and D. Henry, *Solitons and tsunamis*, Z. Naturforsch. **64a** (2009), 65–68.

[9] A. Constantin and R. S. Johnson, *Modelling tsunamis*, J. Phys. A **39** (2006), L215–L217.

[10] A. Constantin and R. S. Johnson, *Propagation of very long water waves, with vorticity, over variable depth, with applications to tsunamis*, Fluid Dynam. Res. **40** (2008), 175–211.

[11] A. Constantin and R. S. Johnson, *On the non-dimensionalisation, scaling and resulting interpretation of the classical governing equations for water waves*, J. Nonl. Math. Phys. **15** (2008), 5873.

[12] A. Constantin and R. S. Johnson, *Addendum: Propagation of very long water waves, with vorticity, over variable depth, with applications to tsunamis*, Fluid Dynam. Res. **42** (2010), Art. No. 038901.

[13] A. Constantin and W. Strauss, *Exact steady periodic water waves with vorticity*, Comm. Pure Appl. Math. **57** (2004), 481–527.

[14] A. Constantin and W. Strauss, Pressure beneath a Stokes wave, *Comm. Pure Appl. Math.* **53** (2010), 533–557.

[15] W. A. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, D. C. Heath and Co., Boston, Massachussetts, 1965.

[16] W. Craig, Surface water waves and tsunamis, *J. Dynam. Differential Equations* 18 (2006), 525–549.

[17] J. Dieudonné, *Foundations of Modern Analysis*, Academic Press, New York-London, 1969

[18] V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall, New Jersey, 1974.

[19] J. L. Hammack, *A note on tsunamis: their generation and propagation in an ocean of uniform depth*, J. Fluid Mech. **60** (1973), 769–799.

[20] R. S. Johnson, *A Modern Introduction to the Mathematical Theory of Water Waves*, Cambridge University Press, Cambridge, 1997.

[21] M. Lakshmanan, *Integrable nonlinear wave equations and possible connections to tsunami dynamics*, in "Tsunami and Nonlinear Waves" (ed. A. Kundu), Springer, Berlin (2007), pp. 31–49.

[22] P. A. Madsen, D. R. Fuhrman, and H. A. Schaeffer, *On the solitary wave paradigm for tsunamis*, J. Geophys. Res. - Oceans **113** (2008), Art. No. C12012.

[23] H. Segur, *Waves in shallow water with emphasis on the tsunami of 2004*, in "Tsunami and Nonlinear Waves" (ed. A. Kundu), Springer, Berlin (2007), pp. 3-29.

[24] H. Segur, *Integrable models of waves in shallow water*, in "Probability, Geometry and Integrable Systems" (ed. M. Pinsky et al.), Cambridge University Press, Math. Sci. Res. Inst. Publ. (2008), pp. 345–371.

[25] R. Stuhlmeier, *KdV theory and the Chilean tsunami of 1960*, Discrete Contin. Dyn. Syst. Ser. B **12** (2009), 623–632.

[26] N. Zahibo, E. Pelinovsky, T. Talipova, A. Kozelkov, and A. Kurkin, *Analytical and numerical study of nonlinear effects at tsunami modeling*, Appl. Math. Comput. **174** (2006), 795–809.