A NOTE ON THE DARBOUX THEORY OF INTEGRABILITY OF NON–AUTONOMOUS POLYNOMIAL DIFFERENTIAL SYSTEMS

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ABSTRACT. In this work we unfold some differential algebraic aspects of Darboux first integrals of polynomial vector fields. An interesting improvement is that our approach can be applied both to autonomous and non-autonomous vector fields. We give a sufficient and necessary condition for the existence of a Darboux first integral of a specific form for a polynomial vector field with some known algebraic invariant hypersurfaces. For the autonomous case, the classical result of Darboux is obtained as a corollary. For the non-autonomous case our characterization improves a known criterium.

1. INTRODUCTION

One of the most important subjects in mathematical physics is the discussion of integrability of differential equations started by Newton in the 17th century. There is not an unified approach to integrability. There are many different technics for integrating differential equations or proving their non–integrability within some specific framework: Lie symmetries [26], Noether symmetries [4], Lax pairs [14], Painlevé analysis [2], differential Galois theory [24, 28], Darboux's method [10], among others.

A useful approach is the search of invariants and their classification. The first class of invariants are the conserved quantities or first integrals. There are also some other invariants like Jacobi multipliers, tensor invariants or symmetries. The search and classification of those invariant objects of differential equations is still a very active research topic.

In this paper we give some generalization of a classical method started by Darboux in 1878 for the computation of first integrals of polynomial differential systems, [10, 11]. More recently this method has been refined by several authors, Jouanolou, Singer, Schlomiuk, Llibre, Christopher, Zhang [5, 9, 18, 22, 20, 21, 13, 25, 28, 19, 17, 15] among others. This method has been related with some applications concerning limit cycles, centers and bifurcation problems, see for instance [12, 23, 27]. Moreover, several inverse problems for dimension two have been studied in [7, 16, 8].

Here we isolate the differential algebraic nature of the classical Darboux's method on elementary first integrals. More concretely, in Theorems 5 and 8 we present some new results related to the known theorems about Darboux integrability for autonomous and non–autonomous polynomial vector fields in $\mathbb{C}^n$.

The structure of the paper is the following: In Section 2 first we give the basic notions and the known results about Darboux's theory of integrability for autonomous polynomial differential systems in $\mathbb{C}^n$. The new results are presented in Subsection 2.2. More concretely, Theorem 5 improves the known results of Darboux integrability for autonomous polynomial systems. The examples in Subsection 2.3 illustrate these improvements. In Section 3 first we present the basic notions about Darboux first integrals and Jacobi multipliers for a class of non–autonomous vector fields. The new results for non-autonomous vector fields are given in Subsection 3.2, in Theorem 8. At the end we illustrate the theory with an example.

2. Darboux first integrals for autonomous differential systems

2.1. **Basic notions and known results.** First, we will fix the following notation. Let $\mathcal{F}$ be a differential field of characteristic zero, and let us denote by ∂ its derivation. The field of constants of $\mathcal{F}$ is denoted by $C(\mathcal{F})$. We consider $a_1, \cdots, a_r \in \mathcal{F}$. We denote by $\mathcal{W}(a_1, \ldots, a_r)$ the Wronskian matrix

$$\mathcal{W}(a_1, \cdots, a_r) = \begin{pmatrix} a_1 & \cdots & a_r \\ \partial a_1 & \cdots & \partial a_r \\ & \cdots & \\ \partial^{(r-1)} a_1 & \cdots & \partial^{(r-1)} a_r \end{pmatrix}. \tag{1}$$

The following differential algebraic lemma contains the core of Darboux's method for the search of first integrals and Jacobi multipliers.

Lemma 1. *Let us consider elements $f_1, \cdots, f_r, \psi \in \mathcal{F}$ and denote by $a_i = (\partial f_i)f_i^{-1} = \partial \log f_i$. The following conditions are equivalent:*

(i) *The rank of the Wronskian matrices $\mathcal{W}(a_1, \ldots, a_r, \psi)$ and $\mathcal{W}(a_1, \ldots, a_r)$ coincide.*

(ii) *ψ is a non-trivial $C(\mathcal{F})$-linear combination of $a_1, \ldots, a_r$*

$$\psi = \lambda_1 a_1 + \cdots + \lambda_r a_r. \tag{2}$$

(iii) *There exist constants $\lambda_1, \cdots, \lambda_r$ in $C(\mathcal{F})$ such that for any differential extension $\mathcal{F} \subset \mathcal{F}(b_1, \cdots, b_r)$ by elements b_i satisfying $\partial b_i = \lambda_i a_i b_i$ the element $M = b_1 \cdots b_r$ verifies $\partial M = \psi \cdot M$.*

Proof. The equivalence between (i) and (ii) is due to Lemma 1.12 in [29]. The equivalence between (ii) and (iii) is proved using the same argument as in the proof of the classical theorem of Darboux, see [11]. We note that a simple computation shows that $\partial \log M = \sum_{i=1}^{r} \lambda_i a_i$. □

Remark 2.

(i) *If $\psi = 0$, then M is a new constant.*

(ii) *If ∂ represents the Lie derivative with respect a vector field of divergence ψ then M is a Jacobi multiplier.*

We will develop this remark throughout the forthcoming sections.

We consider the $n \times n$ complex polynomial differential systems of the form

$$\dot{x_j} = \frac{dx_j}{dt} = P_j(x_1, \cdots, x_n) = \sum_{0 \le i_1 + \cdots + i_n \le m}^{n} a_{j i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}, \qquad j = 1, \cdots, n \tag{3}$$

with $a_{ji_1\cdots i_n} \in \mathbb{C}$. Note that system (3) has degree $m = \max\{\deg P_1, \cdots, \deg P_n\}$. The associated vector field of system (3) is

$$X = \sum_{i=1}^{n} P_i \frac{\partial}{\partial x_i}. \tag{4}$$

Let $f \in \mathbb{C}[x_1, \cdots, x_n]$ be irreducible. The hypersurface $f = 0$ is an *invariant hypersurface* of the vector field (4) if it satisfies

$$X(f) = \sum_{j=1}^{n} P_j \frac{\partial f}{\partial x_j} = Kf,$$

or, equivalently,

$$\frac{X(f)}{f} = X(\log(f)) = K, \tag{5}$$

with $K \in \mathbb{C}_{m-1}[x_1, \cdots, x_n]$. We stress that if the polynomial system (3) has degree m then we have that $\deg K \leq m-1$, independently of the degree of the hypersurface $f(x_1, \cdots, x_n) = 0$. Note that if an orbit $x(t) = (x_1(t), \cdots, x_n(t))$ of the vector field X has a point on the hypersurface $f = 0$ then the whole orbit is contained in $f = 0$. This is the reason that we say that the algebraic hypersurface is invariant for the vector field X, i.e. it is invariant by its flow.

Assume that $h, g \in \mathbb{C}[x_1, \cdots, x_n]$ and are relatively prime polynomials in the variables $x_1 \cdots x_n$. The function $F(x_1, \cdots, x_n) = \exp(g/h)$ is called an *exponential factor* (see also [5]) of the differential system (3) if there is $K \in \mathbb{C}_{m-1}[x_1, \cdots, x_n]$ that satisfies

$$X(F) = \sum_{j=1}^{n} P_j \frac{\partial F}{\partial x_j} = KF, \tag{6}$$

or equivalently

$$X\left(\frac{g}{h}\right) = K,$$

and we say that K is the *cofactor* of the exponential factor F. If $F = \exp(g/h)$ is an exponential factor with g, h coprimes it is easy to prove that $h = 0$ is an invariant hypersurface of X. The existence of exponential factors with cofactor of degree $\leq m-1$ is related with the concept of multiplicity [9, 19, 20, 21].

Let W be an open subset of $\mathbb{C}^n$. An analytic function $I : W \longrightarrow \mathbb{C}$ is a *first integral* of the vector field X on W if $X(I) = 0$. This implies that I is constant along integral curves of X contained in W.

An analytic function $M : W \longrightarrow \mathbb{C}$ is a *Jacobi multiplier* of the vector filed X on W if it satisfies $X(M) = -(\text{div}X)M$ where $\text{div}X = \sum_{i=1}^{n} \partial P_i/\partial x_i$ is the divergence of the vector field (4). Note that due its definition a Jacobi multiplier is also an invariant hypersurface (maybe not a polynomial) of system (3) with cofactor $-\text{div}X$. Additionally, the existence of a Jacobi multiplier M yields to the divergence free system $MX = (MP_1, \cdots, MP_n)$.

Consider $f_1, \cdots, f_p$ irreducible and coprime polynomials in $\mathbb{C}[x]$ and some $F_1, \cdots, F_q$ exponential factors. Any function of the form

$$I = f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q}, \tag{7}$$

will be called a *Darboux function*. Note that powers and products of exponential factors are also exponential factors. Therefore any Darboux function can be written into the form

$$I = f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp\left(\frac{g}{h}\right). \tag{8}$$

A *Darboux first integral* (*Darboux Jacobi multiplier*) is a first integral (Jacobi multiplier) given by a Darboux function of the form (8).

The following theorem summarizes known results on the subject of Darboux first integrals dating back to Darboux [10], Jouanolou [13], and some more recent works like [5, 6, 19, 27] among others.

Theorem 3. *Assume that the polynomial vector field X in $\mathbb{C}^n$ given by (4) of degree m admits*

(i) *p irreducible and coprimes invariant algebraic hypersurfaces $f_1 = 0, \cdots, f_p = 0$ with cofactors $K_1, \cdots, K_p$.*

(ii) *q exponential factors $F_{p+1}, \cdots, F_{p+q}$ with cofactors $K_{p+1}, \cdots, K_{p+q}$ of degree $\leq m - 1$.*

Then the following statements hold.

(a) *If $p + q = \begin{pmatrix} n + m - 1 \\ n \end{pmatrix} + 1$, then the vector field X has a Darboux first integral.*

(b) *If $p + q = \begin{pmatrix} n + m - 1 \\ n \end{pmatrix} + n$ then the vector field X has a rational first integral.*

(c) *If $p + q = \begin{pmatrix} n + m - 1 \\ n \end{pmatrix}$, then the vector field X has either a Darboux first integral or a Darboux Jacobi multiplier.*

Statement (a) is due to Darboux [10, 11], and statement (b) of Theorem 3 was proved by Jouanolou [13]. For a short proof of statement (b) see [5, 6] for n = 2 and [19] for $n \geq 2$. A similar to statement (c) of Theorem 3 appears in [1] and its proof follows easily.

Remark 4. *In Theorem 3 the number $p + q \geq \begin{pmatrix} n + m - 1 \\ n \end{pmatrix}$ is a sufficient but not necessary condition for the existence of a Darboux first integral or of Jacobi multiplier.*

2.2. **New results for autonomous polynomial differential systems.** Let X be a polynomial vector field in $\mathbb{C}^n$. We consider $K_1, \cdots, K_r$ cofactors of hypersurfaces or of exponential factors of X. We denote by $\mathcal{W}$ the Wronskian matrix of the cofactors defined as

$$\mathcal{W} = \mathcal{W}(K_1, \cdots, K_r) = \begin{pmatrix} K_1 & \cdots & K_r \\ X(K_1) & \cdots & X(K_r) \\ & \cdots & \\ X^{(r-1)}(K_1) & \cdots & X^{(r-1)}(K_r) \end{pmatrix}, \tag{9}$$

where $X^{(l+1)}(K_j) = X(X^{(l)}(K_j))$. Next we are going to use the Wronskian (9) in order to present a result related to the classical Darboux Theorem (3).

Theorem 5. *Assume that the polynomial vector field X in $\mathbb{C}^n$ given by (4) of degree m admits*

  (i) *p irreducible and coprime invariant hypersurfaces, $f_1 = 0, \cdots, f_p = 0$ with cofactors $K_1, \cdots, K_p$.*
  (ii) *q exponential factors $F_{p+1}, \cdots, F_{p+q}$ with cofactors $K_{p+1}, \cdots, K_{p+q}$.*

We further assume that the 1-forms $d\log f_1, \ldots, d\log f_p, d\log F_{p+1}, \cdots, d\log F_{p+q}$ are linearly independent over $\mathbb{C}$.

  (a) *$p + q > \operatorname{rank}(\mathcal{W}(K_1, \cdots, K_{p+q}))$ if and only if the vector field X admits a first integral which is either rational or Darboux.*
  (b) *$p + q > \operatorname{rank}(\mathcal{W}(K_1, \cdots, K_{p+q}, \operatorname{div} X)) + 1$ if and only if the vector field X admits either a rational first integral or a Darboux Jacobi multiplier.*
  (c) *$p + q > \operatorname{rank}(\mathcal{W}(K_1, \cdots, K_{p+q})) + n$ then the vector field X has at least one rational first integral.*

Proof. The proof follows directly from Lemma 1. Let us consider $\mathcal{F} = \mathbb{C}(x_1, \cdots, x_n)$ as a differential field endowed with the derivation X. The field of constants $C(\mathcal{F})$ is the field of rational first integrals of X. Let us prove statement (a). Condition (i) of Lemma 1 is satisfied with $\psi = 0$ (see equation (2)). Assuming that there is not a non-constant rational first integral, then we have that there is a non-trivial linear combination over $\mathbb{C}$, $\lambda_1 K_1 + \cdots + \lambda_{p+q} K_{p+q} = 0$ and therefore $I = f_1^{\lambda_1} \cdots F_{p+q}^{\lambda_{p+q}}$ is a new constant of the extension $\mathcal{F} \subset \mathcal{F}(f_1^{\lambda_1}, \cdots, F_{p+q}^{\lambda_{p+q}})$, and therefore a first integral of X. This function I is not a constant in $\mathbb{C}^n$ because of the linear $\mathbb{C}$-independence of $d\log f_1, \ldots, d\log f_p, d\log F_{p+1}, d\log F_{p+q}$.

Let us prove statement (b). Now, condition (i) of Lemma 1 is satisfied with $\psi = -\operatorname{div} X$. Following the same argument, by Lemma 1(iii) we have that $M = f_1^{\lambda_1} \cdots F_{p+q}^{\lambda_{p+q}}$ verifies that $XM = (-\operatorname{div} X)M$ and so it is a Jacobi multiplier of X.

The proof of statement (c) will be given as a particular case of Theorem 8. $\qquad\square$

Remark 6.

  (i) *We note that the hypothesis of independence guarantees that $d\log I$ is a non zero rational 1-form and therefore I is a non-constant function.*

  (ii) *In Theorem 5 there is no condition about the degree of the cofactors $K_{p+1}, \ldots, K_{p+q}$ of the exponential factors.*

  (iii) *If $\det(\mathcal{W}(K_1, \ldots, K_r)) \neq 0$ means that there is not a Darboux first integral formed by the invariant hypersurfaces and exponential factors that corresponds to the cofactors $K_1, \cdots, K_r$. However, it maybe exist a Darboux first integral formed by a different collection of invariant hypersurfaces and exponential factors.*

2.3. **Examples.**

Example 1. *We consider the quadratic vector field,*

$$X = (\mu x + \beta xy)\frac{\partial}{\partial x} + (\lambda y + \alpha xy)\frac{\partial}{\partial y}.$$

It is clear that $f_1 = x$, $f_2 = y$ are invariant algebraic curves with cofactors $K_1 = \mu + \beta y$ and $K_2 = \lambda + \alpha x$ respectively. We also consider exponential factors $F_3 = e^x$ and $F_4 = e^y$, with cofactors $K_3 = xK_1$ and $K_4 = yK_2$ of degree two. We have $\mathrm{rank}\mathcal{W}_3(K_1, K_2, K_3, K_4) = 3$ and so according to Theorem 5(b) we obtain the Darboux first integral of the form $I = x^\lambda y^{-\mu} e^{\alpha x - \beta y}$. Note that Theorem 3 cannot been applied in this case.

Note that the exponential factor $e^{\alpha x - \beta y}$ is also invariant of X with cofactor $K = \alpha\mu x - \beta\lambda y$ of degree one and we have that $\mathrm{rank}(\mathcal{W}_3(K_1, K_2, K)) = 2$ and also $\lambda K_1 - \mu K_2 + K = 0$. Therefore we can apply both Theorems 3 and 5.

Example 2. *We consider the polynomial differential system*

$$
\begin{aligned}
\dot{x} &= xz(bx^{n_1}y^{n_2}z^{n_3} - n_2) \\
\dot{y} &= -y^{n_2+1}x^{n_1}z^{n_3}(\gamma cx + az) + y(n_3\gamma x + n_1 z) \\
\dot{z} &= \gamma xz(bx^{n_1}y^{n_2}z^{n_3} - n_2),
\end{aligned}
\tag{10}
$$

with n_1, n_2, n_3 positive integers and $a, b, c \in \mathbb{C}$ of degree $n_1 + n_2 + n_3 + 2$. Easy computations shows that system (10) admits the invariant surfaces $f_1 = x$, $f_2 = y$, $f_3 = z$ with cofactors $K_1 = bx^{n_1}y^{n_2}z^{1+n_3} - n_2 z$, $K_2 = -y^{n_2}2x^{n_1}z^{n_3}(\gamma cx + az) + n_1 z + n_3\gamma x$ and $K_3 = \gamma x(bx^{n_1}y^{n_2}z^{n_3} - n_2)$ respectively. Additonally, admits the exponential factor $F = \exp\left(\frac{1}{x^{n_1}y^{n_2}z^{n_3}}\right)$ with cofactor $K = -n_1bz + n_2\gamma cx + n_2 az - \gamma n_3 bx$. We have $\mathrm{rank}(\mathcal{W}_3(K_1, K_2, K_3)) = 2$ and therefore we can apply Theorem 5 (a). We obtain the first integral

$$
H(x, y, z) = x^a y^b z^c \exp\left(\frac{1}{x^{n_1}y^{n_2}z^{n_3}}\right).
$$

Note that for system (10) we have $n = 2$, $m = n_1 + n_2 + n_3 + 2$ and so

$$
\binom{n + m - 1}{n} = \frac{(n_1 + n_2 + n_3 + 2)(n_1 + n_2 + n_3 + 3)}{2}.
$$

Therefore for the application of Theorem 3 we need an arbitrarily big number of invariant surfaces and exponential factors.

3. Darboux first integrals for non–autonomous differential systems

3.1. **Basic notions.** Let U be a connected open subset of $\mathbb{C}$. We consider $\mathcal{C}^\omega(U, \mathbb{C})$ the ring of analytic functions and we denote by $M(U, \mathbb{C}) = \{f/g : f, g \in \mathcal{C}^\omega(U, \mathbb{C})\}$ the field of meromorphic functions in U. We recall that the derivative of a meromorphic function is also a meromorphic function and so the field $M(U, \mathbb{C})$ endowed with the derivation operator d/dt is also a differential field.

We also consider the ring of polynomials in the variables $x_1, \cdots, x_n$ with coefficients in $M(U, \mathbb{C})$ and we denote it by $M(U, \mathbb{C})[x] = M(U, \mathbb{C})[x_1, \cdots, x_n]$. In particular $M(U, \mathbb{C})[x]$ is a domain of unique factorization. In the following we denote by $M(U, \mathbb{C})(x)$ the ring of rational functions in the variables $x_1 \cdots x_n$ and coefficients in $M(U, \mathbb{C})$. Here, we deal with the differential systems of the form

$$
\dot{x}_j = \frac{dx_j}{dt} = P_j(t, x) = \sum_{0 \le i_1 + \cdots + i_n \le m} a_{ji_1\cdots i_n}(t)x_1^{i_1}\cdots x_n^{i_n}, \qquad j = 1, \cdots, n
\tag{11}
$$

with $a_{ji_1\cdots i_n} \in M(U, \mathbb{C})$ and so $P_j \in M(U, \mathbb{C})[x]$ for $j = 1, \cdots, n$. Note that system (11) is a polynomial differential system in the variables $x_1, \cdots x_n$ of *degree*

$m = \max\{\deg P_1, \cdots, \deg P_n\}$. In what follows these systems will be called *non-autonomous polynomial differential systems*. Note that t is the independent variable of system (3) called the *time*.

We associate to the differential system (11) the vector field in $U^* \times \mathbb{C}^n$,

$$X = \frac{\partial}{\partial t} + \sum_{j=1}^{n} P_j \frac{\partial}{\partial x_j}. \tag{12}$$

where U^* is the open subset of $\mathbb{C}$ obtained by removing from U the poles of the coefficients of the polynomials P_i. The vector field X can be interpreted as a derivation of the ring $M(U, \mathbb{C})[x]$ and therefore of the quotient field $M(U, \mathbb{C})(x)$. Thus, we obtain a differential field extension $(M(U, \mathbb{C}), d/dt) \subset (M(U, \mathbb{C})(x), X)$.

Let $f_1, \cdots, f_p \in M(U, \mathbb{C})[x]$ be irreducible and coprime polynomials. We also consider $F = \exp(g/h)$ with $g, h \in M(U, \mathbb{C})[x]$ coprimes.

Any function of the form

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp\left(\frac{g}{h}\right), \tag{13}$$

will be called a *generalized Darboux function*.

Let W be an open subset of $U \times \mathbb{C}^n$. A non constant analytic function $I : W \longrightarrow \mathbb{C}$ is a *first integral* of the vector field X on W if it is constant on all integral curves $x(t) = (t, x_1(t), \cdots, x_n(t))$ of X contained in W. Hence $I(x(t)) = $ constant for all values of t for which the solution $x(t)$ is defined and contained in W. So, $X(I) = 0$ on W.

A nonconstant analytic function $M : W \longrightarrow \mathbb{C}$ is a *Jacobi multiplier* of the vector filed X on W if it satisfies $X(M) = -(\text{div}X)M$ where $\text{div}X = \sum_{i=1}^{n} \partial P_i/\partial x_i$ is the divergence of the vector field (12). Note that due its definition a Jacobi multiplier is also an invariant hypersurface (maybe not a polynomial) of system (11) with cofactor $-\text{div}X$. Additionally, the existence of a Jacobi multiplier M yields to the divergence free system $MX = (M, MP_1, \cdots, MP_n)$.

For non–autonomous systems (11) a *Darboux first integral* is a first integral given by a generalized Darboux function (13) and similarly for *Darboux Jacobi multiplier*.

The theory of Darboux integrability for a 2–dimensional non–autonomous polynomial systems has been developed in [17].

3.2. **New results for non-autonomous polynomial differential systems.** In what follows we consider the differential system X in $U^* \times \mathbb{C}^n$ given by expression (12). First we prove the following lemma.

Lemma 7. *Let W be an open subset of $U \times \mathbb{C}^n$. Let $J_1, \cdots, J_n$ be functionally independent analytic first integrals of X on W.*

(a) *If J_{n+1} is another first integral then there exists analytic functions $C_1, \cdots, C_n$ such that:*

$$dJ_{n+1} = C_1 dJ_1 + \cdots + C_n dJ_n. \tag{14}$$

(b) *The functions $C_1, \ldots, C_n$ are first integrals of n.*

(c) *Assume $J_k = \log(I_k)$ for $k = 1, \ldots, n+1$ where $I_1, \ldots, I_{n+1}$ are Darboux first integrals in which the factors $f_1, \ldots, f_p, F_{p+1}, \ldots, F_{p+q}$ appear, being $d\log f_1, \ldots, d\log f_p, d\log F_{p+1}, \ldots, d\log F_{p+q}$ $\mathbb{C}$-linearly independent. Assume that at least one factor appearing in I_{n+1} does not appear in the $I_1, \ldots, I_n$. Then, the functions C_i are in $M(U, \mathbb{C})(x)$ and at least one of them is not a constant.*

Proof. (a) The vector field X has no zeroes inside W. It follows that at each point of W the 1-forms $dJ_1, \ldots, dJ_n$ are a basis of the space of 1-forms anihilated by X. The functions $C_1, \ldots, C_n$ appearing in (14) are simply the coordinates of dJ_{n+1} in the above basis.

(b) Since the functions J_k are first integrals, the Lie derivatives $\mathrm{Lie}_X(dJ_k)$ vanish. The Lie derivative sasifies the Leibniz formula, and therefore by derivation of (14) we have,

$$0 = \sum_{k=1}^{n} \mathrm{Lie}_X(C_k dJ_k) = \sum_{k=1}^{n} X(C_k) dJ_k.$$

Taking into account that $J_1, \ldots, J_n$ are functionally independent it follows that $X(C_k) = 0$ for $k = 1, \ldots, n$.

(c) First, if the functions I_k are of Darboux type, it follows that $dJ_k = I_k^{-1} dI_k$ are 1-forms,

$$dJ_k = f_{k0} dt + \sum_{j=1}^{n} f_{kj} dx_j$$

were the coefficients f_{ki} are in $M(U, \mathbb{C})(x)$. The functions C_k are the coordinates of dJ_{k+1} in the basis $dJ_1, \ldots, dJ_n$. They are obtained by solving a linear systems of equations with coefficients in $M(U, \mathbb{C})(x)$, and therefore they are in $M(U, \mathbb{C})(x)$. Let us assume now that all the functions C_k are constant. By hypothesis, the 1-forms $dJ_1, \ldots, dJ_{n+1}$ are $\mathbb{C}$-linear combinations of $d\log f_1, \ldots, d\log f_p, d\log F_{p+1}, \ldots, d\log F_{p+q}$. Substituting them in equation (14) it leads us to a non-trivial vanishing $\mathbb{C}$-linear combination of $d\log f_1, \ldots, d\log f_p, d\log F_{p+1}, \ldots, d\log F_{p+q}$. By hypothesis those 1-forms are $\mathbb{C}$-linearly independent, and therefore at least on of the functions C_k is not in $\mathbb{C}$. □

The following theorem generalizes the Darboux theory of integrability for n–dimensional non–autonomous polynomial systems.

Theorem 8. *We assume that the differential system X in $U \times \mathbb{C}^n$ given by (11) of degree m admits*

(i) *p invariant hypersurfaces $f_1 = 0, \cdots, f_p = 0$ with cofactors $K_1, \cdots, K_p$. Here $f_1, \cdots, f_p \in M(U, \mathbb{C})[x]$ are irreducible and coprime polynomials.*

(ii) *q exponential factors $F_{p+1}, \cdots, F_{p+q}$ with cofactors $K_{p+1}, \cdots, K_{p+q}$. Here $F_i = \exp(g_i/h_i)$ with $g_i, h_i \in M(U, \mathbb{C})[x]$ are irreducible and coprimes.*

We further assume that $d\log f_1, \ldots, d\log f_p, d\log F_{p+1}, \ldots, d\log F_{p+q}$ are $\mathbb{C}$-linearly independent as meromorphic 1-forms in $U \times \mathbb{C}^n$.

Then the following statements hold.

(a) *$p + q > \mathrm{rank}(\mathcal{W}(K_1, \cdots, K_r))$ if there is a first integral. This first integral is either rational or Darboux (of the form (13)).*

(b) $p+q > \operatorname{rank}(\mathcal{W}(K_1, \cdots, K_r, div\, X))+1$ *if and only if there is a rational first integral or a Darboux Jacobi Multiplier of X of the form* (13).

(c) *If* $p+q > \operatorname{rank}(\mathcal{W}(K_1, \cdots, K_r))+n$ *for* $j = 1, \cdots, n$ *then the vector field X has at least one rational first integral.*

Proof. Let us consider $\mathcal{F} = M(U, \mathbb{C})(x)$ endowed with the derivation X. The proof of statements (a) and (b) here follows exactly the same argument than Theorem 5 statements (a) and (b).

Let us prove statement (c). Reasoning by *reductio ad absurdum* let us assume that $\operatorname{rank}(\mathcal{W}) < p+q-n$ and there is no rational first integral. Therefore, all minors of $\mathcal{W}$ of order $p+q-n$ vanish. Let us assume that there is a minor of order $p+q-n-1$ that does not vanish otherwise we can forget some of our invariant surfaces or exponential factors. Let us reorder our factors such that $\det(\mathcal{W}(K_1, \cdots, K_{p+q-n-1})) \neq 0$ but for each $j = p+q-n-1, \cdots, p+q$ we have $\det(\mathcal{W}(K_1, \cdots, K_{p+q-n-1}, K_j)) = 0$. The $n+1$ Wronskian matrices considered here are under the hypothesis of statement (a) and therefore they lead to $n+1$ different Darboux first integrals $I_1, \cdots, I_{n+1}$. The application of Lemma 7 to that set of first integrals leads us to the existence of a non–constant rational first integral. This completes the proof. $\square$

Remark 9. *Note that the condition about the existence of a Darboux first integrals and Jacobi multipliers improve the conditions of Theorem 2 in* [17].

Example 3. *The integrability of autonomous three-dimensional Lotka–Volterra systems has been studied by many authors, see for example* [3]. *Here we consider the non–autonomous Lotka–Volterra system:*

$$
\begin{aligned}
\dot{x} &= x\left(\frac{ta_1(t)-1}{t} - \frac{\lambda_2 b_{21}+b_{31}}{\lambda_1}tx + b_{12}y + b_{13}z\right) \\
\dot{y} &= y\left(a_2(t) + b_{21}tx - \frac{\lambda_1 b_{12}+b_{32}}{\lambda_2}y + b_{23}z\right) \\
\dot{z} &= z(-\lambda_1 a_1(t) - \lambda_2 a_2(t) + tb_{31}x + b_{32}y - (\lambda_1 b_{13}+\lambda_2 b_{23})z).
\end{aligned}
\tag{15}
$$

System (15) *admits the invariant hypersurfaces* $f_1 = x$, $f_2 = y$ *and* $f_3 = z$ *with cofactors*

$$
\begin{aligned}
K_1 &= -\frac{(\lambda_2 b_{21}+b_{31})}{\lambda_1}tx + b_{12}y + b_{13}z + \frac{ta_1(t)-1}{t} \\
K_2 &= a_2(t) + b_{21}tx - \frac{\lambda_1 b_{12}+b_{32}}{\lambda_2}y + b_{23}z \\
K_3 &= -\lambda_1 a_1(t) - \lambda_2 a_2(t) + tb_{31}x + b_{32}y - (\lambda_1 b_{13}+\lambda_2 b_{23})z.
\end{aligned}
$$

Note that $\det(\mathcal{W}(K_1, K_2, K_3)) \neq 0$.

System (15) *admits the exponential factor* $F = \exp(\lambda_1 \ln(t))$ *with cofactor* $K = \lambda_1/t$ *and we have that* $\det(\mathcal{W}(K_1, K_2, K_3, K)) = 0$. *System* (15) *admits the Darboux first integral*

$$
H = x^{\lambda_1}y^{\lambda_2}zt^{\lambda_1}.
$$

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