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POLYNOMIAL VECTOR FIELDS IN $\mathbb{R}^3$ WITH INFINITELY MANY LIMIT CYCLES

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Abstract. We provide a method to obtain polynomial vector fields in $\mathbb{R}^3$ having infinitely many limit cycles starting from polynomial vector fields in $\mathbb{R}^2$ having a period annulus. We give two examples of polynomial vector fields in $\mathbb{R}^3$ having infinitely many limit cycles, one of them of degree 2 and the other one of degree 12. The main tools are the Melnikov integral and the Hamiltonian structure.

1. Introduction

A vector field $X : \mathbb{R}^3 \to \mathbb{R}^3$ of the form $X = (P, Q, R)$ is called a *polynomial vector field of degree m* if P, Q and R are polynomials and m is the maximum of the degrees of P, Q and R.

A *limit cycle* of a vector field is an isolated periodic solution in the set of all periodic solutions of this vector field.

In this paper we provide a method to construct polynomial vector fields in $\mathbb{R}^3$ having infinitely many limit cycles in $\mathbb{R}^3$, starting from polynomial vector fields in $\mathbb{R}^2$ having a period annulus. The main tools are the Melnikov integral and the Hamiltonian structure.

It is known that every polynomial vector field in $\mathbb{R}^2$ has finitely many limit cycles, see Écalle [9] and Il'Yashenko [15]. Also there are colateral proofs showing the existence of polynomial vector fields in $\mathbb{R}^3$ having infinitely many limit cycles. For instance in the papers [20, 18, 19] the authors show the existence of a geometric Lorenz attractor in polynomial differential systems near the Lorenz system, and using [13] it follows the existence of polynomial vector fields in $\mathbb{R}^3$ having infinitely many limit cycles. There are other papers showing the existence of infinitely many periodic orbits in polynomial vector fields of degree 2

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and 4 in $\mathbb{R}^3$ (see [6] and [11], respectively) which probably are limit cycles, but there is no proof of this fact.

Of course the polynomial vector fields of degree 2, after the linear ones, are the easiest polynomial vector fields. It is known that the linear systems, or equivalently the polynomial vector fields of degree 1, cannot have limit cycles. Here we provide a constructive method for obtaining polynomial vector fields in $\mathbb{R}^3$ having infinitely many limit cycles. Using this method construct two such polynomial vector fields in $\mathbb{R}^3$, one of degree 2 and the other of degree 12.

The examples of our method consist of perturbing a polynomial vector field in $\mathbb{R}^3$ of degree 2 (simply a *quadratic* vector field) inside the class of polynomial vector fields of degree 2, and a polynomial vector field in $\mathbb{R}^3$ of degree 12 inside the class of polynomial vector fields of degree 6. These unperturbed polynomial vector fields in $\mathbb{R}^3$ are constructed rotating through an axis polynomial vector fields in $\mathbb{R}^2$ having a center. Additionally these unperturbed polynomial vector fields in $\mathbb{R}^3$ correspond to completely integrable Hamiltonian polynomial vector fields in $\mathbb{R}^4$ fixing a level of the first integral different from the Hamiltonian one.

Let X_ε be the real polynomial vector field in $\mathbb{R}^3$ associated to the polynomial differential system of degree 2

$$\begin{aligned} \dot{x} &= -y + xz - \varepsilon y(a_1x + a_2y + a_3z), \\ \dot{y} &= x + yz + \varepsilon x(a_1x + a_2y + a_3z), \\ \dot{z} &= 2(1 - x^2 - y^2) + \frac{z^2}{2}, \end{aligned} \tag{1}$$

where $a_1, a_2, a_3 \in \mathbb{R}$ and $\varepsilon \geq 0$.

Let Y_ε be the real polynomial vector field in $\mathbb{R}^3$ associated to the polynomial differential system of degree 12

$$\begin{aligned} \dot{x} &= -(r^2+1)^2(xz[(r^2-1)^2+z^2] + 2r^2y), \\ \dot{y} &= -(r^2+1)^2(yz[(r^2-1)^2+z^2] - 2r^2x), \\ \dot{z} &= 2r^2(r^2-1)(r^2+1)^2[(r^2-1)^2+z^2] + 2\varepsilon r^2x^2(r^2+2), \end{aligned} \tag{2}$$

where $r^2 = x^2 + y^2$ and $\varepsilon \geq 0$.

Besides the method our main result is the following.

Main Theorem *The following statements hold.*

(a) *For suitable values of* a_1, a_2 *and* a_3 *the polynomial vector field* X_ε *of degree* 2 *defined by system* (1) *has infinitely many limit cycles for* $\varepsilon > 0$ *small enough.*
(b) *The polynomial vector field* Y_ε *of degree* 12 *defined by system* (2) *has infinitely many limit cycles for* $\varepsilon > 0$ *small enough.*

The paper is structured as follows. In section 2 we present our method. In section 3 we describe the dynamics of the unperturbed vector fields X_0 and Y_0. In section 4 we work with the perturbed vector fields X_ε and Y_ε. The proof of the Main Theorem using the Melnikov integral is given in section 5.

2. The method

For more details on the definitions and results presented in this section see section 4.8 of [14] and for more information on Hamiltonian systems see [16].

Consider the polynomial differential system in $\mathbb{R}^2$

$$dr/ds = P(r, z), \quad dz/ds = Q(r, z). \tag{3}$$

Suppose that this system has a period annulus contained in the positive half–plane $r > 0$. Let $F(r, z)$ be a first integral of system (3) defined in the period annulus. It is important in order to apply the method to know a first integral of this system (and consequently an integrating factor) and the explicit expression of its periodic orbits.

The rotation of the region $r > 0$ around the z–axis provides a new system in $\mathbb{R}^3$ which is given in cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, by

$$\dot{r} = dr/ds = P(r, z), \quad \dot{z} = dz/ds = Q(r, z), \quad \dot{\theta} = d\theta/ds = 1, \tag{4}$$

where $(r, z) \in \mathbb{R}^+ \times \mathbb{R} = \{(r, z) \in \mathbb{R}^2 : r > 0\}$ and $\theta \in \mathbb{S}^1$. Thus the phase portrait of system (4) in $\mathbb{R}^3$ contains a region fulfilled of invariant tori. These tori come from the rotation of the periodic orbits of the annulus of system (3). System (4) has the first integral $H(r, z, \theta) = F(r, z)$ defined in $\mathbb{R}^+\times\mathbb{R}\times\mathbb{S}^1$. We note that the functions $H(r, z, \theta)$ and $F(r, z)$ have the same expression. However we write it with different names in order to distinguish the space where they are defined: $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{S}^1$ for H and $\mathbb{R}^+ \times \mathbb{R}$ for F.

Consider the perturbation of system (4)

$$\begin{aligned} \dot r &= P(r,z) + \varepsilon \bar P(r,z,\theta), \\ \dot z &= Q(r,z) + \varepsilon \bar Q(r,z,\theta), \\ \dot \theta &= 1 + \varepsilon \bar R(r,z,\theta), \end{aligned} \tag{5}$$

for some functions $\bar P$, $\bar Q$ and $\bar R$ such that the corresponding vector field in cartesian coordinates is polynomial. Note that system (5) is a perturbation of system (4) for $\varepsilon > 0$ sufficiently small. Our aim is to prove that system (5) has infinitely many limit cycles for a suitable perturbation.

On the other hand taking into account the function F defined above we consider the Hamiltonian analytic function

$$H^\varepsilon(r,z,w,\theta) = F(r,z) + G(w) + \varepsilon H^1(r,z,\theta,w), \tag{6}$$

for some analytic functions G and H^1. We assume that H^1 is 2π–periodic in θ. The Hamiltonian H^ε is a perturbation of the completely integrable Hamiltonian $H^0(r,z,w,\theta) = F(r,z) + G(w)$ for small $\varepsilon > 0$. The unperturbed Hamiltonian system associated to H^0 decouples into two independent systems. The first system is

$$\dot r = -\partial F/\partial z, \quad \dot z = \partial F/\partial r,$$

with F as first integral (say the F–system), and the other system is

$$\dot w = -\partial G/\partial \theta = 0, \quad \dot\theta = \partial G/\partial w = G'(w),$$

with w as first integral (say the w–system). We note that the F–system is equivalent to system (3) possibly after a change of time.

If $G'(w) \neq 0$ then for small ε the equation $H^\varepsilon = F + G + \varepsilon H^1 = h$ allows to write w as a function of r, z, θ and h, i.e. $w = L^\varepsilon(r,z,\theta;h)$. Due to the presence of the small parameter ε we can compute the implicit function L^ε as a power series in ε. Hence we obtain

$$w = L^\varepsilon(r,z,\theta;h) = L^0(r,z;h) + \varepsilon L^1(r,z,\theta;h) + \mathcal{O}(\varepsilon^2),$$

for some functions L^0 and L^1. From these expressions we obtain the *reduced system*

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{\partial L^0}{\partial z}(r,z;h) + \varepsilon \frac{\partial L^1}{\partial z}(r,z,\theta;h) + \mathcal{O}(\varepsilon^2), \\ \frac{dz}{d\theta} &= -\frac{\partial L^0}{\partial r}(r,z;h) - \varepsilon \frac{\partial L^1}{\partial r}(r,z,\theta;h) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{7}$$

Since H^1 is 2π–periodic in θ, L^1 is also 2π–periodic in θ.

For suitable choice of $\bar{P}$, $\bar{Q}$ and $\bar{R}$ in system (5) the reduced system $(dr/d\theta, dz/d\theta)$ associated to (5) is equivalent to system (7), possibly after a change of time.

We assume that some compact region of the phase plane of the F–system is filled with periodic orbits whose periods vary continuously with respect to the F–energy. Let $F(r,z)=c$ be one of these periodic orbits. This orbit provides a periodic orbit $\gamma^* = (r^*(\theta), z^*(\theta))$ of the unperturbed ($\varepsilon = 0$) autonomous system (7) given by $L^0(r,z;c)=0$. This is exactly the periodic orbit $F(r,z)=c$. Additionally the period P^* of the orbit γ^* varies continuously with the curve γ^*.

System (7) can be studied using the Melnikov integral, and the following theorem holds.

Theorem 1 (see section 4.8 of [14])**.** *Let γ^* be the periodic orbit $(r^*(\theta), z^*(\theta))$ of system (7) with $\varepsilon = 0$ having period P^* such that $P^*/2\pi = m/n \in \mathbb{Q}^+$. If the subharmonic Melnikov function*

$$M^{m/n}(\theta^0) = \int_0^{2\pi m} \{L^0(r^*(\theta), z^*(\theta); h), L^1(r^*(\theta), z^*(\theta), \theta+\theta^0; h)\}\, d\theta, \quad (8)$$

where $\{L^0, L^1\} = (\partial L^0/\partial r)(\partial L^1/\partial z) - (\partial L^0/\partial z)(\partial L^1/\partial r)$ is the Poisson bracket, is independent of ε and has j simple zeros in $\theta^0 \in [0, 2\pi)$, then the resonant periodic orbit γ^* for the Poincaré 2π–map of system (7) with $\varepsilon = 0$ breaks into a set of j periodic orbits, each of period m. Moreover, $j = 2km$, $k \in \mathbb{N}$, is necessarily an even multiple of m and precisely km of the fixed points are hyperbolic and km are elliptic.

Theorem 1 is a computable version of the classical Poincaré–Birkhoff Theorem (see [2, 3, 4, 5, 12, 17]).

The hyperbolic fixed points of system (7) obtained from Theorem 1 correspond to hyperbolic fixed points of the reduced system associated to system (5), and therefore they correspond to hyperbolic periodic orbits of system (5). The more pairs (m,n) we have, the more limit cycles we obtain. Therefore if we have infinitely many pairs (m,n), then system (5) has infinitely many limit cycles.

3. The dynamics of the unperturbed vector fields

3.1. **The dynamics of the vector field X_0.**

Consider the real quadratic system

$$\begin{aligned} dx/d\tau &= -b_1 - 2b_2 y + b_1 x^2 - 2b_3 xy - b_1 y^2, \\ dy/d\tau &= b_3 + 2b_2 x + b_3 x^2 + 2b_1 xy - b_3 y^2, \end{aligned}$$

with $b_1^2 + b_3^2 > 0$ and $b_2^2 - (b_1^2 + b_3^2) = q > 0$. This system appears in [7] as one of the families of quadratic systems having a rational first integral of degree 2. After the affine change of variables and rescaling of the time $r = (b_3x + b_1y + b_2)/\sqrt{q}$, $z = 2(b_1x - b_3y)/\sqrt{q}$, $s = \tau\sqrt{q}$, the system becomes

$$dr/ds = rz, \quad dz/ds = 2(1 - r^2) + z^2/2. \tag{9}$$

System (9) in the (r, z)–plane has two centers at $(-1, 0)$ and $(1, 0)$ and no other finite singular points. We note that by an affine change of variables system (9) can be written as system (S1) of [8] (see Theorem 10.1 in there), so these two centers are isochronous (i.e. the periods of all periodic orbits of each center are equal). System (9) has the rational first integral $F_{X_0}(r, z) = 16r/(4(r + 1)^2 + z^2)$ of degree 2, and the rational integrating factor

$$W(r, z) = \frac{32}{(4(r + 1)^2 + z^2)^2}. \tag{10}$$

We note that system (9) has also a polynomial inverse integrating factor given by $V(r, z) = r(4(r + 1)^2 + z^2)$. The zero set of the inverse integrating factor is an important object, and has been studied in the case that the system has a rational first integral (see [10]). Here the zero set of V contains the straight line $r = 0$, corresponding to the level $F_{X_0}^{-1}(0)$. This straight line is invariant under the flow of system (9). The level $F_{X_0}^{-1}(1)$ is formed by the center at $(1, 0)$.

¿From now on we work on the region $r > 0$ where all orbits are periodic of period π except the center itself. Each one of these orbits corresponds to a level set $F_{X_0}^{-1}(c)$, for $0 < c < 1$. The expressions of these periodic orbits are

$$\begin{aligned} r(s) &= \frac{c\sec^2(2s)(2 - c + 2\sqrt{1 - c}\sin(2s))}{(c - 2)^2 + c^2\tan^2(2s)}, \\ z(s) &= \frac{4\sqrt{1 - c}\sec(2s)(2 - c + 2\sin(2s)\sqrt{1 - c})}{(c - 2)^2 + c^2\tan^2(2s)}. \end{aligned} \tag{11}$$

The rotation of the region $r > 0$ around the z–axis (see Figure 1) provides the new system in $\mathbb{R}^3$ that in cylindrical coordinates is given by

$$\dot{r} = dr/ds = rz, \quad \dot{z} = dz/ds = 2(1 - r^2) + z^2/2, \quad \dot{\theta} = d\theta/ds = 1, \tag{12}$$

where $(r, z) \in \mathbb{R}^+ \times \mathbb{R}$ and $\theta \in \mathbb{S}^1$. We note that system (12) corresponds to the vector field X_0 written in cylindrical coordinates.

System (12) has a rational first integral $H_{X_0} : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}$ defined by $H_{X_0}(r, z, \theta) = F_{X_0}(r, z)$.

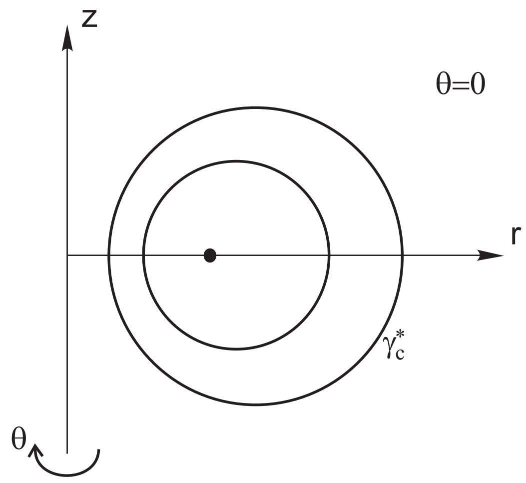

FIGURE 1. The construction of system (12). Each point in a periodic orbit of system (9) is contained in a periodic orbit of system (12), and this last periodic orbit is contained into an invariant 2–dimensional torus of system (12). The center at $(1, 0)$ of system (9) becomes the periodic orbit $r = 1$, $z = 0$ of system (12).

For $h \in [0, 1]$ we define $H^h_{X_0} = H^{-1}_{X_0}(h)$. For system (12) the level $H^0_{X_0}$ would correspond to the z–axis and the level $H^1_{X_0}$ is the periodic orbit $r = 1$, $z = 0$. If $0 < h < 1$, then $H^h_{X_0}$ is an invariant torus of system (12). Since all the periodic orbits of system (9) have period π, all the orbits of system (12) living on an invariant torus are periodic of period 2π.

3.2. **The dynamics of the vector field Y_0.** Consider the linear planar system $\dot r = -z, \dot z = r - 1$. We can do a change of time to obtain the cubic system

$$\dot r = -z((r-1)^2 + z^2), \quad \dot z = (r-1)((r-1)^2 + z^2),$$

associated to the Hamiltonian function $F_{Y_0}(r, z) = ((r-1)^2+z^2)/4$. The restriction of this system to the half–plane $\mathcal{P} = \{(r, z) : r > 0, z \in \mathbb{R}\}$ has a center at the point $(r, z) = (1, 0)$. The period annulus of the system is $\{(r, z) \in \mathbb{R}^2 : 0 < (r-1)^2 + z^2 < 1\}$, and its periodic orbits,

$$\gamma_h = (r(\theta), z(\theta)) = \left(1 + \sqrt{2}\sqrt[4]{h}\cos(2\sqrt{h}\,\theta),\ \sqrt{2}\sqrt[4]{h}\sin(2\sqrt{h}\,\theta)\right), \quad (13)$$

are circles centered at the point $(1, 0)$ with radius $\sqrt{2}\sqrt[4]{h} \in (0, 1)$, F_{Y_0}–energy $h \in (0, 1/4)$, and period $P_h = \pi/\sqrt{h}$.

The rotation of the region $r > 0$ around the z–axis provides a new system in $\mathbb{R}^3$ which in cylindrical coordinates is given by

$$\dot{r} = -z((r-1)^2 + z^2),$$
$$\dot{z} = (r-1)((r-1)^2 + z^2), \tag{14}$$
$$\dot{\theta} = 1.$$

This system is Y_0 written in cylindrical coordinates. We note that Y_0 has the polynomial first integral $H_{Y_0}(r, z, \theta) = F_{Y_0}(r, z)$.

Now we analyze the flow of Y_0 in the space $\mathcal{E} = \{(r, z, \theta) : r > 0, z \in \mathbb{R}, \theta \in [0, 2\pi)\}$ where it is defined. System (14) has a periodic orbit Γ formed by the circle $\{(r, z, \theta) : r = 1, z = 0, \theta \in [0, 2\pi)\}$. This periodic orbit is obtained rotating the point $(1, 0)$ of the half–plane $\mathcal{P}$ of the planar system around the z–axis. Moreover for each circle γ_h system (14) has an invariant 2–dimensional torus $\mathbb{T}^2_h$ obtained rotating γ_h around the z–axis. If $P_h/2\pi = 1/(2\sqrt{h})$ is rational and equal to m/n with m and n coprime, then the invariant torus $\mathbb{T}^2_h$ is fulfilled of periodic orbits of system (14) with period $2\pi m = nP_h$. These periodic orbits give m turns around the z–axis and n turns around the periodic orbit Γ before closing. The number m/n is called the *rotation number* of the torus $\mathbb{T}^2_h$. If the number $P_h/2\pi$ is irrational, then the invariant torus $\mathbb{T}^2_h$ is filled of quasi–periodic orbits dense in $\mathbb{T}^2_h$.

4. The Perturbed Hamiltonian systems and the reduced systems

4.1. **The vector field X_ε.** Consider the general quadratic perturbation of X_0

$$\begin{array}{rcl} \dot{x} = dx/ds & = & -y + xz + \varepsilon P_2(x, y, z), \\ \dot{y} = dy/ds & = & x + yz + \varepsilon R_2(x, y, z), \\ \dot{z} = dz/ds & = & 2(1 - x^2 - y^2) + z^2/2 + \varepsilon Q_2(x, y, z), \end{array} \tag{15}$$

where P_2, Q_2 and R_2 are polynomials of degree 2 in x, y and z. Under the cylindrical change of coordinates system (15) becomes

$$\begin{array}{rcl} \dot{r} = dr/ds & = & rz + \varepsilon \bar{P}(r, z, \theta), \\ \dot{z} = dz/ds & = & 2(1 - r^2) + z^2/2 + \varepsilon \bar{Q}(r, z, \theta), \\ \dot{\theta} = d\theta/ds & = & 1 + \varepsilon \bar{R}(r, z, \theta), \end{array} \tag{16}$$

for some convenient functions $\bar{P}$, $\bar{Q}$ and $\bar{R}$. Dividing the first and second equations by the third one, we obtain the reduced system

$$
\begin{aligned}
\frac{dr}{d\theta} &= rz + \varepsilon[\bar{P}(r,z,\theta) - rz\bar{R}(r,z,\theta)], \\
\frac{dz}{d\theta} &= 2(1-r^2) + \frac{z^2}{2} + \varepsilon\big[\bar{Q}(r,z,\theta) - (2(1-r^2) + \frac{z^2}{2})\bar{R}(r,z,\theta)\big].
\end{aligned} \tag{17}
$$

Now we study how to take the functions $(\bar{P}, \bar{Q}, \bar{R})$ in order to apply the method described in section 2. We consider the Hamiltonian function

$$
H_X^\varepsilon(r,z,w,\theta) = F_{X_0}(r,z) + w + \varepsilon\, wK(r,z,\theta), \tag{18}
$$

where $K(r,z,\theta)$ will be a convenient 2π–periodic function in θ (see (6)). Its associated Hamiltonian system is

$$
\begin{aligned}
r' &= -\frac{\partial H_X^\varepsilon}{\partial z} = \frac{32rz}{(4(r+1)^2+z^2)^2} - \varepsilon w\frac{\partial K}{\partial z}(r,z,\theta), \\
z' &= \frac{\partial H_X^\varepsilon}{\partial r} = \frac{32(2(1-r^2)+z^2/2)}{(4(r+1)^2+z^2)^2} + \varepsilon w\frac{\partial K}{\partial r}(r,z,\theta), \\
w' &= -\frac{\partial H_X^\varepsilon}{\partial \theta} = -\varepsilon w\frac{\partial K}{\partial \theta}(r,z,\theta), \\
\theta' &= \frac{\partial H_X^\varepsilon}{\partial w} = 1 + \varepsilon K(r,z,\theta).
\end{aligned} \tag{19}
$$

The Hamiltonian system (19) for $\varepsilon = 0$ is completely integrable because it has two independent first integrals $H_X^0 = H_X^\varepsilon|_{\varepsilon=0}$ and w in involution, see for more details [1, 2].

The hyperplane $w = 0$ is invariant under the flow of the Hamiltonian system (19) for all ε, because $\dot{w}|_{w=0} = 0$. On $w = 0$ system (19) goes over to

$$
\begin{aligned}
r' &= \frac{32rz}{(4(r+1)^2+z^2)^2}, \\
z' &= \frac{32(2(1-r^2)+z^2/2)}{(4(r+1)^2+z^2)^2}, \\
\theta' &= 1 + \varepsilon K(r,z,\theta),
\end{aligned} \tag{20}
$$

from which we obtain the reduced system

$$\begin{aligned} \frac{dr}{d\theta} &= W\Big[rz - \varepsilon rz K(r,z,\theta)\Big], \\ \frac{dz}{d\theta} &= W\Big[2(1-r^2) + z^2/2 - \varepsilon(2(1-r^2)+z^2/2)K(r,z,\theta)\Big], \end{aligned} \tag{21}$$

where W is the integrating factor defined in (10). Taking $K=\bar{R}$ and $\bar{P}=\bar{Q}=0$ system (21) and system (17) coincide after a change of time given by the integrating factor W. This means that their phase portraits are topologically equivalent. Additionally we choose the function K as

$$K(r,z,\theta) = a_1 r\cos\theta + a_2 r\sin\theta + a_3 z,$$

where $a_1, a_2, a_3 \in \mathbb{R}$. With this expression of K the perturbed system (15) in cartesian coordinates is X_ε, and it becomes a quadratic polynomial differential system.

System (21) is written in the form of system (7) and therefore we can apply the Melnikov integral described in Theorem 1 to it.

4.2. **The vector field Y_ε.** We consider the Hamiltonian analytic function

$$H^\varepsilon(r,z,w,\theta) = \frac{((r-1)^2+z^2)^2}{4} + w + \varepsilon\frac{r^2\cos^2\theta}{1+r}. \tag{22}$$

Here the analytic functions F, G and H^1 of (6) in the phase space $\{(r,z,\theta,w) : r>0, z\in\mathbb{R}, \theta\in[0,2\pi), w\in\mathbb{R}\}$ are $F_{Y_0}(r,z)$, w and $r^2\cos^2\theta/(1+r)$, respectively. Note that H^1 is 2π–periodic in θ and is obtained in a similar way as the function K is obtained in subsection 4.1.

The Hamiltonian equations associated to the Hamiltonian H_Y^ε are

$$\begin{aligned} \dot{r} &= -\frac{\partial H_Y^\varepsilon}{\partial z} &&= -z((r-1)^2+z^2), \\ \dot{z} &= \frac{\partial H_Y^\varepsilon}{\partial r} &&= (r-1)((r-1)^2+z^2) + \varepsilon\frac{r(r+2)\cos^2\theta}{(r+1)^2}, \\ \dot{w} &= -\frac{\partial H_Y^\varepsilon}{\partial \theta} &&= \varepsilon\frac{r^2\sin 2\theta}{r+1}, \\ \dot{\theta} &= \frac{\partial H_Y^\varepsilon}{\partial w} &&= 1. \end{aligned} \tag{23}$$

Note that this system with $\varepsilon=0$ is completely integrable with the two first integrals H_Y^0 and w.

Since the first, second and fourth equations of this system do not contain the variable w, we restrict our attention to study the dynamics of the system

$$\begin{aligned} \dot r &= -z((r-1)^2+z^2), \\ \dot z &= (r-1)((r-1)^2+z^2)+\varepsilon\frac{r(r+2)\cos^2\theta}{(r+1)^2}, \\ \dot\theta &= 1, \end{aligned} \tag{24}$$

which is a perturbation of system (14), in the space $\mathcal{E}$ defined in subsection 3.2. From system (24) we obtain the reduced Hamiltonian system

$$\begin{aligned} \frac{dr}{d\theta} &= -z((r-1)^2+z^2), \\ \frac{dz}{d\theta} &= (r-1)((r-1)^2+z^2)+\varepsilon\frac{r(r+2)\cos^2\theta}{(r+1)^2}. \end{aligned} \tag{25}$$

System (25) is written in the form of system (7) and therefore we can apply the Melnikov integral described in Theorem 1 to it.

5. Proof of the Main Theorem

5.1. **The vector field X_ε.** For proving statement (a) of the Main Theorem we shall apply Theorem 1. The Hamiltonian (6) is the Hamiltonian (18) taking $F=F_{X_0}$, $G=w$ and $H^1=wK$. Hence by the Implicit Function Theorem for small ε and from the equation $H_X^\varepsilon=h$ we can write w as a function of r, z, θ and h. We obtain from section 2 that

$$\begin{aligned} L^0(r,z;h) &= h-F_{X_0}(r,z), \\ L^1(r,z,\theta;h) &= -(h-F_{X_0}(r,z))K(r,z,\theta). \end{aligned}$$

System (7) writes then as

$$\begin{aligned} \frac{\partial r}{\partial\theta} &= -\frac{\partial F_{X_0}}{\partial z}+\varepsilon\left[K\frac{\partial F_{X_0}}{\partial z}-(h-F_{X_0})\frac{\partial K}{\partial z}\right]+\mathcal{O}(\varepsilon^2), \\ \frac{\partial z}{\partial\theta} &= \frac{F_{X_0}}{\partial r}+\varepsilon\left[-K\frac{\partial F_{X_0}}{\partial r}+(h-F_{X_0})\frac{\partial K}{\partial r}\right]+\mathcal{O}(\varepsilon^2). \end{aligned}$$

This system on $w=0$ is the reduced system (21).

The corresponding reduced Hamiltonian system (21) for $\varepsilon=0$ is equivalent to system (9) after the change of time $d\theta=W\,ds$. Hence we can obtain the orbits γ_c^* of the reduced Hamiltonian system from the orbits $(r(s), z(s))$ of system (9) given in (11). Hence

$$\begin{aligned}\int d\theta &= \int W\,ds = \int \frac{32ds}{(4(r(s)+1)^2+z(s)^2)^2}\\ &= \frac{(6-6c+c^2)s - 2\sqrt{1-c}(c-2)\cos 2s + \frac{1}{2}(c-1)\sin 4s}{8}\\ &= \theta(s).\end{aligned}$$

The orbits of system (9) have constant period π, thus for $s \in [0, \pi]$ we get

$$P^*(c) = \int_0^{\pi} W\,ds = \frac{2\pi}{16}(6-6c+c^2).$$

We note that $P^*(c)$ is the period of the periodic orbit γ_c^* of the reduced Hamiltonian system $(-\partial F_{X_0}/\partial z, \partial F_{X_0}/\partial r)$.

Let $\gamma_c^* = (r^*(\theta), z^*(\theta))$ be a periodic orbit of system $(-\partial F_{X_0}/\partial z, \partial F_{X_0}/\partial r)$ surrounding the center $(1,0)$ with period $P^*(c)$. The function $P^*(c)$ is not constant because the reparametrization of the time for passing from system (9) to system $(-\partial F_{X_0}/\partial z, \partial F_{X_0}/\partial r)$ is not constant. For every point p of the periodic orbit γ_c^* of system $(-\partial F_{X_0}/\partial z, \partial F_{X_0}/\partial r)$ let $\Gamma_c^*(p)$ be the orbit of system (20) with $\varepsilon = 0$ passing through p. Note that when p varies in γ_c^* all the orbits $\Gamma_c^*(p)$ are in the same 2–dimensional torus $\mathbb{T}_c^2$. Suppose that for the periodic orbit γ_c^* there exist $m, n \in \mathbb{N}$, $(m,n) = 1$, satisfying the relation $2\pi m = nP^*(c)$. The rotation number of the orbit $\Gamma_c^*(p)$ is m/n for system (20) with $\varepsilon = 0$. Thus when an orbit $\Gamma_c^*(p)$ of system (20) with $\varepsilon = 0$ passing through a point of γ_c^* has given m turns around the z–axis, the orbit γ_c^* has also given n turns around the center $(1,0)$, see Figure 1. Note that the rotation number does not depend on $\Gamma_c^*(p)$ but on $\mathbb{T}_c^2$.

The rotation number m/n of $\Gamma_c^*(p)$ is

$$\frac{m}{n} = \frac{P^*(c)}{2\pi} = \frac{6-6c+c^2}{16} \in \left(\frac{1}{16}, \frac{6}{16}\right), \tag{26}$$

because $c \in (0,1)$. Thus from Theorem 1 the integral (8) becomes

$$\begin{aligned}M^{m/n}(\theta^0) &= \int_0^{2\pi m} \{L^0(r^*(\theta), z^*(\theta); h), L^1(r^*(\theta), z^*(\theta), \theta+\theta^0; h)\}\,d\theta\\ &= \int_0^{\alpha} \{L^0(r(s), z(s); h), L^1(r(s), z(s), \theta(s)+\theta^0; h)\}\,W(s)\,ds,\end{aligned}$$

where α is such that $\theta(\alpha) - \theta(0) = 2\pi m$. The integrant is, up to a non–zero constant,

$$-8(2-c-2\sqrt{1-c}\sin(2s))^2\Big(2a_3(2-2c+(c-2)\sqrt{1-c}\sin(2s))+ \\ c\sqrt{1-c}\cos(2s)[a_1\cos(\theta(s)+\theta^0)+a_2\sin(\theta(s)+\theta^0)]\Big).$$

We can write $M^{m/n}(\theta^0)$ as

$$M^{m/n}(\theta^0) = (a_1\cos\theta^0+a_2\sin\theta^0)I_1+(a_1\sin\theta^0-a_2\cos\theta^0)I_2-16a_3I_3, \quad (27)$$

where

$$\begin{aligned} I_1 &= -8c\sqrt{1-c}\int_0^\alpha \cos(2s)(2-c-2\sqrt{1-c}\sin(2s))^2\cos\theta(s)\,ds,\\ I_2 &= -8c\sqrt{1-c}\int_0^\alpha \cos(2s)(2-c-2\sqrt{1-c}\sin(2s))^2\sin\theta(s)\,ds,\\ I_3 &= \int_0^\alpha (2-c-2\sqrt{1-c}\sin(2s))^2(2-2c+(c-2)\sqrt{1-c}\sin(2s))\,ds. \end{aligned}$$

We claim that $M^{m/n}(\theta^0)$ has two simple zeros for infinitely many rational numbers m/n in a neighborhood of $1/3$ contained in $(1/16, 6/16)$. In order to prove the claim we use the following result.

Lemma 2. *The equation $M^{1/3}(\theta^0) = 0$ has two simple roots in the interval $[0, 2\pi)$ for suitable values of a_1, a_2 and a_3.*

Proof: We note that $m/n = 1/3 \in (1/16, 6/16)\cap\mathbb{Q}$. Now from equation (26) we obtain $c = 3 - 5/\sqrt{3} \in (0, 1)$. The value of α in the expression of $M^{1/3}(\theta^0)$ follows from the equation $\theta(\alpha) - \theta(0) = 2\pi$, and it is $\alpha = 8.68414\ldots$ Hence we obtain $I_1 = 3.40179\ldots$, $I_2 = 0.420336\ldots$ and $I_3 = -2031.19866\ldots$ For suitable values of a_1, a_2 and a_3 and from (27) we get that $M^{1/3}(\theta^0) = 0$ has two simple solutions in $[0, 2\pi)$, see Figure 2 for an example. Therefore the lemma follows. ∎

We remark that the proof of Lemma 2 is analytic in the sense that the three integrals I_1, I_2 and I_3 and the graph in Figure 2 can be obtained with as much precision as we want.

Now we prove the claim. Note that $M^{m/n}(\theta^0)$ is an analytic function of θ^0 and c (through I_1, I_2 and I_3), and therefore of m/n (through (26)). Additionally, from Lemma 2 the equation $M^{1/3}(\theta^0) = 0$ has two simple roots. Thus there exist infinitely many values $m/n \in \mathbb{Q}$ close enough to $1/3$ such that $M^{m/n}(\theta^0) = 0$ has two simple roots. Therefore statement (a) of the Main Theorem follows.

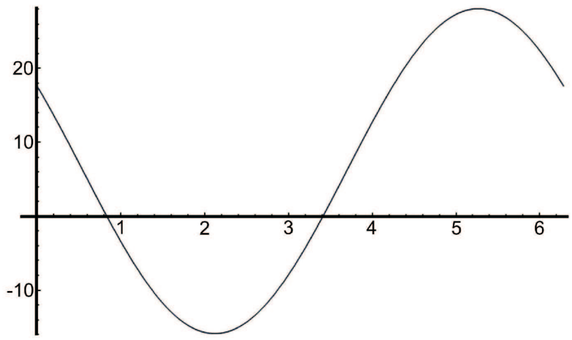

FIGURE 2. The function $M^{1/3}(\theta^0)$ for $a_1 = 4$, $a_2 = -5$ and $a_3 = -0.003$. Its zeros are located at $\theta^0 = 0.833139\ldots$ and $\theta^0 = 3.41206\ldots$

5.2. **The vector field Y_ε.** The functions L^0 and L^1 are for Y_ε

$$L^0(r, z; h) = h - F_{Y_0}(r, z), \quad L^1(r, z, \theta; h) = -\frac{r^2\cos^2\theta}{1 + r}. \tag{28}$$

Let $0 < \delta < 1$ be fixed and small. We consider the annulus

$$A = \{(r, z) \in \mathbb{R}^2 : \delta \le (r - 1)^2 + z^2 \le 1 - \delta\}.$$

This compact annulus is filled with the periodic orbits γ_h of system (25) with $\varepsilon = 0$. The Melnikov integral is

$$\begin{aligned} M^{m/n}(\theta^0) &= \int_0^{2\pi m} \Big\{L^0(r(\theta), z(\theta); h), L^1(r(\theta), z(\theta), \theta + \theta^0; h)\Big\}\, d\theta \\ &= \int_0^{2\pi m} \frac{rz(r + 2)((r - 1)^2 + z^2)\cos^2(\theta + \theta^0)}{(1 + r)^2}\, d\theta. \end{aligned}$$

In this last expression $r = r(\theta)$ and $z = z(\theta)$ are given by (13). Then

$$M^{m/n}(\theta^0) = \int_0^{2\pi m} \frac{N(h,\theta)}{D(h,\theta)}\, \cos^2(\theta + \theta^0)\, d\theta,$$

where

$$\begin{aligned} N(h,\theta) &= 2\sqrt{2}h^{3/4}\left(\sqrt{2}\sqrt[4]{h}\cos(2\sqrt{h}\,\theta) + 1\right) \\ &\quad \left(\sqrt{2}\sqrt[4]{h}\cos(2\sqrt{h}\,\theta) + 3\right)\sin(2\sqrt{h}\theta), \\ D(h,\theta) &= \left(\sqrt{2}\sqrt[4]{h}\cos(2\sqrt{h}\,\theta) + 2\right)^2. \end{aligned}$$

More precisely

$$M^{m/n}(\theta^0) = a(h)\cos^2\theta^0 + b(h)\sin 2\theta^0 + c(h)\sin^2\theta^0,$$

where

$$a(h) = \int_0^{2\pi m} f(h,\theta)\cos^2\theta\, d\theta,$$

$$b(h) = -\int_0^{2\pi m} f(h,\theta)\cos\theta\sin\theta\, d\theta,$$

$$c(h) = \int_0^{2\pi m} f(h,\theta)\sin^2\theta\, d\theta,$$

and $f(h,\theta) = N(h,\theta)/D(h,\theta)$. In the computation of these last integrals we must take into account that $2\sqrt{h} = n/m$.

Since the function $f(h,\theta)$ is odd with respect to the variable θ it follows that $a(h) = c(h) = 0$. Hence $M^{m/n}(\theta^0) = b(h)\sin 2\theta^0$. For suitable values of h we have $b(h) \not\equiv 0$. As an example, take $h = 1/9$, that is, $m = 3$ and $n = 2$, to obtain

$$b(h) = -3\pi\sqrt{\frac{6}{5}(5291 - 966\sqrt{30})}.$$

So the zeros of $M^{m/n}(\theta^0)$ in $[0, 2\pi)$ are always 0, $\pi/2$, π and $3\pi/2$ independently of $h \in (0, 1/4)$. Thus statement (b) of the Main Theorem follows.

We remark that the proof of statement (b) does not need any numerical computation with estimation of its error as the proof of statement (a).

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