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INTEGRABILITY IN NONLINEAR BIOMATHEMATICAL MODELS

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Abstract. We study the integrability of two biomathematical models described by quadratic polynomial differential systems in the plane. These two models can be divided in six families of differential systems. For five of these families we classify all the systems which are Darboux integrable or globally analytic integrable.

1. Introduction and statement of the main results

The nonlinear ordinary differential equations appear in many branches of applied mathematics, physics and, in general, in applied sciences. For a differential system defined on an open subset U of the plane the existence of a first integral determines completely its phase portrait. Since for such differential systems the notion of integrability is based on the existence of a first integral the following natural question arises: *Given a differential system on an open subset U of the plane, that eventually can be the whole plane, how to recognize if this differential system has a first integral*?

In this paper we study the integrability (i.e. the existence of first integrals) in two biological models. The first is due to Swihart et al. [17] in which the behavior of a predator–prey system with different pathches is studied. The second model due to Hadeler and Castillo–Chavez [9] studies the homosexual cohorts in which the effectiveness of education is examined.

These two models where analyzed using the Painlevé property by Meletlidou and Leach [14], and they found some of the systems which can be integrated. In their study they divide the first model into the following three differential systems:

$$
\begin{array}{llll}
(I) & x' = x(1-x-by), & y' = y(a-y+cx),\\
(II) & x' = x(-x-by), & y' = y(1-y+cx),\\
(III) & x' = x(-x-by), & y' = y(-y+cx),
\end{array}
$$

and the second model again into the next three differential systems:

$$
\begin{array}{llll}
(IV) & x' = -x+ay-bxy, & y' = y(c+x-y),\\
(V) & x' = -x+y-xy, & y' = y(c-y),\\
(VI) & x' = -x+ay-xy, & y' = y(c+x),
\end{array}
$$

where $a, b, c \in \mathbb{R}$ and the prime indicates derivative with respect to the time.

For these six families of differential systems we characterize the systems which are Darboux integrable (i.e. which have a Liouvillian first integral [16]) or which have a global analytical first integral. Only system (IV) resist a complete analysis. Thus for that system we only can characterize the previous integrability when the parameter $b \in \{0, 1\}$.

Systems (I)–(VI) have always the invariant algebraic curve $y = 0$. Furhtermore, the systems (I)–(III), systems (IV) with $a = 0$ and systems (VI) with $a = 0$ have also the invariant algebraic curve $x = 0$. These systems are a particular case of the well–known Lotka–Volterra systems in the plane, i.e.,

$$
x' = x(\alpha x + \beta y + \gamma), \quad y' = y(Ax + By + C), \tag{1}
$$

with $\alpha, \beta, \gamma, A, B, C \in \mathbb{R}$. From the Darboux theory of integrability (see Section 2.1) we will see that if systems (1) have another invariant algebraic curve or an exponential factor, then the system has a Liouvillian first integral. Systems (1) are completely studied from the view point of Darboux theory of integrability (see [15] and [3]), and also their analytic integrability (see [12]). This is the

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reason why in this paper the proofs of the results concerning the systems that are a particular cases of a Lotka–Volterra system (1) will be different than the proof of the rest of the systems.

We start with a brief description of the notions and some preliminary results that we will use. We also state the main results of this paper.

2. Basic definitions and results

2.1. **Darboux theory of integrability.** By definition a complex planar *polynomial differential system* or simply a *polynomial system* will be a differential system of the form

$$x' = P(x,y), \quad y' = Q(x,y) \tag{2}$$

where the dependent variables x and y are complex and the independent variable (the time) t is real, P and Q are polynomials in the variables x and y with complex coefficients. In all this paper $m = \max\{\deg P, \deg Q\}$ will denote the *degree* of the polynomial system. The *vector field* X associeted to system (2) is defiend by

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}.$$

Let $f(x,y) \in \mathbb{C}[x,y]$. As usual, $\mathbb{C}[x,y]$ denotes the ring of all complex polynomials in the variables x and y. We say that $f = 0$ is an *invariant algebraic curve* of the vector field X if it satisfies

$$Xf = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf, \tag{3}$$

the polynomial $K = K(x,y) \in \mathbb{C}[x,y]$ is called the *cofactor* of $f = 0$ and has degree $m-1$.

Proposition 1. *We suppose that $f \in \mathbb{C}[x,y]$ and let $f = f_1^{n_1} \cdots f_r^{n_r}$ be its factorization in irreducible factors over $\mathbb{C}[x,y]$. Then, for a polynomial system (2), $f = 0$ is an invariant algebraic curve with cofactor K_f if and only if $f_i = 0$ is an* invariant algebraic curve *for each $i = 1,\ldots,r$ with* cofactor K_{f_i}. *Moreover, $K_f = n_1K_{f_1} + \ldots + n_rK_{f_r}$.*

This proposition can be found in [5].

An *exponential factor* F of a polynomial vector field X of degree m is an exponential function of the form $F = \exp(h/g) \notin \mathbb{C}[x,y]$ satisfying that

$$XF = P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} = LF, \tag{4}$$

for some polynomial $L = L(x,y) \in \mathbb{C}[x,y]$ of degree at most $m-1$. The polynomial L is called the *cofactor* of F.

Proposition 2. *The following statements hold.*

(a) *If $F = \exp(h/g)$ is an exponential factor for the polynomial system (2) and g is not a constant polynomial, then $g = 0$ is an invariant algebraic curve.*
(b) *Eventually e^{x^k} for $k = 1,2,\ldots$ can be exponential factors, coming from the multiplicity of the infinite invariant straight line.*

For a geometrical meaning of the exponential factors and a proof of Proposition 2 see [6].

Let $U \subset \mathbb{C}^2$ be an open set. We say that the non–constant C^1 function $H\colon \mathbb{C}^2 \to \mathbb{C}$ is a *first integral* of the polynomial vector field X on U, if $H(x(t),y(t))$ = constant for all values of t for which the solution $x(t),y(t))$ of X is defined on U. Clearly, H is a first integral of X on U if and only if $XH = 0$ on U.

A non–constant complex function $R\colon \mathbb{C}^2 \to \mathbb{C}$ is an *integrating factor* of the polynomial vector field X on U, of one of the following three equivalent conditions holds

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}, \quad \mathrm{div}\,(RP,RQ) = 0, \quad XR = -R\mathrm{div}(P,Q)$$

on U. As usual, the *divergence* of the vector field X is given by

$$\mathrm{div}\,(P,Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

The next result summarizes the main results about the Darbouxian theory of integrability that we shall use in this paper.

Theorem 3. *Suppose that the polynomial vector field X of degree m defined in $\mathbb{C}^2$ admits p invariant algebraic curves $f_i = 0$ with cofactors K_i, for $i = 1, \dots, p$ and q exponential factors $F_j = \exp(g_j/h_j)$ with cofactors L_j, for $j = 1, \dots, q$. Then the following statements hold.*

(a) There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0 \tag{5}$$

if and only if the function of Darboux type

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} \tag{6}$$

is a first integral of the vector field X.

(b) There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -div\,(P, Q), \tag{7}$$

if and only if the function of Darboux type in (6) is an integrating factor of the vector field X.

(c) If $p + q = [m(m+1)/2] + 1$, then there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that (5) holds. If $p + q = m(m+1)/2$, then there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that (5), or (7) holds.

2.2. **Polynomial first integrals.** All quadratic polynomial differential systems having a polynomial first integral where characterized in [4] using normal forms. Moreover in [8] are provided all their phase portraits. In [2] the authors using the results of [4] and applying the invariant theory provide invariant conditions on the coefficients of any non degenerate quadratic system (i.e. quadratic systems having finitely many singular points) in order to determine if it has or not a polynomial first integral without using any normal form. Thus, in [2] and using the invariants θ, μ_0, K, M, B_1, B_3, R_j for $j = 1, \dots, 8$, $\mathcal{F}_1$, $\mathcal{F}_2$, and a cubic polynomial $\Phi(z)$ (see [2] for the definition of these invariants) the authors provide the following result.

In the rest of the paper we shall use the following notations $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}^-$ and $\mathbb{Q}^+$, which are the set of positive integers, integers, rationals, negative rationals and positive rationals respectively.

Theorem 4. *A non-degenerate quadratic system has a polynomial first integral if and only if either*

$\mathcal{A}$) it is Hamiltonian, or it is not Hamiltonian but one of the following sets of conditions holds:

$\mathcal{B}$) If $\theta = \mu_0 = 0$, $KM \neq 0$ then $R_1 = 0$ and, either

- *$\mathcal{B}_1$) $R_2 < 0$, $R_3 = 0$, $B_3 \neq 0$, $\mathcal{F}_1 = -2k$, $k \in \mathbb{N}$, or*
- *$\mathcal{B}_2$) $R_2 < 0$, $R_3 = 0$, $B_3 = 0$, $\mathcal{F}_1 \in \mathbb{Q}^-$, or*
- *$\mathcal{B}_3$) $R_2 > 0$, $B_3 \neq 0$, $\mathcal{F}_1 = -\dfrac{p}{q} \in \mathbb{Q}^-$, $\mathcal{F}_2 = \dfrac{r^2}{q^2} \in \mathbb{Q}^+$, $\dfrac{p+r}{2q} \in \mathbb{N}$, $\mathcal{F}_2 < \mathcal{F}_1^2$, or*
- *$\mathcal{B}_4$) $R_2 > 0$, $B_3 = 0$, $\mathcal{F}_1 = -\dfrac{p}{q} \in \mathbb{Q}^-$, $\mathcal{F}_2 = \dfrac{r^2}{s^2} \in \mathbb{Q}^+$, $\mathcal{F}_2 < \mathcal{F}_1^2$, or*
- *$\mathcal{B}_5$) $R_2 = 0$, $R_3 = 0$, $B_3 \neq 0$, $\mathcal{F}_1 \in \mathbb{Q}^-$, $\mathcal{F}_1 < -1$.*

$\mathcal{C}$) If $\theta = \mu_0 = K = M = 0$ then $B_3 = 0$, $R_9 \neq 0$, $\mathcal{F}_3 \in \mathbb{Q}^-$.

$\mathcal{D}$) If $\theta\mu_0 \neq 0$ then $B_1 = R_4 = 0$, $\Phi(z)$ has three roots in $\mathbb{Q}^-$ and, either

- *$\mathcal{D}_1$) $R_5 \neq 0$, $B_3 = 0$, or*
- *$\mathcal{D}_2$) $R_5 \neq 0$, $B_3 \neq 0$, $R_6 = 0$, $R_7 = 0$, or*
- *$\mathcal{D}_3$) $R_5 = 0$, $R_8 = 0$.*

Now using Theorem 4 we can determine if a given quadratic polynomial differential system has a polynomial first integral and its expression. The determination is a little tedious because we need to evaluate all the mentioned invariants.

2.3. **Liouvillian first integrals.** A *Liouvillian first integral* is a first integral H which is a Liouvillian function, that is, roughly speaking it can be obtained "by quadratures" of elementary functions, see [16] for a precise definition.

An integrating factor R is called *Darboux* if it is of the form in (6) with $\lambda_i, \mu_j \in \mathbb{Z}$ for $i = 1, \dots, p$ and $j = 1, \dots, q$.

To prove the results related with Liouvillian first integrals we have the following result proved in [16].

Theorem 5. *If system* (2) *has a Liouvillian first integral, then it has a Darboux integrating factor.*

2.4. **Analytic first integrals.** A global analytic first integral $f = f(x, y)$ of system (2) is a first integral described by a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ which is analytic in the variables x and y.

A local analytic first integral $f = f(x, y)$ of system (2) is a first integral which is an analytic function defined in a neighborhood of some point of $\mathbb{R}^2$.

We introduce in this subsection two auxiliary results that will be used through the paper. We will denote by $Df(0)$ the Jacobian matrix of system (2) at $(x, y) = (0, 0)$ and by Df the Jacobian matrix of system (2) at an arbitrary point $(x, y) = (\bar{x}, \bar{y})$ that will be explicitly specified.

The following result is due to Poincaré (see [1]) and its proof can be found in [7]. Throughout the paper $\mathbb{Z}^+$ will denote the set of non–negative integers.

Theorem 6. *Assume that the eigenvalues* λ_1 *and* λ_2 *of* Df *at some singular point* $(\bar{x}, \bar{y})$ *do not satisfy any resonance condition of the form* $\lambda_1 k_1 + \lambda_2 k_2 = 0$ *for* $k_1, k_2 \in \mathbb{Z}^+$ *with* $k_1 + k_2 > 0$. *Then system* (2) *has no local analytic first integrals.*

The proof of the following result can be found in [11].

Theorem 7. *Assume that the eigenvalues* λ_1 *and* λ_3 *of* Df *at some singular point* $(x, y) = (\bar{x}, \bar{y})$ *satisfy that* $\lambda_1 = 0$ *and* $\lambda_2 \neq 0$. *Then system* (2) *has no local analytic first integrals if the singular point* $(\bar{x}, \bar{y})$ *is isolated.*

2.5. **Main results.** We state a result for each class of the systems (I)–(VI), where we characterize for every class the systems having Liouvillian or global analytic first integrals. The explicit expressions for the integrating factors can be found in the proof of each one of the theorems. Their proofs are given in separate sections one section for each class of systems. For classes (I)–(III), class (IV) with $a \in \{0, (1-b)(1+c)\}$ and class (VI) with $a \in \{0, -1-c\}$, because they are particular classes of Lotka–Volterra systems in the plane we only provide explicitly the integrable ones, the Liouvillian first integrals and the global analytic first integrals. For class (IV) with $a \notin \{0, (1-b)(1+c)\}$, and class (VI) with $a \notin \{0, -1-c\}$ we additionally provide their polynomial first integrals, their Darboux polynomials and their exponential factors. The reason of separating the classes of systems (I)–(VI) in these two groups is that the Liouvillian integrability of the Lotka–Volterra quadratic polynomial differential systems have been completely studied in [15] and [3] while the global analytic first integrability has been completely studied in [12].

For completeness of the paper we provide an appendix where the results needed from [15], [3] or [12] are given. We also provide an appendix with a result in [13] which characterize the global analytic first integrals for a quadratic–linear polynomial differential system on the plane having a singular point.

Theorem 8. *The following holds for systems (I).*

(a) The unique Liouvillian integrable systems (I) are the ones having parameters (a, b, c) *in the following list. For every subsystem we provide their integrating factor.*

$(a, b, a(b-1)-1)$	$(a, 1, 0)$	$(0, 0, c)$
$(1/(b-2), b, (3-2b)/(b-2))$	$(2, (3+2c)/(2+c), c)$	$(a, 2+1/a, -1/2)$
$(a, -1, -2-a)$	$(a, 1/2, -2-a)$	$(1/2, b, (3-2b)/(b-2))$
$(a, 2+1/a, 1)$	$(-7/3, 11/7, -2/3)$	$(-4/7, -2, -10/7)$
$(3, 10/3, -11/4)$	$(-7/4, 10/7, 2)$	$(-3/7, 2/3, -11/7)$
$(1/3, 11/4, -10/3)$	$(-3/2, 7/3, -1/2)$	$(-4/3, -1, -5/3)$
$(2, 5/2, -7/4)$	$(-3/4, 5/3, 1)$	$(-2/3, 1/2, -7/3)$
$(1/2, 7/4, -5/2)$	$(2, 7/6, -3/4)$	$(-1/6, -3, -1/2)$
$(4, -1, -7)$	$(-6, 1/2, 3)$	$(1/2, 3/4, -7/6)$
$(1/4, 7, 1)$	$(2, 2/3, -15/8)$	$(1/3, 15/7, -1/2)$
$(-8/7, -1, 2)$	$(3, 1/2, -15/7)$	$(1/2, 15/8, -2/3)$

$(-7/8, -2, 1)$	$(6, 5/3, 1)$	$(-2/3, 1/2, -5/6)$
$(1/2, -5, -5/2)$	$(-3/2, 5/6, -1/2)$	$(1/6, -1, -5/3)$
$(2, 5/2, 5)$	$(-6, 1/2, 1)$	$(1/2, 1/2, -7/6)$
$(1/2, 7, 1)$	$(2, 7/6, -1/2)$	$(-1/6, -1, -1/2)$
$(2, -1, -7)$	$(3, 11/6, 4)$	$(-5/6, 4/5, -2/3)$
$(1/5, -2, -11/5)$	$(-6/5, 2/3, -4/5)$	$(1/3, -4, -11/6)$
$(5, 11/5, 2)$	$(2, 2/3, -20/13)$	$(1/33, 20/7, -1/2)$
$(-13/7, -1, 2)$	$(3, 1/2, -20/7)$	$(1/2, 20/13, -2/3)$
$(-7/13, -2, 1)$	$(2, -1, -1/2)$	$(1/2, 1/2, 1)$
$(2, 5/2, -2/3)$	$(-3/2, -2, -1/2)$	$(3, -1, -5/3)$
$(-2/3, 1/2, 2)$	$(1/2, 2/3, -5/2)$	$(1/3, 5/3, 1)$
$(2, 7/2, -1/2)$	$(-5/2, -1, -1/2)$	$(2, -1, -7/5)$
$(-2/5, 1/2, 1)$	$(1/22, 1/2, -7/2)$	$(1/2, 7/5, 1)$
$(-4/3, 9/4, -2/3)$	$(-5/4, -2, -7/4)$	$(3, 7/3, -9/5)$
$(-4/5, 7/4, 2)$	$(-3/4, 2/3, -9/4)$	$(1/3, 9/5, -7/3)$
$(-9/4, 14/9, -3/4)$	$(-5/9, -3, -13/9)$	$(4, 13/4, -14/5)$
$(-9/5, 13/9, 3)$	$(-4/9, 3/4, -14/9)$	$(1/4, 14/5, -13/4)$
$(-3/2, 10/3, -1/2)$	$(-7/3, -1, -5/3)$	$(2, 5/2, -10/7)$
$(-3/7, 5/3, 1)$	$(-2/3, 1/2, -10/3)$	$(1/2, 10/7, -5/2)$
$(-5/2, 13/5, -1/2)$	$(-8/5, -1, -7/5)$	$(2, 7/2, -13/8)$
$(-5/8, 7/5, 1)$	$(-2/5, 1/2, -13/5)$	$(1/2, 13/8, -7/2)$
$(-10/3, 17/10, -2/3)$	$(-7/10, -2, -13/10)$	$(3, 13/3, -17/7)$
$(-10/7, 13/10, 2)$	$(-3/10, 2/3, -17/10)$	$(1/3, 17/7, -13/3)$
$(-(2l+1)/(2l-1), -1, 1)$	$(2, 1/2, -4l/(2l+1))$	$(1/2, 4l/(2l-1), -1/2)$
$(1/2, 4l/(2l+1), -1/2)$	$((1-2l)/(2l+1), -1, 1)$	$(2, 1/2, -4l/(2l-1))$
$(0, b, -1)$	$(a, 0, -1)$	$(-1, 1/2, -1)$
$(a, 1, -1)$		

The first 102 subsystems correspond to $c \neq -1$. Additionally the fourth has $b \notin \{1, 2\}$, the fifth has $c \neq -2$, the sixth has $a \neq 0$, the seventh and eight have $a \neq -1$, the ninth has $b \notin \{1, 2\}$, and the tenth has $a \neq 0$.

There are six subsystems corresponding to the cases 103 to 108 for which we do not provide the integrating factor because their third invariant algebraic curve is of degree $2l$ where l can be any positive integer. The explicit expression of these invariant algebraic curves can be computed from [10].

The cases 110 and 112 have $a \neq 0$.

(b) *Systems (I) have a polynomial first integral if and only if $a, b \in \mathbb{Q}^-$ and $c = ab - a - 1$. Moreover these integrals are the unique global analytic first integrals of systems (I).*

Theorem 9. *The following holds for systems (II).*

(a) *It has a Liouvillian first integral if and only if $b = 1$, or if $b \neq 1$ and $c = -1$.*
(b) *It has no global analytic first integrals.*

Theorem 10. *The following holds for systems (III).*

(a) *It is Liouvillian integrable.*
(b) *It has a polynomial first integral $H = x^{|c+1|}y^{|1-b|}((c+1)x + (b-1)y)^{|-bc-1|}$ if and only if b and a are rational and the signs of $c+1$, $1-b$ and $-bc-1$ are equal. Moreover these are the unique global analytic first integrals of system (III).*

Theorem 11. *The following holds for systems (IV) with $b \in \{0, 1\}$ and $a = 0$.*

(a) *It has a Liouvillian first integral if and only if $c = -1$.*
(b) *It has no global analytic first integrals.*

Theorem 12. *The following holds for systems (IV) with $b \in \{0, 1\}$ and $a = (1-b)(1+c) \neq 0$.*

(a) *It has a Liouvillian first integral if and only if $b = 0$.*
(b) *It has a global analytic first integral $H = y(x-y)^c e^x$ if and only if $b = 0$ and $c \in \mathbb{Q}^+ \cup \{0\}$. It has no polynomial first integrals.*

Theorem 13. *The following holds for systems Systems (VI) with $a = 0$.*

(a) *It is Liouvillian integrable.*
(b) *It has a global analytic first integral $H = yx^c e^{x+y}$ if and only if $c \in \mathbb{Q}^+ \cup \{0\}$. It has no polynomial first integrals.*

Theorem 14. *The following holds for systems (VI) with $a = -1 - c \neq 0$.*
(a) *It is Liouvillian integrable.*
(b) *It has no global analytic first integrals.*

Now we start the results related with the systems that are not Lotka–Volterra. Those are sytems (IV) with $a \notin \{0, (1-b)(1+c)\}$, systems (V) and systems (VI) with $a \notin \{0, -1-c\}$. We start with systems (V).

Theorem 15. *The following holds for systems (V).*
(a) *It is Liouvillian integrable.*
(b) *It has no global analytic first integrals.*

Theorem 16. *Systems (IV) with $b \in \{0, 1\}$ and $a \notin \{0, (1-b)(1+c)\}$ have neither Liouvillian first integrals, nor global analytic first integrals.*

Theorem 17. *Systems (VI) with $a \notin \{0, -1-c\}$ have neither Liouvillian first integrals, nor global analytic first integrals.*

The rest of the paper is devoted to the proofs of Theorems 8–17. We recall that we provide in all the proofs the explicit expressions for the first integrals. Theorems 8, 9 and 10 are proved in Sections 3, 4 and 5, respectively. Theorems 11, 12, 13, 14 and 15 are all of them proved in Section 6, while Theorems 16 and 17 are proved in Sections 7 and 8, respectively.

3. Proof of Theorem 8

We consider different cases.

Case 1: $c \neq -1$. In this case doing the change of variables

$$(x, y) \mapsto \Big(\frac{1}{c+1} X, Y\Big), \tag{8}$$

we have that system (I) becomes equation (40) with

$$\bar{A} = a, \quad \bar{B} = \frac{1}{c+1}, \quad \bar{C} = 1 - b.$$

In view of Proposition 28 and setting (when they are well defined)

$$p = -a - c - 1, \quad q = -\frac{1}{c+1} - \frac{1}{1-b}, \quad r = b - 1 - \frac{1}{a},$$

we have that system (8) has a Liouvillian first integral if and only if

$(\mathbf{a}, \mathbf{b}, \mathbf{-1 + a(b-1)})$ (Case 1 of [3]) has the integrating factor

$$R = X^{\frac{b+c}{1-b}} \left(-1 - c + X + (-1+b)\, Y\right)^{\frac{2+b\,(-1+c)}{-1+b}}.$$

$(\mathbf{a}, \mathbf{1}, \mathbf{0})$ (Case 2 of [3]) then $R^{-1} = Y^{1-\frac{1}{a}} X^2 \left(-a + Y\right)^{\frac{1}{a}}$.

$(\mathbf{0}, \mathbf{0}, \mathbf{c})$ (Case 2 of [3]) then $R = X^{-1}\, Y^{-2} (1 + c - X)^{-1-c}$.

$(\mathbf{1/(b-2)}, \mathbf{b}, \mathbf{(3-2b)/b-2})$ with $b \notin \{1, 2\}$ (Case 3 of [3]). Then R^{-1} is equal to

$$X\, Y \sqrt{\left(-1 + b + (-2+b)\, X\right)^2 + 2\, \left(1 + b\, (-1+X) - 2\, X\right) Y (b-1)(b-2) + Y^2 (b-1)^2 (b-2)^2}.$$

$(\mathbf{2}, \mathbf{(3+2c)/(2+c)}, \mathbf{c})$ with $c \neq -2$ (Case 4 of [3]). Then

$$R = X^{-2} Y^{-1+\frac{1}{2+c}} \left((2+c)^2 X^2 + 2\, (1+c)\, (2+c)\, X\, Y + (1+c)^2\, (-2+Y)\, Y\right)^{\frac{c}{4+2\,c}}.$$

$(\mathbf{a}, \mathbf{2 + 1/a}, \mathbf{-1/2})$ with $a \neq 0$ (Case 4 of [3]). Then $R = Y^{-2} X^{-1+a} \left(-2\, a^2\, X + (a-Y)^2\right)^{-\left(\frac{1}{2}\right)-a}$.

$(\mathbf{a}, \mathbf{-1}, \mathbf{-2-a})$ with $a \neq -1$ (Case 4 of [3]). Then $R = \left((1 + a + X)^2 - 2\, X\, Y\right)^{-\frac{1}{2}+\frac{1}{1+a}} X^{\frac{1}{1+a}}\, Y^{\frac{2+a}{1+a}}$.

$(\mathbf{a}, \mathbf{1/2}, \mathbf{-2-a})$ with $a \neq -1$ (Case 4 of [3]). Then

$$R = X^{-2} Y^{-1+\frac{1}{a}} \left(a (1 + a + X)^2 - Y (1+a)^2\right)^{-\frac{2+a}{2\,a}}.$$

$(\mathbf{1/2, b, (3-2b)/(b-2)})$ with $b \notin \{1,2\}$ (Case 4 of [3]), then

$$R = Y^{-2} X^{-1+\frac{1}{2-b}} \left((-1+b)\ X + (b-2)(X + (-1+b)\ Y)^2\right)^{\frac{b}{-4+2\,b}}.$$

$(\mathbf{a, 2+1/a, 1})$ with $a \neq 0$ (Case 4 of [3]), then

$$R = X^{-2+\frac{1}{1+a}}\, Y^{-1+\frac{1}{1+a}} \left((a-Y)^2(1+a) + a\, X\, Y\right)^{\frac{1}{2}-\frac{1}{1+a}}.$$

$(\mathbf{-7/3, 11/7, -2/3})$ (Case 8 of [3]), then R^{-1} is equal to

$$X^{-\frac{1}{6}} \sqrt{Y} \left(-259308\, X^3 + 63\, X\ (-28+Y)\ (7+3\,Y)^2 + 4\,(7+3\,Y)^4 - 37044\, X^2\ (-7+5\,Y)\right)^{\frac{5}{6}}.$$

$(\mathbf{-4/7, -2, -10/7})$ (Case 8 of [3]), then R^{-1} is equal to

$$Y^{\frac{-1}{6}} X^{\frac{1}{3}} \left(4\,(3+7\,X)^4 - 63\,(3+7\,X)^2\ (-1+28\,X)\ Y + 37044\, X\ (-5+7\,X)\ Y^2 - 259308\, X\, Y^3\right)^{\frac{5}{6}}.$$

$(\mathbf{3, 10/3, -11/4})$ (Case 8 of [3]), then

$$R^{-1} = \sqrt{X}\, Y^{\frac{1}{3}} \left(324\, X^4 + 9604\,(-3+Y)^3\, Y + 2646\, X^2\, Y\ (-5+4\,Y) + 189\, X^3\ (3+16\,Y)\right.$$
$$\left. +1029\, X\, Y\ (-180 + Y\ (-69+16\,Y))\right)^{\frac{5}{6}}.$$

$(\mathbf{-7/4, 10/7, 2})$ (Case 8 of [3]) then R^{-1} is equal to

$$X^{-\frac{1}{6}} Y^{\frac{1}{3}} \left(87808\, X^3\, Y + 27\,(7+4\,Y)^4 + 9408\, X^2\, Y\ (-35+12\,Y) + 63\, X\,(7+4\,Y)^2\ (-7+48\,Y)\right)^{\frac{5}{6}}.$$

$(\mathbf{-3/7, 2/3, -11/7})$ (Case 8 of [3]), then R^{-1} is equal to

$$Y^{-\frac{1}{6}} \sqrt{X} \left(27\,(4+7\,X)^4 - 63\ (-48+7\,X)\ (4+7\,X)^2\, Y - 9408\ (-12+35\,X)\ Y^2 + 87808\, Y^3\right)^{\frac{5}{6}}.$$

$(\mathbf{1/3, 11/4, -10/3})$ (Case 8 of [3]), then

$$R^{-1} = X^{\frac{1}{3}} \sqrt{Y} \left(256\, X\,(7+3\,X)^3 + 336\, X\ (-980 + 3\, X\ (161+48\,X))\ Y\right.$$
$$\left. +3528\, X\ (35+36\,X)\ Y^2 + 3087\ (-7+48\,X)\ Y^3 + 64827\, Y^4\right)^{\frac{5}{6}}.$$

$(\mathbf{-3/2, 7/3, -1/2}$ (Case 9 of [3]), then

$$R^{-1} = \sqrt{X}\, Y^{\frac{2}{3}} \left(108\, X^2 + (3+2\,Y)^3 + 6\, X\ (-18 + Y\ (30+Y))\right)^{\frac{5}{6}}.$$

$(\mathbf{-4/3, -1, -5/3})$ (Case 9 of [3]), then

$$R^{-1} = X^{\frac{1}{3}} \sqrt{Y} \left((2+3\,X)^3 - 6\ (-1 + 6\, X\ (-5+3\,X))\ Y + 108\, X\, Y^2\right)^{\frac{5}{6}}.$$

$(\mathbf{2, 5/2, -7/4})$ (Case 9 of [3]), then

$$R^{-1} = X^{\frac{2}{3}}\, Y^{\frac{1}{3}} \left(8\, X^3 + 27\,(-2+Y)^2\, Y + 18\, X\, Y\ (10+3\,Y) + 6\, X^2\ (1+6\,Y)\right)^{\frac{5}{6}}.$$

$(\mathbf{-3/4, 5/3, 1})$ (Case 9 of [3]), then

$$R^{-1} = \sqrt{X}\, Y^{\frac{1}{3}} \left(-288\, X^2\, Y - 2(3+4\,Y)^3 + X\ \left(27 + 720\, Y - 384\, Y^2\right)\right)^{\frac{5}{6}}.$$

$(\mathbf{-2/3, 1/2, -7/3})$ (Case 9 of [3]), then

$$R^{-1} = X^{\frac{2}{3}} \sqrt{Y} \left(-2(4+3\,X)^3 + \left(-384 + 720\, X + 27\, X^2\right)\ Y - 288\, Y^2\right)^{\frac{5}{6}}.$$

$(\mathbf{1/2, 7/4, -5/2})$ (Case 9 of [3]), then

$$R^{-1} = X^{\frac{1}{3}}\, Y^{\frac{2}{3}} \left(-32\, X\,(3+2\,X)^2 - 144\, X\ (-5+2\,X)\ Y - 27\ (-1+8\,X)\ Y^2 - 54\, Y^3\right)^{\frac{5}{6}}.$$

$(\mathbf{2, 7/6, -3/4})$ (Case 10 of [3]), then

$$R^{-1} = X^2\, Y^{\frac{1}{3}} \left(216\, X^3 + 18\, X\ (-2+Y)\ Y + (-2+Y)^2\, Y + 54\, X^2\ (-1+2\,Y)\right)^{\frac{1}{6}}.$$

$(\mathbf{-1/6, -3, -1/2})$ (Case 10 of [3]), then

$$R^{-1} = X^{\frac{7}{6}}\, Y^2 \left(4\, X^2 + (1+6\,Y)^3 - 2\, X\ (2 + 9\, Y\ (2+3\,Y))\right)^{\frac{1}{6}}.$$

$(\mathbf{4, -1, -7})$ (Case 10 of [3]), then

$$R^{-1} = X^{\frac{1}{3}}\, Y^{\frac{7}{6}} \left((6+X)^3 - 2\ (27 + 2\, X\ (9+X))\ Y + 4\, X\, Y^2\right)^{\frac{1}{6}}.$$

$(\mathbf{-6, 1/2, 3})$ (Case 10 of [3]), then

$$R^{-1} = X^2\, Y^{\frac{7}{6}} \left(54\, X^3 + 432\, X\ (6+Y) - 96\,(6+Y)^2 - 27\, X^2\ (24+Y)\right)^{\frac{1}{6}}.$$

$(\mathbf{1/4, 7, 1})$ (Case 10 of [3]), then

$$R^{-1} = X^{\frac{1}{3}}\, Y^2 \left(-96\, X\,(1+6\,X)^2 + 432\, X\ (1+6\,X)\ Y - 27\ (1+24\,X)\ Y^2 + 54\, Y^3\right)^{\frac{1}{6}}.$$

$(\mathbf{1/4, 7, 1})$ (Case 10 of [3]), then

$$R^{-1} = X^{\frac{7}{6}}\, Y^{\frac{1}{3}} \left(-32\, X^2\, Y - 18\,(-1+4\,Y)^3 - 3\, X\ (3+16\, Y\ (-3+8\,Y))\right)^{\frac{1}{6}}.$$

$(\mathbf{2, 2/3, -15/8})$ (Case 11 of [3]), then

$$R^{-1} = X^2 Y^{-\frac{1}{3}} \left(27\, X^2\ (7+8\,X) + 98\ (7+9\,X)\ Y - 343\, Y^2\right)^{\frac{5}{6}}.$$

$(\mathbf{1/3, 15/7, -1/2})$ (Case 11 of [3]), then

$$R^{-1} = X^{-\frac{1}{6}} Y^2 \left(686\, X^2 + 216\, Y^3 + 7\, X\ (-49+9\, Y\ (14+3\,Y))\right)^{\frac{5}{6}}.$$

$(\mathbf{-8/7, -1, 2})$ (Case 11 of [3]), then

$$R^{-1} = X^{-\frac{1}{3}}\, Y^{-\frac{1}{6}} (216 + 7\, Y\ (27 + 7\, X\ (18 - 7\, X + 14\, Y)))^{\frac{5}{6}}.$$

$(\mathbf{3, 1/2, -15/7})$ (Case 11 of [3]), then

$$R^{-1} = X^2 Y^{-\frac{1}{6}} \left(-54\, X^3 - 432\, X\, Y + 27\, X^2\, Y + 192\ (-3+Y)\ Y\right)^{\frac{5}{6}}.$$

$(\mathbf{1/2, 15/8, -2/3})$ (Case 11 of [3]), then

$$R^{-1} = X^{-\frac{1}{3}} Y^2 \left(-576 X^2 + 48(4-9Y)X + 27(1-2Y)Y^2\right)^{5/6}.$$

$(\mathbf{-7/8, -2, 1})$ (Case 11 of [3]), then

$$R^{-1} = X^{-\frac{1}{6}}\, Y^{-\frac{1}{3}} (-18 + X\ (9 + 16\ (-9 + 4\, X - 12\, Y)\ Y))^{\frac{5}{6}}.$$

$(\mathbf{6, 5/3, 1})$ (Case 12 of [3]), then

$$R = X^{-2} Y^{-\frac{5}{6}} \left(12\, X\,(-6+Y)^2 + 4\,(-6+Y)^3 + 9\, X^2\, Y\right)^{\frac{1}{6}}.$$

$(\mathbf{-2/3, 1/2, -5/6})$ (Case 12 of [3]), then

$$R = X^{-\frac{5}{3}}\, Y^{-2} \left(-4\,(-1+6\,X)^3 + 12\,(1-6\,X)^2\, Y + 9\, Y^2\right)^{\frac{1}{6}}.$$

$(\mathbf{1/2, -5, -5/2})$ (Case 12 of [3]), then

$$R = X^{-\frac{5}{6}}\, Y^{-\frac{5}{3}} \left(X\,(3+2\,X)^2 - 72\, X\ (2+X)\ Y + 432\ (1+X)\ Y^2 - 864\, Y^3\right)^{\frac{1}{6}}.$$

$(\mathbf{-3/2, 5/6, -1/2})$ (Case 12 of [3]), then

$$R = X^{-2}\, Y^{-\frac{5}{3}} \left(-108\, X^2 + 12\, X\,(3+2\,Y)^2 - (3+2\,Y)^3\right)^{\frac{1}{6}}.$$

$(\mathbf{1/6, -1, -5/3})$ (Case 12 of [3]), then

$$R = X^{-\frac{5}{6}}\, Y^{-2} \left(-(2+3\,X)^3 + 12\,(2+3\,X)^2\, Y - 108\, X\, Y^2\right)^{\frac{1}{6}}.$$

$(\mathbf{2, 5/2, 5})$ Case 12 of [3]), then

$$R = X^{-\frac{5}{3}}\, Y^{-\frac{5}{6}} \left(-108\, Y - (2\,X + 3\ (-4+Y))\ (2\,X + 3\,Y)^2\right)^{\frac{1}{6}}.$$

$(\mathbf{-6, 1/2, 1})$ (Case 13 of [3]), then

$$R^{-1} = X^2\, Y^{\frac{7}{6}} \left(3\, X^2\, Y + 24\, X\ (6+Y) - 8\,(6+Y)^2\right)^{\frac{1}{6}}.$$

$(\mathbf{1/2, 1/2, -7/6})$ (Case 13 of [3]), then

$$R^{-1} = X^{\frac{1}{3}}\, Y^2 \left(-8\, X\,(1+6\,X)^2 + 24\, X\ (1+6\,X)\ Y + 3\, Y^2\right)^{\frac{1}{6}}.$$

$(\mathbf{1/2, 7, 1})$ (Case 13 of [3]), then

$$R^{-1} = X^{\frac{7}{6}}\, Y^{\frac{1}{3}} \left(3\, X - 8\ (-3+X)\ X\, Y - 48\ (-3+2\,X)\ Y^2 - 288\, Y^3\right)^{\frac{1}{6}}.$$

$(\mathbf{2}, \mathbf{7/6}, \mathbf{-1/2})$ (Case 13 of [3]), then

$$R^{-1} = X^2\, Y^{\frac{1}{3}} \left(18\, X^2 + 6\, X\, (-2+Y)\, Y + (-2+Y)^2\, Y\right)^{\frac{1}{6}}.$$

$(\mathbf{-1/6}, \mathbf{-1}, \mathbf{-1/2})$ (Case 13 of [3]), then

$$R^{-1} = X^{\frac{7}{6}}\, Y^2\, (1 + 6\, Y + 2\, X\, (-2 + 2\, X + 3\, Y\, (-2 + 3\, Y)))^{\frac{1}{6}}.$$

$(\mathbf{2}, \mathbf{-1}, \mathbf{-7})$ (Case 13 of [3]), then $R^{-1} = X^{\frac{1}{3}}\, Y^{\frac{7}{6}}\, (X\, (X - 2\, Y)\, (6 + X - 2\, Y) + 18\, Y)^{\frac{1}{6}}$.
$(\mathbf{3}, \mathbf{11/6}, \mathbf{4})$ (Case 14 of [3]), then

$$R = X^{-2}\, Y^{-\frac{5}{6}} \Big(1296\, X^4 + 3000\, X\, (-3+Y)^2\, Y + 625\, (-3+Y)^3\, Y + 2700\, X^2\, Y\, (-7 + 2\, Y) + 2160\, X^3\, (-3 + 2\, Y)\Big)^{\frac{1}{6}}.$$

$(\mathbf{-5/6}, \mathbf{4/5}, \mathbf{-2/3})$ (Case 14 of [3]), then

$$R = X^{-\frac{11}{6}}\, Y^{-2} \left(-16875\, X^3 - 45\, X\, (5 + 4\, Y)\, (5 + 6\, Y)^2 + (5 + 6\, Y)^4 + 3375\, X^2\, (5 + 8\, Y)\right)^{\frac{1}{6}}.$$

$(\mathbf{1/5}, \mathbf{-2}, \mathbf{-11/5})$ (Case 14 of [3]), then

$$R = X^{-\frac{5}{6}}\, Y^{-\frac{11}{6}} \left((6 + 5\, X)^4 - 45\, (4 + 5\, X)\, (6 + 5\, X)^2\, Y + 3375\, X\, (8 + 5\, X)\, Y^2 - 16875\, X\, Y^3\right)^{\frac{1}{6}}.$$

$(\mathbf{-6/5}, \mathbf{2/3}, \mathbf{-4/5})$ (Case 14 of [3]), then

$$R = X^{-2}\, Y^{-\frac{11}{6}} \left(216\, (1 - 5\, X)^4 - 180\, (1 - 5\, X)^2\, (-3 + 10\, X)\, Y - 150\, (-3 + 20\, X)\, Y^2 + 125\, Y^3\right)^{\frac{1}{6}}.$$

$(\mathbf{1/3}, \mathbf{-4}, \mathbf{-1/6})$ (Case 14 of [3]), then

$$R = X^{-\frac{5}{6}}\, Y^{-2} \Big(X\, (5 + 6\, X)^3 - 120\, X\, (5 + 6\, X)^2\, Y + 900\, X\, (35 + 36\, X)\, Y^2 - 9000\, (5 + 12\, X)\, Y^3 + 135000\, Y^4\Big)^{\frac{1}{6}}.$$

$(\mathbf{5}, \mathbf{11/5}, \mathbf{2})$ (Case 14 of [3]), then R is equal to

$$X^{-\frac{11}{6}}\, Y^{-\frac{5}{6}} \left(216\, (-5+Y)^4 + 125\, X^3\, Y + 150\, X^2\, Y\, (-20 + 3\, Y) + 180\, X\, (-5+Y)^2\, (-10 + 3\, Y)\right)^{\frac{1}{6}}.$$

$(\mathbf{2}, \mathbf{2/3}, \mathbf{-20/13})$ (Case 15 of [3]), then

$$R^{-1} = Y^{-\frac{7}{6}} X^2 \left(648\, X^4 - 36\, X^2\, (7 + 6\, X)\, Y + 98\, (7 + 12\, X)\, Y^2 - 343\, Y^3\right)^{\frac{5}{6}}.$$

$(\mathbf{1/33}, \mathbf{20/7}, \mathbf{-1/2})$ (Case 15 of [3]), then

$$R^{-1} = X^{-\frac{1}{6}} Y^2 \left(686\, X^2 + 216\, Y^3\, (-1 + 3\, Y) - 7\, X\, (49 + 12\, Y\, (-14 + 3\, Y))\right)^{\frac{5}{6}}.$$

$(\mathbf{-13/7}, \mathbf{-1}, \mathbf{2})$ (Case 15 of [3]), then

$$R^{-1} = X^{-\frac{7}{6}}\, Y^{-\frac{1}{6}} (648 + X\, (-216 + 7\, Y\, (-36 + 7\, X\, (24 - 7\, X + 14\, Y))))^{\frac{5}{6}}.$$

$(\mathbf{3}, \mathbf{1/2}, \mathbf{-20/7})$ (Case 15 of [3]), then

$$R^{-1} = Y^{-\frac{1}{6}} X^2 \left(144\, X^3\, (13 + 7\, X) - 507\, (169 + 4\, X\, (26 + X))\, Y + 28561\, Y^2\right)^{\frac{5}{6}}.$$

$(\mathbf{1/2}, \mathbf{20/13}, \mathbf{-2/3})$ (Case 15 of [3]), then

$$R^{-1} = X^{-\frac{7}{6}} Y^2 \left(-85683\, X^3 + 1008\, Y^4 + 516\, X\, Y^2\, (-13 + 12\, Y) - 2197\, X^2\, (-13 + 24\, Y)\right)^{\frac{5}{6}}.$$

$(\mathbf{-7/13}, \mathbf{-2}, \mathbf{1})$ (Case 15 of [3]), then

$$R^{-1} = X^{-\frac{1}{6}}\, Y^{-\frac{7}{6}} (1008 - 13\, Y\, (-144 + 13\, X\, (12 + 13\, Y\, (24 - 13\, X + 39\, Y))))^{\frac{5}{6}}.$$

$(\mathbf{2}, \mathbf{-1}, \mathbf{-1/2})$ (Case 16 of [3]), then $R^{-1} = X^{\frac{1}{3}}\, Y^{\frac{1}{3}}\, \left(X^2 + Y + X\, (-3 + Y)\, Y\right)$.
$(\mathbf{1/2}, \mathbf{1/2}, \mathbf{1})$ (Case 16 of [3]), then $R^{-1} = X^{\frac{1}{3}}\, Y^{\frac{1}{3}}\, \left(X + (-3 + X)\, X\, Y + Y^2\right)$.
$(\mathbf{2}, \mathbf{5/2}, \mathbf{-2/3})$ (Case 17 of [3]), then $R^{-1} = \sqrt{X}\, Y^{\frac{1}{4}}\, \left(8\, X^2 + 16\, X\, Y - (-2 + Y)^2\, Y\right)$.
$(\mathbf{-3/2}, \mathbf{-2}, \mathbf{-1/2})$ (Case 17 of [3]), then

$$R^{-1} = X^{\frac{1}{4}}\, \sqrt{Y}\, (1 + 4\, X\, (-1 + X - 2\, Y\, (2 + Y))).$$

$(\mathbf{3}, \mathbf{-1}, \mathbf{-5/3})$ (Case 17 of [3]), then

$$R^{-1} = X^{\frac{1}{4}}\, Y^{\frac{1}{4}}\, \left(X^3 - 4\, (2 + X\, (4 + X))\, Y + 4\, X\, Y^2\right).$$

$(\mathbf{-2/3, 1/2, 2})$ (Case 17 of [3]), then

$$R^{-1} = \sqrt{X}\,Y^{\frac{1}{4}}\,\left(4 + 3\,Y\,\left(4 + 2\,\left(-4 + X\right)\,X + 3\,Y\right)\right).$$

$(\mathbf{1/2, 2/3, -5/2})$ (Case 17 of [3]), then

$$R^{-1} = X^{\frac{1}{4}}\,\sqrt{Y}\,\left(X\,(3 + 2\,X)^2 - 24\,X\,Y + 6\,Y^2\right).$$

$(\mathbf{1/3, 5/3, 1})$ (Case 17 of [3]), then

$$R^{-1} = X^{\frac{1}{4}}\,Y^{\frac{1}{4}}\,\left(9\,X^2\,Y + 4\,Y^3 + 6\,X\,\left(1 + 2\,\left(-2 + Y\right)\,Y\right)\right).$$

$(\mathbf{2, 7/2, -1/2})$ (Case 18 of [3]), then

$$R^{-1} = X^{\frac{1}{3}}\,Y^{\frac{1}{3}}\,\left(8\,X^2 - 4\,X\,\left(-6 + Y\right)\,Y + (-2 + Y)^3\,Y\right).$$

$(\mathbf{-5/2, -1, -1/2})$ (Case 18 of [3]), then

$$R^{-1} = X^{-\frac{2}{3}}Y^{\frac{1}{3}}\,\left(-1 + 2\,X\,\left(3 + 2\,Y + 2\,X\,\left(-3 + 2\,X - 2\,Y\,\left(3 + Y\right)\right)\right)\right).$$

$(\mathbf{2, -1, -7/5})$ (Case 18 of [3]), then

$$R^{-1} = Y^{-\frac{2}{3}}X^{\frac{1}{3}}\,\left(X^4 - 2\,X^2\,\left(2 + 3\,X\right)\,Y + 4\,\left(2 + 3\,X\,\left(2 + X\right)\right)\,Y^2 - 8\,X\,Y^3\right).$$

$(\mathbf{-2/5, 1/2, 1})$ (Case 18 of [3]), then

$$R^{-1} = Y^{-\frac{2}{3}}X^{\frac{1}{3}}\,\left(32 + 5\,Y\,\left(125\,X^2\,Y - 20\,X\,\left(2 + 15\,Y\right) + 4\,\left(12 + 5\,Y\,\left(6 + 5\,Y\right)\right)\right)\right).$$

$(\mathbf{1/22, 1/2, -7/2})$ (Case 18 of [3]), then

$$R^{-1} = X^{\frac{1}{3}}\,Y^{\frac{1}{3}}\,\left(4\,X\,(5 + 2\,X)^3 - 100\,X\,\left(15 + 2\,X\right)\,Y + 625\,Y^2\right).$$

$(\mathbf{1/2, 7/5, 1})$ (Case 18 of [3]), then

$$R = X^{-\frac{2}{3}}Y^{\frac{1}{3}}\,\left(500\,X^3\,Y + 32\,Y^4 + 40\,X\,Y^2\,\left(-5 + 6\,Y\right) + 25\,X^2\,\left(25 + 12\,Y\,\left(-5 + 2\,Y\right)\right)\right).$$

$(\mathbf{-4/3, 9/4, -2/3})$ (Case 19 of [3]), then

$$R^{-1} = X^{\frac{1}{3}}\,\sqrt{Y}\,\left(576\,X^2 + (4 + 3\,Y)^3 + 96\,X\,\left(-4 + 9\,Y\right)\right).$$

$(\mathbf{-5/4, -2, -7/4})$ (Case 19 of [3]), then

$$R^{-1} = X^{\frac{1}{6}}\,Y^{\frac{1}{3}}\,\left((3 + 4\,X)^3 - 96\,X\,\left(-9 + 4\,X\right)\,Y + 576\,X\,Y^2\right).$$

$(\mathbf{3, 7/3, -9/5})$ (Case 19 of [3]), then

$$R^{-1} = \sqrt{X}\,Y^{\frac{1}{6}}\,\left(27\,X^3 + 108\,X^2\,Y + 64\,(-3 + Y)^2\,Y + 144\,X\,Y\,\left(6 + Y\right)\right).$$

$(\mathbf{-4/5, 7/4, 2})$ (Case 19 of [3]), then

$$R^{-1} = X^{\frac{1}{3}}\,Y^{\frac{1}{6}}\,\left(576 + 5\,Y\,\left(400\,X^2 + 120\,X\,\left(-12 + 5\,Y\right) + 9\,\left(48 + 5\,Y\,\left(12 + 5\,Y\right)\right)\right)\right).$$

$(\mathbf{-3/4, 2/3, -9/4})$ (Case 19 of [3]), then

$$R^{-1} = \sqrt{X}\,Y^{\frac{1}{3}}\,\left(9\,(5 + 4\,X)^3 - 600\,\left(-5 + 12\,X\right)\,Y + 2000\,Y^2\right).$$

$(\mathbf{1/3, 9/5, -7/3})$ (Case 19 of [3]), then

$$R^{-1} = X^{\frac{1}{6}}\,\sqrt{Y}\,\left(125\,X\,(4 + 3\,X)^2 + 900\,X\,\left(-8 + 3\,X\right)\,Y + 2160\,X\,Y^2 + 576\,Y^3\right).$$

$(\mathbf{-9/4, 14/9, -3/4})$ (Case 20 of [3]), then

$$R^{-1} = X^{-\frac{1}{2}}Y^{\frac{1}{3}}\,\left(419904\,X^3 + 972\,X\,(9 + 4\,Y)^2 - (9 + 4\,Y)^4 + 34992\,X^2\,\left(-9 + 8\,Y\right)\right).$$

$(\mathbf{-5/9, -3, -13/9})$ (Case 20 of [3], then

$$R^{-1} = X^{\frac{1}{6}}Y^{-\frac{1}{2}}\,\left((4 + 9\,X)^4 - 972\,X\,(4 + 9\,X)^2\,Y + 34992\,X\,\left(-8 + 9\,X\right)\,Y^2 - 419904\,X\,Y^3.\right)$$

$(\mathbf{4, 13/4, -14/5})$ (Case 20 of [3]), then

$$R^{-1} = X^{\frac{1}{3}}\,Y^{\frac{1}{6}}\,\big(256\,X^4 + 2304\,X^3\,Y + 6561\,(-4 + Y)^3\,Y + 7776\,X^2\,\left(-2 + Y\right)\,Y \\ +11664\,X\,Y\,\left(-24 + \left(-6 + Y\right)\,Y\right)\big).$$

$(\mathbf{-9/5, 13/9, 3})$ (Case 20 of [3]), then

$$R^{-1} = X^{-\frac{1}{2}} Y^{\frac{1}{6}} \left(419904 + 5\,Y\, \left(91125\,X^3 + 24300\,X^2\,(-18+5\,Y) + 2160\,X\,(9+5\,Y)^2 + 64\,(18+5\,Y)\,(162+5\,Y\,(18+5\,Y))\right)\right).$$

$(\mathbf{-4/9, 3/4, -14/9})$ (Case 20 of [3]), then

$$R^{-1} = X^{\frac{1}{3}} Y^{-\frac{1}{2}} \left(64\,(5+9\,X)^4 + 10800\,(5+9\,X)^2\,Y - 121500\,(-5+18\,X)\,Y^2 + 455625\,Y^3\right).$$

$(\mathbf{1/4, 14/5, -13/4})$ (Case 20 of [3]), then

$$R^{-1} = X^{\frac{1}{6}}\,Y^{\frac{1}{3}} \left(625\,X\,(9+4\,X)^3 + 9000\,X\,(-243+4\,X\,(27+8\,X))\,Y + 97200\,X\,(9+8\,X)\,Y^2 + 933120\,X\,Y^3 + 419904\,Y^4\right).$$

$(\mathbf{-3/2, 10/3, -1/2})$ (Case 21 of [3]), then

$$R^{-1} = X^{\frac{1}{4}}\,\sqrt{Y}\,\left(324\,X^2 + (3+2\,Y)^4 + 36\,X\,(-9+2\,Y\,(12+Y))\right).$$

$(\mathbf{-7/3, -1, -5/3})$ (Case 21 of [3]), then

$$R^{-1} = X^{-\frac{3}{4}} Y^{\frac{1}{4}}\,\left((2+3\,X)^4 - 36\,X\,(-2+3\,X\,(-8+3\,X))\,Y + 324\,X^2\,Y^2\right).$$

$(\mathbf{2, 5/2, -10/7})$ (Case 21 of [3]), then

$$R^{-1} = \sqrt{X} Y^{-\frac{3}{4}}\,\left(16\,X^4 + 96\,X^3\,Y + 81\,(-2+Y)^2\,Y^2 + 216\,X\,Y^2\,(4+Y) + 72\,X^2\,Y\,(1+3\,Y)\right).$$

$(\mathbf{-3/7, 5/3, 1})$ (Case 21 of [3]), then R^{-1} is equal to

$$X^{\frac{1}{4}} Y^{-\frac{3}{4}} \left(324 + 7\,Y\,\left(3087\,X^2\,Y + 42\,X\,(-9+14\,Y\,(-12+7\,Y)) + 4\,(6+7\,Y)\,(18+7\,Y\,(6+7\,Y))\right)\right).$$

$(\mathbf{-2/3, 1/2, -10/3})$ (Case 21 of [3]), then

$$R^{-1} = \sqrt{X}\,Y^{\frac{1}{4}}\,\left(4\,(7+3\,X)^4 - 294\,(-98+3\,X\,(56+3\,X))\,Y + 21609\,Y^2\right).$$

$(\mathbf{1/2, 10/7, -5/2})$ (Case 21 of [3]), then R^{-1} is equal to

$$X^{-\frac{3}{4}} \sqrt{Y} \left(2401\,X^2\,(3+2\,X)^2 + 16464\,(-3+X)\,X^2\,Y + 2646\,X\,(-1+4\,X)\,Y^2 + 3024\,X\,Y^3 + 324\,Y^4\right).$$

$(\mathbf{-5/2, 13/5, -1/2})$ (Case 22 of [3]), then R^{-1} is equal to

$$X^{-\frac{2}{3}} Y^{\frac{1}{3}} \left(-125000\,X^3 + (5+2\,Y)^6 + 50\,X\,(5+2\,Y)^3\,(-15+4\,Y) + 2500\,X^2\,(75+2\,(-45+Y)\,Y)\right).$$

$(\mathbf{-8/5, -1, -7/5})$ (Case 22 of [3]), then

$$R^{-1} = X^{-\frac{5}{3}} Y^{\frac{2}{3}} \left((2+5\,X)^6 - 50\,X\,(2+5\,X)^3\,(-4+15\,X)\,Y + 2500\,X^2\,(2+15\,X\,(-6+5\,X))\,Y^2 - 125000\,X^3\,Y^3\right).$$

$(\mathbf{2, 7/2, -13/8})$ (Case 22 of [3]) , then

$$R^{-1} = X^{\frac{1}{3}} Y^{-\frac{5}{3}} \left(64\,X^6 + 960\,X^5\,Y + 15625\,(-2+Y)^3\,Y^3 + 2000\,X^3\,Y^2\,(3+10\,Y) + 400\,X^4\,Y\,(4+15\,Y) + 12500\,X\,Y^3\,(-18+Y\,(-7+3\,Y)) + 2500\,X^2\,Y^2\,(2+3\,Y\,(-2+5\,Y))\right).$$

$(\mathbf{-5/8, 7/5, 1})$ (Case 22 of [3]), then

$$R^{-1} X^{-\frac{2}{3}} Y^{-\frac{5}{3}} \left(15625 + 16\,Y\,\left(256000\,X^3\,Y^2 + 20\,X\,(5+8\,Y)^3\,(-5+12\,Y) + 800\,X^2\,Y\,(25+48\,Y\,(-15+8\,Y)) + (5+4\,Y)\,(25+8\,Y\,(5+8\,Y))\,(75+8\,Y\,(15+8\,Y))\right)\right).$$

$(\mathbf{-2/5, 1/2, -13/5})$ (Case 22 of [3]), then

$$R^{-1} = X^{\frac{1}{3}} Y^{-\frac{2}{3}} \left((8+5\,X)^6 - 320\,(-12+5\,X)\,(8+5\,X)^3\,Y + 12800\,(384+5\,X\,(-144+5\,X))\,Y^2 + 4096000\,Y^3\right).$$

$(\mathbf{1/2, 13/8, -7/2})$ (Case 22 of [3]), then

$$R^{-1} = X^{-\frac{5}{3}} Y^{\frac{1}{3}} \left(262144\,X^6 + 15625\,Y^6 + 983040\,X^5\,(2+Y) + 50000\,X\,Y^4\,(-4+3\,Y) + 256000\,X^3\,(16+Y\,(3+Y)\,(-12+5\,Y)) + 40000\,X^2\,Y^2\,(8+3\,Y\,(-4+5\,Y)) + 102400\,X^4\,(48+Y\,(28+15\,Y))\right).$$

$(\mathbf{-10/3, 17/10, -2/3})$ (Case 23 of [3]), then

$$R^{-1} = X^{-\frac{3}{2}}Y^{\frac{1}{4}}\big(81000000\,X^4 + 21600000\,X^3\,(-5+3\,Y) - 27000\,X^2\,(-20+Y)\,(10+3\,Y)^2 -1200\,X\,(10+3\,Y)^4 + (10+3\,Y)^6\big)\,.$$

$(\mathbf{-7/10, -2, -13/10})$ (Case 23 of [3]), then

$$R^{-1} = X^{-\frac{3}{4}}Y^{\frac{3}{2}}\big((3+10\,X)^6 - 1200\,X\,(3+10\,X)^4\,Y + 27000\,X\,(3+10\,X)^2\,(-1+20\,X)\,Y^2 -21600000\,X^2\,(-3+5\,X)\,Y^3 + 81000000\,X^2\,Y^4\big)\,.$$

$(\mathbf{3, 13/3, -17/7})$ (Case 23 of [3]), then

$$R^{-1} = X^{\frac{1}{4}}Y^{-\frac{3}{4}}\big(729\,X^6 + 14580\,X^5\,Y + 1000000\,(-3+Y)^4\,Y^2 + 24300\,X^4\,Y\,(-4+5\,Y) + 270000\,X^2\,Y^2\,(12+Y\,(-24+5\,Y)) + 27000\,X^3\,Y\,(-9+4\,Y\,(-12+5\,Y)) +900000\,X\,Y^2\,(72+Y\,(33+2\,(-8+Y)\,Y))\big)\,.$$

$(\mathbf{-10/7, 13/10, 2})$ (Case 23 of [3]), then

$$R^{-1} = X^{-\frac{3}{2}}Y^{-\frac{3}{4}}\big(81000000 + 7\,Y\,\big(168070000\,X^4\,Y + 28812000\,X^3\,Y\,(-20+7\,Y) + 7560\,X\,(10+7\,Y)^4 + 88200\,X^2\,(10+7\,Y)^2\,(-5+21\,Y) +81\,(20+7\,Y)\,(100+7\,Y\,(10+7\,Y))\,(300+7\,Y\,(30+7\,Y))\big)\big)\,.$$

$(\mathbf{-3/10, 2/3, -17/10})$ (Case 23 of [3]), then

$$R^{-1} = X^{\frac{1}{4}}Y^{-\frac{3}{2}}\big(81\,(7+10\,X)^6 + 52920\,(7+10\,X)^4\,Y - 617400\,(-21+5\,X)\,(7+10\,X)^2\,Y^2 -201684000\,(-7+20\,X)\,Y^3 + 1176490000\,Y^4\big)\,.$$

$(\mathbf{1/3, 17/, -13/3})$ (Case 23 of [3]), then

$$R^{-1} = X^{-\frac{3}{4}}Y^{\frac{1}{4}}\big(117649\,X^2\,(10+3\,X)^4 + 1008420\,X^2\,\big(-4000+3\,X\,\big(550+240\,X+27\,X^2\big)\big)\,Y + 10804500\,X^2\,(80+9\,X\,(16+3\,X))\,Y^2 + 61740000\,X\,(-5+3\,X\,(8+3\,X))\,Y^3 +66150000\,X\,(8+9\,X)\,Y^4 + 340200000\,X\,Y^5 + 81000000\,Y^6\big)\,.$$

There are also six cases which correspond to Case 24 of [3]. Those are cases 103 to 109 in the table given in the statement of Theorem 8. Those six systems have an irreducible invariant algebraic curve of degree $2l$ where l can be any positive integer. The explicit expression of these invariant algebraic curves can be computed from [10].

Furthermore, also in view of Proposition 28(b) (see the Appendix) we obtain that it has a global analytic first integral if and only if $a = -p/q$, $b = -p_1/q_1$, $c = -1 + p(p_1+q_1)/(qq_1)$ with $p, q, p_1, q_1 \in \mathbb{N}$ and $pp_1 - qq_1 \geq 0$, then $H = x^{pq_1}y^{qq_1}(pp_1 + pq_1 - qq_1x + p_1qy + qq_1y)^{pp_1-qq_1}$. Note that this first integral is polynomial.

Case 2*:* $c = -1$. In this case we consider three different subcases.

Subcase 2.1*:* $a = 0$. In this case system (I) becomes

$$x' = x(1-x-by), \quad y' = y(-x-y). \tag{9}$$

Doing the change of variables $(x, y) \to (-x, -y)$ system (9) becomes

$$x' = x(1+x+by), \quad y' = y(x+y)$$

which is equation (44) of the Appendix with $\bar{A} = 1$ and $\bar{B} = b$. In view of Proposition 32 we get that system is always Liouvillian integrable. The Liouvillian first integral is $H = \big(x + y\log x - by\log y + 1\big)/y$.

Subcase 2.2*:* $a \neq 0$ and $b \neq 1$. In this case doing the change of variables $(x, y) \to (y, x)$ system (I) becomes

$$x' = x(a-x-y), \quad y' = y(1-y-bx). \tag{10}$$

Doing also the rescaling $(x, y, t) \mapsto (aX/(1-b), aY, T/a)$, we obtain that system (10) is system (40) of the Appendix with $\bar{A} = 1/a$, $\bar{B} = 1/(1-b)$, $\bar{C} = 0$. In view of Proposition 28 we get that system (10) is Liouvillian integrable if and only if $b = 0$ and in this case the integrating factor is $R = Y^{-1+a}(-1+aY)^{-a}X^{-2}$, or $a = -1$, $b = 1/2$ and the integrating factor is $R =$

$X^{-2}Y^{-2}\sqrt{(1+Y)^2-2X}$. Furthermore, it also follows from Proposition 28(b) that it has no global analytic first integrals.

Subcase 2.3*:* $a \neq 0$ and $b = 1$. In this case system (I) becomes

$$x' = x(a - x - y), \quad y' = y(1 - x - y). \tag{11}$$

If $a = 1$ then system (11) becomes system $x' = x$, $y' = y$, which has the first integral $H = x/y$.

If $a \neq 1$, doing the change of variables $x = -1/X$, $y = -Y/X$, $dt = XdT/(1-a)$, we get that system (11) becomes $\dot{X} = X(aX + Y + 1)/(a - 1)$, $\dot{Y} = XY$, which is integrable with first integral $H = Y^a(aX + aY + 1)^{1-a}$. This completes the proof of the theorem.

4. Proof of Theorem 9

We consider different cases.

Case 1: $b \neq 1$. In this case doing the change of variables $(x, y) \to (y, x)$ we get that system (II) becomes

$$x' = x(1 - x + cy), \quad y' = y(-y - bx). \tag{12}$$

Doing now the rescaling of variables $(x, y, t) \mapsto (X/(1-b), Y, T)$, we obtain that system (12) becomes system (40) with $\bar{A} = 0$, $\bar{B} = 1/(1-b)$, $\bar{C} = c + 1$. Then it follows from Proposition 28 that system (12) has a Liouvillian first integral if and only if $c = 0$ and then the integrating factor is $R = X^{-1}Y^{-2}(b - 1 + X)^{b-1}$, or if $c = 1$, $b = 2$ and in this case the integrating factor is $R = X^{-1}Y^{-2}\sqrt{(1+X)^2 - 2XY}$. It also follows also from Proposition 28 that it has no global analytic first integrals.

Case 2: $b = 1$. In this case system (II) becomes

$$x' = x(-x - y), \quad y' = y(1 - y + cx) \tag{13}$$

Doing the change of variables $(x, y) \to (-y, -x)$ we obtain that system (13) becomes $x' = x(1 + x - cy)$, $y' = y(x + y)$ which is system (44) with $\bar{A} = 1$ and $\bar{B} = -c$. It follows from Proposition 32 that it is Liouvillian integrable with Liouvillian first integral $H = \big(x + y\log x + cy\log y + 1\big)/y$.

5. Proof of Theorem 10

We consider different cases.

Case 1*:* $c = 0$. In this case system (III) becomes

$$x' = x(-x - by), \quad y' = -y^2. \tag{14}$$

Doing the change of variables $(x, y) \to (-x, -y)$ system (14) becomes

$$x' = x(x + by), \quad y' = y^2. \tag{15}$$

with is system (41) with $\bar{A} = 1$, $\bar{B} = b$, $\bar{C} = 0$ and $\bar{D} = 1$. In view of Proposition 29(b) system (15) is Liouvillian integrable. The integrating factor is $R = x^{-2}y^{b-2}$. Also in view of Proposition 29 it has no global analytic first integrals. This concludes the proof of the theorem in this case.

Case 2*:* $c \neq 0$. In this case doing the change of variables $(x, y) \to (X/c, -Y)$, we obtain that system (III) becomes $X' = X(-X/c + bY)$, $Y' = Y(X + Y)$. System (III) is homogeneous quadratic and thus it is integrable with the inverse of the integrating factor equal to $XY((1+c)X + c(1-b)Y)$. From Proposition 31(b) of the Appendix, it has a global analytic first integral if and only if a and b are rational, $c + 1$, $1 - b$ and $-bc - 1$ have the same sign and in this case the first integral is the stated in the theorem.

6. Proof of Theorems 11, 12, 13, 14 and 15

We first prove Theorem 11.

Proof of Theorem 11. Doing the rescaling of the variables $(x, y, t) \mapsto (-x, -y, -t)$ we have that system (IV) with $b \in \{0, 1\}$ and $a = 0$ becomes

$$x' = x(1 - by), \quad y' = y(-c + x - y). \tag{16}$$

Therefore, system (16) is system (40) with $\bar{A} = -c$, $\bar{B} = 0$, $\bar{C} = 1 - b$. Then it follows from Proposition 28(a) that system (16) has a Liouvillian first integral if and only if $b = 0$, $c = -1$ and then the integrating factor is $R = y^{-1}(x-y)^{-1}$. It also follows also from Proposition 28 that it has no global analytic first integrals. □

Now we prove Theorem 12.

Proof of Theorem 12. System (IV) with $b \in \{0,1\}$ and $a = (1-b)(1+c) \neq 0$ becomes

$$x' = -x + (1-b)(1+c)y - bxy, \quad y' = y(c+x-y).$$

Doing the change of variables $X = x + (b-1)y$, $Y = y$, it becomes

$$X' = -X(1+Y), \quad Y' = Y(c+X-bY). \tag{17}$$

We consider two different cases.

Case 1: $b = 1$. In this case doing the change of variables $(X, Y, t) \mapsto (-X_1, -Y_1, -T)$, we obtain that system (17) becomes system (40) with $\bar{A} = -c$, $\bar{B} = 0$, $\bar{C} = 0$. Then it follows from Proposition 28(a) in the Appendix that system (17) has no Liouvillian first integrals. It also follows also from Proposition 28 (b) that it has no global analytic first integrals.

Case 2: $b = 0$. In this case system (17) becomes

$$X' = -X(1+Y), \quad Y' = Y(c+X), \tag{18}$$

which clearly has the first integral $H = YX^c e^{Y+X}$. This first integral is always Liouvillian and it is global analytic if and only if $c \in \mathbb{Q}^+$. This completes the proof of the theorem. □

Now we prove Theorem 13.

Proof of Theorem 13. System (VI) with $a = 0$ becomes $x' = x(-1-y)$, $y' = y(c+x)$, which is system (18). The theorem follows now from Case 2 in the proof of Theorem 12. □

Now we prove Theorem 14.

Proof of Theorem 14. System (VI) with $a = -1-c$ with $c \neq -1$ becomes $x' = -x-(1+c)y-xy$, $y' = y(c+x)$. Introducing the change of variables $X = x+y$, $Y = y$, we obtain system

$$X' = -X, \quad Y' = Y(c+X-Y). \tag{19}$$

It is straightforward to verify that system (19) is system (42) in the Appendix with $\bar{D} = -1$, $\bar{A} = 1$, $\bar{B} = -1$ and $\bar{C} = c \neq -1$. It follows from Proposition 30(a) that it is always Liouvillian integrable with integrating factor $R = X^{-c-1}Y^{-2}e^{-X}$.

Furthermore, it also follows from Proposition 30(b) that it has no global analytic first integrals. This completes the proof of the theorem. □

Now we prove Theorem 15.

Proof of Theorem 15. It follows by direct computations that system (V) with $c \neq -1$ is integrable with first integral $H = y^{1/c}(y-c)^{-(c+1)/c}(y-(1+c)x)$, and that if $c = -1$ it has the first integral $H = x/y + \log(y/(y+1))$. Both are Liouvillian first integrals and none of them are global analytic. This completes the proof of the theorem. □

7. Proof of Theorem 16

In this section we prove Theorem 16. For this we divide it into several different partial results, related respectively with the existence of polynomial first integrals, Darboux polynomials, exponential factors, Liouvillian first integrals and global analytic first integrals.

Proposition 18. *Systems (IV) with $b \in \{0,1\}$ and $a \notin \{0, (1-b)(1+c)\}$ has no polynomial first integrals.*

Proof. The proof follows using Theorem 4. With it we can determine if system (IV) has a polynomial first integral and its expression. The determination is a little tedious because we need to evaluate all the mentioned invariants in Theorem 4. In any case, doing that, we obtain the non-existence of polynomial first integrals. □

The next result for system (VI) will be useful also for system (IV).

Proposition 19. *The unique irreducible Darboux polynomial with non–zero cofactor of systems (VI) with $a \notin \{0, -1-c\}$ is y.*

Proof. It follows by direct computations that $f_1 = y$ is the unique irredubicle Darboux polynomial of degree one of system (VI) with $a \notin \{0, -1-c\}$ and with non–zero cofactor. Now we shall see that there are no irreducible Darboux polynomials with non–zero cofactor of degree greater than one. We proceed by contradiction. Let f be a Darboux polynomial of system (VI) with non–zero cofactor which is irreducible with degree $n \geq 2$. Then it satisfies

$$(-x + ay - xy)\frac{\partial f}{\partial x} + y(c + x)\frac{\partial f}{\partial y} = (\alpha_0 + \alpha_1 x + \alpha_2 y)f. \tag{20}$$

If we denote by $\hat{f}$ the restriction of f to $y = 0$, we obtain that $\hat{f} \neq 0$ (otherwise would be reducible) and satisfies (20) restricted to $y = 0$, that is $-xd\hat{f}/dx = (\alpha_0 + \alpha_1 x)\hat{f}$, whose solution is $\hat{f} = Cx^{-\alpha_0}e^{-\alpha_1 x}$, with $C \in \mathbb{C} \setminus \{0\}$. From the fact that $\hat{f}$ must be a polynomial we obtain that $\alpha_1 = 0$ and $\alpha_0 = -k$ with $k \in \mathbb{N}$. Now we introduce the change of variables $X = x + y$, $Y = y$. If we set $g(X, Y) = f(x, y)$ we get that g satisfies

$$(-X + (a + c + 1)Y)\frac{\partial g}{\partial X} + Y(c + X - Y)\frac{\partial g}{\partial Y} = (\alpha_0 + \alpha_2 Y)g. \tag{21}$$

We write $g = \sum_{j=0}^{n} g_j(X, Y)$ where each g_j is a homogeneous polynomial of degree j. Without loss of generality we can assume $g_n \neq 0$. Furthermore computing the terms in (21) of degree $n + 1$ we get that g_n satisfies $Y(X - Y)\partial g_n/\partial Y = \alpha_2 Y g_n$ that is $g_n = c_n(X - Y)^{-\alpha_2}K(X)$. Since g_n is a homogeneous polynomial of degree n we get $g_n = c_n X^{n-m}(X - Y)^m$, $\alpha_2 = -m$, $1 \leq m \leq n$ with $c_n \in \mathbb{C} \setminus \{0\}$. We shall show that $m = n$. It follows from (21) that g_{n-1} satisfies

$$\begin{aligned}&(-X + (a + c + 1)Y)(X - Y)^{m-1}[c_n(n - m)X^{n-m-1}(X - Y) + mc_n X^{n-m}]- \\ &Ycc_n mX^{n-m}(X - Y)^{m-1} + Y(X - Y)\frac{\partial g_{n-1}}{\partial Y} = \\ &- kc_n X^{n-m}(X - Y)^m - mY g_{n-1}.\end{aligned} \tag{22}$$

We claim that $g_{n-1} = (X - Y)^{m-1}h_{n-1}$. Clearly this is true if $m = 1$. Assume that $m > 1$. From (22) we have that g_{n-1} must be divisible by $X - Y$, we write it as $g_{n-1} = (X - Y)^l h_l$ with $1 \leq l < m-1$ and h_l not divisible by $X-Y$. We will reach a contradiction. In fact, after simplifying by $(X - Y)^l$ we get that h_l satisfies

$$\begin{aligned}&(-X + (a + c + 1)Y)(X - Y)^{m-1-l}[c_n(n - m)X^{n-m-1}(X - Y) + mc_n X^{n-m}]- \\ &Ycc_n mX^{n-m}(X - Y)^{m-1-l} + Y(X - Y)\frac{\partial h_l}{\partial Y} = \\ &- kc_n X^{n-m}(X - Y)^{m-l} - (m - l)Y h_l,\end{aligned}$$

and since $l < m - 1$ we get a contradiction with the fact that h_l is not divisible by $X - Y$. Hence the claim is proved. Moreover h_{n-1} satisfies

$$\begin{aligned}&(-X + (a + c + 1)Y)[c_n(n - m)X^{n-m-1}(X - Y) + mc_n X^{n-m}] - Ycc_n mX^{n-m} \\ &+ Y(X - Y)\frac{\partial h_{n-1}}{\partial Y} = -kc_n X^{n-m}(X - Y) - Y h_{n-1}.\end{aligned}$$

Evaluating this expression on $Y = 0$ we get $-c_n nX^{n-m+1} = -kc_n X^{n-m+1}$, that is $k = n$. Thus we have, after simplifying by Y that

$$\begin{aligned}&(X - Y)\frac{\partial h_{n-1}}{\partial Y} + h_{n-1} \\ &= (a + c + 1)nc_n X^{n-m-1}(X - Y) + (a + c + 1)c_n mX^{n-m-1}Y - (c + 1)c_n mX^{n-m}.\end{aligned}$$

Solving this linear differential equation we obtain

$$h_{n-1} = K_0(X)(X - Y) + (a + c + 1)c_n(n - m)(X - Y)\log(X - Y)X^{n-m-1} + ac_n mX^{n-m}.$$

Since h_{m-1} must be a homogeneous polynomial of degree $n-m$ and $c_n(a+c+1) \neq 0$ we must have $m = n$. Then h_{n-1} is a constant and therefore $K_0(X) = 0$. Hence, $\alpha_0 = \alpha_2 = -n$, $g_n = c_n(X-Y)^n$ and $g_{n-1} = ac_n n(X - Y)^{n-1}$. If $n = 1$ then $g_0 = ac_1$ and $\alpha_0 = -1$. Since $g_0\alpha_0 = 0$ (see (21)) and $a \neq 0$, $c_1 \neq 0$ we get a contradiction.

Now we assume $n \geq 2$. Computing the terms in (21) of degree $n-1$ we get that g_{n-2} satisfies

$$(-X+(a+c+1)Y)(X-Y)^{n-2}ac_n n(n-1) - Ycac_n n(n-1)(X-Y)^{n-2} + Y(X-Y)\frac{\partial g_{n-2}}{\partial Y}$$
$$= -n^2 ac_n (X-Y)^{n-1} - nYg_{n-2},$$

which can be written as

$$Y(X-Y)\frac{\partial g_{n-2}}{\partial Y} + nYg_{n-2} + ac_n n^2 (X-Y)^{n-1} + ac_n n(n-1)(-X+(a+1)Y)(X-Y)^{n-2} = 0.$$

Evaluating it on $Y=0$ we obtain that $ac_n n X^{n-1} = 0$. Since $a \neq 0$ and $c_n \neq 0$ we obtain a contradiction. □

Proposition 20. *The unique irreducible Darboux polynomial with non–zero cofactor of systems (IV) with $a \notin \{0, (1-b)(1+c)\}$ is y.*

Proof. It follows by direct computations that $f_1 = y$ is the unique irreducible Darboux polynomial of degree one of system (IV) with $a \notin \{0, (1-b)(1+c)\}$ and with non–zero cofactor. Now we shall see that there are no irreducible Darboux polynomials with non–zero cofactor of degree greater than one. We proceed by contradiction. Let f be a Darboux polynomial of system (IV) with non–zero cofactor which is irreducible with degree $n \geq 2$. Then it satisfies

$$(-x+ay-bxy)\frac{\partial f}{\partial x} + y(c+x-y)\frac{\partial f}{\partial y} = (\alpha_0 + \alpha_1 x + \alpha_2 y)f. \tag{23}$$

If we denote by $\hat{f}$ the restriction of f to $y=0$, we obtain that $\hat{f} \neq 0$ (otherwise would be reducible) and satisfies (23) restricted to $y=0$, that is $-xd\hat{f}/dx = (\alpha_0+\alpha_1 x)\hat{f}$ whose solution is $\hat{f} = Cx^{-\alpha_0}e^{-\alpha_1 x}$ with $C \in \mathbb{C}\setminus\{0\}$. Since $\hat{f}$ must be a polynomial we obtain in particular that $\alpha_1 = 0$ and $\alpha_0 = -k$, $k \in \mathbb{N}$. We consider different cases.

Case 1*:* $b=1$. In this case if we write $f = \sum_{j=0}^{n} f_j(x,y)$ where each f_j is a homogeneous polynomial of degree j we have without loss of generality that $f_n \neq 0$. Computing the terms in (23) of degree $n+1$ we get that f_n satisfies

$$-xy\frac{\partial f_n}{\partial x} + y(x-y)\frac{\partial f_n}{\partial y} = \alpha_2 y f_n.$$

Eliminating y in the partial differential equation and solving it we get $f_n(x,y) = x^{-\alpha_2}h_n((y + x\log x)/x)$, where h_n is an arbitrary function in the variable $(y+x\log x)/x$. Since f_n must be a homogeneous polynomial of degree n, we obtain

$$f_n = c_n x^{-\alpha_2}, \quad \alpha_2 = -n, \quad c_n \in \mathbb{C}\setminus\{0\}. \tag{24}$$

It follows from (23) computing the terms of degree n that f_{n-1} satisfies

$$(-x+ay)c_n n x^{n-1} - xy\frac{\partial f_{n-1}}{\partial x} + y(x-y)\frac{\partial f_{n-1}}{\partial y} = -kc_n x^n - nyf_{n-1}. \tag{25}$$

Evaluating (25) on $y=0$ we get, after simplifying by $c_n x^n$ that $k=n$. Then $\alpha_0 = -n$, and after simplifying by y, f_{n-1} satisfies $anc_n x^{n-1} - x\partial f_{n-1}/\partial x + (x-y)\partial f_{n-1}/\partial y = -nf_{n-1}$. Solving this partial differential equation we get that $f_{n-1}(x,y) = -anc_n x^{n-1} + x^n h_{n-1}((y+x\log x)/x)$, where h_{n-1} is an arbitrary function in the variable $(y+x\log x)/x$. Since f_{n-1} must be a homogeneous polynomial of degree $n-1$, we get that $h_{n-1} = 0$, and consequently, $f_{n-1} = -anc_n x^{n-1}$. If $n=1$ then $\alpha_0 = -1$ and $f_0 = -c_1 a \neq 0$ a contradiction with the fact that $\alpha_0 f_0 = 0$ (see (23)). Furthermore, if $n \geq 2$ then computing the terms of degree $n-1$ in (23) we get

$$-(-x+ay)an(n-1)c_n x^{n-2} - xy\frac{\partial f_{n-2}}{\partial x} + y(x-y)\frac{\partial f_{n-2}}{\partial y} = n^2 ac_n x^{n-1} - nyf_{n-2}.$$

Evaluating it on $y=0$ we get, after simplifying by x^{n-1} that $nac_n = 0$, a contradiction since $a \neq 0$, $c_n \neq 0$. This concludes the proof in this case.

Case 2*:* $b=0$. We introduce the change of variables $X = x-y$, $Y=y$, $A = a-1-c \neq 0$ and $A \neq -1-c$. If we set $g(X,Y) = f(x,y)$ we get that g satisfies $(-X(1+Y)+AY)\partial g/\partial X + Y(c+X)\partial g/\partial Y = (\alpha_0+\alpha_2 Y)g$, which is equation (20). It follows then by Proposition 19 that the unique irreducible Darboux polynomials with non–zero cofactor is Y. This concludes the proof of the lemma. □

Proposition 21. *Systems (IV) with $b \in \{0,1\}$ and $a \notin \{0,(1-b)(1+c)\}$ has exponential factors if and only if $b=0$, being in this case of the form $F=e^x$ with cofactor $L=ay-x$.*

Proof. Let F be an exponential factor. In view of Propositions 2 and 20 we have that F has the form $\exp(h/y^{n_1})$ with $h \in \mathbb{C}[x,y]$ and n_1 is a non-negative integer and where h is coprime with y. We assume that $n_1>0$ and we will reach a contradiction. We have that, after simplifying by the common factor $\exp(h/y^{n_1})$ and after multiplying by y^{n_1}, h satisfies

$$(26) \qquad (-x+ay-bxy)\frac{\partial h}{\partial x}+y(c+x-y)\frac{\partial h}{\partial y}-n_1(c+x-y)h=Ly^{n_1}.$$

Since $n_1>0$, taking $y=0$ in (26) and denoting by $\hat{h}$ the restriction of h to $y=0$ we have that $-xd\hat{h}/dx-n_1(c+x)\hat{h}=0$, that is $h=C_1e^{-n_1x}x^{-n_1c}$ with $C_1\in\mathbb{C}$. Since $\hat{h}$ must be a polynomial we obtain $\hat{h}=0$, a contradiction with the fact that h is coprime with y. Therefore, $n_1=0$ and $F=\exp(h)$ satisfying

$$(27) \qquad (-x+ay-bxy)\frac{\partial h}{\partial x}+y(c+x-y)\frac{\partial h}{\partial y}=L=\beta_0+\beta_1x+\beta_2y.$$

Evaluating (27) on $x=y=0$ we obtain $\beta_0=0$. Now we write $h=\sum_{j=0}^n h_j(y)x^j$, where $h_j(y)$ is a polynomial in y. We claim that

$$(28) \qquad h_j(y)=0 \quad j=2,\dots,n \quad \text{that is} \quad h(x,y)=h_0(y)+xh_1(y).$$

If $n=1$ there is nothing to prove. We can assume that $n\geq 2$. Then computing in (27) the terms of degree $n+1$ in x we get $yh_n'(y)=0$, that is $h_n(y)=c_n\in\mathbb{C}$. Furthermore, computing the terms in (27) of degree n in x we get $-n(1+by)c_n+yh_{n-1}'(y)=0$. Solving this differential equation we get that $h_{n-1}(y)=c_{n-1}+bnc_ny+nc_n\log y$. Since h_{n-1} must be a polynomial in y we get that $c_n=0$ and consequently $h_{n-1}=c_{n-1}\in\mathbb{C}$. This proves (28) with $j=n$. Proceeding by backwards induction we obtain that $h_j=0$ for $j=2,\dots,n$ and (28) is proved.

Using (28) and imposing that it satisfies (27) we obtain $(-x+ay-bxy)h_1(y)+y(c+x-y)(h_0'(y)+xh_1'(y))=\beta_1x+\beta_2y$. Computing the terms of degree two in x we get $h_1'(y)=0$ and hence $h_1=c_1\in\mathbb{C}$. Then we get $(-x+ay-bxy)c_1+y(c+x-y)h_0'(y)=\beta_1x+\beta_2y$. Solving this differential equation we get $h_0(y)=c_0+(\beta_1+c_1)x\log y/(c+x)+(ac_1-\beta_2-bc_1x)\log(-c-x+y)-(\beta_1+c_1)x\log(-c-x+y)/(c+x)$. Since h_0 must be a polynomial we get $\beta_1=-c_1$, $bc_1=0$ and $\beta_2=ac_1$. So $h_0(y)=c_0$. We consider two cases.

(1) If $b=1$ then $c_1=0$. This implies that $L=0$. Then from (27) h must be a polynomial first integral of system (IV), a contradiction with Proposition 18.
(2) If $b=0$ then $L=\beta_1x-a\beta_1y$ and $h=c_0-\beta_1x$ with $\beta_1\in\mathbb{C}$.

Note that e^{c_0} is a constant and can be removed. Moreover, if $e^{-\beta_1x}$ is a Darboux polynomial with cofactor $L=\beta_1(x-ay)$, then e^x is another exponential factor with cofactor $L=ay-x$. This concludes the proof of the proposition. □

Proposition 22. *Systems (IV) with $b\in\{0,1\}$ and $a\notin\{0,(1-b)(1+c)\}$ have no Liouvillian first integrals.*

Proof. We consider two cases.

Case 1*:* $b=1$. In this case it follows from Propositions 20 and 21 that system (IV) has one invariant Darboux polynomial $f_1=y$ with cofactor $K=c+x-y$, and that it has no exponential factors. Then since the divergence of system (IV) is $-1+c+x-3y$, we have $\lambda K=\lambda(c+x-y)=1-c-x+3y$, which is not possible. Then applying Theorems 3(b) and 5 we get the non-existence of Liouvillian first integrals.

Case 2*:* $b=0$. In this case it follows from Propositions 20 and 21 that system (IV) has one invariant Darboux polynomial $f_1=y$ with cofactor $K=c+x-y$ and one exponential factor $F=e^x$ with cofactor $L=ay-x$. Then again since the divergence of system (IV) is $-1+c+x-2y$, we have $\lambda K+\mu L=\lambda(c+x-y)+\mu(ay-x)=1-c-x+2y$ if and only if $a\neq 1$, $c=a-1$, $\lambda=(2-a)/(a-1)$ and $\mu=1/(a-1)$, which is not possible since $a\neq 1+c$. This completes the proof of the proposition. □

Proposition 23. *Systems (IV) with $b\in\{0,1\}$ and $a\notin\{0,(1-b)(1+c)\}$ have no analytic first integrals.*

Proof. We note that the eigenvalues of $Df(0,0)$ are -1 and c. We consider different cases.

Case 1*:* $c=0$. In this case, since $a\notin\{0,(1-b)(1+c)\}$ we get that $a\notin\{0,1-b\}$. Then, since $(a,b)\neq(1,0)$ the origin is isolated and since the eigenvalues of $Df(0,0)$ are $-1,0$ in view of Theorem 7 we conclude that system (IV) has no local analytic first integrals in a neighborhood of the origin. Consequently they have no global analytic first integrals.

Case 2*:* $c\neq 0$. We consider different subcases.

Subcase 2.1*:* $c\notin\mathbb{Q}^+$. Then for any $k_1,k_2\in\mathbb{N}\cup\{0\}$ with $k_1+k_2>0$ we have $k_1\lambda_1+k_2\lambda_2=-k_1+k_2c\neq 0$ and in view of Theorem 6 we have that systems (IV) have no local analytic first integrals and hence they have no global analytic first integrals.

Subcase 2.2*:* $c\in\mathbb{Q}^+$. Now we consider two subcases.

Subcase 2.2.1*:* $b=1$. In this case we have that

$$(\bar x_1,\bar y_1)=\frac{1}{2}\big(-1+a-c-\sqrt{4c+(c+a-1)^2},-1+a+c-\sqrt{4c+(c+a-1)^2}\big),$$
$$(\bar x_2,\bar y_2)=\frac{1}{2}\big(-1+a-c+\sqrt{4c+(c+a-1)^2},-1+a+c+\sqrt{4c+(c+a-1)^2}\big),$$

are also singular points of system (IV), among $(x,y)=(0,0)$. The eigenvalues of $(\bar x_1,\bar y_1)$ are λ_1,λ_2 that satisfy

$$\lambda_1\lambda_2=\frac{1}{2}\big(4c+(c+a-1)^2-(c+a-1)\sqrt{(c+a-1)^2+4c}\big). \tag{29}$$

On the other hand, the eigenvalues of $(\bar x_2,\bar y_2)$ are λ_3,λ_4 that satisfy

$$\lambda_3\lambda_4=\frac{1}{2}\big(4c+(c+a-1)^2+(c+a-1)\sqrt{(c+a-1)^2+4c}\big). \tag{30}$$

Now we consider different cases:

(1) If $c+a-1\geq 0$, then it follows from (30) that $\lambda_3\lambda_4>0$. Therefore, given $k_1,k_2\in\mathbb{N}\cup\{0\}$ with $k_1+k_2>0$, since λ_3 and λ_4 have the same sign we have that $k_1\lambda_3+k_2\lambda_4\neq 0$. Then, in view of Theorem 6 we have that systems (IV) have no local analytic first integrals and hence they have no global analytic first integrals.

(2) If $c+a-1<0$, then it follows from (29) that $\lambda_1\lambda_2>0$. Therefore, given $k_1,k_2\in\mathbb{N}\cup\{0\}$ with $k_1+k_2>0$, since λ_1 and λ_2 have the same sign we have that $k_1\lambda_1+k_2\lambda_2\neq 0$. Then, in view of Theorem 6 we have that systems (IV) have no local analytic first integrals and hence they have no global analytic first integrals.

This completes the proof of the lemma in this case.

Subcase 2.2.2*:* $b=0$. Since $b=0$, then $a\neq 1$. Then system (IV) becomes $x'=-x+ay$, $y'=y(c+x-y)$. This system has $(\bar x,\bar y)=c(a,1)/(1-a)$ as a singular point among $(x,y)=(0,0)$. The eigenvalues of $Df(\bar x,\bar y)$ satisfy that $\lambda_1\lambda_2=c>0$. Therefore, given $k_1,k_2\in\mathbb{N}\cup\{0\}$ with $k_1+k_2>0$, since λ_1 and λ_2 have the same sign we have that $k_1\lambda_1+k_2\lambda_2\neq 0$. Then, in view of Theorem 6 we have that systems (IV) have no local analytic first integrals and hence they have no global analytic first integrals. □

Proof of Theorem 16. The proof of Theorem 16 is immediate from Propositions 22 and 23. □

8. Proof of Theorem 17

In this section we prove Theorem 17. As we did in Section 8 we divide it into several partial results.

Proposition 24. *Systems (VI) with $a\notin\{0,-1-c\}$ has no polynomial first integrals.*

Proof. The proof follows using Theorem 4. With it we can determine if system (VI) has a polynomial first integral and its expression. The determination is a little tedious because we need to evaluate all the mentioned invariants in Theorem 4. In any case, doing that, we obtain the non-existence of polynomial first integrals. □

Proposition 25. *Systems (VI) with $a\notin\{0,-1-c\}$ has exponential factors of the form $F=e^{-(x+y)}$ with cofactor $L=x-(a+c)y$.*

Proof. Suppose that F is an exponential factor. In view of Propositions 2 and 19 we have that F has the form $\exp(h/y^{n_1})$ with $h \in \mathbb{C}[x, y]$ and n_1 is a non-negative integer and where h is coprime with y. We assume that $n_1 > 0$ and we will reach a contradiction. We have that h satisfies

$$(-x + ay - xy)\frac{\partial h}{\partial x} + y(c + x)\frac{\partial h}{\partial y} - n_1(c + x)h = Ly^{n_1}, \tag{31}$$

where we have simplified the common factor $\exp(h/y^{n_1})$ and multiplied by y^{n_1}. Since $n_1 > 0$, taking $y = 0$ in (31) and denoting by $\hat{h}$ the restriction of h to $y = 0$ we have that $-xd\hat{h}/dx - n_1(c+x)\hat{h} = 0$, that is $\hat{h} = C_1 e^{-n_1 x} x^{-n_1 c}$ with $C_1 \in \mathbb{C}$. Since $\hat{h}$ must be a polynomial we obtain $\hat{h} = 0$, a contradiction with the fact that h is coprime with y. Therefore, $n_1 = 0$ and $F = \exp(h)$ satisfying

$$(-x + ay - xy)\frac{\partial h}{\partial x} + y(c + x)\frac{\partial h}{\partial y} = L = \beta_0 + \beta_1 x + \beta_2 y. \tag{32}$$

Evaluating (32) on $x = y = 0$ we obtain $\beta_0 = 0$. Now we write $h = \sum_{j=0}^{n} h_j(y)x^j$, where $h_j(y)$ is a polynomial in y. We claim that

$$h = h_0(y) + xh_1(y). \tag{33}$$

If $n = 1$ the claim (33) is clear. Hence we can assume that $n \geq 2$. Then computing in (32) the terms of degree $n + 1$ in x we get $yh'_n(y) = 0$, that is $h_n(y) = c_n \in \mathbb{C}$. Furthermore computing the terms in (32) of degree n in x we get $-n(1+y)c_n + yh'_{n-1}(y) = 0$. Solving this differential equation we get that $h_{n-1}(y) = c_{n-1} + nc_n y + nc_n \log y$. Since h_{n-1} must be a polynomial in y we get that $c_n = 0$ and consequently $h_{n-1} = c_{n-1} \in \mathbb{C}$. Proceeding by backwards induction we obtain that $h_j = 0$ for $j = 2, \ldots, n$. Then (33) is proved.

Imposing that h satisfies (32) we obtain $(-x + ay - xy)h_1(y) + y(c + x)(h'_0(y) + xh'_1(y)) = \beta_1 x + \beta_2 y$. Computing the terms of degree two in x we get $h'_1(y) = 0$ and hence $h_1 = c_1 \in \mathbb{C}$. Then $(-x + ay - xy)c_1 + y(c + x)h'_0(y) = \beta_1 x + \beta_2 y$. Solving this differential equation we get $h_0(y) = c_0 + (\beta_1 + c_1)x \log y/(c+x) + (\beta_2 - ac_1 + c_1 x)y/(c+x)$. Since $h_0(y)$ must be a polynomial in y we have $c_1 = -\beta_1$ and $\beta_2 - ac_1 = c_1 c$, that is, $\beta_2 = -(a+c)\beta_1$. Therefore, $L = \beta_1(x - (a+c)y)$ and $h = c_0 - \beta_1(y + x)$ with $\beta_1 \in \mathbb{C}$. Thus $F = e^{-(x+y)}$ is an exponential factor with $L = x - (a + c)y$. This completes the proof of the proposition. □

Proposition 26. *Systems (VI) with $a \notin \{0, -1 - c\}$ have no Liouvillian first integrals.*

Proof. It follows from Propositions 19 and 25 that system (VI) has one invariant Darboux polynomial $f_1 = y$ with cofactor $K = c + x$ and one exponential factor $F = e^{-(x+y)}$ with cofactor $L = x - (a + c)y$. Then again since the divergence of system (VI) is $-1 - y + c + x$, we have

$$\lambda K + \mu L = \lambda(c + x) + \mu(x - (a + c)y) = 1 + y - c - x$$

if and only if

$$\lambda c = 1 - c, \quad \lambda + \mu = -1, \quad -(a + c)\mu = 1. \tag{34}$$

Since $a \notin \{0, -1 - c\}$, we get that (34) has no solutions. Then it follows from Theorems 3(b) and 5 that system (VI) has no Liouvillian first integrals. □

Proposition 27. *Systems (VI) with $a \notin \{0, -1 - c\}$ have no global analytic first integrals.*

Proof. We note that the eigenvalues of $Df(0, 0)$ are -1 and c. We consider different cases.

Case 1*:* $c = 0$. In this case the origin is isolated and in view of Theorem 7 we conclude that system (VI) with $a \neq 0$ and $c = 0$ has no local analytic first integrals in a neighborhood of the origin.

Case 2*:* $c \neq 0$. We consider two different subcases.

Subcase 2.1*:* $c \notin \mathbb{Q}^+$. Then for any $k_1, k_2 \in \mathbb{N} \cup \{0\}$ with $k_1 + k_2 > 0$ we have $k_1\lambda_1 + k_2\lambda_2 = -k_1 + k_2 c \neq 0$ and in view of Theorem 6 we have that system (IV) has no local analytic first integrals.

Subcase 2.2*:* $c \in \mathbb{Q}^+$. In this case we consider two subcases.

Subcase 2.2.1*:* $c \neq -a$. In this case system (VI) has also $(\bar{x}, \bar{y}) = -c(1, 1/(a + c))$ as a singular point. The eigenvalues of $Df(\bar{x}, \bar{y})$ satisfy $\lambda_1\lambda_2 = c > 0$. Therefore, given $k_1, k_2 \in \mathbb{N} \cup \{0\}$ with $k_1 + k_2 > 0$, since λ_1 and λ_2 have the same sign we have that $k_1\lambda_1 + k_2\lambda_2 \neq 0$. Then, in view of

Theorem 6 we have that system (VI) has no local analytic first integrals and it also has no global analytic first integrals.

Subcase 2.2.2*:* $c = -a$. In this case system (VI) becomes

$$x' = -x + y(a - x), \quad y' = -y(a - x). \tag{35}$$

Doing the linear change of variables $Y = x + y$, $X = x$ we obtain that system (35) becomes

$$X' = -(a + 1)X + aY - XY + X^2, \quad Y' = -X. \tag{36}$$

Now doing the rescaling $(X, Y, t) \to (-X, -Y, -t)$ then (36) is

$$X' = (a + 1)X - aY - XY + X^2, \quad Y' = X, \tag{37}$$

which is system (45) with $\bar{B} = a + 1$, $\bar{C} = -a$, $\bar{D} = 1$, $\bar{E} = -1$ and $\bar{F} = 0$. In view of Proposition 33 we get that system (37) has no global analytic first integrals. Therefore since all the changes that we made are linear, we conclude that system (VI) with $c = -a$ has no global analytic first integrals. □

Proof of Theorem 17. The proof of Theorem 17 follows directly from Propositions 26 and 27. □

9. Appendix: Known results

9.1. **Known results for the Lotka–Volterra systems.** In this subsection we introduce the results of [3] concerning the Lotka–Volterra systems that we will use. Given a 3-dimensional Lotka–Volterra system

$$\dot{x} = x(\bar{C}y + z), \quad \dot{y} = y(x + \bar{A}z), \quad \dot{z} = z(\bar{B}x + y), \tag{38}$$

which can be thought as the planar projective version of the planar Lotka–Volterra system

$$\dot{x} = x(-\bar{B}x + (\bar{C} - 1)y + 1), \quad \dot{y} = y((1 - \bar{B})x - y + \bar{A}), \tag{39}$$

we associate to it two equivalent systems if $\bar{A}\bar{B}\bar{C} = 0$, or five equivalent systems if $\bar{A}\bar{B}\bar{C} \neq 0$. The first two are obtained doing circular permutation of the variables x, y, z and of the parameters $\bar{A}, \bar{B}, \bar{C}$, i.e.,

$$(x, y, z, \bar{A}, \bar{B}, \bar{C}) \to (y, z, x, \bar{B}, \bar{C}, \bar{A})$$

and the remaining three cases can be obtained doing the transformation

$$(x, y, z, \bar{A}, \bar{B}, \bar{C}) \to (\bar{B}x, \bar{A}z, \bar{C}y, 1/\bar{C}, 1/\bar{B}, 1/\bar{A}),$$

and the two new transformations obtained from this one doing the above circular permutations. We say that all these Lotka–Volterra systems are *E equivalent*. All the results in [3] are stated modulo these E equivalent classes. To obtain the full classes as we do in the present paper, and since we work with system (39) instead of (38) we will use the fact that all the Darboux polynomials in the projective must be homogeneous in the variables x, y, z, for more details see [3].

Proposition 28. *For system*

$$x' = x(-\bar{B}x + (\bar{C} - 1)y + 1), \quad y' = y((1 - \bar{B})x - y + \bar{A}) \tag{40}$$

with $\bar{A}, \bar{B}, \bar{C} \in \mathbb{R}$, *and setting*

$$p = -\bar{A} - \frac{1}{\bar{B}}, \quad q = -\bar{B} - \frac{1}{\bar{C}}, \quad r = -\bar{C} - \frac{1}{\bar{A}},$$

the following holds.

(a) It has a Liouvillian first integral if and only if either it has an algebraic curve $f = 0$ *different from* $x = 0$ *and* $y = 0$*, or the triple* $(\bar{A}, \bar{B}, \bar{C})$ *falls, up to E equivalences to one of the cases of the following list.*

(a.1) If $\bar{A}\bar{B}\bar{C} = -1$*, then* $f = x - \bar{C}y + \bar{A}\bar{C}$.

(a.2) If $\bar{B} = 1$ *and* $\bar{C} = 0$*, then* $f = y - \bar{A}$*. This case has two additional E-equivalent systems.*

(a.3) If $p = 1$*,* $q = 1$ *and* $r = 1$ *(and consequently* $\bar{A}\bar{B}\bar{C} = 1$*),* $f = \bar{A}^2(\bar{B}x - 1)^2 - 2\bar{A}(\bar{B}x + 1)y + y^2$.

(a.4) If $\bar{A} = 2$ *and* $q = 1$*, then* $f = (x - \bar{C}y)^2 - 2\bar{C}^2y$*. This case has five additional E-equivalent systems.*

(a.5) If $(\bar{A}, \bar{B}, \bar{C}) = (-7/3, 3, -4/7)$*, then* $f = -259308x^3 - 185220x^2y + 259308x^2 + 567xy^3 - 13230xy^2 - 71001xy - 86436x + 324y^4 + 3024y^3 + 10584y^2 + 16464y + 9604$.

(a.6) If $(\bar{A},\bar{B},\bar{C})=(-3/2,2,-4/3)$, then $f=108x^2+6xy^2+180xy-108x+8y^3+36y^2+54y+27$.
(a.7) If $(\bar{A},\bar{B},\bar{C})=(2,4,-1/6)$, then $f=216x^3+108x^2y-54x^2+18xy^2-36xy+y^3-4y^2+4y$.
(a.8) If $(\bar{A},\bar{B},\bar{C})=(2,-8/7,1/3)$, then $f=216x^3+189x^2+882xy-343y^2+686y$.
(a.9) If $(\bar{A},\bar{B},\bar{C})=(6,1/2,-2/3)$, then $f=9x^2y+12xy^2-144xy+432x+4y^3-72y^2+432y-864$.
(a.10) If $(\bar{A},\bar{B},\bar{C})=(-6,1/2,1/2)$, then $f=3x^2y+24xy+144x-8y^2-96y-288$.
(a.11) If $(\bar{A},\bar{B},\bar{C})=(3,1/5,-5/6)$, then $f=1296x^4+4320x^3y-6480x^3+5400x^2y^2-18900x^2y+3000xy^3-18000xy^2+27000xy+625y^4-5625y^3+16875y^2-16875y$.
(a.12) If $(\bar{A},\bar{B},\bar{C})=(2,-13/7,1/3)$, then $f=648x^4-216x^3y-252x^2y+1176xy^2-343y^3+686y^2$.
(a.13) If $(\bar{A},\bar{B},\bar{C})=(2,2,2)$, then $f=x^2+xy^2-3xy+y$.
(a.14) If $(\bar{A},\bar{B},\bar{C})=(2,3,-3/2)$, then $f=8x^2+16xy-y^3+4y^2-4y$.
(a.15) If $(\bar{A},\bar{B},\bar{C})=(2,2,-5/2)$, then $f=8x^2-4xy^2+24xy+y^4-6y^3+12y^2-8y$.
(a.16) If $(\bar{A},\bar{B},\bar{C})=(-4/3,3,-5/4)$, then $f=576x^2+864xy-384x+27y^3+108y^2+144y+64$.
(a.17) If $(\bar{A},\bar{B},\bar{C})=(-9/4,4,-5/9)$, then $f=419904x^3+279936x^2y-314928x^2+15552xy^2+69984xy+78732x-256y^4-2304y^3-7776y^2-11664y-6561$.
(a.18) If $(\bar{A},\bar{B},\bar{C})=(-3/2,2,-7/3)$, then $f=324x^2+72xy^2+864xy-324x+16y^4+96y^3+216y^2+216y+81$.
(a.19) If $(\bar{A},\bar{B},\bar{C})=(-5/2,2,-8/5)$, then $f=125000x^3-5000x^2y^2+225000x^2y-187500x^2-1600xy^4-6000xy^3+15000xy^2+87500xy+93750x-64y^6-960y^5-6000y^4-20000y^3-37500y^2-37500y-15625$.
(a.20) If $(\bar{A},\bar{B},\bar{C})=(-10/3,3,-7/10)$, then $f=81000000x^4+64800000x^3y-108000000x^3-243000x^2y^3+3240000x^2y^2+29700000x^2y+54000000x^2-97200xy^4-1296000xy^3-6480000xy^2-14400000xy-12000000x+729y^6+14580y^5+121500y^4+540000y^3+1350000y^2+1800000y+1000000$.
(a.21) If $(\bar{A},\bar{B},\bar{C})=(-(2l+1)/(2l-1),1/2,2)$, $l=1,2,\ldots$. In this case the degree of f is unbounded and it is not easy to provide the explicit expression of f.

Cases (a.5) to (a.21) provide six E–equivalent systems, with the exception of case (a.13) which provides only two E-equivalent systems.

(b) It has a global analytic first integral if and only if $\bar{A}=-p/q$, $\bar{C}=1+p_1/q_1$, $\bar{B}=qq_1/(p(p_1+q_1))$ with $p,q,p_1,q_1\in\mathbb{N}$ and $pp_1-qq_1\geq 0$, then

$$H=x^{pq_1}y^{qq_1}(pp_1+pq_1-qq_1x+p_1qy+qq_1y)^{pp_1-qq_1}.$$

Proposition 29. *For system*

$$x'=x(\bar{A}x+\bar{B}y+\bar{C}),\quad y'=\bar{D}y^2 \tag{41}$$

with $\bar{A}^2+\bar{B}^2+\bar{D}^2\neq 0$, the following holds.

(a) It has a Liouvillian first integral. If $\bar{D}\neq 0$ it has the exponential factor $\exp(1/y)$. If $\bar{D}=0$ then y is a first integral.
(b) It has a global analytic first integral if and only if:
(b.1) $\bar{D}=0$ and the first integral is y;
(b.2) $\bar{A}=\bar{B}=0$ and the first integral is $x^{|\bar{D}|}$;
(b.3) $\bar{A}=0$ $\bar{B}>0$ and $\bar{D}<0$ and the first integral is $y^{\bar{B}}/x^{\bar{D}}$;
(b.4) $\bar{A}=0$ $\bar{B}<0$ and $\bar{D}>0$ and the first integral is $x^{\bar{D}}/y^{\bar{B}}$.

Proposition 30. *For system*

$$x'=\bar{D}x,\quad y'=y(\bar{A}x+\bar{B}y+\bar{C}) \tag{42}$$

with $\bar{A}^2+\bar{B}^2+\bar{D}^2\neq 0$, the following holds.

(a) It is Liouvillian integrable. If $\bar{D}\neq 0$ it has the exponential factor $\exp(x)$. If $\bar{D}=0$ then x is a first integral.
(b) It has a global analytic first integral if and only if:
(b.1) $\bar{D}=0$ and the first integral is x;
(b.2) $\bar{D}>0$, $\bar{B}=\bar{C}=0$ and the first integral is $e^{-\bar{A}x}y^{\bar{D}}$;
(b.3) $\bar{D}<0$, $\bar{B}=\bar{C}=0$ and the first integral is $e^{\bar{A}x}y^{-\bar{D}}$;

(b.4) $\bar{D}/\bar{C} = -p/q$ *with* $p, q \in \mathbb{N}$, $\bar{B} = 0$, *and the first integral is* $e^{\bar{A}qx/\bar{C}} x^q y^p$.

Proposition 31. *For system*

$$x' = x(\bar{A}x + \bar{B}y), \quad y' = y(x + y). \tag{43}$$

The following holds.

- (a) *It is a quadratic homogeneous system and consequently it is integrable. If* $\bar{A} \neq 1$ *and* $\bar{B} \neq 1$ *then the function* $xy((\bar{A} - 1)x + (\bar{B} - 1)y)$ *is the inverse of an integrating factor of the system. If* $\bar{A} = \bar{B} = 1$ *then the straight line* $x + y = 0$ *is given by singular points. So, in* $\mathbb{R}^2 \setminus \{x + y = 0\}$ *the system has the first integral* $H = x/y$.
- (b) *It has a global analytic first integral if and only if* $(\bar{A}, \bar{B}) \neq (1, 1)$, $\bar{A} - 1$, $(1 - \bar{B})\bar{A}$ *and* $\bar{B} - \bar{A}$ *have the same sign with* $\bar{A}, \bar{B} \in \mathbb{Q}$. *Then* $H = x^{|a-1|} y^{|a(1-b)|}((a-1)x - (b-1)y)^{|b-a|}$.

Proposition 32. *For system*

$$x' = x(\bar{A}x + \bar{B}y + 1), \quad y' = y(x + y). \tag{44}$$

The following holds.

- (a) *It has a Liouvillian first integral if and only if:*
  - (a.1) $\bar{B} = 0$ *and it has the invariant straight line* $\bar{A}x + 1 = 0$ *if* $\bar{A} \neq 0$, *and the exponential factor* $\exp(x)$ *if* $\bar{A} = 0$.
  - (a.2) *If* $\bar{B} \neq 0$ *and* $\bar{A} = 1$ *it has the integrating factor* $1/(xy^2)$.
  - (a.3) *If* $(\bar{A}, \bar{B}) = (1/2, -1)$ *then it has the invariant algebraic curve* $xy + 1 + x + x^2/4 = 0$.
  - (a.4) *If* $(\bar{A}, \bar{B}) = (1/2, 1/2)$ *then it has the exponential factor* $\exp((2 + x)^2/y)$.
- (b) *It has no global analytic first integrals.*

9.2. **Known results for quadratic–linear polynomial differential systems.** We consider the system

$$x' = \bar{B}x + \bar{C}y + \bar{D}x^2 + \bar{E}xy + \bar{F}y^2, \quad y' = x. \tag{45}$$

It was proved in [13] the following result.

Proposition 33. *The quadratic–linear polynomial differential systems* (45) *having a global analytic first integral* H *are the following ones.*

- (a) *Systems* (45) *with* $\bar{B} = \bar{D} = \bar{F} = 0$ *and* $\bar{C}\bar{E} \neq 0$ *having* $H = (\bar{C} + \bar{E}x)^{2\bar{C}} \exp(\bar{E}^2 y^2 - 2\bar{E}x)$.
- (b) *Systems* (45) *with* $\bar{B} = \bar{E} = \bar{F} = 0$ *and* $\bar{C}\bar{D} \neq 0$ *having* $H = (\bar{C} + 2\bar{C}\bar{D}y + 2\bar{D}^2x^2) \exp(-2\bar{D}y)$.
- (c) *Systems* (45) *with* $\bar{E} = \bar{B} = 0$ *and* $\bar{F} \neq 0$ *having*

$$H = (2\bar{F}(\bar{C}\bar{D} + \bar{F}) + 4\bar{D}^3\bar{F}x^2 + 4\bar{D}\bar{F}(\bar{C}\bar{D} + \bar{F})y + 4\bar{D}^2\bar{F}^2y^2) \exp[-2\bar{D}(y + \bar{C}/(2\bar{F}))].$$

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