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ON THE SET OF PERIODS FOR THE MORSE–SMALE DIFFEOMORPHISMS ON THE DISC WITH n HOLES

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Abstract. For every homological class we present a complete description of the minimal Lefschetz set of periods for the Morse–Smale diffeomorphisms without periodic points in the boundary defined on the disc with n holes $\mathbb{D}_n$ for $n = 1, 2, 3, 4, 5$ by using the Lefschetz zeta function. The technical applied for obtaining these results also works for $n > 5$.

1. Introduction

We deal with discrete dynamical systems defined by a self–diffeomorphism f on a given compact manifold $\mathbb{M}$ with or without boundary. The periodic orbits play an important role in the dynamics of these systems. In discrete dynamical systems often the topological information can be used to study qualitative and quantitative properties of the system. The best known example in this direction are the results contained in the paper entitle *Period three implies chaos* for continuous self–maps on the interval, see [16].

For continuous self–maps on compact manifolds one of the most useful tools for proving the existence of fixed points, or more generally of periodic points, is the Lefschetz Fixed Point Theorem and its improvements, see for instance [1, 2, 6, 7, 8, 9, 14, 17, 19]. The Lefschetz zeta function $\mathcal{Z}_f(t)$ simplifies the study of the periodic points of f. This function is a generating function for the Lefschetz numbers of all iterates of f.

In this paper we put our attention in the class of discrete smooth dynamical systems defined by the *Morse–Smale diffeomorphisms on the disc with holes without periodic points in the boundary.* First we recall the definition of a Morse–Smale diffeomorphism.

We denote by Diff($\mathbb{M}$) the space of $\mathcal{C}^1$ diffeomorphisms on a compact manifold $\mathbb{M}$. This space is a topological space endowed with the topology of the supremum with respect to f and its differential Df. In this work all the diffeomorphisms will be $\mathcal{C}^1$.

Let f^m be the m–th iterate of $f \in$ Diff($\mathbb{M}$). A point $x \in \mathbb{M}$ is a *nonwandering point* of f if for any neighborhood $\mathcal{U}$ of x there is a positive integer m such that $f^m(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$. We denote by $\Omega(f)$ the set of nonwandering points of f.

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Assume that $x \in \mathbb{M}$. If $f(x) = x$ and the derivative of f at x, $Df(x)$, has all its eigenvalues disjoint from the unit circle, then x is called a *hyperbolic fixed point*.

Let y be a periodic point of period p. Then y is a *hyperbolic periodic point* if y is a hyperbolic fixed point of f^p. We call the set $\{y, f(y), \ldots, f^{p-1}(y)\}$ the *periodic orbit* of the periodic point y.

Suppose that d is the metric on $\mathbb{M}$ induced by the norm of the supremum, and p is a hyperbolic fixed point of f. Then the *stable manifold* of x is $W^s(x) = \{y \in \mathbb{M} : d(x, f^m(y)) \to 0 \text{ as } m \to \infty\}$, and the *unstable* one is $W^u(x) = \{y \in \mathbb{M} : d(x, f^{-m}(y)) \to 0 \text{ as } m \to \infty\}$. We extend these notions to a hyperbolic periodic point x of period p as follows. The stable and unstable manifolds are defined as the stable and unstable manifolds of x under f^p.

A diffeomorphism $f : \mathbb{M} \to \mathbb{M}$ is *Morse–Smale* if

(1) $\Omega(f)$ is finite,
(2) all periodic points are hyperbolic,
(3) for each $x, y \in \Omega(f)$, $W^s(x)$ and $W^u(y)$ have a transversal intersection.

Condition (1) implies that $\Omega(f)$ is the set of all periodic points of f.

We say that two diffeomorphisms $f, g \in \text{Diff}(\mathbb{M})$ are *topologically equivalent* if and only if there exists a homeomorphism $h : \mathbb{M} \to \mathbb{M}$ such that $h \circ f = g \circ h$. A diffeomorphism f is *structurally stable* if there exists a neighborhood $\mathcal{U}$ of f in $\text{Diff}(\mathbb{M})$ such that each $g \in \mathcal{U}$ is topologically equivalent to f. The class of Morse–Smale diffeomorphisms is structurally stable inside the class of all diffeomorphisms (see [21, 22, 23]). Therefore it is interesting to understand the dynamics of this class of diffeomorphims.

Several authors have published papers analyzing the relationships between the dynamics of the Morse–Smale diffeomorphisms and the topology of the manifold where they are defined, see for instance [3, 4, 5, 9, 10, 11, 12, 20, 22, 24, 25, 26]. In the class of all diffeomorphisms the Morse–Smale have a relative simple dynamics. Thus the sets of all their periodic orbits are finite, and their structure are preserved under small perturbations.

Let $\mathbb{D}_n$ be the disc with n holes. Our main objective is to describe the minimal Lefschetz sets of periods of the Morse–Smale diffeomorphisms without periodic points in the boundary of $\mathbb{D}_n$ for $n = 1, 2, 3, 4, 5$ (see Theorem 7) and show that the algorithm used for obtaining these results extends to $n > 5$.

In section 2 we provide the definitions of Lefschetz number and Lefschetz zeta function. Additionally we recall a fundamental result on this last function which works for the Morse–Smale diffeomorphisms without periodic points in the boundary, see Theorem 2. This result will be one of the main tools of this paper.

The definition of minimal Lefschetz set of periods for Morse–Smale diffeomorphisms is given in section 3. This definition already has been used for studying the periodic orbits of the Morse–Smale diffeomorphisms on the compact connected surfaces without boundary [18], and on the n–dimensional torus [13].

As we will see the characteristic polynomials of the homomorphisms induced in the homological groups of $\mathbb{D}_n$ by the Morse–Smale diffeomorphisms are product of cyclotomic polynomials. We define them and describe their basic properties in section 4. These polynomials also will play a key role in the proof of our results.

In section 5 we compute the Lefschetz zeta function for the Morse–Smale diffeomorphisms without periodic points on the boundary in $\mathbb{D}_n$ for $n = 1, 2, 3, 4, 5$. Finally in section 6 we prove our main result Theorem 7.

2. Lefschetz zeta function

Probably the main goal of Lefschetz's work in the 1920's was to link the homology class of a given map with an earlier work on the indices of Brouwer on the continuous self–maps on compact manifolds. These two notions provide equivalent definitions for the Lefschetz numbers, and from their comparison, can be obtained information about the existence of fixed points.

Let $\mathbb{M}$ be a n–dimensional manifold. We denote by $H_k(\mathbb{M}, \mathbb{Q})$ for $k = 0, 1, ..., n$ the homological groups of $\mathbb{M}$ with coefficients in $\mathbb{Q}$. Each of these groups is a finite linear space over $\mathbb{Q}$.

Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ it induces $n+1$ linear maps $f_{*k} : H_k(\mathbb{M}, \mathbb{Q}) \to H_k(\mathbb{M}, \mathbb{Q})$. Every linear map f_{*k} is given by a $n_k \times n_k$ matrix with integer entries where n_k is the dimension of $H_k(\mathbb{M}, \mathbb{Q})$ for $k = 0, 1, ..., n$.

The *Lefschetz number* $L(f)$ of f is defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \text{trace}(f_{*k}).$$

One of the main results connecting the algebraic topology with the fixed point theory is the Lefschetz Fixed Point Theorem:

Theorem 1. *Let $f : \mathbb{M} \to \mathbb{M}$ be a continuous map on a compact manifold without fixed points in the boundary and $L(f)$ be its Lefschetz number. If $L(f) \neq 0$ then f has a fixed point.*

For a proof of Theorem 1 see [6].

Our aim is to obtain information on the set of periods of f. To this purpose it is useful to have information on the whole sequence $\{L(f^m)\}_{m=0}^{\infty}$ of the Lefschetz numbers of all iterates of f. Thus we define the *Lefschetz zeta function* of f as

$$\mathcal{Z}_f(t) = \exp\left(\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m\right).$$

This function generates the whole sequence of Lefschetz numbers, and it may be independently computed through

$$\mathcal{Z}_f(t) = \prod_{k=0}^{n} \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}}, \tag{1}$$

where $n = \dim \mathbb{M}$, $n_k = \dim H_k(\mathbb{M}, \mathbb{Q})$, I_{n_k} is the $n_k \times n_k$ identity matrix, and we take $\det(I_{n_k} - t f_{*k}) = 1$ if $n_k = 0$. Note that the expression (1) is a rational function in t. So the information on the infinite sequence of integers $\{L(f^m)\}_{m=0}^{\infty}$ is contained in two polynomials with integer coefficients, for more details see [9].

Let f be a diffeomorphism on a compact manifold $\mathbb{M}$ having finitely many hyperbolic periodic orbits none of the them in the boundary. If γ is a hyperbolic periodic orbit of period p, then for each $x \in \gamma$ let E_x^u denotes the subspace of $T_x\mathbb{M}$ generated by the eigenvectors of $Df^p(x)$ corresponding to the eigenvalues whose moduli are greater than one. Let E_x^s be the subspace of $T_x\mathbb{M}$ generated by the remaining eigenvectors.

We define the orientation type Δ of γ to be $+1$ if $Df^p(x) : E^u_x \to E^u_x$ preserves orientation, and -1 if it reverses orientation. The *index* u of γ is the dimension of E^u_x for some $x \in \gamma$. We note that the definitions of Δ and u do not depend on the periodic point x, only depend on the periodic orbit γ. Finally we associated the triple (p, u, Δ) to the periodic orbit γ.

For f the *periodic data* is defined as the collection composed by all triples (p, u, Δ), where a same triple can occur more than once provided it corresponds to different periodic orbits. Franks [9] proved the following result.

Theorem 2. *Let f be a C^1 map on a compact manifold having finitely many hyperbolic periodic orbits none of them in the boundary, and let Σ be the period data of f. Then the Lefschetz zeta function of f satisfies*

$$\mathcal{Z}_f(t) = \prod_{(p,u,\Delta)\in\Sigma} (1 - \Delta t^p)^{(-1)^{u+1}}. \tag{2}$$

We note that the original formulation of Theorem 2 was stated for any arbitrary compact manifold but going into its proof we find that, in the case of having a manifold with boundary, the arguments based on Smale [26] only work if the map f has no periodic points in the boundary. Additionally we present a counter-example to Theorem 2 in the case of having a manifold with boundary and a map with some periodic points in it. Consider the two–dimensional closed disk and let f be a map with two fixed points p and q in the boundary and the orbits of the rest of points have α-limit p and ω-limit q respectively. Such a map can be C^1 with only two hyperbolic points one attractor (with index 0) and the other repeller (with index 2). This makes the right hand side of (2) equal to $\frac{1}{(1-t)^2}$ but the left hand side, the Lefschetz zeta function of a map of a disk into itself always is equal to $\frac{1}{1-t}$ because the disk is a space contractible to a point.

Clearly the Morse–Smale diffeomorphisms on $\mathbb{D}_n$ without periodic points on the boundary satisfy the hypotheses of Theorem 2.

3. Minimal Lefschetz set of periods of Morse–Smale diffeomorphisms

The statement of Theorem 2 allows us to define the minimal Lefschetz set of periods for a C^1 map on a compact manifold having finitely many hyperbolic periodic points. From now on in the case of domains with boundary we assume that the map has no periodic points on it. Such a map has a Lefschetz zeta function of the form (2).

Note that in general the expression of one of these Lefschetz zeta functions is not unique as product of the elements of the form $(1 \pm t^p)^{\pm 1}$. For instance the following Lefschetz zeta function for a Morse–Smale diffeomorphism on $\mathbb{T}^4$ can be written in four different ways in the form given by (2):

$$\mathcal{Z}_f(t) = \frac{(1-t^3)^2(1+t^3)}{(1-t)^6(1+t)^3} = \frac{(1-t^3)(1-t^6)}{(1-t)^6(1+t)^3} = \frac{(1-t^3)(1-t^6)}{(1-t)^3(1-t^2)^3} = \frac{(1-t^3)^2(1+t^3)}{(1-t)^3(1-t^2)^3}.$$

According with Theorem 2, the first expression will provide the periods $\{1, 3\}$ for f, the second the periods $\{1, 3, 6\}$, the third the period $\{1, 2, 3, 6\}$, and finally the fourth the periods $\{1, 2, 3\}$. Then for this Lefschetz zeta function $\mathcal{Z}_f(t)$ we will define its *minimal Lefschetz set of periods* as

$$\mathrm{MPer}_L(f) = \{1, 3\} \cap \{1, 3, 6\} \cap \{1, 2, 3, 6\} \cap \{1, 2, 3\} = \{1, 3\}.$$

In general for the Lefschetz zeta function $\mathcal{Z}_f(t)$ of a C^1 map f on a compact manifold having finitely many hyperbolic periodic points, we define its *minimal Lefschetz set of periods* as the intersection of all sets of periods forced by the finitely many different representations of $\mathcal{Z}_f(t)$ as products of the form $(1 \pm t^p)^{\pm 1}$.

The *minimal set of periods* of a Morse–Smale diffeomorphism f defined on a compact manifold $\mathbb{M}$ is

$$\text{MPer}_{ms}(f) = \bigcap_{h \sim f} \text{Per}(h), \tag{3}$$

where h runs over all the Morse–Smale diffeomorphisms of $\mathbb{M}$ which are homotopic to f.

For Morse–Smale diffeomorphisms and from the definition of minimal Lefschetz set of periods it follows always that

$$\text{MPer}_L(f) \subseteq \text{MPer}_{ms}(f).$$

We note that when the minimal Lefschetz set of periods of a map is empty but the Lefschetz zeta function is not equal to one, we always have some information about the periods of the map. Indeed, suppose that $\mathcal{Z}_f(t) = 1 + t^2$, since

$$1 + t^2 = \frac{1 - t^4}{1 - t^2} = \frac{1 - t^4}{(1+t)(1-t)},$$

we know that $\text{MPer}_L(f) = \emptyset$, but clearly by Theorem 2 the map f has either the period 2, or the periods 2 and 4, or the periods 1 and 4, due to the different expressions of its $\mathcal{Z}_f(t)$.

Remark 3. *Assume that n is an even positive integer. The expressions*

$$\begin{aligned} 1 - t^n &= (1 + t^{n/2})(1 - t^{n/2}), \\ 1 + t^n &= \frac{1 - t^{2n}}{1 - t^n} = \frac{1 - t^{2n}}{(1 + t^{n/2})(1 - t^{n/2})}, \end{aligned}$$

show that the even period n never will appear in the minimal Lefschetz set of periods.

4. Cyclotomic polynomials

By definition the n–th cyclotomic polynomial is given by

$$c_n(t) = \prod_k (w_k - t),$$

where $w_k = e^{2\pi i k/n}$ and k runs over all the relative primes smaller than or equal to n. For more details about these polynomials see [15]. An alternative way to express $c_n(t)$ is

$$c_n(t) = \frac{1 - t^n}{\prod_{\substack{d|n \\ d < n}} c_d(t)}. \tag{4}$$

Let $\varphi(n)$ be the degree of $c_n(t)$. Then $n = \sum_{d|n} \varphi(d)$. So $\varphi(n)$ is the *Euler function*, which satisfies $\varphi(n) = n \prod_{\substack{p|n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right)$. Therefore, if the prime decomposition of n is $p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then $\varphi(n) = \prod_{j=1}^{k} p_i^{\alpha_j - 1}(p_j - 1)$.

From the formula (4), we have

$$c_n(t) = \prod_{d|n} (1 - t^d)^{\mu(n/d)}, \tag{5}$$

where μ is the *Möbius function*, i.e.

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } k^2|m \text{ for some } k \in \mathbb{N}, \\ (-1)^r & \text{if } m = p_1 \cdots p_r \text{ distinct primes factors.} \end{cases}$$

TABLE 1. The first thirty cyclotomic polynomials.

$c_1(t) = 1 - t$	$c_2(t) = 1 + t$	$c_3(t) = \frac{1 - t^3}{1 - t}$
$c_4(t) = 1 + t^2$	$c_5(t) = \frac{1 - t^5}{1 - t}$	$c_6(t) = \frac{1 + t^3}{1 + t}$
$c_7(t) = \frac{1 - t^7}{1 - t}$	$c_8(t) = 1 + t^4$	$c_9(t) = \frac{1 - t^9}{1 - t^3}$
$c_{10}(t) = \frac{1 + t^5}{1 + t}$	$c_{11}(t) = \frac{1 - t^{11}}{1 - t}$	$c_{12}(t) = \frac{1 + t^6}{1 + t^2}$
$c_{13}(t) = \frac{1 - t^{13}}{1 - t}$	$c_{14}(t) = \frac{1 + t^7}{1 + t}$	$c_{15}(t) = \frac{(1 - t^{15})(1 - t)}{(1 - t^3)(1 - t^5)}$
$c_{16}(t) = 1 + t^8$	$c_{17}(t) = \frac{1 - t^{17}}{1 - t}$	$c_{18}(t) = \frac{1 + t^9}{1 + t^3}$
$c_{19}(t) = \frac{1 - t^{19}}{1 - t}$	$c_{20}(t) = \frac{1 + t^{10}}{1 + t^2}$	$c_{21}(t) = \frac{(1 - t^{21})(1 - t)}{(1 - t^3)(1 - t^7)}$
$c_{22}(t) = \frac{1 + t^{11}}{1 + t}$	$c_{23}(t) = \frac{1 - t^{23}}{1 - t}$	$c_{24}(t) = \frac{1 + t^{12}}{1 + t^4}$
$c_{25}(t) = \frac{1 - t^{25}}{1 - t^5}$	$c_{26}(t) = \frac{1 + t^{13}}{1 + t}$	$c_{27}(t) = \frac{1 - t^{27}}{1 - t^9}$
$c_{28}(t) = \frac{1 + t^{14}}{1 + t^2}$	$c_{29}(t) = \frac{1 - t^{29}}{1 - t}$	$c_{30}(t) = \frac{(1 + t^{15})(1 + t)}{(1 + t^3)(1 + t^5)}$

Proposition 4. *For $0 < k < n$, w_k is a root of $c_n(t)$ if and only if w_k^{-1} is also a root of $c_n(t)$.*

Proof. If $w_k = e^{2\pi ik/n}$ then $w_k^{-1} = e^{-2\pi ik/n} = e^{2\pi i(n-k)/n}$ with $n-k$ and n coprime if and only if k and n are coprime. Then the proposition follows directly from the definition of $c_n(t)$. □

In what follows there are some elementary properties of the cyclotomic polynomials (cf. [15]).

(p1) If p is prime, then $c_p(t) = (1-t^p)/(1-t)$.
(p2) If $p = 2r$ with r odd, then $c_{2r}(t) = c_r(-t)$.
(p3) If $p = 2^n$, then $c_p(t) = 1 + t^{2^{n-1}}$.
(p4) If $p = r^\alpha$ with r prime, then $c_p(t) = c_r(t^{r^{\alpha-1}}) = (1-t^{r^\alpha})/(1-t^{r^{\alpha-1}})$.
(p5) If $p = 2^n r$ with r odd and $n > 1$, then $c_n(t) = c_{2r}(t^{2^{n-1}})$.
(p6) For all n $c_n(0) = 1$ and the leading term of c_n is 1 if $n \geq 2$.
(p7) The degree of c_n is even for $n > 2$.

5. $\mathcal{Z}_f$ for Morse–Smale diffeomorphisms on $\mathbb{D}_n$

We start recalling the homological groups of the two–dimensional disc with n holes $\mathbb{D}_n$ with coefficients in $\mathbb{Q}$. More precisely $H_0(\mathbb{D}_n,\mathbb{Q}) = \mathbb{Q}$, $H_1(\mathbb{D}_n,\mathbb{Q}) = \mathbb{Q} \oplus \overset{n)}{\cdots} \oplus \mathbb{Q}$, and $H_2(\mathbb{D}_n,\mathbb{Q}) = 0$ for every $n \geq 0$.

Given a continuous map $f : \mathbb{D}_n \to \mathbb{D}_n$ it induces linear maps $f_{*k} : H_k(\mathbb{D}_n,\mathbb{Q}) \to H_k(\mathbb{D}_n,\mathbb{Q})$, $k = 0,1,2$. It is well known that $f_{*0} = (1)$ is the identity map over $\mathbb{Q}$, and $f_{*2} = 0$. The linear map f_{*1} is given by a $n \times n$ matrix with integer entries and by the expression (1) the Lesfchetz zeta function of f has the form

$$\mathcal{Z}_f(t) = \prod_{k=0}^{2} \det(I_{n_k} - tf_{*k})^{(-1)^{k+1}} = \frac{\det(I_{n_1} - tf_{*1})}{1-t}. \tag{6}$$

Let $f : \mathbb{D}_n \to \mathbb{D}_n$ be a Morse–Smale diffeomorphism. It is known (cf. [24]) that if a homotopy class admits a Morse–Smale diffeomorphism, then the linear maps corresponding to its homology must be *quasi–unipotent*, i.e. all their eigenvalues are roots of unity. Consequently the characteristic polynomials associated to the matrix f_{*1} must be product of cyclotomic polynomials.

The next result from [18] links expression (6) with the characteristic polynomial of f_{*1} and hence according with the previous paragraph with the cyclotomic polynomials. Since its proof is shorter we provide it.

Proposition 5. *If f_{*_1} is quasi–unipotent, then*

$$\det(I_{n_1} - tf_{*1}) = (-1)^{1+\det(f_{*1})} \det(f_{*1} - tI_{n_1}). \tag{7}$$

Proof. Let $p(t) = \det(f_{*1} - tI_{n_1})$ be the characteristic polynomial of f_{*1} and $q(t) = \det(I_{n_1} - tf_{*1})$. They are related by $q(t) = t^{n_1}p(t^{-1})$. Since f_{*1} is quasi–unipotent, $p(t)$ is a product of cyclotomic polynomials. Therefore it follows that if w is a root of $p(t)$, then $w^{n_1} = 1$. So from Proposition 4 $q(w) = 0$. By the properties of the cyclotomic polynomials $|p(0)| = 1$, therefore

$$\begin{array}{ll} \det(I_{n_1} - tf_{*1}) = \det(f_{*1} - tI_{n_1}) & \text{if } f \text{ is orientation preserving,} \\ \det(I_{n_1} - tf_{*1}) = -\det(f_{*1} - tI_{n_1}) & \text{if } f \text{ is orientation reversing.} \end{array}$$

So the proof is completed. □

From Proposition 5 it follows that for computing all the possible values of $\det(I_{n_1} - tf_{*1})$ for a certain n_1 fixed, it is sufficient to consider the product of all cyclotomic polynomials such that the degree of the product be equal to n_1.

The following result shows the Lefschetz zeta functions which we shall need in the proof of the Theorem 7 where the minimal Lefschetz set of periods are described.

Proposition 6. *For $n = 1, 2, \ldots, 5$ let f be a Morse–Smale diffeomorphisms on $\mathbb{D}_n$. Then, up to the sign, the zeta function of Lefschetz $\mathcal{Z}_f$ is one of the following functions:*

(a) 1, $\frac{1+t}{1-t}$ *if* $n = 1$;

(b) $1-t$, $\frac{1-t^3}{(1-t)^2}$, $\frac{1+t^2}{1-t}$, $\frac{1+t^3}{(1+t)(1-t)}$, $1+t$, $\frac{(1+t)^2}{1-t}$ *if* $n = 2$;

(c) $(1-t)^2$, $\frac{(1+t)^3}{1-t}$, $\frac{1-t^3}{1-t}$, $1+t^2$, $\frac{1+t^3}{1+t}$, $(1-t)(1+t)$, $\frac{(1+t)(1-t^3)}{(1-t)^2}$, $\frac{(1+t)(1+t^2)}{1-t}$, $\frac{1+t^3}{1-t}$, $(1+t)^2$ *if* $n = 3$;

(d) $(1-t)^3$, $\frac{1-t^5}{(1-t)^2}$, $\frac{1+t^4}{1-t}$, $\frac{1+t^5}{(1+t)(1-t)}$, $\frac{1+t^6}{(1+t^2)(1-t)}$, $\frac{(1+t)^4}{1-t}$, $(1+t)^3$, $1-t^3$, $\frac{(1+t)(1-t^3)}{1-t}$, $(1-t)(1+t^2)$, $(1+t)(1+t^2)$, $\frac{(1-t)(1+t^3)}{1+t}$, $1+t^3$, $(1+t)(1-t)^2$, $(1-t)(1+t)^2$, $\frac{(1+t)^2(1-t^3)}{(1-t)^2}$, $\frac{(1+t)^2(1+t^2)}{1-t}$, $\frac{(1+t)(1+t^3)}{1-t}$, $\frac{(1-t^3)^2}{(1-t)^3}$, $\frac{(1+t^2)^2}{1-t}$, $\frac{(1+t^3)^2}{(1-t)(1+t)^2}$, $\frac{(1-t^3)(1+t^3)}{(1-t)^2(1+t)}$, $\frac{(1+t^2)(1-t^3)}{(1-t)^2}$, $\frac{(1+t^2)(1+t^3)}{(1-t)(1+t)}$ *if* $n = 4$;

(e) $(1-t)^4$, $\frac{(1+t)^5}{1-t}$, $\left(\frac{1-t^3}{1-t}\right)^2$, $\frac{(1+t)(1-t^3)^2}{(1-t)^3}$, $(1+t^2)^2$, $\frac{(1+t)(1+t^2)^2}{1-t}$, $\left(\frac{1+t^3}{1+t}\right)^2$, $\frac{(1+t^3)^2}{(1-t)(1+t)}$, $(1-t)^2(1+t)^2$, $(1+t)^3(1-t)$, $\frac{1-t^5}{1-t}$, $\frac{(1+t)(1-t^5)}{(1-t)^2}$, $1+t^4$, $\frac{(1+t)(1+t^4)}{1-t}$, $\frac{1+t^5}{1+t}$, $\frac{1+t^5}{1-t}$, $\frac{1+t^6}{1+t^2}$, $\frac{(1+t)(1+t^6)}{(1-t)(1+t^2)}$, $(1+t)(1-t)^3$, $(1+t)^4$, $(1-t)(1-t^3)$, $(1+t)(1-t^3)$, $\frac{(1+t)^2(1-t^3)}{1-t}$, $(1-t)^2(1+t^2)$, $(1-t)(1+t)(1+t^2)$, $(1+t)^2(1+t^2)$, $\frac{(1-t)^2(1+t^3)}{1+t}$, $(1-t)(1+t^3)$, $(1+t)(1+t^3)$, $(1+t)^2(1-t)^2$, $(1-t)(1+t)(1+t)^2$, $\frac{(1+t^2)(1+t)^3}{1-t}$, $\frac{(1+t)^2(1+t^3)}{1-t}$, $\frac{(1-t^3)(1+t)^3}{(1-t)^2}$, $\frac{(1-t^3)(1+t^2)}{1-t}$, $\frac{(1-t^3)(1+t^3)}{(1-t)(1+t)}$, $\frac{(1-t^3)(1+t)(1+t^2)}{(1-t)^2}$, $\frac{(1-t^3)(1+t^3)}{(1-t)^2}$, $\frac{(1+t^2)(1+t^3)}{1+t}$, $\frac{(1+t^2)(1+t^3)}{1-t}$ *if* $n = 5$.

Proof. According to the form of the homological groups of the disk with n holes $\mathbb{D}_n$ described in Section 5, the Lefschetz zeta function is given by expression (6) in the form $\mathcal{Z}_f(t) = \frac{\det(I_{n_1} - tf_{*1})}{1-t}$. By Proposition 5 $\det(I_{n_1} - tf_{*1}) = (-1)^{1+\det(f_{*1})} \det(f_{*1} - tI_{n_1})$. So, up to the sign, $\mathcal{Z}_f(t) = \frac{\det(f_{*1} - tI_{n_1})}{1-t}$. By [24] if a homotopy class admits a Morse–Smale diffeomorphism, then the linear maps corresponding to its homology must be *quasi–unipotent*, i.e. all their eigenvalues are roots of unity. Consequently the characteristic polynomials associated to the matrix f_{*1} must be product of cyclotomic polynomials. Therefore

$$\mathcal{Z}_f(t) = \frac{\prod_k c_k(t)}{1-t} \tag{8}$$

where the product comes over k's such that $\sum_k \varphi(k) = n$. Using expression (8) and Table 4 the result is obtained. □

6. Proof of the main result

The aim of this section is to prove our main result.

Theorem 7. *Let f be a Morse–Smale diffeomorphisms on $\mathbb{D}_n$ without periodic points in the boundary for $n = 1, 2, \dots, 5$. Then,*

(a) *If $n = 1$ then* $\mathrm{MPer}_L(f) = \{1\}$ *if* $|\mathcal{Z}_f(t)| = 1, \frac{1+t}{1-t}$.

(b) *If $n = 2$ then* $\mathrm{MPer}_L(f)$ *is*

$\{1\}$ *if* $|\mathcal{Z}_f(t)| = 1-t, 1+t, \frac{(1+t)^2}{1-t}, \frac{1+t^2}{1-t}$;

$\{3\}$ *if* $|\mathcal{Z}_f(t)| = \frac{1+t^3}{(1+t)(1-t)}$;

$\{1,3\}$ *if* $|\mathcal{Z}_f(t)| = \frac{1-t^3}{(1-t)^2}$.

(c) *If $n = 3$ then* $\mathrm{MPer}_L(f)$ *is*

$\{\emptyset\}$ *if* $|\mathcal{Z}_f(t)| = (1-t)(1+t), 1+t^2$;

$\{1\}$ *if* $|\mathcal{Z}_f(t)| = (1-t)^2, (1+t)^2, \frac{(1+t)^3}{1-t}, \frac{(1+t)(1+t^2)}{1-t}$;

$\{1,3\}$ *if* $|\mathcal{Z}_f(t)| = \frac{1-t^3}{1-t}, \frac{1+t^3}{1+t}, \frac{(1+t)(1-t^3)}{(1-t)^2}, \frac{1+t^3}{1+t}$.

(d) *If $n = 4$ then* $\mathrm{MPer}_L(f)$ *is*

$\{1\}$ *if* $|\mathcal{Z}_f(t)| = (1-t)^3, (1+t)^3, (1-t)(1+t)^2, (1+t)(1-t)^2, \frac{(1+t)^4}{1-t}, \frac{(1-t^3)(1+t^3)}{(1-t)^2(1+t)}; (1-t)(1+t^2), (1+t)(1+t^2), \frac{(1+t)^2(1+t^2)}{1-t}, \frac{(1+t^2)^2}{1-t}, \frac{1+t^4}{1-t}, \frac{1+t^6}{(1+t^2)(1-t)}$;

$\{3\}$ *if* $|\mathcal{Z}_f(t)| = 1-t^3, 1+t^3, \frac{(1+t^2)(1+t^3)}{(1-t)(1+t)}$;

$\{5\}$ *if* $|\mathcal{Z}_f(t)| = \frac{1+t^5}{(1+t)(1-t)}$;

$\{1,3\}$ *if* $|\mathcal{Z}_f(t)| = \frac{(1+t)(1-t^3)}{1-t}, \frac{(1-t)(1+t^3)}{1+t}, \frac{(1+t)^2(1-t^3)}{(1-t)^2}, \frac{(1+t)(1+t^3)}{1-t}, \frac{(1-t^3)^2}{(1-t)^3}, \frac{(1+t^3)^2}{(1-t)(1+t)^2}, \frac{(1+t^2)(1-t^3)}{(1-t)^2}$;

$\{1,5\}$ *if* $|\mathcal{Z}_f(t)| = \frac{1-t^5}{(1-t)^2}$.

(e) *If $n = 5$ then* $\mathrm{MPer}_L(f)$ *is*

$\emptyset$ *if* $|\mathcal{Z}_f(t)| = (1-t)^2(1+t)^2, \frac{(1-t^3)(1+t^3)}{(1-t)(1+t)}, (1+t^2)^2, (1-t)(1+t)(1+t^2), 1+t^4, \frac{1+t^6}{1+t^2}$;

$\{1\}$ *if* $|\mathcal{Z}_f(t)| = (1-t)^4, (1-t)^2(1+t)^2, (1+t)^3(1-t), (1+t)(1-t)^3, (1+t)^4, \frac{(1-t^3)(1+t^3)}{(1-t)^2}, \frac{(1+t)^5}{1-t}, (1-t)^2(1+t^2), (1+t)^2(1+t^2), \frac{(1+t^2)(1+t)^3}{1-t}, \frac{(1+t)(1+t^2)^2}{1-t}, \frac{(1+t)(1+t^4)}{1-t}, \frac{(1+t)(1+t^6)}{(1-t)(1+t^2)}$;

$\{3\}$ *if* $|\mathcal{Z}_f(t)| = \frac{(1+t^3)^2}{(1-t)(1+t)}$;

$\{1,3\}$ *if* $|\mathcal{Z}_f(t)| = (1-t)(1-t^3), (1+t)(1-t^3), (1-t)(1+t^3), (1+t)(1+t^3), \left(\frac{1-t^3}{1-t}\right)^2, \frac{(1+t)(1-t^3)^2}{(1-t)^3}, \left(\frac{1+t^3}{1+t}\right)^2, \frac{(1+t)^2(1-t^3)}{1-t}, \frac{(1-t)^2(1+t^3)}{1+t}, \frac{(1+t)^2(1+t^3)}{1-t}, \frac{(1-t^3)(1+t)^3}{(1-t)^2}, \frac{(1-t^3)(1+t^2)}{1-t}, \frac{(1-t^3)(1+t)(1+t^2)}{(1-t)^2}, \frac{(1+t^2)(1+t^3)}{1+t}, \frac{(1+t^2)(1+t^3)}{1-t}$;

$\{1,5\}$ *if* $|\mathcal{Z}_f(t)| = \frac{1-t^5}{1-t}, \frac{(1+t)(1-t^5)}{(1-t)^2}, \frac{1+t^5}{1+t}, \frac{1+t^5}{1-t}$.

Proof. By the Remark 3 we do not take care of the even periods. For the cases such that the Lefschetz zeta function $\mathcal{Z}_f(t)$ has a unique representation as products of the form $(1 \pm t^p)^{\pm 1}$, or finitely many different representations as products of the form $(1 \pm t^p)^{\pm 1}$ forcing the same set of periods, the result is a direct consequence of Proposition 6 and the application of Theorem 2. The remainder cases where the Lefschetz zeta function has finitely many different representations as products of the form $(1 \pm t^p)^{\pm 1}$ forcing different sets of periods, by definition of the minimal Lefschetz set of periods, we take the intersection of the finite number of sets of forced periods:

For $n = 2$ and $|\mathcal{Z}_f(t)| = \frac{1+t^3}{(1+t)(1-t)} = \frac{1+t^3}{1-t^2}$, $\text{MPer}_L(f) = \{1,3\} \cap \{3\} = \{3\}$.
For $n = 3$ and $|\mathcal{Z}_f(t)| = (1-t)(1+t) = 1 - t^2$, $\text{MPer}_L(f) = \{1\} \cap \emptyset = \emptyset$.
Assume $n = 4$.

If $|\mathcal{Z}_f(t)| = \frac{1+t^5}{(1-t)(1+t)} = \frac{1+t^5}{1-t^2}$, then $\text{MPer}_L(f) = \{1,5\} \cap \{5\} = \{5\}$.
If $|\mathcal{Z}_f(t)| = \frac{(1+t^2)(1+t^3)}{(1-t)(1+t)} = \frac{(1+t^2)(1+t^3)}{(1-t^2)}$, then $\text{MPer}_L(f) = \{1,3\} \cap \{3\} = \{3\}$.
If $|\mathcal{Z}_f(t)| = \frac{(1-t^3)(1+t^3)}{(1-t)^2(1+t)} = \frac{(1-t^3)(1+t^3)}{(1-t)(1-t^2)} = \frac{(1-t^6)}{(1-t)^2(1+t)} = \frac{(1-t^6)}{(1-t)(1-t^2)}$, then $\text{MPer}_L(f) = \{1,3\} \cap \{1,3\} \cap \{1\} \cap \{1\} = \{1\}$.

Suppose $n = 5$.

If $|\mathcal{Z}_f(t)| = \frac{(1-t^3)(1+t^3)}{(1-t)(1+t)} = \frac{1-t^6}{1-t^2} = \frac{(1-t^3)(1+t^3)}{1-t^2} = \frac{(1-t^6)}{(1-t)(1+t)}$, then $\text{MPer}_L(f) = \{1,3\} \cap \emptyset \cap \{3\} \cap \{1\} = \emptyset$.
If $|\mathcal{Z}_f(t)| = (1-t)^2(1+t)^2 = (1-t)(1+t)(1-t^2) = (1-t^2)^2$, then $\text{MPer}_L(f) = \{1\} \cap \{1\} \cap \emptyset = \emptyset$.
If $|\mathcal{Z}_f(t)| = \frac{(1-t^3)(1+t^3)}{(1-t)^2} = \frac{1-t^6}{(1-t)^2}$, then $\text{MPer}_L(f) = \{1,3\} \cap \{1\} = \{1\}$.
If $|\mathcal{Z}_f(t)| = (1-t)(1+t)^3 = (1-t^2)(1+t)^2$, then $\text{MPer}_L(f) = \{1\} \cap \{1\} = \{1\}$.
If $|\mathcal{Z}_f(t)| = (1-t)(1+t)(1+t^2) = (1-t^2)(1+t^2)$, then $\text{MPer}_L(f) = \{1\} \cap \emptyset = \emptyset$.
If $|\mathcal{Z}_f(t)| = \frac{(1+t^3)^2}{(1-t)(1+t)} = \frac{(1+t^3)^2}{1-t^2}$, then $\text{MPer}_L(f) = \{1,3\} \cap \{3\} = \{3\}$.

□

Clearly the algorithm used for computing the minimal Lefschetz set of periods for $n = 1, 2, 3, 4, 5$ extends to values of $n > 5$.

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