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ON THE PERIODIC SOLUTIONS OF A CLASS OF DUFFING DIFFERENTIAL EQUATIONS

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Dedicated to Jean Mawhin on the occasion of his 70th birthday.

ABSTRACT. In this work we study the periodic solutions, their stability and bifurcation for the class of Duffing differential equation $x'' + cx' + a(t)x + b(t)x^3 = \lambda h(t)$, where $c > 0$ is a constant, λ is a real parameter, $a(t)$, $b(t)$ and $h(t)$ are continuous T–periodic functions. Our results are proved using the averaging method of first order.

1. Introduction and statement of the main result

Classes of Duffing differential equations have been investigated by many authors. They are interested in the existence of periodic solutions, in their multiplicity, stability, etc. See for example the works [1, 2, 3, 5] and specially the survey of J. Mawhin [4].

In this paper we study the class of Duffing differential equations of the form

$$x'' + cx' + a(t)x + b(t)x^3 = \lambda h(t), \tag{1}$$

where $c > 0$ is a constant, λ is a real parameter, $a(t)$, $b(t)$ and $h(t)$ are continuous T–periodic functions. This equation was also studied by Chen and Li in [2]. In their work they consider the following additional conditions: $b(t) > 0$, $h(t) > 0$ and $a(t)$ satisfies

$$a(t) \le \frac{\pi^2}{T^2} + \frac{c^2}{4}, \quad \text{and} \quad a_0 = \frac{1}{T}\int_0^T a(t)dt > 0. \tag{2}$$

They analyse the stability and the exact multiplicity of the T–periodic solutions of differential equation (1).

In this paper we also study the existence and the stability of periodic solutions of Duffing differential equation (1) under minimal conditions on the functions $a(t)$, $b(t)$ and $h(t)$. In fact we only need that $a(t)$, $b(t)$ and $h(t)$ are continuous T–periodic functions and that $a_0 b_0 > 0$ where $b_0 = \dfrac{1}{T}\displaystyle\int_0^T b(t)dt$. So our results on the periodic solutions of equation (1) are more general than

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the results obtained in [2], and contain as a particular case those results. Moreover our proof is shorter and simpler.

Chen and Li in [1] also studied the Duffing differential equation (1) in a very particular case, i. e. $a(t) = a > 0$, $b(t) = 1$ and $c > 0$, a, c constants. They obtain sharp bounds for $h(t)$ in such away that (1) has exactly three T-periodic solutions.

Equation (1) is equivalent to the non–autonomous planar differential system

$$\begin{aligned} x' &= y, \\ y' &= -cy - a(t)x - b(t)x^3 + \lambda h(t). \end{aligned} \tag{3}$$

To state our main result we consider this system defined for $(x, y) \in D = \{(x, y) : -r \leq x, y \leq r\} \subset \mathbb{R}^2$ where r is an arbitrary real number as large as we want.

Let h_0 be the averaged value of the function $h(t)$ over interval $[0, T]$, see (2). We define the constant $\lambda_0 = \dfrac{1}{\varepsilon^3}\sqrt{\dfrac{4a_0^3}{27b_0h_0^2}}$, where $\varepsilon > 0$.

Now we state our main result.

Theorem 1. *Consider Duffing differential equation* (1) *in* $D \times \mathbb{R}$ *where* $c > 0$ *is a constant,* $a(t)$*,* $b(t)$ *and* $h(t)$ *are continuous* T*–periodic functions and* $a_0b_0 > 0$*. Then for* $\varepsilon > 0$ *sufficiently small the following statements hold.*

(a) *For* $|\lambda| > \lambda_0$ *equation* (1) *has one* T*–periodic solution* α_1 *with initial condition* $(x_1, 0)$*. For* $|\lambda| < \lambda_0$ *equation* (1) *has three* T*–periodic solutions* α_i *with initial conditions* $(x_i, 0)$ *for* $i = 1, 2, 3$*. For* $|\lambda| = \lambda_0$ *equation* (1) *has two* T*–periodic solutions* α_1 *and* $\alpha_2 = \alpha_3$*.*

(b) *Assume that* a_0 *and* b_0 *are positive. Then* α_1 *is unstable for all* λ*. If* $|\lambda| < \lambda_0$ *then* $x_1 < x_2 < x_3$*,* α_2 *is stable and* α_3 *is unstable.*

(c) *Assume that* a_0 *and* b_0 *are negative. Then* α_1 *is stable for all* λ*. If* $|\lambda| < \lambda_0$ *then* $x_1 > x_2 > x_3$*,* α_2 *is unstable and* α_3 *is stable.*

In section 2 we present the proof of Theorem 1. In section 3 we compare our results with the results of Chen and Li [2].

2. Proof of Theorem 1

We assume the Duffing system (3) with $(x, y) \in D$. As $a(t)$, $b(t)$ and $h(t)$ are continuous T–periodic functions and $(x, y) \in D$, then the functions of the right-hand side of system (3) are bounded. Doing the next rescaling of variables and functions

$$(x, y) = (\varepsilon X, \varepsilon^2 Y),\ c = \varepsilon C,\ a(t) = \varepsilon^2 A(t),\ \lambda = \varepsilon^3 \Lambda, \tag{4}$$

system (3) becomes

$$\begin{aligned} X' &= \varepsilon Y, \\ Y' &= \varepsilon(-CY - A(t)X + b(t)X^3 + \Lambda h(t)). \end{aligned} \tag{5}$$

Note that system (5) is in the normal form for applying the averaging theory of first order, see the appendix. Moreover, the functions of the right-hand side of system (5) remain bounded. So the assumptions of Theorem 3 are satisfied.

Let A_0 be the averaged value of the function $A(t)$. Then the averaged system of system (5) is

$$\begin{aligned} X' &= \varepsilon Y, \\ Y' &= \varepsilon(-CY - A_0 X + b_0 X^3 + \Lambda h_0). \end{aligned} \tag{6}$$

The equilibrium points of system (6) have the coordinate Y equal to zero, and the coordinate X is a root of the cubic polynomial equation

$$p(X) = b_0 X^3 - A_0 X + \Lambda h_0 = 0. \tag{7}$$

The number of real zeroes of this cubic equation are determined by its discriminant

$$\Delta = b_0(4A_0^3 - 27 b_0 h_0^2 \Lambda^2).$$

That is, if $\Delta > 0$ then (7) has three real simple roots; if $\Delta = 0$ then (7) has multiple roots and all roots are real; and if $\Delta < 0$ then (7) has one simple real root and two conjugate complex non–real roots. We shall see that the sign of Δ, when it is different from zero, provides the minimal number of periodic solutions of the differential system (5), and consequently this number also provides the minimal number of T–periodic solutions of the original system (3) going back through the rescalings.

We write the averaged function associated system (5) as

$$f(X, Y) = (Y, -CY - A_0 X + b_0 X^3 + \Lambda h_0).$$

To guarantee the existence of periodic solution for system (5) we need, by Theorem 3 (see the appendix), that the Jacobian of f in each equilibrium point (X^*, Y^*) of system (6) be non–zero. Note that the Jacobian of f is given by

$$\det(Jf(X^*, Y^*)) = \begin{vmatrix} 0 & 1 \\ -A_0 - 3b_0 X^{*2} & -C \end{vmatrix} = -(-A_0 - 3b_0 X^{*2}) = -p'(X^*). \tag{8}$$

Case 1: Assume that $\Delta = 0$. Then, since $b_0 \neq 0$ we have that $\Lambda = \pm\sqrt{\dfrac{4A_0^3}{27 b_0 h_0^2}} = \pm\Lambda_0$. Since $A_0 \neq 0$ the polynomial $p(X)$ has two real roots, one simple root X_1 and one root X_2 with multiplicity two. More precisely if $\Lambda = \Lambda_0$ then $X_1 = -2\sqrt{\dfrac{A_0}{3b_0}} < 0$ and $X_2 = \sqrt{\dfrac{A_0}{3b_0}} > 0$. If $\Lambda = -\Lambda_0$ then $X_1 = 2\sqrt{\dfrac{A_0}{3b_0}} > 0$ and $X_2 = -\sqrt{\dfrac{A_0}{3b_0}} < 0$. Notice that $(X_1, 0)$ is an equilibrium point of system (6) satisfying $\det(Jf(X_1, 0)) \neq 0$. Theorem 3(a) guarantees that there exists one T–periodic solution of system (5) associated to this equilibrium point. We note that Theorem 3(a) does not provide information if the equilibrium $(X_2, 0)$ of system (6) produces a periodic solution of system (5). But later on will see that this is the case.

Case 2: Assume that $\Delta \neq 0$. Then it is easy to check that $\Delta > 0$ if and only if $b_0A_0 > 0$ and $|\Lambda| < \Lambda_0$, and consequently $\Delta < 0$ if and only if $b_0A_0 > 0$ and $|\Lambda| > \Lambda_0$.

Let $A_0b_0 > 0$ be. When $|\Lambda| < \Lambda_0$, the discriminant Δ is positive and equation (7) has three real simple roots. When $|\Lambda| > \Lambda_0$ the discriminant Δ is negative and equation (7) has only one real root. In this case each root X^* of equation (7) provides an equilibrium point $(X^*, 0)$ of the averaged system such that the Jacobian $\det(Jf|_{(X^*,0)})$ is non–zero. Then, for each of these equilibrium points $(X^*, 0)$ Theorem 3(a) guarantees that there exists one T–periodic solution of system (5). In short, for $|\Lambda| < \Lambda_0$ the three roots of equation (7) provide three T–periodic solutions of system (5) and for $|\Lambda| > \Lambda_0$ the unique real root of (7) provides one T–periodic solution of system (5). By the rescalings (4) with $\lambda_0 = \varepsilon^3\Lambda_0$ the original system (3) has three T–periodic solutions for $|\lambda| < \lambda_0$ and one T–periodic solution for $|\lambda| > \lambda_0$.

To study the stability of the previous T–periodic solutions we shall apply Theorem 3(b). Indeed it is sufficient to know the stability of the equilibrium points $(X^*, 0)$ of system (6). Then we need to study the sign of the real part of the eigenvalues associated to each equilibrium point $(X^*, 0)$. The eigenvalues of $(X^*, 0)$ are given by the zeroes of the characteristic polynomial

$$\lambda^2 - \mathrm{tr}(Jf(X^*, 0))\lambda + \det(Jf(X^*, 0)) = 0, \tag{9}$$

where $\mathrm{tr}(Jf(X^*, 0)) = -C < 0$ is the trace of Jacobian matrix (8) at the equilibrium point $(X^*, 0)$. The eigenvalues associated with equilibrium point $(X^*, 0)$ are

$$\lambda_+ = \frac{-C + \sqrt{C^2 - 4\det(Jf(X^*, 0))}}{2},$$

and

$$\lambda_- = \frac{-C - \sqrt{C^2 - 4\det(Jf(X^*, 0))}}{2}.$$

For $A_0b_0 > 0$ we distinguish two cases either $b_0 > 0$ and $A_0 > 0$, or $b_0 < 0$ and $A_0 < 0$.

Case A: Assume $b_0 > 0$ and $A_0 > 0$. When $|\Lambda| < \Lambda_0$ the polynomial $p(X)$ has three real simple roots X_1, X_2 and X_3 satisfying $X_1 < X_2 < X_3$. As $b_0 > 0$ we have $p'(X_1) > 0$, $p'(X_2) < 0$ and $p'(X_3) > 0$. The sign of Jacobian (8) in each equilibrium point $(X_i, 0)$ $i = 1, 2, 3$ is negative at $(X_1, 0)$ and $(X_3, 0)$ and positive at $(X_2, 0)$. Since the eigenvalues associated with $(X_1, 0)$ satisfy $\lambda_- < 0 < \lambda_+$, then $(X_1, 0)$ is unstable. The eigenvalues associated with $(X_2, 0)$ satisfy either $\lambda_- < \lambda_+ < 0$ when $C^2 - 4\det(Jf(X_2, 0)) > 0$, or the eigenvalues are complex conjugate with real part negative when $C^2 - 4\det(Jf(X_2, 0)) < 0$. Then in both cases $(X_2, 0)$ is stable. Finally, the eigenvalues associated with $(X_3, 0)$ satisfy $\lambda_- < 0 < \lambda_+$, then $(X_3, 0)$ is unstable.

When $|\Lambda| > \Lambda_0$ the polynomial $p(X)$ has one real root X_1. Since $b_0 > 0$ we have $p'(X_1) > 0$ and $\det(Jf(X_1, 0)) < 0$. Then, the eigenvalues associated to equilibrium point $(X_1, 0)$ satisfy $\lambda_- < 0 < \lambda_+$ and $(X_1, 0)$ is unstable.

When $\Lambda = \Lambda_0$ ($\Lambda = -\Lambda_0$) then X_1 is the real simple root of $p(X)$ and X_2 the double real root. We have that $X_1 < X_2$ ($X_1 > X_2$) and $p'(X_1) > 0$ in both cases. The equilibrium point $(X_1, 0)$ of system (6) is unstable in both cases. In short, when $|\Lambda| = \Lambda_0$ the equilibrium point $(X_1, 0)$ is unstable.

Case B: Assume $b_0 < 0$ and $A_0 < 0$. When $|\Lambda| < \Lambda_0$ the polynomial $p(X)$ has three real simple roots X_1, X_2 and X_3 satisfying $X_1 > X_2 > X_3$. In a similar way as above we obtain that $(X_1, 0)$ and $(X_3, 0)$ are stable and $(X_2, 0)$ is unstable. When $|\Lambda| > \Lambda_0$ the polynomial $p(X)$ has one simple real root X_1 and the equilibrium point $(X_1, 0)$ is stable. And when $|\Lambda| = \Lambda_0$ then X_1 is the real simple root of $p(X)$. We have that $p'(X_1) < 0$ and the equilibrium point $(X_1, 0)$ is stable.

By Theorem 3(b) the (in)stability of equilibrium points of system (6) provide the (in)stability of corresponding periodic solutions of system (5), and through the rescalings (4) the (in)stability of the periodic solution of original system (3).

The points x_i which appear in the statement of Theorem 1 are $x_i = \varepsilon X_i$ for $i = 1, 2, 3$. So Theorem 1 is proved when $|\lambda| < \lambda_0$ and $|\lambda| > \lambda_0$. For $|\lambda| = \lambda_0$ it is proved only with respect to the T–periodic solution α_1. But once there exist the two T–periodic solutions α_2 and α_3 when $|\lambda| < \lambda_0$ and such two T–periodic solutions do not exist for $|\lambda| > \lambda_0$, by continuity of the solutions with respect to the initial conditions and parameters it follows that $\alpha_2 = \alpha_3$ when $|\lambda| = \lambda_0$. This completes the proof of Theorem 1.

3. Comparison with the results of Chen and Li

In this section we first state the main result of [2] on the number of periodic solutions and their (in)stability. After we compare their results with our results.

Theorem 2. *Assume that* (2) *holds,* $b(t) > 0$ *and* $h(t) > 0$ *for all* t. *There is a* $\lambda_0 > 0$ *such that*

(a) *system* (3) *has a unique* T*–periodic solution which is unstable for* $|\lambda| > \lambda_0$*;*
(b) *system* (3) *has exactly three* T*–periodic solutions for* $|\lambda| < \lambda_0$*, one is stable and the remaining two are unstable;*
(c) *system* (3) *has exactly two* T*–periodic solutions; both of them are unstable for* $\lambda = \pm\lambda_0$.

We would highlight the hypotheses of Theorem 1, we just consider the periodicity and continuity of functions $a(t)$, $b(t)$ and $h(t)$ together with $a_0 b_0 > 0$, and we analyse system (3) in a rectangle around the origin as large as we want. In Theorem 2 beside periodicity and continuity of functions $a(t)$, $b(t)$ and $h(t)$, are necessary $b(t) > 0$, $h(t) > 0$ and that the condition (2) holds. So in [2] only a particular subclass of equation (1) with a_0 and b_0 positive has been studied.

We remark that in our case λ_0 is given by the expression $\lambda_0 = \dfrac{1}{\varepsilon^3}\sqrt{\dfrac{4a_0^3}{27b_0h_0^2}}$, with $\varepsilon > 0$. In [2] any expression for λ_0 is given.

4. Appendix

4.1. Averaging Theory of First Order. Now we shall present the basic results from averaging theory that we need for proving the results of this paper.

The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for the proof see Theorems 11.5 and 11.6 of Verhulst [6].

Consider the differential equation

$$\dot{\mathbf{x}} = \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{10}$$

with $\mathbf{x} \in D$, where D is an open subset of $\mathbb{R}^n$, $t \geq 0$. Moreover we assume that both $F_1(t,\mathbf{x})$ and $F_2(t,\mathbf{x},\varepsilon)$ are $T-$periodic in t. We also consider in D the averaged differential equation

$$\dot{\mathbf{y}} = \varepsilon f_1(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{11}$$

where

$$f_1(\mathbf{y}) = \frac{1}{T}\int_0^T F_1(t,\mathbf{y})dt.$$

Under certain conditions, equilibrium solutions of the averaged equation turn out to correspond with $T-$periodic solutions of equation (10).

Theorem 3. *Consider the two initial value problems* (10) *and* (11). *Suppose:*

- (i) *F_1, its Jacobian $\partial F_1/\partial x$, its Hessian $\partial^2 F_1/\partial x^2$, F_2 and its Jacobian $\partial F_2/\partial x$ are defined, continuous and bounded by a constant independent of ε in $[0,\infty) \times D$ and $\varepsilon \in (0,\varepsilon_0]$.*
- (ii) *F_1 and F_2 are $T-$periodic in t (T independent of ε).*

Then the following statements hold.

- (a) *If p is an equilibrium point of the averaged equation* (11) *and*

$$\det\left(\frac{\partial f_1}{\partial \mathbf{y}}\right)\bigg|_{\mathbf{y}=p} \neq 0,$$

then there exists a $T-$periodic solution $\varphi(t,\varepsilon)$ of equation (10) *such that $\varphi(0,\varepsilon) \to p$ as $\varepsilon \to 0$.*

(b) *The stability or instability of the limit cycle* $\varphi(t,\varepsilon)$ *is given by the stability or instability of the equilibrium point* p *of the averaged system* (11). *In fact the singular point* p *has the stability behavior of the Poincaré map associated to the limit cycle* $\varphi(t,\varepsilon)$.

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References

[1] H. Chen and Yi Li, *Stability and exact multiplicity of periodic solutions of Duffing equations with cubic nonlinearities.* Proc. Amer. Math. Soc. **135** (2007), no. 12, 3925–3932.

[2] H. Chen and Yi Li, *Bifurcation and stability of periodic solutions of Duffing equations.* Nonlinearity **21**(2008), no. 11, 2485–2503.

[3] G. Duffing, *Erzwungen Schwingungen bei vernäderlicher Eigenfrequenz und ihre technisch Bedeutung*, Sammlung Viewg Heft 41/42, Viewg, Braunschweig, 1918.

[4] J. Mawhin, *Seventy-five years of global analysis around the forced pendulum equation.* In: Equadiff **9** (1997), Brno. Proceedings, Brno: Masaryk University, 1997, 115–145.

[5] R. Ortega, *Stability and index of periodic solutions of an equation of Duffing type.* Boo. Uni. Mat. Ital B **3**(1989), 533–546.

[6] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems,* Universitext, Springer, 1991.

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