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PERIODIC STRUCTURE OF TRANSVERSAL MAPS ON $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$ AND $\mathbb{S}^p \times \mathbb{S}^q$

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Abstract. A $\mathcal{C}^1$ map $f : \mathbb{M} \to \mathbb{M}$ is called transversal if for all $m \in \mathbb{N}$ the graph of f^m intersects transversally the diagonal of $M \times M$ at each point (x, x) being x a fixed point of f^m. Let $\mathbb{C}\mathrm{P}^n$ be the n–dimensional complex projective space, $\mathbb{H}\mathrm{P}^n$ be the n–dimensional quaternion projective space and $\mathbb{S}^p \times \mathbb{S}^q$ be the product space of the p–dimensional with the q–dimensional spheres, $p \neq q$. Then for the cases $\mathbb{M}$ equal to $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$ and $\mathbb{S}^p \times \mathbb{S}^q$ we study the set of periods of f by using the Lefschetz numbers for periodic points.

1. Introduction and statement of the main results

We consider the discrete dynamical system $(\mathbb{M}, f)$ where $\mathbb{M}$ is a topological space and $f : \mathbb{M} \to \mathbb{M}$ be a continuous map. A point x is called *fixed* if $f(x) = x$, and *periodic* of *period* k if $f^k(x) = x$ and $f^i(x) \neq x$ if $0 \leq i < k$. By $\mathrm{Per}(f)$ we denote the *set of periods* of all the periodic points of f.

If $x \in \mathbb{M}$ the set $\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$ is called the *orbit* of the point x. Here f^n means the composition of n times f with itself. To study the dynamics of the map f is to study all the different kind of orbits of f. Of course if x is a periodic point of f of period k, then its orbit is $\{x, f(x), f^2(x), \ldots, f^{k-1}(x)\}$, and it is called a *periodic orbit.*

In this work we study the possible sets of periods for transversal maps defined on $\mathbb{C}\mathrm{P}^n$ (the n–dimensional complex projective space), on $\mathbb{H}\mathrm{P}^n$ (the n–dimensional quaternion projective space) and on $\mathbb{S}^p \times \mathbb{S}^q$ (the product space of the n–dimensional with the m–dimensional spheres) when $n \neq m$. Here a *transversal map* is a $\mathcal{C}^1$ map $f : \mathbb{M} \to \mathbb{M}$ defined on a $\mathcal{C}^1$ differentiable manifold such that $f(\mathbb{M}) \subset \mathrm{Int}(\mathbb{M})$ and for all $m \in \mathbb{N}$ at each point x fixed by f^m we have that $\det(I - Df^m(x)) \neq 0$, i.e. 1 is not an eigenvalue of $Df^m(x)$. Note that if f is transversal then for all $m \in \mathbb{N}$ the graph of f^m intersects transversally the diagonal $\{(y, y) : y \in \mathbb{M}\}$ at each point (x, x) being x fixed by f^m.

The periodic orbits play an important role in the general dynamics of the system, for studying them we can use topological information. Perhaps the best known

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example in this direction are the results contained in the seminal paper entitle *Period three implies chaos* for continuous self–maps on the interval, see [11]. In the case of transversal maps different studies appear in the literature in general from other point of view, see for instance Franks [7, 8]; Matsuoka [15]; Babenko y Bogatyi [1]; Casasayas, Llibre and Nuñes [3]; Llibre, Paraños and Rodriguez [12], Llibre and Swanson [14] and Fagella and Llibre [6].

Probably the main contribution of the Lefschetz's work in 1920's was to link the homology class of a given map with an earlier work on the indices of Brouwer on the continuous self–maps on compact manifolds. These two notions provide equivalent definitions for the Lefschetz numbers, and from their comparison, can be obtained information about the existence of fixed points.

Let $\mathbb{M}$ be an n–dimensional manifold. We denote by $H_k(\mathbb{M}, \mathbb{Q})$ for $k = 0, 1, ..., n$ the homological groups with coefficients in $\mathbb{Q}$. Each of these groups is a finite linear space over $\mathbb{Q}$.

Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ there exist $n + 1$ induced linear maps $f_{*k} : H_k(\mathbb{M}, \mathbb{Q}) \to H_k(\mathbb{M}, \mathbb{Q})$ for $k = 0, 1, \ldots, n$ by f. Every linear map f_{*k} is given by an $n_k \times n_k$ matrix with integer entries, where n_k is the dimension of $H_k(\mathbb{M}, \mathbb{Q})$.

Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ on a compact n–dimensional manifold $\mathbb{M}$, its *Lefschetz number* $L(f)$ is defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \text{trace}(f_{*k}). \tag{1}$$

One of the main results connecting the algebraic topology with the fixed point theory is the *Lefschetz Fixed Point Theorem* which establishes the existence of a fixed point if $L(f) \neq 0$, see for instance [2].

Our aim is to characterize the possible sets of periods of f. For doing that we can consider the Lefschetz number of f^m but, in general, it is not true that $L(f^m) \neq 0$ implies that f has a periodic point of period m; it only implies the existence of a periodic point with period a divisor of m. Thus, we shall use the *Lefschetz numbers for periodic points* introduced in [4] (see also [13]) for analyzing if a given period belongs to the set of periods of a self–map. More precisely, for every $m \in \mathbb{N}$ we define the *Lefschetz number of period m* $l(f^m)$ as follows

$$l(f^m) = \sum_{r|m} \mu(r) L(f^{m/r}),$$

where $\sum_{r|m}$ denotes the sum over all positive divisors r of m, and μ is the *Möebius function* defined by

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } k^2 | m \text{ for some } k \in \mathbb{N}, \\ (-1)^r & \text{if } m = p_1 \ldots p_r \text{ distinct prime factors.} \end{cases}$$

By the inversion formula, for more details see for instance [17],

$$L(f^m) = \sum_{r|m} l(f^r).$$

The Lefschetz number of period m will play an important role after showing the following result for transversal maps (see [13] and [10]).

Theorem 1. *Let $f : \mathbb{M} \to \mathbb{M}$ be a transversal map and $\mathbb{M}$ be a compact manifold. Suppose that $l(f^m) \neq 0$ for some $m \in \mathbb{N}$.*

(a) *If m is odd then $m \in \mathrm{Per}(f)$.*
(b) *If m is even then $\left\{\frac{m}{2}, m\right\} \cap \mathrm{Per}(f) \neq 0$.*

The results on transversal maps on arbitrary compact manifolds stated in Theorem 1 are in general difficult to apply as a consequence of the computation of $l(f^m)$. Thus, if the homological rational groups are simple then these computations become easier and this is the case for the manifolds studied in the present paper. Our main results are the following.

In what follows the spaces $\mathbb{H}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ are identified to real differentiable manifolds of dimension and $2n$ and $4n$ respectively.

Theorem 2. *Let f be a transversal map defined on $\mathbb{C}P^n$ (respectively $\mathbb{H}P^n$) with $n \geq 1$, and let $f_{*2} = (a)$ (respectively $f_{*4} = (a)$) be. Then the following statements hold.*

(a) *$1 \in \mathrm{Per}(f)$ if either $a = -1$ and n odd, or $a = 0$, or $a = 1$.*
(b) *$\{1, 2\} \cap \mathrm{Per}(f) \neq \emptyset$ if $a = -1$ and n even.*
(c) *$\{1, 3, 5, 7, \ldots\} \subset \mathrm{Per}(f)$ and $\mathrm{Per}(f)$ contains infinitely many even periods if either $a > 1$, or $a < -1$ and n even.*
(d) *$\mathrm{Per}(f)$ contains infinitely many even periods if $a < -1$ and n odd.*

Theorem 2 will be proved in section 3

Theorem 3. *Let f be a transversal map defined on $\mathbb{S}^p \times \mathbb{S}^q$ with $1 \leq p < q$, and let $f_{*p} = (a)$, $f_{*q} = (b)$ and $f_{*p+q} = (c)$ be. Assume that $\{a, b, c\} \neq \{-1, 0, 1\}$ and that if at least two of the elements of the set $\{a, b, c\}$ are equal in absolute value and larger than or equal to the absolute value of the third, we have that $|A| = |B| \geq |C| > 1$ if $\{(A, r), (B, s), (C, t)\} = \{(a, p), (b, q), (c, p + q)\}$. Then there exists a positive integer m^* such that all the odd integers larger than m^* belong to $\mathrm{Per}(f)$, and there are infinitely many even periods contained in $\mathrm{Per}(f)$.*

Theorem 3 will be proved in section 4 while in section 2 we present some auxiliary results which help us to computed the Lefschetz numbers for periodic points.

We note that for transversal maps on $\mathbb{S}^{p_1} \times \cdots \times \mathbb{S}^{p_k}$ such that all their groups of homology are either 0 or $\mathbb{Q}$ we can study their set of periods in the same way than in the proof of Theorem 3.

The set of periods for $\mathcal{C}^1$ maps on $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{S}^p \times \mathbb{S}^q$ having all their periodic orbits hyperbolic have been studied in [9]. We recall that if x is a periodic point of period k of f, then x is *hyperbolic* if $df^k(x)$ has no eigenvalues in the unit circle of the complex plain. Of course these results cannot be applied to the transversal maps of Theorems 2 and 3 because transversal maps do not need to have all their periodic points hyperbolic.

2. Lefschetz numbers for periodic points

Let $f : \mathbb{M} \to \mathbb{M}$ be a transversal map and suppose that the rational homology of $\mathbb{M}$ is of the form

$$H_q(\mathbb{M}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in J \cup \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

where J is a finite subset of natural numbers with cardinality less than or equal to the dimension n of $\mathbb{M}$.

Let (a_j) be the 1×1 integer matrix defined by the induced homology endomorphism $f_{*j} : H_j(\mathbb{M},\mathbb{Q}) \to H_j(\mathbb{M},\mathbb{Q})$ for each $j \in J$. We have that $H_0(\mathbb{M},\mathbb{Q}) \approx \mathbb{Q}$ and that f_{*0} is the identity, because $\mathbb{M}$ is assumed connected (see [16] for more details). Then $L(f^m) = 1 + \sum_{j\in J}(-1)^j a_j^m$ for all $m \in \mathbb{N}$. Note that

$$\begin{aligned}\sum_{r|m}\mu(r) &= 1 - \Big(\sum_{1\le i\le r} 1\Big) + \Big(\sum_{1\le i<j\le r} 1\Big) - \ldots + (-1)^r \\ &= 1 - \binom{r}{1} + \binom{r}{2} - \ldots + (-1)^r\binom{r}{r} \\ &= (1-1)^r = 0,\end{aligned}$$

where $m = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_r^{\alpha_r}$ with $p_1 \ldots p_r$ distinct primes. Thus, if $m > 1$ the Lefschetz number of period m is

$$l(f^m) = \sum_{r|m}\mu(r)\left(1 + \sum_{j\in J}(-1)^j a_j^{m/r}\right) = \sum_{j\in J}(-1)^j\sum_{r|m}\mu(r)a_j^{m/r}.$$

For each $m > 1$ we define the polynomial

$$Q_m(x) = \sum_{r|m}\mu(r)x^{m/r}. \tag{2}$$

Then, if $m > 1$ we can write

$$l(f^m) = \sum_{j\in J}(-1)^j Q_m(a_j). \tag{3}$$

Due to Theorem 1 we must study when $l(f^m)$ is zero or not. We do that by analyzing the polynomials $Q_m(x)$ and evaluating them at a_j. For doing this we shall use as a key point the properties of Q_m described in the next proposition proved in [10]. Let $\mathbb{N}$ be the set of positive integers.

Proposition 4. *Let $m \in \mathbb{N}$. Then the following properties hold.*

(a) *If m is odd, then Q_m is an odd function.*
(b) *If $4|m$, then Q_m is an even function.*
(c) *If $2|m$ and $4 \not| m$, then $Q_m(x) = Q_{m/2}(x^2) - Q_{m/2}(x)$.*
(d) *$Q_m(0) = 0$.*
(e) *If $m > 1$, then $Q_m(1) = 0$.*
(f) *If $m > 2$, then $Q_m(-1) = 0$.*
(g) *For all $i \in \mathbb{N}$ we have $Q_m^{(i)}(1) \ge 0$, where $Q_m^{(i)}(x)$ denotes the i–th derivative of $Q_m(x)$ with respect to the variable x.*
(h) *$Q_m(x)$ is positive and increasing in $(1,+\infty)$.*
(i) *If m is even, then the function $Q_m(x)$ is positive and decreasing in $(-\infty,-1)$. Furthermore, if $2|m$ and $4 \not| m$ we have that $Q_m(x) \le Q_m(-x)$ for all $x \in [1,+\infty)$.*
(j) *If $m > 2$, then $Q_m(1.6) > 2$.*

3. Transversal maps on $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$

3.1. **Case $\mathbb{C}P^n$.** For $n \geq 1$ let $f : \mathbb{C}P^n \to \mathbb{C}P^n$ be a $\mathcal{C}^1$ map. The homological groups of $\mathbb{C}P^n$ over $\mathbb{Q}$ are of the form

$$H_q(\mathbb{C}P^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, 2, 4, ..., 2n\}, \\ 0 & \text{otherwise.} \end{cases}$$

The induced linear maps are $f_{*q} = (a^{q/2})$ for $q \in \{0, 2, 4, ..., 2n\}$ with $a \in \mathbb{Z}$, and $f_{*q} = (0)$ otherwise (see for more details [18, Corollary 5.28]).

From (3) we have that

$$l(f^m) = \sum_{j=0}^{n} Q_m(a^j), \tag{4}$$

for $m > 1$.

3.2. **Case $\mathbb{H}P^n$.** For $n \geq 1$ let $f : \mathbb{H}P^n \to \mathbb{H}P^n$ be a $\mathcal{C}^1$ map. The homological groups of $\mathbb{H}P^n$ over $\mathbb{Q}$ are of the form

$$H_q(\mathbb{H}P^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, 4, 8, ..., 4n\}, \\ 0 & \text{otherwise.} \end{cases}$$

The induced linear maps are $f_{*q} = (a^{q/4})$ for $q \in \{0, 4, 8, ..., 4n\}$ with $a \in \mathbb{Z}$, and $f_{*q} = (0)$ otherwise (see for more details [18, Corollary 5.33]).

Proof of Theorem 2. Assume $a = 1$. From (1) we get that $L(f) = n + 1$, and from (3) and Proposition 4(e) we have that $l(f^m) = 0$ for $m > 1$.

Suppose $a = 0$. From (1) we get that $L(f) = 1$, and from (3) and Proposition 4(e) we have that $l(f^m) = 0$ for $m > 1$.

Assume $a = -1$ and n odd. From (1) we get that $L(f) = [1 - (-1)^n]/2$. From (3) we obtain that $l(f^2) = L(f^2) - L(f)$. Since $L(f^2) = 1 + \sum_{q=1}^{n} (-1)^{2q} = n + 1$ it follows that $l(f^2) \neq 0$. From (3) and statements (e) and (f) of Proposition 4 we have that $l(f^m) = 0$ for $m > 2$. In short, by using Theorem 1 we have proved statement (a).

Suppose $a = -1$ and n even. From the previous paragraph we have that $L(f) = 0$, $l(f^2) = n+1$ and $l(f^m) = 0$ for $m > 2$. By Theorem 1 we have proved statement (b).

Assume that $a > 1$. From (1) we get that $L(f) = (1 - a^{n+1})/(1 - a) \neq 0$, and from (3) and Proposition 4(h) we have that $l(f^m) > 0$ for $m > 1$. So by Theorem 1(a) we obtain that $\{1, 3, 5, 7, \dots\} \subset \mathrm{Per}(f)$. By applying Theorem 1(b) to $l(f^m)$ with $m = 4, 16, 64, \dots$ we get that infinitely many even periods belongs to $\mathrm{Per}(f)$. Therefore statement (c) for $a > 1$ is proved.

Suppose that $a < -1$. From (1) we get that $L(f) = (1 + (-1)^n)/2$. Let m be even. From (4) and by statements (h) and (i) of Proposition 4 we obtain that $l(f^m) > 0$. Repeating the arguments of the previous paragraph it follows that infinitely many even periods belongs to $\mathrm{Per}(f)$. note that this result is independent of the parity of n, and consequently statement (d) is proved. Additionally now we assume that m is odd and n is even. By (4) and statements (a) and (e) of Proposition 4 we get that

$$l(f^m) = \sum_{k=1}^{n} (-1)^k Q_m(|a|^k).$$

By Proposition 4(h) we have that $Q_m(|a|^{2l}) - Q_{|a|^{2l-1}} > 0$ for any positive integer l. Therefore $l(f^m) > 0$ because n is even. Hence by Theorem 1(a) we obtain $\{1, 3, 5, 7, \ldots\} \subset \mathrm{Per}(f)$ and the proof of statement (c) is completed. □

4. Transversal maps on $\mathbb{S}^p \times \mathbb{S}^q$

For $1 \le p < q$, let $f : \mathbb{S}^p \times \mathbb{S}^q \to \mathbb{S}^p \times \mathbb{S}^q$ be a $\mathcal{C}^1$ map. The homological groups of $\mathbb{S}^p \times \mathbb{S}^q$ over $\mathbb{Q}$ are of the form

$$H_q(\mathbb{S}^p \times \mathbb{S}^q, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, p, q, p+q\}, \\ 0 & \text{otherwise.} \end{cases}$$

The induced linear maps are $f_{*0} = (1)$, $f_{*p} = (a)$, $f_{*q} = (b)$ with $a, b \in \mathbb{Z}$, $f_{*p+q} = (c)$, where c is the degree of the map f and $f_{*i} = (0)$ for $i \in \{0, ..., p+q\}$ and $i \ne 0, p, q, p+q$ (see for more details [5]).

From (4) and for $m > 1$ we get

$$l(f^m) = (-1)^p Q_m(a) + (-1)^q Q_m(b) + (-1)^{p+q} Q_m(c), \tag{5}$$

because $Q_m(0) = Q_m(1) = 0$ due to statements (d) and (e) of Proposition 4.

Proposition 5. *Let f be a transversal map on $\mathbb{S}^p \times \mathbb{S}^q$ with $f_{*p} = (a)$, $f_{*q} = (b)$ and $f_{*p+q} = (c)$. Suppose that the assumptions (i) and (ii) in the statement of Theorem 3 hold. Then there exists some m^* such that $l(f^m) \ne 0$ for all $m \ge m^*$.*

Proof. First we assume that exists in the set $\{a, b, c\}$ one element in absolute value larger than the absolute value of the others two, let a be such element. If such element is b or c the arguments of the proof would be the same. Since the polynomial $Q_m(x)$ has dominant term x^m (see (2)), from the expression of $l(f^m)$ given in (5) we get that there exists a positive integer m_1^* such that in the expression (5) we have that a^m for $m \ge m_1^*$ dominates all the other terms. Therefore $l(f^m) \ne 0$ for $m \ge m_1^*$.

Suppose that in the set $\{a, b, c\}$ there are two elements such that their absolute values are equal and larger than the absolute value of the third. Let a and b such two values. If a and b are not such two values the proof is similar. If in the expressions $(-1)^p Q_m(a)$ and $(-1)^q Q_m(b)$ the coefficients of a^m and b^m respectively are the same, the argument of the above case can be applied for obtaining that there exists a positive integer m_2^* such that $l(f^m) \ne 0$ for $m \ge m_2^*$. Consider now that the mentioned coefficients of a^m and b^m have different sign, then the dominant term in the expression (5) of $l(f^m)$ is c^m and since $|c| > 1$, we obtain that $l(f^m) \ne 0$ for $m > m_3^*$ for some positive integer m_3^*.

Finally assume that $|a| = |b| = |c|$. So $|a| > 1$. Clearly from the expression of $l(f^m)$ given in (5) and by statements (a), (h) and (i) of Proposition 4, it follows that $l(f^m) \ne 0$ for $m > 1$. For $m^* = \max\{m_1^*, m_2^*, m_3^*\}$ the statement of the proposition follows. □

After Proposition 5 it remains to study $l(f^m)$ in the case that the set $\{a, b, c\} \subset \{-1, 0, 1\}$. But from statements (d), (e) and (f) of Proposition 4 it follows that in such a case $l(f^m) = 0$ for $m > 2$.

Proof of Theorem 3. Under the assumptions of Theorem 3 and from Proposition 5 there exists a positive integer m^* such that $l(f^m) \ne 0$ for $m \ge m^*$. By Theorem 1 it follows that all the odd integers larger than m^* belong to $\mathrm{Per}(f)$. Moreover repeating the arguments of the proof of Theorem 2 we get that there are infinitely many even periods contained in $\mathrm{Per}(f)$. □

References

[1] I.K. Babenko and S.A. Bogatyi, The behavior of the index of periodic points under iterations of a mapping, *Math. USSR Izvestiya* **38** (1992), 1–26.

[2] R.F. Brown, *The Lefschetz fixed point theorem*, Scott, Foresman and Company, Glenview, IL, 1971.

[3] J. Casasayas, J. Llibre and A. Nunes, Periodic orbits of transversal maps, *Math. Proc. Camb. Phil. Soc.* **118** (1995), 161–181.

[4] A. Dold, Fixed points indices of iterated maps, *Invent. Math.* **74** (1983), 419–435.

[5] C.T. Dodson and P.E. Parker, *A user's guide to algebraic topology*, Kluwer Acad. Publ., 1996.

[6] N. Fagella and J. Llibre, Periodic points of holomorphic maps via Lefschetz numbers, *Trans. Amer. Math. Soc.* **352** (2000), 4711–4730.

[7] J. Franks, *Homology and dynamical systems*, CBSM Regional Conf. Ser. in Math. **49**, Amer. Math. Soc., Providence, R.I. 1982.

[8] J. Franks, *Period doubling and the Leschetz formula*, *Trans. Amer. Math. Soc.* **287** (1985), 275–283.

[9] J.L.G. Guirao and J. Llibre, $\mathcal{C}^1$ self–maps on $\mathbb{S}^n$, $\mathbb{S}^n \times \mathbb{S}^m$, $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$ with all their periodic orbits hyperbolic, to appear in *Taiwanese J. of Math.*

[10] A. Guillamón, X. Jarque, J. Llibre, J. Ortega and J. Torregrosa, Periods for transversal maps via Lefschetz numbers for periodic points, *Trans. Math. Soc.* **347** (1995), 4779–4806.

[11] T.Y. Li and J. Yorke, Period three implies chaos, *Amer. Math. Monthly* **82** (1975), 985–992.

[12] J. Llibre, J. Paraños and J.A. Rodriguez, Periods for transversal maps on compact manifolds with a given homology, *Houston J. Math.* **24** (1998), 397–407.

[13] J. Llibre, Lefschetz numbers for periodic points, *Contemporary Math.* **152**, Amer. Math. Soc., Providence, RI, (1993), 215–227.

[14] J. Llibre and R. Swanson, Periodic points for transversal maps on surfaces, *Houston J. Math.* **19** (1993), 395–403.

[15] T. Matsuoka, The number of periodic points of smooth maps, *Erg. Th. & Dyn. Sys.* **9** (1989), 153–163.

[16] J.R. Munkres, Elements of Algebraic Topology, Addison–Wesley, 1984.

[17] I. Niven and H.S. Zuckerman, An introduction to the theory of numbers, fourth edition, John Wiley & Sons, New York, 1980.

[18] J.W. Vicks, *Homology theory. An introduction to algebraic topology*, Springer-Verlag, New York, 1994. Academic Press, New York, 1973.

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