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UNIVERSAL CENTERS AND COMPOSITION CONDITIONS

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Abstract. In this paper we characterize the universal centers of the ordinary differential equations $d\rho/d\theta = \sum_{i=1}^{\infty} a_i(\theta)\rho^{i+1}$, where $a_i(\theta)$ are trigonometric polynomials, in terms of the composition conditions. These centers are closely related with the classical Poincaré center problem for planar analytic differential systems.

Additionally we show that the notion of universal center is not invariant under changes of variables, and we also provide different families of universal centers. Finally we characterize all the universal centers for the quadratic polynomial differential systems.

1. Introduction and statement of the main results

We consider the analytic ordinary differential equation

$$\frac{d\rho}{d\theta} = \sum_{i=1}^{\infty} a_i(\theta)\rho^{i+1}, \tag{1}$$

on the cylinder $(\rho, \theta) \in \mathbb{R} \times \mathbb{S}^1$ in a neighborhood of $\rho = 0$ and where $a_i(\theta)$ are trigonometric polynomials in θ. When all $a_i(\theta)$ are identically zero, we say that (1) is a *trivial* center. We shall denote the derivative of ρ with respect θ by $d\rho/d\theta$ or ρ'. We can solve equation (1) by the Picard iteration method and find a solution which is unique with the prescribed initial value $\rho(0) = \rho_0$, where $|\rho_0|$ is small enough. The differential equation (1) is determined by its coefficients $a = (a_1(\theta), a_2(\theta), \ldots)$.

We say that equation (1) determines a *center* if for any sufficiently small initial value $\rho(0)$ the solution of (1) satisfies $\rho(0) = \rho(2\pi)$. The *center problem* for equation (1) is to find conditions on the coefficients a_i under which this equation has a center.

The center problem and an explicit expression for the first return map of the differential equation (1) has been studied by Brudnyi in [12, 14] for a more general class of equations. The expression of the first return map is

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given in terms of the following iterated integrals of order k

$$I_{i_1\dots i_k}(a) := \int\cdots\int_{0\le s_1\le\cdots\le s_k\le 2\pi} a_{i_k}(s_k)\cdots a_{i_1}(s_1)\,ds_k\cdots ds_1, \tag{2}$$

where, by convention we assume that for $k=0$ the previous expression is equal to 1. By the Ree formula [29] the linear space generated by all such functions is an algebra. A linear combination of iterated integrals of order $\le k$ is called an *iterated polynomial of degree k*. Iterated integrals appear in a similar context in the study of several differential equations, see for instance [2, 18, 19, 21, 23]. Let $\rho(\theta;\rho_0;a)$ with $\theta\in[0,2\pi]$ be the solution of equation (1) corresponding to a with initial value $\rho(0;\rho_0;a)=\rho_0$. Then $P(a)(\rho_0) := \rho(2\pi;\rho_0;a)$ is the first return map of this equation and in [12, 14] it is proved the following.

Theorem 1. *For sufficiently small initial values ρ_0 the first return map $P(a)$ is an absolute convergent power series $P(a)(\rho_0) = \rho_0 + \sum_{n=1}^{\infty} c_n(a)\rho_0^{n+1}$, where*

$$c_n(a) = \sum_{i_1+\cdots+i_k=n} c_{i_1\dots i_k} I_{i_1\dots i_k}(a), \quad \text{and}$$

$$c_{i_1\dots i_k} = (n-i_1+1)\cdot(n-i_1-i_2+1)\cdot(n-i_1-i_2-i_3+1)\cdots 1.$$

By Theorem 1, the center set $\mathcal{C}$ of the differential equation (1) is determined by the system of polynomial equations $c_n(a)=0$, for $n=1,2,\dots$.

The following definition is given in [13]: The differential equation (1) has a *universal center* if for all positive integers $i_1,\dots,i_k$ with $k\ge 1$ the iterated integral $I_{i_1\dots i_k}(a)=0$.

The set of universal centers will be denoted by $\mathcal{U}$. It is known that, in general, $\mathcal{U}\neq\mathcal{C}$, see for instance Proposition 9.

We say that differential equation (1) satisfies the *composition conditions* if there is a trigonometric polynomial q and there are polynomials $p_i\in\mathbb{R}[z]$, for $i\ge 1$ such that

$$\tilde{a}_i = p_i\circ q, \quad i\ge 1, \quad \text{where} \quad \tilde{a}_i(\theta) = \int_0^\theta a_i(s)ds. \tag{3}$$

The first time that such conditions appear in the literature was in the paper of Alwash and Lloyd [5]. The composition conditions are studied in several papers, see for instance [1, 3, 4, 5, 8, 9, 10, 11].

Now we consider the differential equation (1) with a finite number of terms, i.e.

$$\frac{d\rho}{d\theta} = \sum_{i=1}^{n} a_i(\theta)\rho^{i+1}. \tag{4}$$

Brudnyi in Corollary 1.19 of [14] proved that equation (4) with all a_i trigonometric polynomials has a universal center if and only if it satisfies the composition conditions (3) for $1 \leq i \leq n$. Therefore the universality condition for a center of (4) is equivalent to composition conditions (3) for all a_i with $i = 1, 2, \ldots, n$. In this paper we generalize this result to the differential equation (1), i.e. note that now the equation has infinitely many a_i's. In short the result of [14] for polynomial differential equations in ρ as (4) is extended to analytic differential equations in ρ as (1). In particular this extension allow us to complete the characterization of the universal quadratic centers in the plane, see Proposition 9.

One of the main results of this paper is:

Theorem 2. *Any center of the differential equation* (1) *is universal if and only if equation* (1) *satisfies the composition condition.*

Theorem 2 is proved in subsection 2.1.

A related result with Theorem 2 was given by Brudnyi in section 1.8 of [14] providing sufficient conditions in order that if equation (1) restricted to the first n terms has a universal center, then equation (1) has also a universal center.

In what follows we present two big families of universal centers.

Given an angle $\alpha \in [0, \pi)$, we say that the differential equation (1) is *α-symmetric* if its flow is symmetric with respect to the straight line $\theta = \alpha$. Obviously, this is equivalent to that equation (1) is invariant under the change of variables $\theta \mapsto 2\alpha - \theta$. Any differential equation (1) which is α-symmetric has a center, due to the symmetry.

We say that the differential equation (1) is of *separable variables* if the function on the right-hand side of equation (1) splits as product of two functions of one variable, one depending on ρ and the other on θ, that is, $\frac{d\rho}{d\theta} = a(\theta)\, b(\rho)$. In such a case there is only one center condition which is $\int_0^{2\pi} a(\theta)\, d\theta = 0$.

Theorem 3. *If the differential equation* (1) *has a center which is either α-symmetric, or of separable variables, then it is universal.*

Theorem 3 is proved in subsection 2.3.

Another family of universal centers is the following one.

Proposition 4. *Let m and n be positive integers with $m \geq 2$, and let $f(\theta)$ and $g(\theta)$ be trigonometric polynomials. The differential equation*

$$\frac{d\rho}{d\theta} = \frac{f(\theta)\rho^m}{1 + g(\theta)\rho^n} \tag{5}$$

has a universal center if and only if there exists a nonconstant trigonometric polynomial $q(\theta)$ and two polynomials $p_0, p_1 \in \mathbb{R}[z]$ such that

$$\int_0^\theta f(t)\, dt = p_0(q(\theta)), \quad \int_0^\theta f(t)g(t)\, dt = p_1(q(\theta)).$$

Proposition 4 is proved in subsection 2.3.

As we will show in the proof of the next result there is a change of variable passing the differential equation (6) to the differential Abel equation (7), but the existence of a universal center is not equivalent in both equations. Therefore the notion of universal center is not invariant under changes of variables.

Theorem 5. *Let $f(\theta)$ and $g(\theta)$ be trigonometric polynomials and $s \geq 2$ be an integer.*

(a) *If the differential equation*

$$\frac{dr}{d\theta} = \frac{f(\theta)r^s}{1+g(\theta)r^{s-1}} \tag{6}$$

has a universal center, the corresponding Abel equation

$$\frac{d\rho}{d\theta} = \big((s-1)f(\theta) - g'(\theta)\big)\,\rho^2 - (s-1)f(\theta)g(\theta)\rho^3 \tag{7}$$

has also a universal center.

(b) *There are Abel equations* (7) *having a universal center but the center of the corresponding equation* (6) *is not universal.*

Theorem 5 is proved in subsection 3.1.

In the particular case of the differential equation (1) with only two terms, i.e. the Abel equation

$$\frac{d\rho}{d\theta} = a_1(\theta)\rho^2 + a_2(\theta)\rho^3, \tag{8}$$

where $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials, we have the following relation between centers of equation (8) and the composition condition. This relation was studied in [7] where it is proved the following result.

Proposition 6. *All centers of equation* (8) *when $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials of degree* 1 *and* 2 *are universal, and consequently they satisfy the composition condition.*

Let $p \in \mathbb{R}^2$ be a singular point of an analytic differential system in $\mathbb{R}^2$, and assume that p is a center. Recall that p is a *center* if there exists a neighborhood U of p such that all the orbits of $U \setminus \{p\}$ are periodic. Without loss of generality we can assume that p is at the origin of coordinates. Then after a linear change of variables and a constant rescaling of the time variable, if necessary, the system can be written in one of the following three forms:

$$\dot{x} = -y + P(x,y), \quad \dot{y} = x + Q(x,y); \tag{9}$$

$$\dot{x} = y + P(x,y), \quad \dot{y} = Q(x,y); \tag{10}$$

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y); \tag{11}$$

where $P(x,y)$ and $Q(x,y)$ are real analytic functions without constant and linear terms defined in a neighborhood of the origin. In what follows a center of an analytic differential system in $\mathbb{R}^2$ is called *non-degenerate*, *nilpotent* or *degenerate* if after an affine change of variables and a constant rescaling of the time it can be written as system (9), (10) or (11), respectively.

The classical non-degenerate center problem arises from the study of the planar analytic differential systems first studied by Poincaré [28] and later by Liapunov [27] and other authors, see [6, 20, 22, 25, 26]. In the case of a non-degenerate singular point the system can be written into the form

$$\dot{x} = -y + \sum_{i=2}^{\infty} P_i(x, y), \quad \dot{y} = x + \sum_{i=2}^{\infty} Q_i(x, y), \tag{12}$$

where P_i and Q_i are homogeneous polynomials of degree i. Poincaré proved that the origin of system (12) is a center if and only if the coefficients of P and Q satisfy a certain infinite system of algebraic equations called the Poincaré-Liapunov constants. We note that taking polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ system (12) becomes

$$\dot{r} = \sum_{s=2}^{\infty} f_s(\theta) r^s, \quad \dot{\theta} = 1 + \sum_{s=2}^{\infty} g_s(\theta) r^{s-1}, \tag{13}$$

where

$$\begin{aligned} f_i(\theta) &= \cos\theta\, P_i(\cos\theta, \sin\theta) \,+\, \sin\theta\, Q_i(\cos\theta, \sin\theta), \\ g_i(\theta) &= \cos\theta\, Q_i(\cos\theta, \sin\theta) \,-\, \sin\theta\, P_i(\cos\theta, \sin\theta). \end{aligned}$$

We remark that f_i and g_i are homogeneous trigonometric polynomials of degree $\leq i+1$ in the variables $\cos\theta$ and $\sin\theta$. In the region $\mathcal{R} = \{(r, \theta) : \dot{\theta} > 0\}$ the differential system (13) is equivalent to the differential equation

$$\frac{dr}{d\theta} = \frac{\displaystyle\sum_{s=2}^{\infty} f_s(\theta) r^s}{1 + \displaystyle\sum_{s=2}^{\infty} g_s(\theta) r^{s-1}}. \tag{14}$$

We can expand the right-hand side of (14) as an analytic series in r to obtain equation (1) whose coefficients are trigonometric polynomials. This reduces the center problem for the planar differential system (12) to the center problem for the class of differential equations (1). This reduction explains the importance of the study of the center problem for the differential equation (1).

We say that a *non-degenerate center* (9) *is universal* if its associated equation (14) has a universal center.

The first statement of the next result is well-known but we provide here a new proof. The second statement shows that any non-degenerate center after a convenient change of variables is universal.

Theorem 7. *The following statements hold.*

(a) *The differential system* (12) *has centers which are not universal, and consequently the differential equation* (1) *also has centers which are not universal.*
(b) *Any non-degenerate center after a convenient analytic change of variables becomes universal.*

Theorem 7 is proved in subsection 3.2.

We remark that the notion of universal center given only for non-degenerate centers can be extended to nilpotent centers transforming system (10) into a similar equation (14) but working with generalized Liapunov polar coordinates, see [27]. Then, in a similar way to the proof of statement (b) of Theorem 7 for the non-degenerate centers, we would prove that any nilpotent center after a convenient analytic change of variables (given by Strózyna and Żołądek [30]) and a non-constant (in general) time rescaling becomes universal.

The next result provides two families of non-degenerate universal centers.

The center at the origin of system (9) is *time–reversible* when its flow is symmetric with respect to a straight line passing through it.

Let $H = (x^2 + y^2)/2 + H_{s+1}(x, y)$ be a polynomial with $H_{s+1}(x, y)$ a homogeneous polynomial of degree $s + 1 \geq 3$. In what follows we deal with the Hamiltonian systems:

$$\dot{x} = -y - \frac{\partial H_{s+1}}{\partial y}, \quad \dot{y} = x + \frac{\partial H_{s+1}}{\partial x}. \tag{15}$$

Theorem 8. *The following statements hold.*

(a) *Any time–reversible system* (9) *has a universal center at the origin.*
(b) *Any Hamiltonian system* (15) *has a universal center at the origin.*

Theorem 8 is proved in section 4.

We note that Theorem 8 for $s = 2, 3$ was already proved by Blinov in [7].

The study of non-degenerate centers (9) has a long history starting at the beginning of the last century, see for instance [20, 22, 25, 26]. The first systems studied where the quadratic systems, i.e. systems (9) with homogeneous P and Q of degree 2. Such systems can be written in the Dulac-Kapteyn form with five parameters using an appropriate rotation of the plane:

$$\begin{array}{rcl} \dot{x} & = & -y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} & = & x + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2. \end{array} \tag{16}$$

It has been established by Dulac [20] that the centers of these systems are described by:

(a) Lotka-Volterra component: $\lambda_3 = \lambda_6$;
(b) Symmetric component: $\lambda_2 = \lambda_5 = 0$;
(c) Hamiltonian component: $\lambda_4 = \lambda_5 = 0$;
(d) Darboux component: $\lambda_5 = \lambda_4 + 5\lambda_3 - 5\lambda_6 = \lambda_3\lambda_6 - 2\lambda_6^2 - \lambda_2^2 = 0$.

In [14] the following universal quadratic centers were detected:

(i) A linear two-dimensional subspace in the Lotka-Volterra component: $\lambda_3 - \lambda_6 = 4\lambda_2 + \lambda_5 = 4\lambda_3 + \lambda_4 = 0$, and

(ii) symmetric component: $\lambda_2 = \lambda_5 = 0$.

In fact many other quadratic centers are universal as we state in the following proposition.

Proposition 9. *The quadratic system* (16) *has a universal center at the origin if and only if one of the following conditions holds.*

(a) *Lotka-Volterra component:* $\lambda_3 = \lambda_6$ *and* $\lambda_2\lambda_4(\lambda_4^2 - 3\lambda_5^2) + \lambda_5(\lambda_4^3 - \lambda_4\lambda_5^2 + 3\lambda_6\lambda_4^2 - \lambda_6\lambda_5^2) = 0$.

(b) *Symmetric component:* $\lambda_2 = \lambda_5 = 0$.

(c) *Hamiltonian component:* $\lambda_4 = \lambda_5 = 0$.

(d) *The Darboux component:* $\lambda_5 = \lambda_4 + 5\lambda_3 - 5\lambda_6 = \lambda_3\lambda_6 - 2\lambda_6^2 - \lambda_2^2 = 0$, *is universal only when* $\lambda_2 = 0$ *and, then the system belongs to the symmetric component.*

Proposition 9 is proved in section 4.

2. Universal centers of equation (1)

This section is divided into three subsections. In the first one we mainly do the proof of Theorem 2. In the second we characterize when the differential equation (1) satisfies the composition condition. In the last subsection we provide families of universal centers.

2.1. Proof of Theorem 2.

The proof of this result is based on several properties of iterated integrals that we summarize below.

From (3) we have

$$\tilde{a}_i(\theta) := \int_0^\theta a_i(t)\, dt \qquad \text{for } i \geq 1.$$

We recall that equation (1) satisfies the composition condition if there exist polynomials $p_i(z) \in \mathbb{R}[z]$ and a nonconstant trigonometric polynomial $q(\theta)$ such that

$$\tilde{a}_i(\theta) = p_i\left(q(\theta)\right), \qquad \text{for all } i \geq 1. \tag{17}$$

Given a k-vector of indexes $i_1 i_2 \ldots i_k$, we define

$$\tilde{I}_{i_1 i_2 \ldots i_k}(\theta) = \int \cdots \int_{0 \leq s_1 \leq \cdots \leq s_k \leq \theta} a_{i_k}(s_k) \cdots a_{i_1}(s_1)\, ds_k \cdots ds_1. \tag{18}$$

We remark that $\tilde{I}_{i_1 i_2 \ldots i_k}(2\pi) = I_{i_1 i_2 \ldots i_k}(a)$, as defined in (2). If we denote by $\vec{i} = i_1 i_2 \ldots i_k$, by the definition of (18), it is clear that

$$\tilde{I}_{\vec{i}j}(\theta) = \int_0^\theta \tilde{I}_{\vec{i}}(t)\, a_j(t)\, dt. \tag{19}$$

By convention we have that $\tilde{I}_{\emptyset}(\theta) = 1$.

The "if" part of Theorem 2 is proved by manipulation of the iterated integrals. However the "only if" part of Theorem 2 is based upon the algebraic structures of these iterated integrals. Moreover we introduce some notation for working with these iterated integrals.

We denote by $\mathbb{R}(\theta)$ the field of all rational functions quotient of two trigonometric polynomials with coefficients in $\mathbb{R}$. We remark that $\mathbb{R}(\theta)$ is a simple transcendental extension of $\mathbb{R}$ as it can be seen from the fact that we have a field isomorphism φ between $\mathbb{R}(\theta)$ and $\mathbb{R}(T)$, the field of quotients of polynomials with coefficients in $\mathbb{R}$ and independent variable T. We can define $\varphi : \mathbb{R}(\theta) \longrightarrow \mathbb{R}(T)$ as the field homomorphism such that $\varphi(\cos\theta) = (1-T^2)/(1+T^2)$ and $\varphi(\sin\theta) = 2T/(1+T^2)$. Any field homomorphism is injective and this φ is surjective because we have that $\varphi^{-1}(T) = \sin\theta/(1+\cos\theta)$.

The following lemma establishes that, given any index $i \geq 1$, the function $\tilde{a}_i(\theta)$ belongs to the field $\mathbb{R}(\theta)$ when equation (1) has a universal center.

Lemma 10. *If the differential equation* (1) *has a universal center, then* $\tilde{a}_i(\theta)$ *is a trigonometric polynomial for all* $i \geq 1$.

Proof. Given an index $i \geq 1$ we consider the expansion in Fourier series of the coefficient $a_i(\theta)$ of ρ^{i+1} in equation (1). We denote by d_i the degree of the trigonometric polynomial $a_i(\theta)$.

$$a_i(\theta) = \frac{c_{0i}}{2} + \sum_{\ell=1}^{d_i} \big(c_{\ell i}\cos(\ell\theta) + s_{\ell i}\sin(\ell\theta)\big),$$

where c_{0i}, $c_{\ell i}$, $s_{\ell i}$ are real numbers defined by

$$\begin{aligned} c_{0i} &= \frac{1}{\pi}\int_0^{2\pi} a_i(\theta)\, d\theta, \\ c_{\ell i} = \frac{1}{\pi}\int_0^{2\pi} a_i(\theta)\, \cos(\ell\theta)\, d\theta, &\quad s_{\ell i} = \frac{1}{\pi}\int_0^{2\pi} a_i(\theta)\, \sin(\ell\theta)\, d\theta, \end{aligned} \tag{20}$$

for $\ell \geq 1$. We have that

$$\begin{aligned} \tilde{a}_i(\theta) &= \int_0^\theta a_i(t)\, dt \\ &= \int_0^\theta \left(\frac{c_{0i}}{2} + \sum_{\ell=1}^{d_i} \big(c_{\ell i}\cos(\ell t) + s_{\ell i}\sin(\ell t)\big) \right) dt \\ &= \frac{c_{0i}}{2}\,\theta + \sum_{\ell=1}^{d_i} \left(\frac{c_{\ell i}}{\ell}\sin(\ell\theta) + \frac{s_{\ell i}}{\ell}(1-\cos(\ell\theta)) \right). \end{aligned}$$

Since $\tilde{a}_i(2\pi) = I_i(a)$ and (1) has a universal center, we have $\tilde{a}_i(2\pi) = 0$. Therefore we conclude that $c_{0i} = 0$ and $\tilde{a}_i(\theta)$ is a trigonometric polynomial. □

Given an equation (1) we denote by $\Sigma(a)$ the minimal field containing all the functions $\tilde{a}_i(\theta)$ and $\mathbb{R}$. We remark that $\Sigma(a)$ is the quotient field of the polynomial domain formed by all the linear combinations with coefficients in $\mathbb{R}$ of monomials of the form

$$\tilde{a}_{i_1}^{m_1}(\theta)\, \tilde{a}_{i_2}^{m_2}(\theta)\, \ldots \tilde{a}_{i_k}^{m_k}(\theta)\,, \tag{21}$$

where $i_j \geq 1$ and $m_j \geq 0$ for $j = 1, 2, \ldots, k$.

As a consequence of the Ree's formula [29], see also [16], we have that any of these monomials (21) can be written as a linear combination of iterated integrals of order $m_1 + m_2 + \ldots + m_k$. In our notation the Ree's formula establishes how to write the product of two iterated integrals $\tilde{I}_{\vec{i}}(\theta)$ and $\tilde{I}_{\vec{j}}(\theta)$ as a summation of all the iterated integrals indexed by the shuffle products of the indexes $\vec{i}$ and $\vec{j}$.

Lemma 11 (Ree's formula)**.** *Given two sets of indexes $\vec{i} = i_1 i_2 \ldots i_r$ and $\vec{j} = j_1 j_2 \ldots j_s$, the following property holds:*

$$\tilde{I}_{\vec{i}}(\theta)\, \tilde{I}_{\vec{j}}(\theta) \,=\, \sum_{\sigma} \tilde{I}_{\sigma(\vec{i},\vec{j})}(\theta),$$

where the sum runs over all $\sigma(\vec{i},\vec{j})$ of (r,s)-shuffles.

We recall that a (r,s)-shuffle is a permutation σ of $r+s$ letters with $\sigma^{-1}(1) < \sigma^{-1}(2) < \ldots < \sigma^{-1}(r)$ and $\sigma^{-1}(r+1) < \sigma^{-1}(r+2) < \ldots < \sigma^{-1}(r+s)$.

For instance, Ree's formula gives that

$$\begin{aligned}
&\tilde{I}_{i_1}\tilde{I}_{i_2} \,=\, \tilde{I}_{i_1 i_2} + \tilde{I}_{i_2 i_1},\\
&\tilde{I}_{i}^{m} \,=\, m!\, \tilde{I}_{ii\cdots^{(m)} i},\\
&\tilde{I}_{i_1}\tilde{I}_{i_2 i_3} \,=\, \tilde{I}_{i_1 i_2 i_3} + \tilde{I}_{i_2 i_1 i_3} + \tilde{I}_{i_2 i_3 i_1},\\
&\tilde{I}_{i_1 i_2}\tilde{I}_{i_3 i_4} \,=\, \tilde{I}_{i_1 i_2 i_3 i_4} + \tilde{I}_{i_1 i_3 i_2 i_4} + \tilde{I}_{i_1 i_3 i_4 i_2} + \tilde{I}_{i_3 i_1 i_2 i_4} + \tilde{I}_{i_3 i_1 i_4 i_2} + \tilde{I}_{i_3 i_4 i_1 i_2},
\end{aligned}$$

where we have omitted the dependence on θ for the sake of simplicity.

As a direct consequence of Lemma 11 we have the following result which is applied to the monomials of the form (21).

Lemma 12. *There exist nonnegative integer numbers n_j for $j = 1, 2, \ldots, L$, such that*

$$\tilde{a}_{i_1}^{m_1}(\theta)\, \tilde{a}_{i_2}^{m_2}(\theta)\, \ldots \tilde{a}_{i_k}^{m_k}(\theta) \,=\, \sum_{j=1}^{L} n_j\, \tilde{I}_{\sigma_j(\vec{i})}(\theta),$$

where σ_j runs over all the permutations of the vector

$$\vec{i} = i_1 i_1 \cdots^{(m_1)} i_1\, i_2 i_2 \cdots^{(m_2)} i_2\, \ldots\, i_k i_k \cdots^{(m_k)} i_k$$

and $L = (m_1 + m_2 + \ldots + m_k)!$

Given an equation (1) we recall that $\Sigma(a)$ denotes the minimal field containing all the functions $\tilde{a}_i(\theta)$ and $\mathbb{R}$. We consider two polynomials $A(\theta)$ and $B(\theta)$ of $\Sigma(a)$, that is two functions formed by linear combinations of monomials of the form (21) with coefficients in $\mathbb{R}$. The following result shows that, when equation (1) has a universal center, the property (19) for iterated integrals can be extended in some sense to the polynomials $A(\theta)$ and $B(\theta)$.

Lemma 13. *Consider two polynomials $A(\theta), B(\theta) \in \Sigma(a)$. If the differential equation* (1) *has a universal center, then*

$$\int_0^{2\pi} A(\theta)\, B'(\theta)\, d\theta \;=\; 0.$$

Proof. We will show this result assuming that $A(\theta)$ is a monomial of the form (21) and $B(\theta) = \tilde{a}_b(\theta)$, where b is an index with $b \geq 1$. Then the result follows from the linear properties of integration. Hence we consider $A(\theta) = \tilde{a}_{i_1}^{m_1}(\theta)\, \tilde{a}_{i_2}^{m_2}(\theta) \ldots \tilde{a}_{i_k}^{m_k}(\theta)$ and $B'(\theta) = a_b(\theta)$ and we have that

$$\int_0^{\theta} A(t)\, B'(t)\, dt \;=\; \int_0^{\theta} \tilde{a}_{i_1}^{m_1}(t)\, \tilde{a}_{i_2}^{m_2}(t)\, \ldots \tilde{a}_{i_k}^{m_k}(t)\, a_b(t)\, dt.$$

By Lemma 12 we have that

$$\tilde{a}_{i_1}^{m_1}(\theta)\, \tilde{a}_{i_2}^{m_2}(\theta) \ldots \tilde{a}_{i_k}^{m_k}(\theta) \;=\; \sum_{j=1}^{L} n_j\, \tilde{I}_{\sigma_j(\vec{i})}(\theta),$$

where σ_j runs over all the permutations of the vector

$$\vec{i} = i_1 i_1 \cdots^{(m_1)} i_1\; i_2 i_2 \cdots^{(m_2)} i_2\; \ldots\; i_k i_k \cdots^{(m_k)} i_k$$

and $L = (m_1 + m_2 + \ldots m_k)!$ Thus we have

$$\int_0^{\theta} A(t)\, B'(t)\, dt \;=\; \sum_{j=1}^{L} n_j \int_0^{\theta} \tilde{I}_{\sigma_j(\vec{i})}(t)\, a_b(t)\, dt.$$

We consider each one of the integrals in the summation. By the relation (19) we get that

$$\int_0^{\theta} \tilde{I}_{\sigma_j(\vec{i})}(t)\, a_b(t)\, dt \;=\; \tilde{I}_{\sigma_j(\vec{i})b}(\theta).$$

Since equation (1) has a universal center, we have that $\tilde{I}_{\sigma_j(\vec{i})b}(2\pi) = 0$. Therefore

$$\int_0^{2\pi} A(t)\, B'(t)\, dt \;=\; \sum_{j=1}^{L} n_j \cdot 0 \;=\; 0.$$

□

We need the previous lemmas to show that $\Sigma(a)$ is an intermediate field between $\mathbb{R}$ and $\mathbb{R}(\theta)$. Then we will be able to apply Lüroth's Theorem in the proof of Theorem 2. For a proof of Lüroth's Theorem see [31] (page 218). We have adapted the statement of Lüroth's Theorem to our purposes.

Theorem 14 (Lüroth's Theorem)**.** *Every intermediate field $\Sigma(a)$ with $\mathbb{R} \subsetneq \Sigma(a) \subsetneq \mathbb{R}(\theta)$ is a simple transcendental extension, that is, $\Sigma(a) = \mathbb{R}(q(\theta))$, where $q(\theta)$ is a nonconstant quotient of trigonometric polynomials.*

We first state a couple of lemmas related with the degree of a trigonometric polynomial. We recall that the degree d of a trigonometric polynomial is the maximum positive integer such that either the monomial $\cos(d\theta)$ or $\sin(d\theta)$ appear in its Fourier development. Let $A(\theta)$ be a trigonometric polynomial whose Fourier development is

$$A(\theta) = \frac{c_0}{2} + \sum_{\ell=1}^{d} \big(c_\ell \cos(\ell\theta) + s_\ell \sin(\ell\theta)\big).$$

The degree of A is the maximum positive integer such that $c_d^2 + s_d^2 \neq 0$. Then the polynomial $\cos^2\theta + \sin^2\theta$ is of degree 0.

Lemma 15. *Let $A(\theta)$ and $B(\theta)$ be two trigonometric polynomials of degrees d and $\bar{d}$, respectively. The following statements hold.*

(a) *The trigonometric polynomial $A'(\theta)$ is of degree d.*
(b) *The trigonometric polynomial $A(\theta)B(\theta)$ is of degree $d+\bar{d}$.*

Proof. The Fourier development of $A'(\theta)$ is

$$A'(\theta) = \sum_{\ell=1}^{d} \big(-\ell c_\ell \sin(\ell\theta) + \ell s_\ell \cos(\ell\theta)\big),$$

which is also of degree d.

The Fourier development of $B(\theta)$ is

$$B(\theta) == \frac{\bar{c}_0}{2} + \sum_{\ell=1}^{\bar{d}} \big(\bar{c}_\ell \cos(\ell\theta) + \bar{s}_\ell \sin(\ell\theta)\big).$$

Thus, the product of $A(\theta)$ and $B(\theta)$ gives:

$$\begin{aligned} A(\theta)B(\theta) &= (c_d\cos(d\theta) + s_d\sin(d\theta))\big(\bar{c}_{\bar{d}}\cos(\bar{d}\theta) + \bar{s}_{\bar{d}}\sin(\bar{d}\theta)\big) + \cdots \\ &= \frac{(c_d\bar{c}_{\bar{d}} - s_d\bar{s}_{\bar{d}})\cos((d+\bar{d})\theta) + (c_d\bar{s}_{\bar{d}} + s_d\bar{c}_{\bar{d}})\sin((d+\bar{d})\theta)}{2} + \cdots \end{aligned}$$

where the dots denote terms of lower degree. We remark that

$$c_d\bar{c}_{\bar{d}} - s_d\bar{s}_{\bar{d}} = 0 \quad \text{and} \quad c_d\bar{s}_{\bar{d}} + s_d\bar{c}_{\bar{d}} = 0$$

if and only if the following linear system of equations is verified:

$$\begin{pmatrix} \bar{c}_{\bar{d}} & -\bar{s}_{\bar{d}} \\ \bar{s}_{\bar{d}} & \bar{c}_{\bar{d}} \end{pmatrix} \begin{pmatrix} c_d \\ s_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of this system is $\bar{c}_{\bar{d}}^2 + \bar{s}_{\bar{d}}^2$ which is different from 0 by hypothesis. Hence, the system has only one solution which needs to be $c_d = s_d = 0$, which is a contradiction with $A(\theta)$ being of degree d. We conclude that the product of the two polynomials is of degree $d+\bar{d}$. □

The thesis of Lüroth's Theorem 14 is about fields generated by a nonconstant quotient of trigonometric polynomials. Let $q(\theta) = N(\theta)/D(\theta)$ this generator, where $N(\theta)$ and $D(\theta)$ are trigonometric polynomials which we can assume to be coprime. The following lemma shows that if the field $\mathbb{R}(q(\theta))$ contains a trigonometric polynomial, then there is a nontrivial algebraic relationship between $N(\theta)$ and $D(\theta)$.

Lemma 16. *Let $q(\theta) = N(\theta)/D(\theta)$ with $N(\theta)$ and $D(\theta)$ trigonometric polynomials which are coprime. Let $\tilde{a}(\theta)$ be a nonconstant trigonometric polynomial with $\tilde{a}(\theta) \in \mathbb{R}(q(\theta))$. Then, either $D(\theta)$ is a nonzero constant or there exists a homogeneous real polynomial $R(X,Y) \in \mathbb{R}[X,Y]$ of degree $k \geq 1$ such that $R(N(\theta), D(\theta)) \equiv 1$.*

Proof. If $D(\theta)$ is a nonzero constant, then the thesis holds. We assume that $D(\theta)$ is nonconstant for the rest of the proof. Since $\tilde{a}(\theta) \in \mathbb{R}(q(\theta))$, there exist two real polynomials $s(z), r(z) \in \mathbb{R}[z]$ of degree k_1 and k_2 respectively, which we can assume to be coprime and such that

$$\tilde{a}(\theta) = \left.\frac{s(z)}{r(z)}\right|_{z=q(\theta)} = \frac{s\left(N(\theta)/D(\theta)\right)}{r\left(N(\theta)/D(\theta)\right)}.$$

We define $S(X,Y)$ and $R(X,Y)$ homogeneous polynomials in $\mathbb{R}[X,Y]$ of degree k_1 and k_2 respectively such that

$$\begin{aligned} s\left(N(\theta)/D(\theta)\right) &= S(N(\theta), D(\theta))/D(\theta)^{k_1}, \\ r\left(N(\theta)/D(\theta)\right) &= R(N(\theta), D(\theta))/D(\theta)^{k_2}. \end{aligned}$$

Therefore,

$$\tilde{a}(\theta) = \frac{D(\theta)^{k_2-k_1} S(N(\theta), D(\theta))}{R(N(\theta), D(\theta))}.$$

Since $\tilde{a}(\theta)$ is a trigonometric polynomial and s and r are coprime, we obtain that $k_2 \geq k_1$ and that $R(N(\theta), D(\theta)))$ is a constant, that we can choose without loss of generality equal to 1. We get that $k_2 \geq 1$ because $\tilde{a}(\theta)$ is nonconstant. □

We have that any Moëbius transformation of $q(\theta)$ generates the same field. That is, if we consider

$$\eta(\theta) = \frac{\mu_{11}\, q(\theta) + \mu_{12}}{\mu_{21}\, q(\theta) + \mu_{22}} = \frac{\mu_{11} N(\theta) + \mu_{12} D(\theta)}{\mu_{21} N(\theta) + \mu_{22} D(\theta)},$$

where μ_{ij} are real numbers with $\mu_{11}\mu_{22} - \mu_{12}\mu_{21} \neq 0$, we have that $\mathbb{R}(q(\theta)) = \mathbb{R}(\eta(\theta))$. The following proposition shows that when the field $\mathbb{R}(q(\theta))$ contains a nonconstant trigonometric polynomial then we can choose, by a Moëbius transformation, a generator which is either a trigonometric polynomial or the function $\tan(m\theta)$ with m a positive integer.

Proposition 17. *Let $\tilde{a}(\theta)$ be a nonconstant trigonometric polynomial with $\tilde{a}(\theta) \in \mathbb{R}(q(\theta))$. Then, either there exists a nonconstant trigonometric polynomial $p(\theta)$ such that $\mathbb{R}(q(\theta)) = \mathbb{R}(p(\theta))$, or $\mathbb{R}(q(\theta)) = \mathbb{R}(\tan(m\theta))$ with m a positive integer.*

Proof. By Lemma 16 if $D(\theta)$ is constant, we are done because $q(\theta)$ is a trigonometric polynomial. We assume that $D(\theta)$ is not a constant for the rest of the proof. By Lemma 16 again, there exists a homogeneous polynomial $R(X,Y)$ such that $R(N(\theta),D(\theta)) \equiv 1$. We can assume that this polynomial $R(X,Y) \in \mathbb{R}[X,Y]$ is one of the lowest degree k satisfying this property. Since $R(X,Y)$ is a homogeneous polynomial of degree k, with $k \geq 1$, by Euler's formula for homogeneous polynomials we have that

$$XR_X(X,Y) + YR_Y(X,Y) \;=\; kR(X,Y), \tag{22}$$

where the subindexes denote partial derivatives.

We assume that there exists a value θ^* such that $R_X(N(\theta^*),D(\theta^*)) = R_Y(N(\theta^*),D(\theta^*)) = 0$. By formula (22) we would have that

$$N(\theta^*)R_X(N(\theta^*),D(\theta^*)) + D(\theta^*)R_Y(N(\theta^*),D(\theta^*)) \;=\; kR(N(\theta^*),D(\theta^*)).$$

Since $R(N(\theta^*),D(\theta^*)) = 1$ and $k \geq 1$, we get a contradiction. Therefore, there is no value θ^* such that $R_X(N(\theta^*),D(\theta^*)) = R_Y(N(\theta^*),D(\theta^*)) = 0$.

We derive the relationship $R(N(\theta),D(\theta)) \equiv 1$ with respect to θ and we obtain

$$R_X(N(\theta),D(\theta))N'(\theta) + R_Y(N(\theta),D(\theta))D'(\theta) \;\equiv\; 0.$$

By this relation and by the previous paragraph, we have that all the zeroes of $R_X(N(\theta),D(\theta))$ need to be zeroes of $D'(\theta)$ and that all the zeroes of $R_Y(N(\theta),D(\theta))$ need to be zeroes of $N'(\theta)$. Indeed, we have that there exists a trigonometric polynomial $\bar{p}(\theta)$ such that

$$N'(\theta) \;=\; \bar{p}(\theta)\, R_Y(N(\theta),D(\theta)), \quad D'(\theta) \;=\; -\bar{p}(\theta)\, R_X(N(\theta),D(\theta)). \tag{23}$$

We denote by d_N and by d_D the degrees of the polynomials $N(\theta)$ and $D(\theta)$, respectively. By a Moëbius transformation, if necessary, we can assume that $d_N \geq d_D$, because $\mathbb{R}\,(N(\theta)/D(\theta)) = \mathbb{R}\,(D(\theta)/N(\theta))$. By the second of the relations in (23) and by Lemma 15 we have the following relation among the degrees

$$d_D \;=\; \deg(\bar{p}) \,+\, d_N(k-1). \tag{24}$$

We consider three cases: $k=1$, $k=2$ and $k>2$.

Case $k = 1$. We have that $d_D = \deg(\bar{p})$ and that $R(X,Y)$ is a homogeneous polynomial of degree 1. Let α and β be real constants with $R(X,Y) = \alpha X + \beta Y$. The relations (23) become

$$N'(\theta) \;=\; \bar{p}(\theta)\cdot\beta, \qquad D'(\theta) \;=\; -\bar{p}(\theta)\cdot\alpha.$$

We denote by $p(\theta)$ a trigonometric polynomial with $p'(\theta) = \bar{p}(\theta)$. We have that $p(\theta)$ is periodic because $\bar{p}(\theta)$ is already the derivative of a trigonometric polynomial (either $N'(\theta)$ or $D'(\theta)$). Hence, we get that

$$N(\theta) \;=\; \beta p(\theta) + \gamma, \qquad D(\theta) \;=\; -\alpha p(\theta) + \delta,$$

where α, β, γ and δ are real numbers with $\beta\delta + \alpha\gamma \neq 0$. If this value was zero, we would have that the quotient $N(\theta)$ over $D(\theta)$ would be a constant. Thus,

$$q(\theta) = \frac{\beta p(\theta) + \gamma}{-\alpha p(\theta) + \delta}.$$

We conclude that, by a Moëbius transformation $\mathbb{R}(q(\theta)) = \mathbb{R}(p(\theta))$ where $p(\theta)$ is a trigonometric polynomial.

Case $k = 2$. By relation (24) we have that $d_D = \deg(\bar{p}) + d_N$. We recall that $d_N \geq d_D$ by assumption. Thus, we have that $\deg(\bar{p}) = 0$, and hence $\bar{p}(\theta) = c$ constant, and $d_D = d_N$. The relations (23) become

$$N'(\theta) = cR_Y(N(\theta), D(\theta)), \qquad D'(\theta) = -cR_X(N(\theta), D(\theta)).$$

We see that

$$\begin{aligned} & N'(\theta)D(\theta) - N(\theta)D'(\theta) \\ & \quad = c\,(R_Y(N(\theta), D(\theta))D(\theta) + N(\theta)R_X(N(\theta), D(\theta))) \\ & \quad = c \cdot 2 \cdot R(N(\theta), D(\theta)) = 2c, \end{aligned}$$

where we have used Euler's formula for homogeneous polynomials (22) (with $k = 2$) and that $R(N(\theta), D(\theta)) \equiv 1$. Hence,

$$q'(\theta) = \frac{N'(\theta)D(\theta) - N(\theta)D'(\theta)}{D(\theta)^2} = \frac{2c}{D(\theta)^2}.$$

Since $R(X, Y)$ is a homogeneous polynomial of degree k, we have that $R(X, Y)/Y^k = \bar{R}(X/Y)$ with $\tilde{R}(z)$ a polynomial in $\mathbb{R}[z]$ of degree k. As we are in the case $k = 2$, we have that $\bar{R}(q) = \bar{b}_0 + \bar{b}_1 q + \bar{b}_2 q^2$ with $\bar{b}_0$, $\bar{b}_1$ and $\bar{b}_2$ real numbers. We note that if $\bar{b}_1^2 - 4\bar{b}_0\bar{b}_2 \geq 0$ then $R(X, Y)$ factorizes in two linear polynomials with real coefficients. Thus, there would be a product of two linear polynomials in $N(\theta)$ and $D(\theta)$ equivalently equal to 1, which is a contradiction with Lemma 15. Therefore, we have that

$$\frac{1}{D(\theta)^2} = \frac{R(N(\theta), D(\theta))}{D(\theta)^2} = \bar{R}(q(\theta)) = \bar{b}_0 + \bar{b}_1 q(\theta) + \bar{b}_2 q(\theta)^2.$$

Consequently, $q(\theta)$ satisfies the following Riccati equation with constant coefficients:

$$q'(\theta) = b_0 + b_1 q(\theta) + b_2 q(\theta)^2,$$

where $b_i = 2c\bar{b}_i$ are real numbers, $i = 0, 1, 2$. We also have that $b_1^2 - 4b_0 b_2 < 0$ which implies that b_0 and b_2 are both different from zero. We denote by $\Delta^2 := -b_1^2 + 4b_0 b_2$ and the solution of the above Riccati equation is

$$q(\theta) = \frac{(\Delta + b_1 C)\tan(\Delta\theta/2) + (\Delta C - b_1)}{-2b_2 C\tan(\Delta\theta/2) + 2b_2},$$

where C is any real constant. We observe that since $q(\theta)$ is a rational trigonometric function, we have that $\Delta = 2m$ with $m \geq 1$ a positive integer. Hence, by a Moëbius transformation, we deduce that $\mathbb{R}(q(\theta)) = \mathbb{R}(\tan(m\theta))$.

Case $k > 2$. By relation (24) we have that $d_D = \deg(\bar{p}) + d_N(k-1)$ and that $d_N \geq d_D$ by assumption. These two facts imply a contradiction because $\deg(\bar{p}) \geq 0$. □

Proof of Theorem 2. We first show the "if" part of the statement. We assume that equation (1) satisfies the composition condition (17) and we will show that (1) has a universal center. We take an iterated integral $\tilde{I}_{\vec{i}}(\theta)$ of order k, and doing induction on k we shall prove that there exists a polynomial $P_{\vec{i}}(z) \in \mathbb{R}[z]$ such that $\tilde{I}_{\vec{i}}(\theta) = P_{\vec{i}}(q(\theta))$ and $P_{\vec{i}}(q(0)) = 0$.

When the order is $k = 1$, given any index $i \geq 1$, we have that

$$\tilde{I}_i(\theta) = \int_0^\theta a_i(t)\, dt = \tilde{a}_i(\theta).$$

Thus, since the equation satisfies the composition condition, there exists a polynomial $p_i(z) \in \mathbb{R}[z]$ such that $\tilde{I}_i(\theta) = p_i(q(\theta))$. Note that $p_i(q(0)) = 0$.

We assume that the statement holds for order k and we want to show that it holds for order $k+1$. We take $\vec{i} = i_1 i_2 \ldots i_k$ and by induction hypothesis we have that there exists a polynomial $P_{\vec{i}}(z) \in \mathbb{R}[z]$ such that $I_{\vec{i}}(\theta) = P_{\vec{i}}(q(\theta))$. We consider any index $j \geq 1$ and by (19) we have that

$$\tilde{I}_{\vec{i}j}(\theta) = \int_0^\theta \tilde{I}_{\vec{i}}(t)\, a_j(t)\, dt.$$

Since $\tilde{a}_j'(\theta) = a_j(\theta)$ and by the composition condition there exists a polynomial $p_j(z) \in \mathbb{R}[z]$ such that $\tilde{a}_j(\theta) = p_j(q(\theta))$, we get that

$$\tilde{I}_{\vec{i}j}(\theta) = \int_0^\theta P_{\vec{i}}(q(t))\, p_j'(q(t))\, q'(t)\, dt = P_{\vec{i}j}(q(\theta)) - P_{\vec{i}j}(q(0)),$$

where $P_{\vec{i}j}(z)$ is a polynomial and a primitive of the polynomial $P_{\vec{i}}(z)p_j'(z)$, that is, $P_{\vec{i}j}'(z) = P_{\vec{i}}(z)p_j'(z)$. Without loss of generality we can assume that $P_{\vec{i}j}(z)$ satisfies $P_{\vec{i},j}(q(0)) = 0$.

The equation (1) has a universal center if any iterated integral $I_{\vec{i}}(a) = 0$. Given any $\vec{i}$ we have proved that there exists a polynomial $P_{\vec{i}}(z) \in \mathbb{R}[z]$ such that $\tilde{I}_{\vec{i}}(\theta) = P_{\vec{i}}(q(\theta))$ and $P_{\vec{i}}(q(0)) = 0$. Thus

$$I_{\vec{i}}(a) = \tilde{I}_{\vec{i}}(2\pi) = P_{\vec{i}}(q(2\pi)).$$

Since $q(\theta)$ is a trigonometric polynomial, we have that $q(2\pi) = q(0)$ and, since $P_{\vec{i}}(q(0)) = 0$, we obtain that $I_{\vec{i}}(a) = 0$.

Now, we prove the "only if" part of the statement. We want to show that when (1) has a universal center, then equation (1) satisfies the composition condition. If equation (1) has a trivial center, that is, $a_i(\theta) \equiv 0$ for all i, then the composition condition trivially holds taking any polynomial $q(\theta)$ and all the $p_i \equiv 0$. We assume that equation (1) has a nontrivial universal center from now on. As before we denote by $\Sigma(a)$ the minimal field containing all the functions $\tilde{a}_i(\theta)$ and $\mathbb{R}$. In order to apply Theorem 14, we need to show that $\Sigma(a)$ is an intermediate field between $\mathbb{R}$ and $\mathbb{R}(\theta)$. By definition

$\mathbb{R} \subset \Sigma(a)$, but if these two fields were equal we would have that all the functions $\tilde{a}_i(\theta)$ would be constant and, since $\tilde{a}_i'(\theta) = a_i(\theta)$, we would have a trivial universal center. We have already discarded this possibility. Thus we have that $\mathbb{R} \subsetneq \Sigma(a)$.

By Lemma 10 we have that $\Sigma(a) \subset \mathbb{R}(\theta)$. If these two fields were equal, then we would have that the polynomials $A(\theta) = \sin\theta$ and $B(\theta) = \cos\theta$ belong to $\Sigma(a)$. We consider the integral described in Lemma 13. We have that

$$\int_0^{2\pi} A(\theta)\, B'(\theta)\, d\theta \;=\; -\int_0^{2\pi} \sin^2\theta\, d\theta \;=\; -\pi,$$

in contradiction with the thesis of Lemma 13 (this integral should be 0). Therefore we deduce that $\Sigma(a) \subsetneq \mathbb{R}(\theta)$.

Applying Theorem 14, we have that there exists a nonconstant quotient of trigonometric polynomials $q(\theta)$ such that $\Sigma(a) = \mathbb{R}\,(q(\theta))$. Since $\Sigma(a)$ contains at least one $\tilde{a}_i$ which is a non-constant trigonometric polynomial, by Proposition 17 we have that either there exists a nonconstant trigonometric polynomial $p(\theta)$ such that $\Sigma(a) = \mathbb{R}(p(\theta))$ or $\Sigma(a) = \mathbb{R}(\tan(m\theta))$, with $m \geq 1$ an integer. To discard this second possibility, we remark that if $\tilde{a}(\theta)$ is a nonconstant trigonometric polynomial and $\tilde{a}(\theta) \in \mathbb{R}(\tan(m\theta))$, then $\tilde{a}'(\theta)$ also belongs to $\mathbb{R}(\tan(m\theta))$. This fact comes from the derivative of $\tan(m\theta)$ with respect to θ which is $m(1+\tan(m\theta)^2)$. Since $\tilde{a}(\theta) \in \mathbb{R}(\tan(m\theta))$ there exists a rational function $s(z)/r(z) \in \mathbb{R}(z)$ such that

$$\tilde{a}(\theta) \;=\; \left.\frac{s(z)}{r(z)}\right|_{z=\tan(m\theta)}.$$

We see that

$$\tilde{a}'(\theta) \;=\; \left.\frac{d}{dz}\left(\frac{s(z)}{r(z)}\right)\right|_{z=\tan(m\theta)} m(1+\tan(m\theta)^2) \in \mathbb{R}(\tan(m\theta)).$$

We consider Lemma 13 and we take $A(\theta) = \tilde{a}'(\theta)$ and $B(\theta) = \tilde{a}(\theta)$. We have that

$$\int_0^{2\pi} A(\theta)B'(\theta)d\theta \;=\; \int_0^{2\pi} \left(\tilde{a}'(\theta)\right)^2 d\theta \neq 0,$$

in contradiction with Lemma 13. In conclusion, $\Sigma(a) = \mathbb{R}(p(\theta))$ with $p(\theta)$ a trigonometric polynomial. □

The following statement is a consequence of Theorem 2.

Lemma 18. *If the differential equation* (1)

$$\frac{d\rho}{d\theta} \;=\; \sum_{i\geq 1} a_i(\theta)\rho^{i+1},$$

where $a_i(\theta)$ are trigonometric polynomials for all i has a universal center, then the differential equation

$$\frac{d\rho}{d\theta} = \sum_{i\geq 1} \lambda_i \, a_i(\theta)\rho^{i+1}, \tag{25}$$

has a universal center for arbitrary real numbers λ_i.

Proof. If equation (1) has a universal center, by Theorem 2 there exists a nonconstant trigonometric polynomial $q(\theta)$ and polynomials $p_i(z) \in \mathbb{R}[z]$ for all i such that $\tilde{a}_i(\theta) = p_i(q(\theta))$. Hence for any value of λ_i we have that

$$\int_0^\theta \lambda_i a_i(t)dt \;=\; \lambda_i \tilde{a}_i(\theta) \;=\; \lambda_i p_i(q(\theta)).$$

So equation (25) has a universal center. □

The centers of Lemma 18 when all λ_i's are equal are related with the persistent centers defined in [17].

2.2. **The composition condition in equation (1).** In this subsection we consider a differential system of the form

$$\frac{d\rho}{d\theta} = \mathcal{F}(\rho,\theta), \tag{26}$$

where $\mathcal{F}(\rho,\theta)$ is an analytic function in a neighborhood of $\rho = 0$ which takes the form (1) when it is expanded in power series of ρ. Now we are interested in translating the composition condition in terms of the function $\mathcal{F}(\rho,\theta)$. We define $G(\rho,\theta) := \displaystyle\int_0^\theta \mathcal{F}(\rho,\xi)\, d\xi$.

Proposition 19. *The differential equation* (26) *satisfies the composition condition if and only if the function $G(\rho,\theta)$ can be written as $G(\rho,\theta) = A(\rho, q(\theta))$, where $q(\theta)$ is a trigonometric polynomial and $A(\rho, z)$ is analytic in ρ in a neighborhood of $\rho = 0$ and the coefficient of each power of ρ is a polynomial in z.*

Proof. From definition of $G(\rho,\theta)$ we have

$$\begin{aligned} G(\rho,\theta) &= \int_0^\theta \mathcal{F}(\rho,\xi)\, d\xi = \int_0^\theta \left(\sum_{i\geq 1} a_i(\xi)\rho^{i+1} \right) d\xi \\ &= \sum_{i\geq 1} \left(\int_0^\theta a_i(\xi)d\xi \right) \rho^{i+1} = \sum_{i\geq 1} \tilde{a}_i(\theta)\rho^{i+1}, \end{aligned}$$

where we have interchanged the integral sign by the summation sign which is possible because $\mathcal{F}(\rho,\xi)$ is an analytic function, and consequently uniformly convergent. If we take into account the composition condition $\tilde{a}_i(\theta) =$

$p_i(q(\theta))$ where $p_i \in \mathbb{R}[z]$ for all i and q is a trigonometric polynomial, we have

$$G(\rho,\theta) = \sum_{i\geq 1} p_i(q(\theta))\rho^{i+1} = A(\rho, q(\theta)),$$

where $A(\rho, z)$ is an analytic function in a neighborhood of $\rho = 0$ and the coefficients of the powers of ρ are polynomials in z. □

2.3. **Families of centers which are universal centers.** In this subsection we will see that the α-symmetric centers and the separable variables centers are universal.

Proof of Theorem 3. We first consider the α-symmetric case. We have that equation (1) is α-symmetric and by the change $\theta \to \theta - \alpha$ we can assume, without loss of generality, that the equation is α-symmetric with $\alpha = 0$. Since the equation is now invariant under the change $\theta \to -\theta$ we deduce that $a_i(-\theta) = -a_i(\theta)$ for all $i \geq 1$. We recall that, given an index $i \geq 1$, $a_i(\theta)$ is the coefficient of ρ^{i+1} in the right-hand side of equation (1). We denote by d_i the degree of the trigonometric polynomial $a_i(\theta)$ and we consider the expansion in Fourier series of $a_i(\theta)$:

$$a_i(\theta) \; = \; \frac{c_{0i}}{2} \; + \; \sum_{\ell=1}^{d_i} c_{\ell i} \cos(\ell\theta) \; + \; s_{\ell i} \sin(\ell\theta),$$

where the coefficients are defined in (20). Since $a_i(-\theta) = -a_i(\theta)$ for all $\theta \in \mathbb{R}$, we have that $a_i(-\theta) + a_i(\theta) \equiv 0$ and, hence

$$a_i(-\theta) \, + \, a_i(\theta) \; = \; c_{0i} \, + \, \sum_{\ell=1}^{d_i} 2c_{\ell i} \cos(\ell\theta) \; \equiv \; 0.$$

We deduce that $c_{0i} = 0$ and $c_{\ell i} = 0$ for $\ell \geq 1$. In this way we have that $a_i(\theta) = \sum_{\ell=1}^{d_i} s_{\ell i} \sin(\ell\theta)$, and therefore $\tilde{a}_i(\theta) = \int_0^\theta a_i(t)\, dt$ is

$$\tilde{a}_i(\theta) \; = \; \sum_{\ell=1}^{d_i} \frac{s_{\ell i}}{\ell} \left(1 - \cos(\ell\theta)\right) \; = \; \sum_{\ell=1}^{d_i} \frac{s_{\ell i}}{\ell} \left(1 - T_\ell(\cos\theta)\right),$$

where $T_\ell(z)$ is the Chebyshev polynomial of first kind and degree ℓ. We conclude that $\tilde{a}_i(\theta)$ is of the form $\tilde{a}_i(\theta) = p_i(\cos\theta)$, where

$$p_i(z) \; := \; \sum_{\ell=1}^{d_i} \frac{s_{\ell i}}{\ell} \left(1 - T_\ell(z)\right)$$

is a polynomial with real coefficients. Thus the differential α-symmetric equation (1) satisfies the composition condition and by Theorem 2, we have that it has a universal center.

In the case of separable variables we have that the equation takes the form $d\rho/d\theta = a(\theta)\, b(\rho)$, where $a(\theta)$ is a trigonometric polynomial and $b(\rho)$ is an analytic function in a neighborhood of $\rho = 0$ with $b(0) = b'(0) = 0$. We have that this equation has a center if and only if $\int_0^{2\pi} a(t)\, dt = 0$. Hence the function $q(\theta) := \int_0^{\theta} a(t)\, dt$ is a trigonometric polynomial and the differential equation satisfies the composition condition (with this q and all the polynomials $p_i(z)$ equal to the identity z). By Theorem 2 the differential equation has a universal center. □

Proof of Proposition 4. If $f(\theta)$ is identically zero, we have that equation (5) is a trivial universal center. We assume that $f(\theta)$ is not identically zero. We develop equation (5) in power series of ρ and we see that the coefficient of ρ^{m+in} is $(-1)^i f(\theta) g(\theta)^i$ for $i \geq 0$. By Theorem 2 we have that equation (5) has a universal center if and only if there exist a nonconstant trigonometric polynomial $q(\theta)$ and polynomials $p_i \in \mathbb{R}[z]$ such that

$$\int_0^{\theta} f(t) g^i(t)\, dt = p_i(q(\theta)) \quad \text{for } i \geq 0.$$

We will see that the two first of these conditions imply all the rest. Taking the derivative with respect to θ of the first two conditions we get that

$$f(\theta) = p_0'(q(\theta)) q'(\theta), \quad f(\theta) g(\theta) = p_1'(q(\theta)) q'(\theta).$$

We divide the second identity with respect to the first one and we get that

$$g(\theta) = \frac{p_1'}{p_0'}(q(\theta)).$$

Thus $g(\theta)$ is a rational function of $q(\theta)$, but since $g(\theta)$ is a trigonometric polynomial we conclude that p_0' divides p_1' and, hence there exists a polynomial $h \in \mathbb{R}[z]$ such that $g(\theta) = h(q(\theta))$. Therefore we obtain the integral

$$\int_0^{\theta} f(t) g^i(t)\, dt = \int_0^{\theta} p_0'(q(t)) q'(t) h^i(q(t))\, dt = p_i(q(\theta)),$$

where $p_i(z)$ is a primitive of the polynomial $p_0'(z) h^i(z)$ for all $i \geq 0$. □

3. Invariance under change of variables

In this section we will see that the universal centers are in general not invariant under changes of variables. Moreover there exist appropriate coordinates where any non-degenerate center of system (12) is a universal center.

3.1. **Rational transformations.** If we apply the Cherkas transformation (see [15])

$$\rho = \frac{r^{s-1}}{1 + r^{s-1} g(\theta)}, \quad \text{whose inverse is} \quad r = \frac{\rho^{1/(s-1)}}{(1 - \rho g(\theta))^{1/(s-1)}}, \tag{27}$$

to the differential equation (6) we obtain the Abel differential equation (7). By the regularity of the Cherkas transformation and its inverse at $r = \rho = 0$, equation (6) has a center if and only if equation (7) has a center.

Blinov in [7] studied the centers corresponding to the Hamiltonian systems (15) and the time–reversible symmetric centers of the systems

$$\dot{x} = -y + P_s(x, y), \quad \dot{y} = x + Q_s(x, y),$$

with P_s and Q_s homogeneous polynomials of degree $s \geq 2$. He showed that these centers, passing first to polar coordinates and second doing Cherkas transformation, become centers of an Abel equation of the form (7). Finally he proved that these last centers are universal. We prove that this Hamiltonian and symmetric centers are universal already for equation (6), see Theorem 8. Moreover, we will show that when a center of equation (6) is universal, then the corresponding center of equation (7) is also universal, but the converse is not true, see Theorem 5.

We remark that for $s = 2$ and $s = 3$ Blinov showed that the above Hamiltonian and symmetric centers are universal once they are written as the Abel differential equation (7), and that the other centers of these Abel equations in general are not universal, see [7].

We will see that the notion of universal center is not invariant under changes of variables.

Proposition 20. *The following statements hold.*

(a) *A rational transformation does not maintain a center to be universal for equation* (1)*.*

(b) *The transformation $\rho = r^{s-1}/(1 + r^{s-1}g(\theta))$, where $s \geq 2$ and $g(\theta)$ is any trigonometric polynomial, does not maintain a center to be universal.*

Proof. (a) We consider $R(\rho, \theta) = \rho/(1 + \tilde{a}_1(\theta)\rho + \tilde{a}_2(\theta)\rho^2)$ where $\tilde{a}_i(\theta)$ are trigonometric polynomials for $i = 1, 2$. We take the equation

$$\frac{d\rho}{d\theta} = \frac{-\partial R/\partial\theta}{\partial R/\partial\rho} = \frac{\rho^2 a_1(\theta) + \rho^3 a_2(\theta)}{1 - \rho^2 \tilde{a}_2(\theta)},$$

where $a_i(\theta) = \tilde{a}_i'(\theta)$, for $i = 1, 2$. We have that this equation has a center at the origin because it has a well-defined first integral $R(\rho, \theta)$ in a neighborhood of $\rho = 0$. We expand the previous equation in power series of ρ and we get $d\rho/d\theta = \rho^2 a_1(\theta) + \rho^3 a_2(\theta) + h.o.t.$ We integrate the coefficients in θ and we get $\tilde{a}_1(\theta)$ and $\tilde{a}_2(\theta)$ which are not, in general, composite polynomials of the same trigonometric polynomial $q(\theta)$, due to their arbitrariness. Take for instance $\tilde{a}_1(\theta) = \sin\theta$ and $\tilde{a}_2(\theta) = \cos\theta$. Then the equation has not a universal center. We take the rational transformation $r = R(\rho, \theta)$, and we have that $dr/d\theta = 0$, which is a trivial universal center.

(b) We take the following Abel differential equation

$$\frac{d\rho}{d\theta} = \sin\theta\rho^2 + \cos\theta\sin\theta\rho^3,$$

which has a universal center because the integration of the coefficients give $-\cos\theta$ for ρ^2 and $-\cos^2\theta/2$ for ρ^3. These two expressions are both polynomials of $\cos\theta$ and, therefore, we have a universal center. We consider the transformation $\rho = r/(1 - r\mu\sin\theta)$, where μ is any nonzero real number. After this change we get an equation of the form

$$\frac{dr}{d\theta} = (\sin\theta - \mu\cos\theta)\, r^2 + \sin\theta\cos\theta\, r^3 + h.o.t.$$

If we integrate the coefficients we obtain $-\cos\theta - \mu\sin\theta$ for r^2 and $-\cos^2\theta/2$ for r^3. These two polynomials do not satisfy the composition condition because $\mu \neq 0$. Hence the resulting equation has not a universal center. □

Proof of Theorem 5. (a) Differential equation (6) after the change (27) becomes the Abel equation (7). If $f(\theta) \equiv 0$, we have that both equations have universal centers. We assume that $f(\theta) \not\equiv 0$. By Proposition 4, we have that equation (6) has a universal center if and only if there exists a nonconstant trigonometric polynomial $q(\theta)$ and two polynomials $p_0, p_1 \in \mathbb{R}[z]$ such that

$$\int_0^\theta f(t)\, dt = p_0(q(\theta)), \quad \int_0^\theta f(t)g(t)\, dt = p_1(q(\theta)). \tag{28}$$

We assume that (6) has a universal center and we want to prove the existence of two polynomials $A_0(z), A_1(z) \in \mathbb{R}[z]$ such that:

$$\int_0^\theta \big((s-1)f(t) - g'(t)\big)\, dt = A_0(q(\theta)), \qquad \int_0^\theta f(t)g(t)dt = A_1(q(\theta)).$$

This fact implies that the Abel equation (7) has a universal center. We take $A_1(z)$ to be $p_1(z)$ and the second condition is verified. Taking the derivatives of the identities (28) with respect to θ we get

$$f(\theta) = p_0'(q(\theta))\, q'(\theta), \qquad f(\theta)\, g(\theta) = p_1'(q(\theta))\, q'(\theta).$$

We divide the second condition by the first one and we obtain that

$$g(\theta) = \frac{p_1'}{p_0'}(q(\theta)).$$

Thus $g(\theta)$ is a rational function of $q(\theta)$ but since $g(\theta)$ is a trigonometric polynomial we conclude that p_0' divides p_1'. Hence there exists a polynomial $h \in \mathbb{R}[z]$ such that $g(\theta) = h(q(\theta))$. Therefore

$$\begin{aligned} \int_0^\theta \big((s-1)f(t) - g'(t)\big)\, dt &= (s-1)p_0(q(\theta)) - \int_0^\theta h'(q(t))q'(t)dt \\ &= (s-1)p_0(q(\theta)) - h(q(\theta)) + K, \end{aligned}$$

where $K = h(q(0))$. Taking $A_0(z) = (s-1)p_0(z) - h(z) + K$ the statement follows.

(b) We take the Abel differential equation (7) with $s = 2$ and $f(\theta) = -2(\nu_1\cos\theta - \nu_2\cos(2\theta) + \nu_2\sin\theta - \nu_1\sin(2\theta))$ and $g(\theta) = \nu_1 + 2\nu_2\cos\theta + \nu_1\cos(2\theta) + \nu_2\sin(2\theta)$, where ν_1 and ν_2 are nonzero real numbers. The Abel differential equation (7) defined by these polynomials has a universal center because the integration of its coefficients $f(\theta)-g'(\theta)$ and $-f(\theta)g(\theta)$ are both two polynomials in $\sin\theta$. However the corresponding equation (6) has not a universal center because the integral

$$\int_0^{2\pi} f(t)g^2(t)dt = -4\pi\nu_1^2\nu_2 \neq 0,$$

and the vanishing of this integral is a necessary condition in order that equation (6) be universal. □

3.2. **Analytic transformations.**

Proof of Theorem 7. Statement (a) follows immediately from the results of Proposition 9.

Now we shall prove statement (b). Its proof is obtained taking into account the analytic change of variables that transforms any center of system (12) into the Poincaré normal form. According to Poincaré [28] there exists a near–identity analytic change of coordinates $(u,v) = \phi(x,y) = (x + o(|(x,y)|), y + o(|(x,y)|))$, transforming system (12) with a center at the origin into the normal form

$$\dot{u} = -v\,[1+\psi(u^2+v^2)]\ ,\ \dot{v} = u[1+\psi(u^2+v^2)]\ , \tag{29}$$

with ψ an analytic function near the origin such that $\psi(0) = 0$. It is clear that the transformed system (29) is time–reversible, that is symmetric with respect to an axis through the origin. In this case it is symmetric with respect the both axes $u = 0$ and $v = 0$. In fact taking polar coordinates $u = r\cos\theta$, $v = r\sin\theta$ system (29) becomes $\dot{r} = 0$, and $\dot{\theta} = -1 - \psi(r^2)$ and therefore we have $dr/d\theta = 0$ which is a trivial universal center. □

In summary the notion of universal center depends on the variables in which the system (12) is expressed. In a similar way that the time–reversible condition or the Hamiltonian condition of a system are not invariant under changes of variables. In fact we only can use these conditions as sufficient conditions in order to have a center but not as an invariant characterization of any center.

4. Universal analytic centers in the plane

In this section we consider a system in the plane of the form (12) and we study when its corresponding differential equation (14) has a universal center.

Proof of Theorem 8. (a) A time-reversible symmetry for system (12) with respect to a straight line through the origin with slope $m = \tan\alpha$ with $\alpha \in (-\pi/2, \pi/2]$ is given by

$$(x,y,t) \mapsto \left(\frac{x+2my-m^2x}{1+m^2}, \frac{-y+2mx+m^2y}{1+m^2}, -t\right).$$

We denote by φ the change from polar coordinates to cartesian coordinates, that is $\varphi(r,\theta) = (r\cos\theta, r\sin\theta)$. Given m we denote by ψ the symmetry in the plane with respect to a straight line of slope m passing through the origin, that is $\psi(x,y) = \left(\frac{x+2my-m^2x}{1+m^2}, \frac{-y+2mx+m^2y}{1+m^2}\right)$. It is easy to check that $\varphi^{-1}\circ\psi\circ\varphi(r,\theta) = (r, 2\alpha-\theta)$. Thus if system (12) is time–reversible, then its corresponding equation (14) after the change to polar coordinates is α–symmetric. Then any time–reversible center is universal by Theorem 3.

(b) In polar coordinates system (15) becomes equation (6). After the change $R = r^{s-1}$ we obtain

$$\frac{dR}{d\theta} = \frac{(s-1)f(\theta)R^2}{1+g(\theta)R}, \tag{30}$$

where $f(\theta) = -h'_{s+1}(\theta)$ and $g(\theta) = (s+1)h_{s+1}(\theta)$ with $h_{s+1}(\theta)$ the trigonometric polynomial defined by $h_{s+1}(\theta) = H_{s+1}(\cos\theta, \sin\theta)$. We remark that the change $R = r^{s-1}$ modifies nor the center conditions of (6) neither the universal center conditions.

In a more general context, whenever one takes $g'(\theta) = kf(\theta)$ with k a nonzero real number, equation (30) has a center, see [24]. Moreover this center is universal because

$$\int_0^\theta \frac{f(\xi)R^2}{1+g(\xi)R}d\xi = \frac{1}{k}\int_0^\theta \frac{g'(\xi)R^2}{1+g(\xi)R}d\xi = \frac{R}{k}\log\left(\frac{1+Rg(\theta)}{1+Rg(0)}\right),$$

and in Proposition 19 we obtain $G(r,\theta) = \frac{r^{s-1}}{k}\log\left(\frac{1+r^{s-1}g(\theta)}{1+r^{s-1}g(0)}\right)$. So equation (30) has a universal center when $g'(\theta) = kf(\theta)$ with $k \neq 0$. Since for system (15) we have $g'(\theta) = -(s+1)\,f(\theta)$, it has a universal center at the origin. □

Proof of Proposition 9. Writing system (16) in polar coordinates we obtain

$$\frac{dr}{d\theta} = \frac{f(\theta)r^2}{1+g(\theta)r}, \tag{31}$$

with

$$\begin{aligned} f(\theta) &= -\lambda_3\cos^3\theta + (3\lambda_2+\lambda_5)\cos^2\theta\sin\theta \\ &\quad +(2\lambda_3+\lambda_4+\lambda_5)\cos\theta\sin^2\theta - \lambda_2\sin^3\theta, \\ g(\theta) &= \lambda_2\cos^3\theta + (3\lambda_3+\lambda_4)\cos^2\theta\sin\theta \\ &\quad -(3\lambda_2+\lambda_5)\cos\theta\sin^2\theta - \lambda_6\sin^3\theta. \end{aligned}$$

Now we analyze the universal conditions for each center component of quadratic systems described in the introduction.

(a) Lotka Volterra component. We put $\lambda_3 = \lambda_6$ in equation (31) and we have that it is a center. A simple computation shows that

$$\begin{aligned} I_4 = \tilde{a}_4(2\pi) &= \int_0^{2\pi} a_4(\theta)\, d\theta \\ &= \frac{\pi}{64}\left(\lambda_2\lambda_4(\lambda_4^2 - 3\lambda_5^2) + \lambda_5(\lambda_4^3 - \lambda_4\lambda_5^2 + 3\lambda_6\lambda_4^2 - \lambda_6\lambda_5^2)\right), \end{aligned}$$

where we recall that $a_4(\theta)$ is the coefficient of r^5 in the expansion in power series of r of the function which defines equation (31). In this case we have that $a_4(\theta) = -f(\theta)g(\theta)^3$.

When $I_4 = 0$, some computations show that equation (31) is α–symmetric with $\alpha = \arctan(\lambda_5/\lambda_4)$. So, by Theorem 3 we have that the center is universal. From Proposition 4 this symmetry is equivalent to say that there exist two polynomials $p_0(z), p_1(z) \in \mathbb{R}[z]$ such that

$$\int_0^{\theta} f(t)\, dt = p_0(q(\theta)), \qquad \int_0^{\theta} f(t)g(t)\, dt = p_1(q(\theta)),$$

with $q(\theta) = \lambda_4 \sin\theta - \lambda_5 \cos\theta$.

(b) Symmetric component. When $\lambda_2 = \lambda_5 = 0$ equation (31) is α–symmetric with $\alpha = \pi/2$ and, thus it has a universal center.

(c) Hamiltonian component. When $\lambda_4 = \lambda_5 = 0$, we have that equation (31) has a universal center as a consequence of Theorem 8(b).

(d) The Darboux component. We put $\lambda_5 = \lambda_4 + 5\lambda_3 - 5\lambda_6 = \lambda_3\lambda_6 - 2\lambda_6^2 - \lambda_2^2 = 0$ in equation (31). If $\lambda_6 = 0$ we have that the center is symmetric and we have analyzed this case in the second second. If $\lambda_6 \neq 0$ we have that

$$I_4 = \tilde{a}_4(2\pi) = \int_0^{2\pi} a_4(\theta) d\theta = -\frac{5\pi\lambda_2(\lambda_2^2 + \lambda_6^2)^3}{16\lambda_6^3},$$

where, as before, $a_4(\theta) = -f(\theta)g(\theta)^3$ with the parameters fixed in order to be in the Darboux component. The universal condition forces that $I_4 = 0$, which implies $\lambda_2 = 0$. Consequently a center in the Darboux component is universal only when $\lambda_2 = 0$. In this case equation (31) also belongs to the symmetric component. □

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