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The solution of the Composition Conjecture for Abel equations

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Abstract

Trigonometric Abel differential equations appear when one studies the number of limit cycles and the center-focus problem for certain families of planar polynomial systems. Inside trigonometric Abel equations there is a class of centers, the composition centers, that have been widely studied during these last years. We fully characterize this type of centers. They are given by the couples of trigonometric polynomials for which all the generalized moments vanish and also coincide with the strongly persistent centers. This result solves the so called *Composition Conjecture* for trigonometric Abel differential equations. We also prove the equivalent version of this result for Abel equations with polynomial coefficients.

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1 Introduction and main results

The study of Abel differential equations of the form

$$\dot r = \frac{dr}{d\theta} = A(\theta)\, r^3 + B(\theta)\, r^2 + C(\theta) r,$$

provides a useful tool for knowing either the number of limit cycles of certain planar polynomial differential equations or for studying the center-focus problem for them, see for instance [3, 15, 19]. These equations also turn out to be interesting from the point of view of applications, see [16, 18].

In this paper we consider Abel differential equations of the form

$$\dot r = A(\theta)\, r^3 + B(\theta)\, r^2, \tag{1}$$

defined on the cylinder $(r,\theta)\in\mathbb{R}\times\mathbb{R}/(2\pi\mathbb{Z})$, with A and B being trigonometric polynomials and we concentrate on the *center-focus* problem. Recall that this problem consists on characterizing A and B to ensure that all the solutions $r=r(\theta,r_0)$, with initial condition $r(0,r_0)=r_0$ and $|r_0|$ small enough are 2π-periodic. For short, if this property holds we will say that the Abel equation has a *center.*

One of the main motivations for considering the above problem is that the center-focus problem for planar polynomial equations with homogeneous non-linearities can be reduced to it, see [13, 19]. Our results solve the so called *Composition Conjecture* that wonders about the relation between a special type of centers, the ones satisfying the *composition condition*, and the annulation of some *moments* computed from A and B, see [2, 4]. To be more precise we introduce some definitions.

When there exist $\mathcal{C}^1$-functions A_1, B_1 and u, with u being 2π-periodic, such that

$$\widetilde{A}(\theta):=\int_0^\theta A(\psi)\,d\psi=A_1(u(\theta))\quad\text{and}\quad\widetilde{B}(\theta):=\int_0^\theta B(\psi)\,d\psi=B_1(u(\theta)),\tag{2}$$

it is said that the corresponding Abel equation satisfies the *composition condition.* This condition was introduced in [3] and ensures that the Abel equation has a center. In this situation we will say that the Abel equation has a *CC-center.*

This condition plays a similar role for Abel equations similar that being Hamiltonian, or reversible with respect to one line, for planar vector fields with homogeneous non-linearities. This is because it can be seen that if one of these systems has one of these types of center, the corresponding Abel equation, constructed from the Cherkas transformation, has a CC-center. Centers for (1) which are no CC-centers are given for instance in [1, 2, 12].

Another interesting family of centers is the class of *persistent centers.* Recall that it is said that equation (1) has a persistent center if the family of equations

$$\dot r=\varepsilon\,A(\theta)\,r^3+B(\theta)\,r^2,\tag{3}$$

has a center for all ε small enough, see [2] and the references therein. It can be seen that this definition is equivalent to say that

$$\dot r=\alpha\,A(\theta)\,r^3+\beta\,B(\theta)\,r^2,\tag{4}$$

has a center for all $\alpha,\beta\in\mathbb{R}$, see [12]. It is known that persistent centers satisfy the following *moment conditions*

$$\int_0^{2\pi}\widetilde{B}^p(\theta)\,A(\theta)\,d\theta=0\tag{5}$$

and

$$\int_0^{2\pi}\widetilde{A}^p(\theta)B(\theta)\;d\theta=0,\tag{6}$$

for all natural number $p \in \mathbb{N} \cup \{0\}$, see [2, 12].

Several authors have tried to relate the above three concepts: CC-centers, persistent centers and moment conditions. For instance it is clear that CC-centers are persistent centers and the corresponding A and B satisfy the moment conditions (5) and (6). In particular, the problem of knowing whether conditions (5), either when A and B are trigonometric polynomials or when A and B are polynomials, imply that the corresponding Abel equation (1) has a CC-center has been known as the *Composition Conjecture.* In the polynomial case it has been showed to be false in [21]. In the trigonometric case, even assuming that (5) and (6) hold, it also turns out to be false, see [12]. The trigonometric counterexample given in that paper is

$$\dot{r} = \big(a\cos(2\theta) + b\sin(2\theta) + c\sin(6\theta)\big)\, r^3 + \frac{1}{32}\cos(3\theta)\, r^2. \tag{7}$$

For $a(a^2 - 3b^2) \neq 0$ it has a center which is not a CC-center but the moment conditions (5) and (6) for the corresponding functions A and B are satisfied. It was constructed from the class of integrable Lotka-Volterra quadratic systems in the plane.

Therefore, to characterize CC-centers, more restrictive conditions that the moment conditions (5) and (6) have to be given. Following [12] we introduce two new concepts.

We will say that equation (1) has a *strongly persistent center* if

$$\frac{dr}{d\theta} = \big(\alpha\, A(\theta) + \beta\, B(\theta)\big)\, r^3 + \big(\gamma\, A(\theta) + \delta\, B(\theta)\big)\, r^2, \tag{8}$$

has a center for all $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and will say that A and B satisfy the *generalized moment conditions* if

$$\int_0^{2\pi} \widetilde{A}^p(\theta)\, \widetilde{B}^q(\theta)\, A(\theta)\, d\theta = 0 \quad \text{and} \quad \int_0^{2\pi} \widetilde{A}^p(\theta)\, \widetilde{B}^q(\theta)\, B(\theta)\, d\theta = 0, \tag{9}$$

for all $p, q \in \mathbb{N} \cup \{0\}$. It is easy to see that for the Abel equation (7),

$$\int_0^{2\pi} \widetilde{A}^3(\theta)\, \widetilde{B}(\theta)\, B(\theta)\, d\theta \neq 0.$$

Therefore the corresponding A and B do not satisfy the generalized moment conditions.

Our main result is:

Theorem 1. *Consider the Abel equation* (1)*. The following statements are equivalent:*

(i) The equation has a strongly persistent center.

(ii) For the corresponding A and B the generalized moment conditions (9) *are satisfied.*

(iii) The equation has a CC-center.

The most difficult step is to show that (ii) implies (iii). Our approach for proving it relies on the Lüroth's Theorem and it is strongly inspired on some of the results of [17].

Remark 2. *Notice that if an Abel equation has a CC-center then there exist infinitely many functions A_1, B_1 and u satisfying* (2)*, because if A_1, A_2, u satisfy the CC-condition, all the triplets $A_1 \circ h, A_2 \circ h, h^{-1} \circ u$, with h being a diffeomorphism satisfy* (2) *as well. As a consequence of the proof of Theorem 1 we will see that the trigonometric CC-centers always admit functions A_1, B_1 and u with A_1, B_1 polynomials and u a trigonometric polynomial.*

Notice that Theorem 1 can be interpreted as the solution of the Composition Conjecture in the trigonometric setting, because it characterizes the CC-centers in terms of the annulation of certain moments associated to A and B. Moreover it also relates both concepts with a more dynamic one, the strongly persistence. It can also be seen that the three classes appearing in the theorem are also equivalent to the class of so called *universal centers*, introduced in [9, 10].

The above concepts have also been widely studied when the functions A and B, instead of being trigonometric polynomials are usual polynomials, see [4, 5, 6, 7, 8, 11, 14, 23]. As we will see in Section 4, the equivalent version of Theorem 1 also holds in this context, see Theorem 7. Therefore we have also solved the composition conjecture in the polynomial setting.

2 Preliminary results

From now on, $\mathbb{R}[x]$ will denote the ring of polynomials with real coefficients and $\mathbb{R}(x)$ its quotient field. Also we will denote by $\mathbb{R}[\theta]$ the ring of trigonometric polynomials with real coefficients and by $\mathbb{R}(\theta)$ its quotient field. It is well known that $\mathbb{R}(\theta)$ is isomorphic to $\mathbb{R}(x)$ by means of the map $\Phi : \mathbb{R}(\theta) \longrightarrow \mathbb{R}(x)$ defined by

$$\Phi(\sin\theta) = \frac{2x}{1+x^2} \quad \text{and} \quad \Phi(\cos\theta) = \frac{1-x^2}{1+x^2}. \tag{10}$$

In particular, this morphism satisfies that

$$\Phi\left((\tan(\theta/2)\right) = \Phi\left(\frac{\sin\theta}{1+\cos\theta}\right) = x.$$

Next lemma characterizes the image by Φ of the set of trigonometric polynomials.

Lemma 3. *It holds that*

$$\Phi(\mathbb{R}[\theta]) = \left\{ \frac{p(x)}{(1+x^2)^n} : p(x) \in \mathbb{R}[x] \text{ and } \deg(p(x)) \leq 2n \right\} =: \mathbb{T}(x).$$

Proof. From the definition of Φ it follows that $\Phi(\mathbb{R}[\theta]) \subset \mathbb{T}(x)$. To prove the converse inclusion it suffices to show that $\frac{x^i}{(1+x^2)^n} \in \Phi(\mathbb{R}[\theta])$ for all $i \le 2n$ and all $n \in \mathbb{N}$. We will prove this fact by induction on n. For $n = 0$ the statement follows because $1 = \Phi(1) \in \Phi(\mathbb{R}[\theta])$. Assume that the statement holds for n and we prove it for $n+1$. Set $i \le 2(n+1)$. If in addition $i \le 2n$ then $\frac{x^i}{(1+x^2)^{n+1}} = \frac{x^i}{(1+x^2)^n} \cdot \frac{1}{1+x^2}$ which belongs to $\mathbb{T}(x)$ by the induction hypothesis. If $i \in \{2n+1, 2n+2\}$ then $\frac{x^i}{(1+x^2)^{n+1}} = \frac{x^{i-2}}{(1+x^2)^n} \cdot \frac{x^2}{1+x^2}$ that also belongs to $\mathbb{T}(x)$, again by the induction hypothesis. $\square$

Given $r, s \in \mathbb{R}(x)$ (respectively $r, s \in \mathbb{R}(\theta)$) we will say that they are *equivalent*, and we write $r \sim s$, if there exists a Möbius transformation γ such that $\gamma(r) = s$. Recall that a Möbius transformation γ is a rational map given by $\gamma(z) = \frac{az+b}{cz+d}$ for some fixed $a, b, c, d \in \mathbb{R}$ such that $ad - bc \neq 0$.

For $\xi \in \mathbb{R}(x)$ (respectively $\xi \in \mathbb{R}(\theta)$) we denote by $\mathbb{R}(\xi)$ the minimum field containing $\mathbb{R}$ and ξ. It is well known that $\mathbb{R}(r) = \mathbb{R}(s)$ if and only if $r \sim s$, see [22].

To state next result we need to introduce some definitions. For $\alpha \in \mathbb{R}$, let

$$\Delta, R_\alpha : \mathbb{R}[x] \times \mathbb{R}[x] \longrightarrow \mathbb{R}[x] \times \mathbb{R}[x]$$

be the maps defined by

$$\Delta(P, Q) = (P + xQ, Q - xP)$$

and

$$R_\alpha(P, Q) = (P \cos\alpha + Q \sin\alpha, -P \sin\alpha + Q \cos\alpha).$$

Easy computations show that both maps commute, that is $\Delta \circ R_\alpha = R_\alpha \circ \Delta$.

Proposition 4. *Consider the equation*

$$P^2 + Q^2 = (1 + x^2)^n \tag{11}$$

with $P, Q \in \mathbb{R}[x]$. The following assertions holds:

(a) If (P, Q) satisfies equation (11) with $n = k$ then $\Delta(P, Q)$ and $\Delta(P, -Q)$ satisfy equation (11) with $n = k + 1$.

(b) If (P, Q) satisfies equation (11) and $\gcd(P, Q) = 1$ *then either* $\gcd(\Delta(P, Q)) = 1$ *and* $\gcd(\Delta(P, -Q)) \neq 1$ *or viceversa.*

(c) For any $n \ge 1$ equation (11) has a solution with $\gcd(P, Q) = 1$.

(d) If (P_1, Q_1) and (P_2, Q_2) are solutions of (11) with $\gcd(P_1, Q_1) = \gcd(P_2, Q_2) = 1$ *then $P_1/Q_1 \sim P_2/Q_2$.*

Proof. To prove (a) assume that $P^2+Q^2=(1+x^2)^k$. Then

$$(P+xQ)^2+(Q-xP)^2=P^2+Q^2+x^2Q^2+x^2P^2=(1+x^2)^{k+1}.$$

To see (b) assume that (P,Q) satisfies equation (11), $\gcd(P,Q)=1$ and $\gcd(\Delta(P,Q))\neq 1$. From (a), $(P+xQ)^2+(Q-xP)^2=(1+x^2)^{n+1}$. Hence it follows that the only common irreducible factor of $P+xQ$ and $Q-xP$ is $1+x^2$ and the same situation holds for $P+x(-Q)$ and $-Q-xP$. Then if $\gcd(\Delta(P,-Q))\neq 1$ we will obtain that $1+x^2$ is a common factor of $P+xQ$ and $P+x(-Q)$. However this implies that $1+x^2$ is a common factor of P and Q contradicting that $\gcd(P,Q)=1$. On the other hand, since

$$(P+xQ)(P-xQ)=P^2-x^2Q^2=P^2+Q^2-(1+x^2)Q^2=(1+x^2)((1+x^2)^{n-1}-Q^2),$$

it follows that either $P+xQ$ or $P-xQ$ is a multiple of $1+x^2$. In the first case we will have that $\gcd(\Delta(P,Q))\neq 1$ and in the second one we will get that $\gcd(\Delta(P,-Q))\neq 1$.

Now we prove (c) inductively. For $n=1$ we have that $P=ax+b$, $Q=cx+d$ with $a^2+c^2=1, b^2+d^2=1$ and $ab+cd=0$. Clearly all the solutions (P,Q) verify that $\gcd(P,Q)=1$ and $\frac{P}{Q}\sim\frac{x}{1}$. Now assume the result holds for $n=k$ and we show it for $n=k+1$. Let (P,Q) satisfying equation (11) with $n=k>1$ and $\gcd(P,Q)=1$. Then from (a) and (b) the result follows.

To see (d) we take a pair (P,Q) satisfying equation (11) with $n=k>1$ and $\gcd(P,Q)=1$ and we look for a pair satisfying (11) with $n=k-1$.

As we have noticed either $P+xQ$ or $P-xQ$ is a multiple of $(1+x^2)$. Assume for example that $P-xQ=(1+x^2)R$ with $R\in\mathbb{R}[x]$. Then we get that

$$Q+xP=Q+x((1+x^2)R+xQ)=(1+x^2)(xR+Q).$$

Thus we will have that also $Q+xP$ is a multiple of $(1+x^2)$. Thus in this case we can consider $\Upsilon(P,Q)=(\frac{P-xQ}{1+x^2},\frac{Q+xP}{1+x^2})\in\mathbb{R}[x]\times\mathbb{R}[x]$. In the other case we can consider $\Upsilon(P,-Q)=(\frac{P+xQ}{1+x^2},\frac{-Q+xP}{1+x^2})$. Note that in both situations we have that $\Delta(\Upsilon(P,Q))=(P,Q)$. Also an easy computation shows that $\Upsilon(P,Q)$ satisfies equation (11) with $n=k-1$. Moreover if $\gcd(\Upsilon(P,Q))\neq 1$ since $(P,Q)=\Delta(\Upsilon(P,Q))$ we obtain that $\gcd(P,Q)\neq 1$ which gives a contradiction.

Now we observe that if (P,Q) and (R,S) satisfy equation (11) and $P/Q\sim R/S$, then easy computations show that necessarily either $(R,S)=R_\alpha(P,Q)$ or $(R,S)=R_\alpha(P,-Q)$ for some $\alpha\in[0,2\pi]$.

Assume that for $n=k+1$ equation (11) has two solutions (P_1,Q_1) and (P_2,Q_2) with $\gcd(P_1,Q_1)=\gcd(P_2,Q_2)=1$.

Suppose also without loss of generality that P_1-xQ_1 and P_2-xQ_2 are multiple of $(1+x^2)$. Then we will have that $(\widehat{P}_1,\widehat{Q}_1):=\Upsilon(P_1,Q_1)$ and $(\widehat{P}_2,\widehat{Q}_2):=\Upsilon(P_2,Q_2)$ are solutions of equation (11) with $n = k$ and $\gcd(\Upsilon(P_1, Q_1)) = \gcd(\Upsilon(P_2, Q_2)) = 1$. Thus from the induction hypothesis we will have that $\widehat{P}_2/\widehat{Q}_2 \sim \widehat{P}_1/\widehat{Q}_1$. From the previous observation we will have that either $(\widehat{P}_2, \widehat{Q}_2) = R_\alpha(\widehat{P}_1, \widehat{Q}_1)$ or $(\widehat{P}_2, \widehat{Q}_2) = R_\alpha(\widehat{P}_1, -\widehat{Q}_1)$. In the first case we obtain

$$(P_2, Q_2) = \Delta(\widehat{P}_2, \widehat{Q}_2) = \Delta(R_\alpha(\widehat{P}_1, \widehat{Q}_1)) = R_\alpha(\Delta(\widehat{P}_1, \widehat{Q}_1)) = R_\alpha(P_1, Q_1)$$

and hence $P_1/Q_1 \sim P_2/Q_2$. In the second case we will have

$$(P_2, Q_2) = \Delta(\widehat{P}_2, \widehat{Q}_2) = \Delta(R_\alpha(\widehat{P}_1, -\widehat{Q}_1)) = R_\alpha(\Delta(\widehat{P}_1, -\widehat{Q}_1)).$$

Since $\Delta(\widehat{P}_1, \widehat{Q}_1) = (P_1, Q_1)$ and $\gcd(P_1, Q_1) = 1$ it follows from (b) that $\gcd(\Delta(\widehat{P}_1, -\widehat{Q}_1)) \neq 1$ and hence the same occurs for $(P_2, Q_2) = R_\alpha(\Delta(\widehat{P}_1, -\widehat{Q}_1))$. This contradicts the fact that $\gcd(P_2, Q_2) = 1$ and shows that this second possibility does not occur. This ends the proof of the proposition. $\square$

Lemma 5. *Let P_n, Q_n be such that* $\gcd(P_n, Q_n) = 1$, $P_n^2(0) + Q_n^2(0) = 1$ *and* $\Phi(\tan(\frac{n\theta}{2})) = \frac{P_n}{Q_n}$. *Then* $P_n^2 + Q_n^2 = (1 + x^2)^n$.

Proof. We prove the lemma by induction. For $n = 1$ we have $\Phi(\tan(\frac{\theta}{2})) = x$. So $P_1 = x$ and $Q_1 = 1$. Thus we have $P_1^2 + Q_1^2 = 1 + x^2$.

Now assume that the lemma holds for $n = k$ and we prove it for $n = k + 1$. First of all note that

$$\tan\left(\frac{(k+1)\theta}{2}\right) = \frac{\tan(\frac{k\theta}{2}) + \tan(\frac{\theta}{2})}{1 - \tan(\frac{k\theta}{2})\tan(\frac{\theta}{2})}$$

and hence

$$\Phi\left(\tan\left(\frac{(k+1)\theta}{2}\right)\right) = \frac{\frac{P_k}{Q_k} + x}{1 - \frac{P_k}{Q_k}x} = \frac{P_k + Q_k x}{Q_k - P_k x} = \frac{P_{k+1}}{Q_{k+1}}.$$

With the notation introduced in the previous lemma we have that $(P_{k+1}, Q_{k+1}) = \Delta(P_k, Q_k)$ and then from Proposition 4.(a) we get that $P_{k+1}^2 + Q_{k+1}^2 = (1+x^2)^{k+1}$. Therefore to prove the result it remains to show that $\gcd(P_{k+1}, Q_{k+1}) = 1$.

If $\gcd(P_{k+1}, Q_{k+1}) \neq 1$ from Proposition 4.(b) we will have that $\gcd(\Delta(P_k, -Q_k)) = 1$. If we write $\Delta(P_k, -Q_k) = (\widehat{P}_{k+1}, \widehat{Q}_{k+1})$ we have

$$\frac{\widehat{P}_{k+1}}{\widehat{Q}_{k+1}} = \frac{P_k - Q_k x}{-Q_k - P_k x} = -\frac{\frac{P_k}{Q_k} - x}{1 + \frac{P_k}{Q_k}x} = -\Phi\left(\tan\left(\frac{(k-1)\theta}{2}\right)\right)$$

and by the induction hypothesis we obtain that $\widehat{P}_{k+1}^2 + \widehat{Q}_{k+1}^2 = (1 + x^2)^{k-1}$ in contradiction with the fact that $\widehat{P}_{k+1}^2 + \widehat{Q}_{k+1}^2 = (1+x^2)^{k+1}$ because $(\widehat{P}_{k+1}, \widehat{Q}_{k+1}) = \Delta(P_k, -Q_k)$. Therefore $\gcd(\widehat{P}_{k+1}, \widehat{Q}_{k+1}) \neq 1$ and hence $\gcd(P_{k+1}, Q_{k+1}) = 1$. This ends the proof of the lemma. $\square$

Next result is also proved in [17]. We include here a proof slightly different.

Theorem 6. *Let $\mathbb{K}$ be a subfield of $\mathbb{R}(\theta)$ containing a non-constant trigonometric polynomial. Then either $\mathbb{K} = \mathbb{R}(\tan(\frac{n\theta}{2}))$ for some $n \in \mathbb{N}$ or $\mathbb{K} = \mathbb{R}(p)$ for some trigonometric polynomial p.*

Proof. By Lüroth's Theorem it holds that $\mathbb{K} = \mathbb{R}(\xi)$ for some quocient of trigonometric polynomials ξ, see [22]. Set $\Phi(\xi) = \frac{p}{q}$, with $p, q \in \mathbb{R}[x]$ and $\gcd(p, q) = 1$, where Φ is defined in (10). By Lemma 3, the hypothesis that $\mathbb{K}$ contains some trigonometric polynomial is translated into the fact that $\mathbb{R}(\frac{p}{q})$ contains some element of the form $\frac{M}{(1+x^2)^n}$, with M a polynomial of degree at most $2n$. Changing p/q, if necessary, by a Möbius transformation we can assume that $\deg(p) > \deg(q)$. Let $R, S \in \mathbb{R}[x]$ be such that $\gcd(R, S) = 1$ and

$$\frac{R(\frac{p}{q})}{S(\frac{p}{q})} = \frac{M}{(1+x^2)^n}.$$

Note that since $\deg(p) > \deg(q)$ necessarily $\deg(S) \geq 1$. Thus we obtain

$$\frac{q^s \widehat{R}(p,q)}{q^r \widehat{S}(p,q)} = \frac{M}{(1+x^2)^n}, \tag{12}$$

where $\widehat{R}, \widehat{S}$ denotes the homogenization of R and S and r, s are the degrees of R and S respectively. We claim that $\gcd(\widehat{S}(p,q), q^s \widehat{R}(p,q)) = 1$. To see this it suffices to show that $\widehat{S}(p,q)$ does not share roots (real or complex) with q^s or $\widehat{R}(p,q)$.

Let $z \in \mathbb{C}$ be a root of $\widehat{S}(p,q)$ and suppose first that z is also a root of q. If $S = \sum_{i=0}^{s} a_i x^i$ with $a_s \neq 0$ then

$$\widehat{S}(p,q) = \sum_{i=0}^{s} a_i p^i q^{s-i} \quad \text{and} \quad \widehat{S}(p,q)(z) = a_s\, p^s(z) = 0.$$

Since $a_s \neq 0$, it holds that $p(z) = 0$ which contradicts that $\gcd(p, q) = 1$. So $q(z) \neq 0$.

Suppose now that z is also a root $\widehat{R}(p,q)$. Since $q(z) \neq 0$ we will obtain that $R(\frac{p(z)}{q(z)}) = S(\frac{p(z)}{q(z)}) = 0$ which contradicts that $\gcd(R, S) = 1$.

Thus from (12) we obtain that $\widehat{S}(p,q) = (1+x^2)^k$ for some $k \geq 0$. Since $\widehat{S}$ is a homogeneous polynomial it decomposes in a product of real irreducible homogeneous polynomials of degrees 1 or 2. So we will have $\prod_{i=1}^{l} \widehat{S}_i(p,q) = (1+x^2)^k$. Clearly this implies that for each i, $\widehat{S}_i(p,q) = (1+x^2)^{k_i}$ for some $k_i \geq 0$. If there is some linear $\widehat{S}_i$ we have that there exists $0 \neq a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that $ap + bq = (1+x^2)^{k_i}$. Set $c, d \in \mathbb{R}$ such that $ad - bc \neq 0$. We will have that

$$\frac{p}{q} \sim \frac{c\frac{p}{q} + d}{a\frac{p}{q} + b} = \frac{cp + dq}{ap + bq} = \frac{cp + dq}{(1+x^2)^{k_i}}.$$

Since $\deg(cp+dq) \leq \deg(p) = \deg(ap+bq) = 2k_i$ we get that $\mathbb{R}(\frac{p}{q})$ admits a generator of the form $\frac{N}{(1+x^2)^{k_i}}$ with $\deg(N) \leq 2k_i$. From Lemma 3 we get that $\mathbb{K} = \mathbb{R}(\xi)$ admits a polynomial generator. So the result follows in this case.

Now suppose that all $\widehat{S}_i$ are quadratic. Then, for each i, $\widehat{S}_i(p,q) = (1+x^2)^{k_i}$ with $\widehat{S}_i$ irreducible. Thus

$$(ap+bq)^2 + c^2q^2 = (1+x^2)^{k_i}$$

for some non-zero real numbers a and c. Therefore, considering

$$\frac{\overline{p}}{\overline{q}} = \frac{ap+bq}{cq}$$

we have that $\frac{\overline{p}}{\overline{q}} \sim \frac{p}{q}$ and $\overline{p}^{\,2} + \overline{q}^{\,2} = (1+x^2)^{k_i}$. Finally, from Proposition 4.(c) and Lemma 5 we obtain that

$$\frac{\overline{p}}{\overline{q}} \sim \Phi\Big(\tan\Big(\frac{k_i\theta}{2}\Big)\Big),$$

and thus that $\xi \sim \tan(\frac{k_i\theta}{2})$, as we wanted to prove. □

3 Proof of Theorem 1

$(i) \Rightarrow (ii)$. The proof of this implication is also contained in [12]. Since equation (1) has a strongly persistent center, the Abel equation (8) has a persistent center. In particular we know that

$$\int_0^{2\pi} \big(\gamma\,\widetilde{A}(\theta) + \delta\,\widetilde{B}(\theta)\big)^k \big(\alpha\,A(\theta) + \beta\,B(\theta)\big)\,d\theta = 0\,,\ k \geq 0. \tag{13}$$

Taking $\beta = 0$ and $\alpha = 1$ we get

$$F(\gamma,\delta) := \int_0^{2\pi} \big(\gamma\,\widetilde{A}(\theta) + \delta\,\widetilde{B}(\theta)\big)^k A(\theta)\,d\theta = \sum_{i=0}^{k} \gamma^i\,\delta^{k-i}\binom{k}{i}\int_0^{2\pi} \widetilde{A}^i(\theta)\,\widetilde{B}^{k-i}(\theta)\,A(\theta)\,d\theta = 0.$$

Since $F(\gamma,\delta)$ is a polynomial in γ and δ we obtain that all its coefficients are zero. Therefore we have proved that for all $k \in \mathbb{N}$, $0 \leq i \leq k$,

$$\int_0^{2\pi} \widetilde{A}^i(\theta)\,\widetilde{B}^{k-i}(\theta)\,A(\theta)\,d\theta = 0.$$

Starting with $\beta = 1$ and $\alpha = 0$ we obtain the other set of conditions.

$(ii) \Rightarrow (iii)$. Assume that all the generalized moments vanish and consider the field $\mathbb{K} := \mathbb{R}(\widetilde{A}(\theta), \widetilde{B}(\theta))$. Notice that since $\int_0^{2\pi} A(\psi)\,d\psi = \int_0^{2\pi} B(\psi)\,d\psi = 0$, the functions $\widetilde{A}$ and $\widetilde{B}$ are trigonometric polynomials. Therefore we can apply Theorem 6 and $\mathbb{K} = \mathbb{R}(\xi)$, with ξ

either a trigonometric polynomial or $\xi = \tan(\frac{n\theta}{2})$ for some $n > 0$. Now we will see that the second possibility does not occur. Assume that

$$\frac{P(\widetilde{A}(\theta), \widetilde{B}(\theta))}{Q(\widetilde{A}(\theta), \widetilde{B}(\theta))} = \tan\left(\frac{n\theta}{2}\right),$$

for some $P, Q \in \mathbb{R}[x, y]$. Derivating with respect to θ we get

$$\frac{(QP_x - PQ_x)(\widetilde{A}(\theta), \widetilde{B}(\theta))A(\theta) + (QP_y - PQ_y)(\widetilde{A}(\theta), \widetilde{B}(\theta))B(\theta)}{Q^2(\widetilde{A}(\theta), \widetilde{B}(\theta))} = \frac{n}{2}\left(1 + \tan^2\left(\frac{n\theta}{2}\right)\right).$$

So

$$(QP_x - PQ_x)(\widetilde{A}(\theta), \widetilde{B}(\theta))A(\theta) + (QP_y - PQ_y)(\widetilde{A}(\theta), \widetilde{B}(\theta))B(\theta) = \frac{n}{2}(P^2 + Q^2)(\widetilde{A}(\theta), \widetilde{B}(\theta)).$$

Note that the integral in the interval $[0, 2\pi]$ of the left side of this equality is zero because is the sum of a finite number of generalized moments, but the right side of the equality is a positive continuous function. This gives the desired contradiction.

So we conclude that $\mathbb{R}(\widetilde{A}(\theta), \widetilde{B}(\theta))$ is generated by a trigonometric polynomial p. Then $\widetilde{A}(\theta) = \frac{R_1}{S_1}(p(\theta))$ and $\widetilde{B}(\theta) = \frac{R_2}{S_2}(p(\theta))$ with $\frac{R_i}{S_i} \in \mathbb{R}(x)$ and $\gcd(R_i, S_i) = 1$ for $i = 1, 2$. We are going to prove that we can choose $S_1 = S_2 = 1$. We prove this fact for S_1. From Lemma 3 we have that

$$\frac{R_1}{S_1}\left(\frac{M}{(1+x^2)^i}\right) = \frac{N}{(1+x^2)^j},$$

with $M, N \in \mathbb{R}[x]$, $\gcd(M, (1+x^2)) = \gcd(N, (1+x^2)) = 1$, $\deg(M) \leq 2i$ and $\deg(N) \leq 2j$. Adding, if necessary, a constant to $p(\theta)$ we can assume that $\deg(M) < 2i$. Now assume to arrive a contradiction that $\deg S_1 \geq 1$. Thus we obtain

$$\frac{(1+x^2)^{is}\widehat{R}(M, (1+x^2)^i)}{(1+x^2)^{ir}\widehat{S}(M, (1+x^2)^i)} = \frac{N}{(1+x^2)^j},$$

where $\widehat{R}$ and $\widehat{S}$ denote the homogenization of R_1 and S_1 and r and s are the corresponding degrees of R_1 and S_1. Arguing as in the proof of Theorem 6 we obtain that $\widehat{S}(M, (1+x^2)^i) = (1+x^2)^k$ for some $k \leq j$. Since $\widehat{S}(M, (1+x^2)^i) = a_s M^s + (1+x^2)^i L$ with $L \in K[x]$, $a_s \neq 0$ and $\gcd(M, (1+x^2)) = 1$ we obtain that $k = 0$ and $\widehat{S}(M, (1+x^2)^i) = 1$. If we decompose the homogeneous polynomial $\widehat{S}$ in its real irreducible components we will obtain that for each one of them, say T,

$$T(M, (1+x^2)^i) \in \mathbb{R}.$$

If $\deg(T) = 2$ this last property does not hold because it is impossible that

$$(aM + b(1+x^2)^i)^2 + c^2(1+x^2)^{2i} \in \mathbb{R},$$

with a, b, c real numbers and $a \neq 0$ and $c \neq 0$. If $\deg(T) = 1$ we obtain $aM + b(1+x^2)^i \in \mathbb{R}$ for some $a, b \in \mathbb{R}$. Since $\deg M < 2i$ the only possibility is $b = 0$ and $M \in \mathbb{R}$. Then the only possible irreducible factor of T is x. Hence $S_1 = x^s$. However since $\gcd(R_1, S_1) = 1$, this implies that $R_1(0) \neq 0$ and $\deg \widehat{R}(M, (1+x^2)^i) = 2ir$. Since

$$\frac{(1+x^2)^{is}\widehat{R}(M,(1+x^2)^i)}{(1+x^2)^{ir}} = \frac{N}{(1+x^2)^j}$$

and $\deg(N) \leq 2j$ we get $s = 0$ and $S_1 = 1$. So, $\widetilde{A} = R_1(p)$. Similarly $\widetilde{B} = R_2(p)$ and the result follows.

$(iii) \Rightarrow (i)$. This implication is trivial because if equation (1) has a CC-center the same holds with equation (8).

4 Polynomial Abel equations

Similarly that in the trigonometric case, for each two real numbers $a < b$ we can consider the problem of giving necessary and sufficient conditions for the two real polynomials $A(t)$ and $B(t)$ to ensure that the solutions of the equation

$$\frac{dx}{dt} = A(t)\, x^3 + B(t)\, x^2, \tag{14}$$

satisfy $x(a) = x(b)$, for all initial conditions close enough to the solution $x = 0$. This question is considered in several papers, see for instance [4, 5, 6, 7, 8, 11, 14, 23]. The notions of center, CC-center, persistent center, strongly persistent center, moment conditions and generalized moment conditions are similar to the ones presented for trigonometric Abel equations. For instance the generalized moment conditions read as

$$\int_a^b \widetilde{A}^p(t)\, \widetilde{B}^q(t)\, A(t)\, dt = 0 \quad \text{and} \quad \int_a^b \widetilde{A}^p(t)\, \widetilde{B}^q(t)\, B(t)\, dt = 0, \tag{15}$$

for all $p, q \in \mathbb{N} \cup \{0\}$ and the condition of having a CC-center like

$$\widetilde{A}(t) := \int_a^t A(s)\, ds = A_1(u(t)) \quad \text{and} \quad \widetilde{B}(t) := \int_a^t B(s)\, ds = B_1(u(t)), \tag{16}$$

for some $\mathcal{C}^1$-functions A_1, B_1 and u, where u is such that $u(a) = u(b)$. The following result solves the Composition Conjecture in this setting.

Theorem 7. *Consider the polynomial Abel equation* (14)*. The following statements are equivalent:*

(i) The equation has a strongly persistent center.

(ii) For the corresponding A and B the generalized moment conditions (15) *are satisfied.*

(iii) The equation has a CC-center.

Remark 8. *When a polynomial Abel equation* (14) *has a CC-center it follows from the proof of Theorem 7 that it is possible to choose A_1, B_1 and u in* (16) *being polynomials.*

Our proof of Theorem 7 is based on the following result, which is quite similar to Theorem 6.

Theorem 9. *Let $\mathbb{K}$ be a subfield of $\mathbb{R}(x)$ containing a non-constant polynomial. Then $\mathbb{K} = \mathbb{R}(p)$ for some polynomial p. Moreover, if a polynomial $t \in \mathbb{K}$ then $t = R(p)$ for some polynomial R.*

Proof. By Lüroth's Theorem there exists a rational function $p/q \in \mathbb{R}(x)$, with $\gcd(p,q) = 1$, such that $\mathbb{K} = \mathbb{R}(\frac{p}{q})$. By using a Möbius transformation, if necessary, we can also assume that $\deg p > \deg q$. By hypothesis there exists $t \in \mathbb{R}[x] \cap \mathbb{K}$. Let $R, S \in \mathbb{R}[x]$ be such that $\frac{R}{S}(\frac{p}{q}) = t$ and $\gcd(R,S) = 1$. Equivalently,

$$\frac{q^s \widehat{R}(p,q)}{q^r \widehat{S}(p,q)} = t,$$

where $\widehat{R}$ and $\widehat{S}$ denote the homogenization of R and S and r, s denote the degrees of R and S, respectively. By using similar arguments that in the proof of Theorem 6 we obtain that $\gcd(\widehat{R}(p,q), \widehat{S}(p,q)) = \gcd(q, \widehat{S}(p,q)) = 1$. Hence $\widehat{S}(p,q) = 1$. Since $\deg p > \deg q$ it follows that $\deg \widehat{S}(p,q) = sp$ and hence $s = 0$ and S is constant. So we can assume $S = 1$. Therefore $\frac{\widehat{R}(p,q)}{q^r} = t$. Since $\gcd(\widehat{R}(p,q), q) = 1$ we get that q is also a constant polynomial. Thus, $t = R(p)$ and the result follows. ☐

Proof of Theorem 7. The proofs of implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are similar to the corresponding ones in the trigonometric case.

Next we show that (2) $\Rightarrow$ (3). By Theorem 9, since $\widetilde{A}, \widetilde{B}$ are polynomials, we have that $\mathbb{R}(\widetilde{A}, \widetilde{B}) = \mathbb{R}(p)$ with $p \in \mathbb{R}[x]$. To prove the implication it suffices to show that $p(a) = p(b)$. We know that

$$p = \frac{P(\widetilde{A}, \widetilde{B})}{Q(\widetilde{A}, \widetilde{B})},$$

for some $P, Q \in \mathbb{R}[x,y]$. Derivating this expression we obtain

$$p' = \frac{(QP_x - PQ_x)(\widetilde{A}, \widetilde{B})A + (QP_y - PQ_y)(\widetilde{A}, \widetilde{B})B}{Q^2(\widetilde{A}, \widetilde{B})}.$$

Since $\widetilde{A}$ and $\widetilde{B}$ are polynomial functions of p we have that

$$Q^2(\widetilde{A}, \widetilde{B}) = Q^2(A_1(p), A_2(p)) =: M(p) := N'(p),$$

for some polynomials A_1, A_2 and M, and $N \in \mathbb{R}(x)$ such that $N' = M$. Thus

$$N'(p)p' = Q^2(\widetilde{A}, \widetilde{B})p' = (QP_x - PQ_x)(\widetilde{A}, \widetilde{B})A + (QP_y - PQ_y)(\widetilde{A}, \widetilde{B})B$$

Integrating both sides of this equality in $[a, b]$ and using that the generalized moments vanish we obtain that $N(p(b)) - N(p(a)) = 0$. Since $N'(p) = Q^2(\widetilde{A}, \widetilde{B}) \geq 0$ we have that $N'(x) \geq 0$ for all x in the interval with extremes $p(a)$ and $p(b)$. Therefore N is increasing on this interval and $p(b) = p(a)$, as we wanted to prove. $\square$

Final remarks and open questions

We have solved the Composition Conjecture for Abel equations in the polynomial and trigonometric polynomial settings. Both results can be easily extended for general equations of the form

$$\dot{r} = \sum_{k \geq 2} A_k(\theta) r^k,$$

having either a finite or an infinite sum, with the natural generalizations of the concepts appearing in this paper. We have only focused on the case of Abel equations because it already presents the main difficulties.

From our point of view, there are at least two problems that deserve to be studied in this context. The first one is to know if all the persistent centers are also CC-centers.

The second one appears only in the polynomial case. It turns out that there is no known example that satisfies both moment conditions (5) and (6), and is not a CC-center. Recall that the example given in [21], with A and B constructed by using some Chebyshev polynomials, is not a CC-center but only the moments (5) vanish. The problem is to know whether such an example exists. The results of [20] seem a good starting point to investigate this question.

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