A CLASS OF GALACTIC POTENTIALS: THEIR PERIODIC ORBITS AND INTEGRABILITY

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ABSTRACT. In this work, applying general results from averaging theory, we find periodic orbits for a class of Hamiltonian systems H whose potential models the motion of elliptic galaxies. Using the above periodic orbits on the energy level $H = h$ we provide information about the non-integrability, in the sense of Liouville–Arnold, of the respective Hamiltonian system generated by H.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Galactic dynamics is a branch of Astrophysics whose development started only around sixty years ago, when it was possible to have a view of the physical world beyond the integrable and near integrable systems [4]. Even the importance of the analysis of galactic potentials, the global dynamics of galaxies is not a simple question and represents a big challenge for the researches in the field [2]. Most of the work in the analysis of galaxies is numerical, in this paper we present an analytical technique, the averaging theory, which allows to find periodic orbits of a differential system.

In the last years, great quantity of the research on galactic dynamics has been focused on models of elliptical galaxies. In most of these models the terms in the potential are of even order, so we have adopted this fact in the Hamiltonian system that we are analyzing. Another important point that appears in these kind of potentials is that the existence of periodic orbits is a useful tool for constructing new and more complicated self consistent models. One way to identify periodic orbits is to localize the central fixed points on the surfaces of constant energy. In [9], the authors study the localization of periodic orbits and their linear stability for a particular two-component galactic potential. In fact, in our days the study of individual orbits in some galactic potentials is a new branch of galactic dynamics [5] .

The calculation of particular orbits in some analytical potentials modeling elliptical galaxies, indicates that relatively small symmetry breaking corrections can increase dramatically the number of stochastic orbits, showing the importance of the study of perturbations of simple models [6]. The class of potentials studied in this paper have not chosen with the aim of modeling some particular galaxies, our objective is to study systems which are generic in their basic properties.

In [10], the authors study the galactic potential

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x^2, y^2).$$

These kind of potentials are important in the modeling of elliptic galaxies, as for instance we can mention the potentials $V_L = \log(1 + x^2 + y^2/q)$ and $V_C = \sqrt{1 + x^2 + y^2/q} - 1$, where the parameter q gives the eccentricity of the elliptic galaxy. In this work we study a perturbation of the above potential taking into account a parameter which plays the role of the eccentricity. Our goal is to study which orbits of the unperturbed system persists under the perturbation, and the main technique that we are using is the averaging theory. More concretely, along this paper we deal with the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + x^2) + \frac{1}{2q}(p_y^2 + y^2) + \varepsilon(ax^4 + bx^2y^2 + cy^4) + O(\varepsilon^2), \tag{1}$$

and its respective Hamilton's equation

$$\begin{aligned}
\dot{x} &= p_x, \\
\dot{y} &= \frac{p_y}{q}, \\
\dot{p}_x &= -x - \varepsilon(4ax^3 + 2bxy^2) + O(\varepsilon^2), \\
\dot{p}_y &= -\frac{y}{q} - \varepsilon(2bx^2y + 4cy^3) + O(\varepsilon^2),
\end{aligned} \tag{2}$$

the matrix of the linear part of this system at the origin of coordinates is

$$M = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \frac{1}{q} \\
-1 & 0 & 0 & 0 \\
0 & -\frac{1}{q} & 0 & 0
\end{pmatrix},$$

with eigenvalues $\pm i, \pm i/q$ where $i = \sqrt{-1}$.

From the above we observe the necessity to split the analysis for the periodic orbits in two cases

- q is an irrational number. Here the linear part of system (2) has two planes foliated by periodic orbits. In the first one the orbits have period 2π, each periodic orbit on this plane is of the form

$$PO_1 = (x_0 \cos t + p_{x_0} \sin t, 0, p_{x_0} \cos t - x_0 \sin t, 0).$$

In the second one, the orbits have period $2\pi q$, each periodic orbit on this plane is of the form

$$PO_2 = (0, y_0 \cos(t/q) + p_{y_0} \sin(t/q), 0, p_{y_0} \cos(t/q) - y_0 \sin(t/q)).$$

- q is a rational number. Here the linear part of system (2) has a 4–dimensional space filled of periodic orbits of period $2\pi r$ if $q = r/s$ with $(r, s) = 1$, where each periodic orbit is of the form

$$\begin{aligned}
PO_3 &= (x_0 \cos t + p_{x_0} \sin t, y_0 \cos(st/r) + p_{y_0} \sin(st/r), \\
&\quad p_{x_0} \cos t - x_0 \sin t, p_{y_0} \cos((st/r) - y_0 \sin(st/r)).
\end{aligned}$$

When q is an irrational number our first main result is :

Theorem 1.1. *For ε sufficiently small and q an irrational number, we have that in every energy level $H = h > 0$ the perturbed Hamiltonian system (2) has*

   (a) *at least one periodic solution $((x(t), y(t), p_x(t), p_y(t))$ such that when $\varepsilon \to 0$, we have that $((x(0), y(0), p_x(0), p_y(0))$ tends to $(\sqrt{2h}, 0, 0, 0)$;*

   (b) *at least one periodic solution $((x(t), y(t), p_x(t), p_y(t))$ such that when $\varepsilon \to 0$, we have that $((x(0), y(0), p_x(0), p_y(0))$ tends to $(0, \sqrt{2hq}, 0, 0)$.*

So, for q irrational, we obtain that in every energy level $H = h > 0$ the perturbed Hamiltonian system has at least 2 periodic orbits.

Unfortunately we cannot obtain periodic solutions when when q is a rational number, see Remark 3.1.

Our second main result concerns to the non–integrability of system (2) in the sense of Liouville-Arnold (see definition 2.1), it states the following.

Theorem 1.2. *The Hamiltonian system (2) is either Liouville–Arnold integrable with two independent first integrals H and J, where the gradients of H and J are linearly dependent on some points of the periodic orbits found in Theorem 1.1; or it is not Liouville–Arnold integrable with any second first integral of class C^1.*

The paper is organized as follows. In section 2 we present the theorem from averaging theory necessary to prove our main results, and the basic results on integrability that we shall use. In section 3 we give the proofs of Theorems 1.1 and 1.2.

2. Some results from averaging theory and integrability

In order to have a self contained paper, in this section we present the basic results from the averaging theory and from integrability that are necessary for proving the main results of this paper.

2.1. **Results from averaging theory.**
We consider the problem of the bifurcation of T–periodic solutions from the differential system

$$\mathbf{x}'(t) = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \tag{3}$$

where the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are of class $\mathcal{C}^2$ functions, T–periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. When $\varepsilon = 0$ we get the unperturbed system

$$\mathbf{x}'(t) = F_0(t, \mathbf{x}). \tag{4}$$

One of the main assumptions on the above system is that it has a submanifold of periodic solutions. A solution of system (3), for ε sufficiently small is given using the averaging theory. For a general introduction to the averaging theory see the books of Sanders and Verhulst [12], and of Verhulst [13].

Let $\mathbf{x}(t, \mathbf{z})$ be the solution of the unperturbed system (4) such that $\mathbf{x}(0, \mathbf{z}) = \mathbf{z}$. We write the linearization of the unperturbed system along the periodic solution $\mathbf{x}(t, \mathbf{z})$ as

$$\mathbf{y}' = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}))\mathbf{y}. \tag{5}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (5), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e. $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

Theorem 2.1. *Let $V \subset \mathbb{R}^k$ be open and bounded, and let $\beta_0 : \mathrm{Cl}(V) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that*

   (i) *$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)), \ \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ and that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha)$ of (4) is T–periodic;*
   (ii) *for each $\mathbf{z}_\alpha \in \mathcal{Z}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (5) such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ has in the upper right corner the $k \times (n-k)$ zero matrix, and in the lower right corner a $(n-k) \times (n-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$.*

We consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^k$

$$\mathcal{F}(\alpha) = \xi \left(\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha)) dt \right). \tag{6}$$

If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $\det\left((d\mathcal{F}/d\alpha)(a)\right) \neq 0$, then there is a T–periodic solution $\varphi(t, \varepsilon)$ of system (3) such that $\varphi(0, \varepsilon) \to \mathbf{z}_a$ as $\varepsilon \to 0$.

Theorem 2.1 goes back to Malkin [7] and Roseau [11], for a shorter proof see [3].

2.2. **Results from integrability.** We start this subsection with the definition of integrability in the sense of Liouville-Arnold.

Definition 2.1. A completely Liouville-Arnold integrable system is a symplectic manifold M of dimension $2n$ together with n functions $I_1, I_2, \cdots, I_n$ in involution, that is $\{I_i, I_j\} = 0$ for all i, j, where $\{,\}$ is the Poisson bracket, and with differentials dI_i independent at each cotangent space $T_x^*(M)$. For Hamiltonian systems with two degrees of freedom (the subject of this paper), we say that a Hamiltonian system with Hamiltonian H is Liouville-Arnold integrable if it possesses a first integral J independent of H (i.e. the gradients of H and J are independent in all points of M, except perhaps in a set of zero Lebesgue measure) and $\{H, J\} = 0$.

We also need the following definition.

Definition 2.2. Considerer the smooth dynamical system generated by the differential equation $\dot{x} = f(x)$, if all solutions are defined for all time t, we say that the global flow is complete.

A periodic solution of an autonomous Hamiltonian system on a fixed level of energy, as in our case, always has one multiplier equal to one. Recall that the multipliers of a periodic solution are the eigenvalues of the monodromy matrix $M(t)$ of the variational equations associated to the periodic solution, when it is evaluated at the respective period of the orbit (remember that $M(0) = Id$.) The next theorem is due to Poincaré [8], it

gives a systematic way to study the non integrability in the sense of Liouville-Arnold, in order to apply it we must find periodic orbits with multipliers different from one.

Theorem 2.2. *If a Hamiltonian system with two degrees of freedom and Hamiltonian H is Liouville–Arnold integrable, and J is a second first integral such that the gradients of H and J are linearly independent at each point of a given periodic orbit, then all the multipliers of the variational equations associated to this periodic orbit are equal to one.*

3. Proof of the main Theorems

In this section we give the proofs of our main results Theorem 1.1 and Theorem 1.2.

3.1. **Proof of Theorem 1.1.** We know that the periodic orbits of a Hamiltonian system always appear in cylinders foliated by periodic orbits, each periodic orbit corresponds to a different value of the energy h, see for more details [1]. In order to have isolated periodic orbits and be able to apply the averaging theory we fix the total energy $H = h$. Computing p_x in the energy level $H = h$ we get

$$p_x = \pm\sqrt{2h - p_y^2/q - x^2 - y^2/q - \varepsilon(2ax^4 + 2bx^2y^2 + 2cy^4) + O(\varepsilon^2)}. \tag{7}$$

The fix value h of the total energy is determined by the initial periodic orbit, which in our case for the periodic orbit PO_1 it corresponds to $h = \frac{1}{2}(p_{x_0}^2 + x_0^2)$, choosing the sign $+$ for p_x, and expanding around $\varepsilon = 0$ we obtain

$$p_x = \sqrt{p_{x_0}^2 - x^2 + x_0^2 - (y^2 + p_y^2)/q} - \varepsilon\frac{ax^4 + bx^2y^2 + cy^4}{\sqrt{p_{x_0}^2 - x^2 + x_0^2 - (y^2 + p_y^2)/q}} + O(\varepsilon^2). \tag{8}$$

The equations of motion on the energy level $H = (p_{x_0}^2 + x_0^2)/2$ are

$$\dot{x} = \sqrt{p_{x_0}^2 - x^2 + x_0^2 - (y^2 + p_y^2)/q} - \varepsilon\frac{ax^4 + bx^2y^2 + cy^4}{\sqrt{p_{x_0}^2 - x^2 + x_0^2 - (y^2 + p_y^2)/q}} + O(\varepsilon^2)$$

$$\dot{y} = \frac{p_y}{q}, \tag{9}$$

$$\dot{p}_y = -\frac{y}{q} - \varepsilon(2bx^2y + 4cy^3) + O(\varepsilon^2).$$

In order to apply Theorem 2.1 to system (9), let

$$\begin{aligned}
\mathbf{x} &= (x, y, p_y), \\
F_0(t, \mathbf{x}) &= \left(\sqrt{p_{x_0}^2 - x^2 + x_0^2 - (y^2 + p_y^2)/q}, \quad p_y/q, \quad -y/q\right), \\
F_1(t, \mathbf{x}) &= \left(-\frac{ax^4 + bx^2y^2 + cy^4}{\sqrt{p_{x_0}^2 - x^2 + x_0^2 - (y^2 + p_y^2)/q}}, \quad 0, \quad -(2bx^2y + 4cy^3)\right).
\end{aligned} \tag{10}$$

The set $\Omega = \{(x, y, p_y)|q(p_{x_0}^2 - x^2 + x_0^2) - y^2 - p_y^2 \neq 0\}$ is an open subset of $\mathbb{R}^3$. Clearly the above functions are of class $C^2(\Omega)$. The V of Theorem 2.1 is the set

$$V = \{\mathbf{z} = (x_0, 0, 0) : |x_0| < \rho\} \quad \text{for some } \rho \text{ large enough.}$$

Let $\mathbf{x}(t, \mathbf{z})$ be the solution of the unperturbed system (4) such that $\mathbf{x}(0, \mathbf{z}) = \mathbf{z}$. The variational equations of the unperturbed system along the periodic solution PO_1 are

$$\mathbf{y}' = D_{\mathbf{x}}F_0(t, \mathbf{x}(t, \mathbf{z}))\mathbf{y}, \tag{11}$$

where $\mathbf{y}$ is a 3×3 matrix.

The fundamental matrix $M(t)$ of the differential system (11) such that $M(0)$ is the identity matrix of $\mathbb{R}^3$ takes the simple form

$$M(t) = \begin{pmatrix} \cos t - x_0 \sin(t)/p_{x_0} & 0 & 0 \\ 0 & \cos(t/q) & \sin(t/q) \\ 0 & -\sin(t/q) & \cos(t/q) \end{pmatrix}, \tag{12}$$

whose inverse is given by

$$M^{-1}(t) = \begin{pmatrix} p_{x_0}/(p_{x_0} \cos t - x_0 \sin t) & 0 & 0 \\ 0 & \cos(t/q) & -\sin(t/q) \\ 0 & \sin(t/q) & \cos(t/q) \end{pmatrix}.$$

An easy computation shows that

$$M^{-1}(0) - M^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\sin^2(\pi/q) & \sin(2\pi/q) \\ 0 & -\sin(2\pi/q) & 2\sin^2(\pi/q) \end{pmatrix}.$$

We observe that this matrix has a couple of zeros in the upper right corner of size 1×2; the determinant of the 2×2 matrix which appears in the lower right corner is $4\sin^2(\pi/q) \neq 0$ because q is an irrational number. Consequently all the assumptions of Theorem 2.1 are satisfied. Therefore we must compute the simple zeroes of the function $\mathcal{F}$ defined in Theorem 2.1. A straightforward computations shows that

$$F_1(t, \mathbf{x}(t, \mathbf{z})) = \left(-\frac{a(x_0 \cos t + p_{x_0} \sin t)^4}{p_{x_0} \cos t - x_0 \sin t}, 0, 0 \right),$$

therefore we get

$$M^{-1}(t)F_1(t, \mathbf{x}(t, \mathbf{z})) = \left(-\frac{ap_{x_0}(x_0 \cos t + p_{x_0} \sin t)^4}{(p_{x_0} \cos t - x_0 \sin t)^2}, 0, 0 \right).$$

Let

$$f_1(x_0) = \frac{1}{2\pi} \int_0^{2\pi} -\frac{ap_{x_0}(x_0 \cos t + p_{x_0} \sin t)^4}{(p_{x_0} \cos t - x_0 \sin t)^2} dt = 3ap_{x_0}(p_{x_0}^2 + x_0^2)/2 = 3ah\sqrt{2h - x_0^2}.$$

In the last equality we have gotten p_{x_0} from the energy relation $h = (p_{x_0}^2 + x_0^2)/2$. So the solutions of $f_1 = 0$ are $x_0 = \pm\sqrt{2h}$, which are simple zeroes. On the other hand we can verify that both zeroes generate the same periodic orbit . This completes the proof of statement (a) of Theorem 1.1.

For the proof of statement (b), as in the previous case we fix the value of the total energy as $H = (p_{x_0}^2 + x_0^2)/2q = h$ determined by the initial periodic orbit PO_2. Computing

p_y from the equation $H = h$ we obtain

$$p_y = \pm\sqrt{2qh - qp_x^2 - qx^2 - y^2 - \varepsilon(2aqx^4 + 2bqx^2y^2 + 2cqy^4) + O(\varepsilon^2)}, \qquad (13)$$

we choose the sign $+$ for p_y and expand arounf $\varepsilon = 0$ getting

$$p_y = \sqrt{2qh - q(p_x^2 + x^2) - y^2} - \varepsilon\frac{q(ax^4 + bx^2y^2 + 2y^4)}{\sqrt{2qh - q(p_x^2 + x^2) - y^2}} + O(\varepsilon^2). \qquad (14)$$

We write the equations of motion on the energy level $H = (p_{x_0}^2 + x_0^2)/2q$ in the order (y, x, p_y), they are given by the system

$$
\begin{aligned}
\dot{y} &= \frac{\sqrt{2qh - q(p_x^2 + x^2) - y^2}}{q} - \varepsilon\frac{ax^4 + bx^2y^2 + 2y^4}{\sqrt{2qh - q(p_x^2 + x^2) - y^2}} + O(\varepsilon^2), \\
\dot{x} &= p_x, \qquad\qquad (15) \\
\dot{p}_y &= -\frac{y}{q} - \varepsilon(2bx^2y + 4cy^3) + O(\varepsilon^2).
\end{aligned}
$$

In order to apply Theorem 2.1 to system (15) we are using the same notations and definitions (with the obvious changes) than in the previous case.

Let $\mathbf{x}(t, \mathbf{z})$ be the solution of the unperturbed system (4) such that $\mathbf{x}(0, \mathbf{z}) = \mathbf{z}$. The variational equations of the unperturbed system along the periodic solution PO_2 are

$$\mathbf{y}' = D_{\mathbf{x}}F_0(t, \mathbf{x}(t, \mathbf{z}))\mathbf{y}, \qquad (16)$$

where $\mathbf{y}$ is a 3×3 matrix.

The fundamental matrix $M(t)$ of the differential system (16) such that $M(0)$ is the identity matrix of $\mathbb{R}^3$ takes the simple form

$$M(t) = \begin{pmatrix} \cos(t/q) - y_0\sin(t/q)/p_{y_0} & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}, \qquad (17)$$

whose inverse is given by

$$M^{-1}(t) = \begin{pmatrix} p_{y_0}/(p_{y_0}\cos(t/q) - y_0\sin(t/q)) & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}.$$

An easy computation shows that

$$M^{-1}(0) - M^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\sin^2(\pi q) & \sin(2\pi q) \\ 0 & -\sin(2\pi q) & 2\sin^2(\pi q). \end{pmatrix}.$$

We observe that this matrix has two zeros in the upper right corner of size 1×2; the determinant of the 2×2 matrix which appears in the lower right corner is $4\sin^2(\pi q) \neq 0$ because q is an irrational number. Consequently all the assumptions of Theorem 2.1 are satisfied. Therefore we must compute the simple zeroes of the function $\mathcal{F}$ defined in Theorem 2.1.

A straightforward computations shows that

$$F_1(t, \mathbf{x}(t, \mathbf{z})) = \left(-\frac{c(y_0 \cos{(t/q)} + p_{y_0} \sin{(t/q)})^4}{p_{y_0} \cos{(t/q)} - y_0 \sin{(t/q)}}, 0, 0 \right),$$

therefore we have

$$M^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z})) = \left(-\frac{c p_{y_0} (y_0 \cos{(t/q)} + p_{y_0} \sin{(t/q)})^4}{(p_{y_0} \cos{(t/q)} - y_0 \sin{(t/q)})^2}, 0, 0) \right).$$

Let

$$\begin{aligned}
f_1(y_0) &= \frac{1}{2\pi} \int_0^{2\pi} \left(-\frac{c p_{y_0} (y_0 \cos{(t/q)} + p_{y_0} \sin{(t/q)})^4}{(p_{y_0} \cos{(t/q)} - y_0 \sin{(t/q)})^2} \right) dt \\
&= 3 c p_{y_0} (p_{y_0}^2 + y_0^2)/2 = 3 c h q^2 \sqrt{2hq - y_0^2}.
\end{aligned}$$

In the last equality we have gotten p_{x_0} from the energy relation $h = \frac{(p_{x_0}^2 + x_0^2)}{2q}$. So the solutions of $f_1 = 0$ are $y_0 = \pm\sqrt{2hq}$, which are simple zeroes. On the other hand we can verify that both zeroes generate the same periodic orbit. This completes the proof of statement **(b)** of Theorem 1.1.

Therefore we have proved that for q an irrational number, in every energy level the perturbed Hamiltonian system has at least 2 periodic orbits, so Theorem 1.1 holds.

Remark 3.1. Using the methods of averaging theory studied in this paper, we could not obtain any periodic orbit for the perturbed case when q is a rational number. We have tried to get some information in two different ways, using cartesian coordinates as in in statement **(a)** and using a modified kind of polar coordinates in two different planes. In the first way we have obtained the variational equations, but unfortunately we could not solve them. In the second way we have obtained that one of the equations that we must solve for obtain the periodic solutions is identically zero.

3.2. **Proof of Theorem 1.2.** We first considerer the 2π–periodic orbit PO_1 given in Section 1. In the proof of Theorem 1.1 we have computed the monodromy matrix for the variational equations along the periodic solution PO_1, it is given by equation (12), from here we can verify easily that in this case the eigenvalues of $M(2\pi)$ (the multipliers) are $\lambda_1 = 1, \lambda_2 = \cos{(2\pi/q)} + i \sin{(2\pi/q)}$ and $\lambda_3 = \cos{(2\pi/q)} - i \sin{(2\pi/q)}$. Since q is an irrational number the last two multipliers are different from 1, applying Theorem 2.2 we obtain that either the Hamiltonian system is not Liouville–Arnold integrable, or it is integrable and the gradients of H and J are linearly dependent on some points of the periodic orbit PO_1.

We can also use the $2\pi q$–periodic orbit PO_2, in this case the multipliers are the eigenvalues of the matrix (17) valuate at $2\pi q$. A straightforward computation shows that they are $\lambda_1 = 1, \lambda_2 = \cos{(2\pi q)} + i \sin{(2\pi q)}$ and $\lambda_3 = \cos{(2\pi q)} - i \sin{(2\pi q)}$. We observe that as as in the previous case, two of them are different from one. This completes the proof of Theorem 1.2.

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