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CYCLICITY OF A SIMPLE FOCUS VIA THE VANISHING MULTIPLICITY OF INVERSE INTEGRATING FACTORS

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Abstract. First we provide new properties about the vanishing multiplicity of the inverse integrating factor of a planar analytic differential system at a focus. After we use this vanishing multiplicity for studying the cyclicity of some simple foci of several classes of planar analytic differential systems.

1. Introduction and statement of the results

We consider planar differential systems

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1}$$

where $P, Q : U \to \mathbb{R}$ are $\mathcal{C}^1$ functions defined in the simple connected open subset U of $\mathbb{R}^2$. A $\mathcal{C}^1$ function $R : U \to \mathbb{R}$ such that

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y} \tag{2}$$

is an *integrating factor* of system (1). The differential systems (1) having an integrating factor in U have a first integral $H : U \to \mathbb{R}$ satisfying that

$$RP = \frac{\partial H}{\partial y}, \qquad RQ = -\frac{\partial H}{\partial x}.$$

As usual a *first integral* $H : U \to \mathbb{R}$ is a function constant on the solutions of the differential system (1).

It is immediate to check that R is an integrating factor of system (1) in U if and only if R is a solution of the linear partial differential equation

$$P\frac{\partial R}{\partial x} + Q\frac{\partial R}{\partial y} = -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) R \tag{3}$$

in U.

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A $\mathcal{C}^1$ function $V : U \to \mathbb{R}$ is an *inverse integrating factor* if V verifies the linear partial differential equation

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V \tag{4}$$

in U. We note that V satisfies (4) in U if and only if $R = 1/V$ satisfies (3) in $U \setminus \Sigma$ where $\Sigma = \{(x, y) \in U : V(x, y) = 0\}$.

In 1996 it was proved in [11] the following result. Assume that the $\mathcal{C}^1$ planar differential system (1) defined in the open subset U of $\mathbb{R}^2$ has an inverse integrating factor $V : U \to \mathbb{R}$. If γ is a limit cycle of system (1) contained in U, then γ is contained in Σ. For an easier proof see [13]. After this result many papers have been published studying different aspects of the limit cycles using the properties of the inverse integrating factor. For a good survey see [8].

First in this paper we provide some new properties on the vanishing multiplicity of the inverse integrating factor of a planar analytic differential system, see Theorem 1. Later on we use this vanishing multiplicity for studying the cyclicity of some foci of several classes of planar polynomial differential systems.

We deal with real planar analytic differential system with a *monodromic singular point* at the origin, i.e. we consider differential systems (1) where $P(x, y)$ and $Q(x, y)$ are real analytic functions in a neighborhood $\mathcal{U}$ of the origin such that $P(0, 0) = Q(0, 0) = 0$, and the origin is either a focus or a center. A *focus* is a singular point such that in a neighborhood of it all the orbits different from the singular point spiral either tending to it or going away from it. A *center* is a singular point having a neighborhood filled of periodic orbits with the unique exception of the singular point.

We will only consider analytic system (1) being the origin a *simple focus*, i.e. the monodromic singular point is one of the following three types: non-degenerate focus, degenerate focus without characteristic directions or nilpotent focus (see the definitions in section 2). System (1) having a simple monodromic singular point, after performing a *generalized* polar blow-up, can be transformed into a differential equation defined over a cylinder blowing up the origin into a periodic orbit. More precisely, performing a generalized polar blow-up, system (1) defined in a neighborhood $\mathcal{U}$ of the origin pass to be defined into a cylinder $C = \left\{(r, \theta) \in \mathbb{R} \times \mathbb{S}^1 : |r| < \delta\right\}$ for a certain $\delta > 0$ sufficiently small. Here, we have considered the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}T$ where $T > 0$ is the constant period associated to the polar change and $\mathbb{Z}T = \{kT : k \in \mathbb{Z}\}$. This change to polar coordinates is a diffeomorphism in $\mathcal{U}\setminus\{(0, 0)\}$ and transforms the origin of coordinates into the circle $r = 0$. In fact, the neighborhood $\mathcal{U}$ is transformed into an annulus contained in the half-cylinder $r \geq 0$, but we can consider its extension to the values in which $r < 0$. In generalized polar coordinates system (1) can be seen as a differential equation over the cylinder C of the form

$$\frac{dr}{d\theta} = \mathcal{F}(r,\theta), \tag{5}$$

where $\mathcal{F}(r,\theta)$ is an analytic function in C. The circle $r=0$ is a particular periodic orbit of the differential equation (5) and, therefore, $\mathcal{F}(0,\theta)=0$ for all $\theta \in \mathbb{S}^1$.

In a neighborhood of $r=0$ we can write the Taylor series

$$\mathcal{F}(r,\theta) = \sum_{i\geq \ell} \mathcal{F}_i(\theta) r^i, \tag{6}$$

where $\mathcal{F}_i(\theta)$ are T–periodic functions and $\mathcal{F}_\ell(\theta) \neq 0$. When the origin of (1) is a focus, the circle $r=0$ is an isolated periodic orbit (i.e. a *limit cycle*) of the differential equation (5), and it is a non–isolated periodic orbit when the origin of (1) is a center. The positive integer ℓ which appears in (6) is called the *vanishing multiplicity* of $\mathcal{F}(r,\theta)$ at $r=0$.

Along the paper we shall work with *inverse integrating factors* $V(r,\theta)$ of the differential equation (5); that is, with functions $V: C \to \mathbb{R}$ which are non–locally zero and which admit either a Taylor or Laurent series in a neighborhood of $r=0$,

$$V(r,\theta) = \sum_{i\geq m} v_i(\theta) r^i\ , \tag{7}$$

with $v_m(\theta) \neq 0$ and $m \in \mathbb{Z}$, satisfying the partial differential equation (4) which in polar coordinates writes

$$\frac{\partial V(r,\theta)}{\partial \theta} + \frac{\partial V(r,\theta)}{\partial r}\, \mathcal{F}(r,\theta) = \frac{\partial \mathcal{F}(r,\theta)}{\partial r}\, V(r,\theta). \tag{8}$$

We remark that since $V(r,\theta)$ is a function defined over the cylinder C it needs to be T–periodic in θ. The integer m which appears in (7) is called the *vanishing multiplicity* of $V(r,\theta)$ at $r=0$.

Let $\Psi(\theta;r_0) = \sum_{i\geq 1} \Psi_i(\theta) r_0^i$ be the flow associated to equation (5) such that $\Psi(0;r_0)=r_0$. We recall that the *Poincaré map* $\Pi : \Sigma \subseteq \mathbb{R} \to \mathbb{R}$ associated to the periodic orbit $r=0$ of the differential equation (5) is defined as $\Pi(r_0) = \Psi(T;r_0) = \sum_{i\geq 1} c_i r_0^i$ where the $c_i := \Psi_i(T)$ are called *Poincaré–Liapunov constants.*

To know the value of m in the simple focus case is useful because the Poincaré map Π has a Taylor series of the form $\Pi(r_0) = r_0 + c_m r_0^m + \mathcal{O}(r_0^{m+1})$ with $c_m \neq 0$, see the details in [8]. In a non–degenerate focus we have that $m = 2j+1$ is odd where the integer $j \geq 1$ is called the *order* of the focus.

The *cyclicity of a focus* of an analytic autonomous differential system in the real plane is the maximum number of limit cycles which can bifurcate from the focus under any analytic perturbation. In general to study the cyclicity of a focus is not an easy problem. In [6] assuming the knowledge of an inverse integrating factor the authors study the cyclicity of a simple focus of an analytic system (1) using the vanishing multiplicity of $V(r,\theta)$ at $r=0$. When an inverse integrating factor is known, they proved that the cyclicity of a non-degenerate focus can be given in terms of the vanishing multiplicity of the inverse integrating factor at the origin. For a nilpotent or a degenerate focus without characteristic directions the maximum number of limit cycles which can bifurcate from the focus is also determined in terms of m only when certain perturbations are taken into account, see [6]. To be more precise, consider an analytic system (1) with a simple focus at the origin and take an analytic perturbation of it having the form

$$\dot{x} = P(x,y) + \bar{P}(x,y,\varepsilon), \qquad \dot{y} = Q(x,y) + \bar{Q}(x,y,\varepsilon), \tag{9}$$

where $\varepsilon \in \mathbb{R}^p$ are the parameters of perturbation, $0 < \|\varepsilon\| << 1$ and the functions $\bar{P}(x,y,\varepsilon)$ and $\bar{Q}(x,y,\varepsilon)$ are analytic for $(x,y) \in \mathcal{U}$ a neighborhood of the origin, analytic near $\varepsilon = 0$ and $\bar{P}(x,y,0) = \bar{Q}(x,y,0) \equiv 0$. We associate to the perturbed system (9) the vector field $\mathcal{X}_\varepsilon = (P(x,y) + \bar{P}(x,y,\varepsilon))\partial_x + (Q(x,y) + \bar{Q}(x,y,\varepsilon))\partial_y$. We are interested in giving a sharp upper bound for the number of limit cycles which can bifurcate from the focus at the origin of system (9) under such a kind of perturbation. This sharp (realizable) upper bound is called the *cyclicity* of the origin of system (1) and will be denoted by $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0)$ along this paper. Of course, these limit cycles are created in a Hopf bifurcation.

In relation with system (9), a perturbed field $(\bar{P}(x,y,\varepsilon), \bar{Q}(x,y,\varepsilon))$ is said to have subdegree s if $(\bar{P}(x,y,\varepsilon), \bar{Q}(x,y,\varepsilon)) = \mathcal{O}(\|(x,y)\|^s)$. In this case, we denote by $\mathcal{X}_\varepsilon^{[s]}$ the vector field associated to such a perturbation.

On the other hand, the perturbed vector field $(\bar{P}(x,y,\varepsilon), \bar{Q}(x,y,\varepsilon))$ is said to be $(1,n)$*–quasi–homogeneous of weighted subdegree* (w_x, w_y) if $\bar{P}(\lambda x, \lambda^n y, \varepsilon)$ $= \mathcal{O}(\lambda^{w_x})$ and $\bar{Q}(\lambda x, \lambda^n y, \varepsilon) = \mathcal{O}(\lambda^{w_y})$. In this case, we denote by $\mathcal{X}_\varepsilon^{[w_x, w_y]}$ the vector field associated (9) under such a perturbation.

Our first result study the relationship between the vanishing multiplicities of $\mathcal{F}(r,\theta)$ and of $V(r,\theta)$, enlarging results from [5, 9, 6] in the sense that we do not assume the knowledge of the explicit expression of an inverse integrating factor and only its existence is used. The following result allows in some cases to know the vanishing multiplicity m of $V(r,\theta)$ at $r=0$, and therefore the cyclicity of a simple focus via the vanishing multiplicity of $\mathcal{F}(r,\theta)$ at $r=0$.

Theorem 1. *We assume that the origin of the analytic differential system (1) is a simple focus. Let ℓ and m be the vanishing multiplicities of $\mathcal{F}(r,\theta)$ and of $V(r,\theta)$ at $r=0$, respectively. Then $m \geq \ell \geq 1$. Moreover the following statements hold.*

(a) *We have $m = \ell$ if and only if $v_k(\theta)$ is constant for $k = m, \ldots, 2\ell - 1$.*
(b) *Assume that $\ell \geq 2 + k$ with k a non–negative integer. If*

$$\int_0^T \mathcal{F}_\ell(\theta)\, d\theta = \int_0^T \mathcal{F}_{\ell+1}(\theta)\, d\theta = \cdots = \int_0^T \mathcal{F}_{\ell+k-1}(\theta)\, d\theta = 0,$$

but
$$\int_0^T \mathcal{F}_{\ell+k}(\theta)\, d\theta \neq 0,$$
then $m = \ell + k$.

Our other results are on the class of planar polynomial differential systems of the form

$$\dot{x} = -y + P_n(x,y),\ \ \dot{y} = x + Q_n(x,y), \tag{10}$$

where P_n and Q_n are real homogeneous polynomials of degree $n \geq 2$.

After performing a change of variables to polar coordinates, system (10) can be transformed into a differential system defined over the cylinder defined by $\{(r,\theta) \in \mathbb{R} \times \mathbb{S}^1\}$ with $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Thus, system (10) becomes

$$\dot{r} = r^n a(\theta), \quad \dot{\theta} = 1 + r^{n-1} b(\theta), \tag{11}$$

with $a(\theta)$ and $b(\theta)$ homogeneous trigonometric polynomials of degree $n+1$ given by

$$\begin{aligned} a(\theta) &= \cos\theta\, P_n(\cos\theta, \sin\theta) + \sin\theta\, Q_n(\cos\theta,\sin\theta),\\ b(\theta) &= \cos\theta Q_n(\cos\theta,\sin\theta) - \sin\theta\, P_n(\cos\theta,\sin\theta). \end{aligned}$$

We study the cyclicity of the focus at the origin of the polynomial differential systems (10) for two subclasses of systems (10), the ones having either $a(\theta)$, or $b(\theta)$ equal to a non–zero constant. Our results are stated in the following two theorems.

Theorem 2. *Assume that the polynomial differential system* (10) *has the function* $a(\theta) = a \in \mathbb{R}$. *Then system* (10) *is of the form*

$$\begin{aligned} \dot{x} &= -y\,(1 + \Lambda_{n-1}(x,y)) + a\,x(x^2+y^2)^{\frac{n-1}{2}},\\ \dot{y} &= x\,(1 + \Lambda_{n-1}(x,y)) + a\,y(x^2+y^2)^{\frac{n-1}{2}}, \end{aligned} \tag{12}$$

where n is odd, and $\Lambda_{n-1}(x,y)$ is an arbitrary homogeneous polynomial of degree $n-1$. The following two statements hold.

(a) *The origin of system* (12) *is a focus if and only if $a \neq 0$. In this case, the origin is the unique singularity of system* (12), *and this system has no periodic orbits.*
(b) *The cyclicity of the focus of system* (12) *is* $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0) = (n-1)/2$.

Theorem 3. *Assume that the polynomial differential system* (10) *has the function* $b(\theta) = b \in \mathbb{R}$. *Then system* (10) *is of the form*

$$\begin{aligned} \dot{x} &= -y + x\,\Omega_{n-1}(x,y) - b\,y(x^2+y^2)^{\frac{n-1}{2}},\\ \dot{y} &= x + y\,\Omega_{n-1}(x,y) + b\,x(x^2+y^2)^{\frac{n-1}{2}}, \end{aligned} \tag{13}$$

where n is odd, and $\Omega_{n-1}(x,y)$ is an arbitrary homogeneous polynomial of degree $n-1$. The following two statements hold.

(a) *The origin of system* (13) *is a center if and only if*

$$\int_0^{2\pi} \Omega_{n-1}(\cos\theta, \sin\theta)\, d\theta = 0. \tag{14}$$

(b) *If the origin of system* (13) *is a focus, then its cyclicity is given by* $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0) = (n-1)/2$.

The proof of the cyclicity in Theorem 3 uses results on the vanishing multiplicity of $V(r,\theta)$ provided in the papers [5, 9, 6], whereas the proof of Theorem 2 additionally also uses Theorem 1.

On the other hand the order of the nondegenerate focus for some classes of polynomial differential systems (10) has been studied in [14, 16]. See also for instance the paper [10] for interesting examples about the relationships between order and cyclicity.

Strózyna and Zoladek proved in [17] that there is an analytic change of variables such that any analytic system with nilpotent linear part can be transformed into a generalized Liénard system

$$\dot{x} = -y \ , \ \dot{y} = a(x) + y\tilde{b}(x) \ ,$$

with $a(x) = a_s x^s(1 + \mathcal{O}(x))$, $s \geq 2$ and $\tilde{b}(0) = 0$. In addition, in the monodromic case $s = 2n-1$ with $n \geq 2$, and after the change $x \mapsto u$ with $u(x) = (2n\int_0^x a(z)dz)^{1/(2n)} = x(a_{2n-1} + \mathcal{O}(x))^{1/(2n)}$ and the reparametrization of the time $t \mapsto \tau$ with $dt/d\tau = u^{2n-1}/a(x) = a_{2n-1}^{-1/(2n)} + \mathcal{O}(x)$, we can simplify even more the above normal form. In short it holds that, to study monodromic singular points, we can reduce our attention to the study of the analytic vector field

$$\dot{x} = -y \ , \ \dot{y} = x^{2n-1} + yb(x) \ , \tag{15}$$

where $b(x) = \sum_{j\geq\beta} b_j x^j$. We call n the *Andreev number* associated to system (15). From here, it is not difficult to characterize the centers of monodromic nilpotent singularities; see [3, 15].

Theorem 4 (Moussu)**.** *Consider the analytic system* (15) *having the origin as a monodromic singular point, i.e. satisfying one of the following conditions: (i)* $\beta > n-1$*; (ii)* $\beta = n-1$ *and* $b_\beta^2 - 4n < 0$*; (iii)* $b(x) \equiv 0$*. Then, the origin is a center if and only if* $b(x)$ *is an odd function.*

In [7] it is proved the nonexistence of an analytic first integral of system (15) in a neighborhood of the origin in the center case with $\beta = n-1$. Here we study the cyclicity at the origin of (15) in the focus case with $\beta = n-1$ under the assumption of the existence of a local analytic inverse integrating factor of (15). Our result is the following one.

Theorem 5. *Consider the analytic nilpotent system* (15) *having a focus at the origin with odd Andreev number* n*, and assume the existence of an inverse integrating factor* $V(x,y)$ *analytic in a neighborhood of the focus. If* $\beta = n-1$ *then the following holds.*

(a) *The cyclicity* $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0)$ *of the origin of system* (15) *has the lower bound* $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0) \geq (n+1)/2 - 1$.

(b) *If only analytic perturbations of system* (15) *with* $(1, n)$*–quasihomogeneous weighted subdegrees* (w_x, w_y) *with* $w_x \geq n$ *and* $w_y \geq 2n-1$ *are taken into account, then no limit cycles can bifurcate from the origin of system* (15)*, that is,* $\mathrm{Cycl}(\mathcal{X}_\varepsilon^{[n,2n-1]}, 0) = 0$.

After Theorem 5, we only need to study the vanishing multiplicity m of analytic inverse integrating factors at nilpotent foci of system (15) in the case $\beta > n-1$. We present some examples in the forthcoming proposition. We emphasize that, in these examples we do not assume the existence of a local analytic inverse integrating factor of (15). We only use statement (a) of Theorem 7 which guarantees the existence of a smooth and non–flat inverse integrating factor $V(r, \theta)$ of the associated differential equation (5) in a neighborhood of $r = 0$.

Proposition 6. *Consider system* (15) *with* $\beta > n-1$ *having a focus at the origin and let* m *be the vanishing multiplicity at* $r = 0$ *of a smooth and non–flat inverse integrating factor of the associated equation* (5)*. Then the following holds.*

(a) *If* β *is even, then* $m = \beta - n + 2$ *and moreover* $\mathrm{Cycl}(\mathcal{X}_\varepsilon^{[n,2n-1]}, 0) = \lfloor(\beta - n + 1)/2\rfloor$.

(b) *Let* β *odd and* $b(x) = b_\beta x^\beta + b_\gamma x^\gamma$ *with* $b_\beta b_\gamma \neq 0$*,* $\gamma > \beta$ *and* γ *even.*

(b.1) *If* $\gamma = \beta + 1$ *and* $\beta \geq n+1$*, then* $m = \beta - n + 3$ *and* $\mathrm{Cycl}(\mathcal{X}_\varepsilon^{[n,2n-1]}, 0) = \lfloor(\beta - n + 2)/2\rfloor$.

(b.2) *If* $\gamma = \beta + 3$ *and* $\beta \geq n+3$*, then* $m = \beta - n + 5$ *and* $\mathrm{Cycl}(\mathcal{X}_\varepsilon^{[n,2n-1]}, 0) = \lfloor(\beta - n + 4)/2\rfloor$.

Here $\lfloor . \rfloor$ *denotes the integer part.*

The paper is organized as follows. Theorems 1, 2 and 3 are proved in sections 3, 4 and 5, respectively. Finally, in section 6 we shall prove Theorem 5 and Proposition 6.

2. Preliminary results

2.1. **Focus without characteristic directions.** We consider an analytic system (1) of the form

$$\dot{x} = P_d(x,y) + \tilde{P}(x,y), \qquad \dot{y} = Q_d(x,y) + \tilde{Q}(x,y), \tag{16}$$

where $d \geq 1$ is an odd number, $P_d(x,y)$ and $Q_d(x,y)$ are homogeneous polynomials of degree d and $\tilde{P}(x,y), \tilde{Q}(x,y) \in \mathcal{O}(\|(x,y)\|^{d+1})$. We assume that $P_d^2(x,y) + Q_d^2(x,y) \not\equiv 0$.

We say that a focus at the origin of system (16) is *non-degenerate* if $d = 1$ and the linear part of system (16) has complex eigenvalues of the form $\alpha \pm \beta\, i$ with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$. The origin of system (16) is a *degenerate singular* *point* if the determinant associated to the linear part of (16) is zero. When $d > 1$, the origin of system (16) is a degenerate singular point.

A *characteristic direction* for the origin of system (16) is a linear factor in $\mathbb{R}[x, y]$ of the homogeneous polynomial $xQ_d(x, y) - yP_d(x, y)$. If there are no characteristic directions, then the origin is a monodromic singular point of system (16). We remark that a non-degenerate focus never has characteristic directions.

If $d \geq 1$ and the origin of system (16) is a focus without characteristic directions, we can perform the polar blow–up $x = r\cos\theta$, $y = r\sin\theta$, which transforms the origin of coordinates to the circle of equation $r = 0$. In these new coordinates, system (16) can be seen as a differential equation (5) over the cylinder C where $\mathcal{F}(r, \theta)$ is an analytic function in C. Clearly, in this situation the period is $T = 2\pi$.

2.2. **Nilpotent focus.** We say that the origin of system (1) is a *nilpotent singular point* if it is a degenerate singularity and it can be written as

$$\dot{x} = y + \tilde{P}(x, y) \ , \ \dot{y} = \tilde{Q}(x, y) \ , \tag{17}$$

with $\tilde{P}(x, y)$ and $\tilde{Q}(x, y)$ analytic functions near the origin without constant and linear terms. The problem of knowing if a nilpotent singularity is monodromic was solved by Andreev [2]. System (17) having a monodromic singular point at the origin can be brought by means of an analytic change of variables to the following Andreev analytic normal form

$$\dot{x} \,=\, y\,(-1 + X_1(x, y)), \quad \dot{y} \,=\, f(x) + y\,\phi(x) + y^2\,Y_0(x, y), \tag{18}$$

where $X_1(0,0) = 0$, $f(x) = x^{2n-1} + \cdots$ and either $\phi(x) \equiv 0$ or $\phi(x) = bx^{\beta} + \cdots$ with $\beta \geq n-1$. Here $n \geq 2$ is called the *Andreev number* associated to (17).

We assume that the origin of system (18) is a nilpotent monodromic singular point with Andreev number n. Then, doing the generalized polar blow-up $(x, y) \mapsto (r, \theta)$ with $(x, y) = (r\,\mathrm{Cs}\theta, r^n\,\mathrm{Sn}\theta)$ system (18) pass to an ordinary analytic differential equation (5) over a cylinder. We recall that the functions $\xi(\theta) = \mathrm{Cs}\,\theta$, $\eta(\theta) = \mathrm{Sn}\,\theta$ are the unique solution of the Cauchy problem

$$\frac{d\xi}{d\theta} = -\eta, \ \frac{d\eta}{d\theta} = \xi^{2n-1}, \qquad \xi(0) = 1, \, \eta(0) = 0.$$

Notice that $\mathrm{Cs}\,\theta$ and $\mathrm{Sn}\,\theta$ are T–periodic with $T = 2\sqrt{\dfrac{\pi}{n}}\dfrac{\Gamma\left(\frac{1}{2n}\right)}{\Gamma\left(\frac{n+1}{2n}\right)}$ where $\Gamma(\cdot)$ denotes the Euler Gamma function.

3. Vanishing multiplicities

The existence, uniqueness and regularity of the inverse integrating factor $V(r, \theta)$ of the differential equation (5) in a neighborhood of $r = 0$ is stated in the following theorem. The existential part of (a) is proved in [4], while the uniqueness part is showed in [5], see also [6]. Statement (b) is showed in [9].

Theorem 7. *Let the origin be a simple monodromic singular point of the analytic differential system* (1)*. Then the following statements hold.*

(a) *If the origin is a focus, then there exists an inverse integrating factor $V(r,\theta)$ of the differential equation* (5) *which is smooth and non–flat in the variable r in a neighborhood of $r=0$. Moreover $V(0,\theta)=0$ for all $\theta\in[0,T)$ and $V(r,\theta)$ is unique, up to a nonzero multiplicative constant.*

(b) *If the origin is a center, then there exists an inverse integrating factor $V(r,\theta)$ of the differential equation* (5) *which is analytic in a neighborhood of $r=0$ and such that $V(0,\theta)\neq 0$ for all $\theta\in[0,T)$. Moreover there is an analytic first integral $H(r,\theta)$ of (5) near $r=0$.*

We can consider a more general situation in which $V(r,\theta)$ is either smooth (C^∞) and non–flat in a neighborhood of $r=0$, or it has a finite order pole at $r=0$. Thus function $V(r,\theta)$ has a Laurent series representation of the form (7) with $v_m(\theta)\neq 0$ and $m\in\mathbb{Z}$. Actually, in [5] it is proved that $v_m(\theta)\neq 0$ for $\theta\in[0,T)$. Moreover, in [6] it is shown that if $m\leq 0$, then the origin of system (1) is a center. In [9] it is also proved the following result.

Theorem 8. *We assume that the origin of the differential system* (1) *is a simple monodromic singularity. Let $V(r,\theta)$ be an inverse integrating factor of the corresponding equation* (5)*, which has a Laurent expansion in a neighborhood of $r=0$ of the form* (7)*. Then the origin is a center if and only if*

$$\int_0^T \frac{\mathcal{F}(r,\theta)}{V(r,\theta)}d\theta\equiv 0,$$

for all $r\geq 0$ sufficiently small.

We note that Theorems 7 and 8 will be used in the proofs of our Theorems 2 and 3.

The next simple consequence of Theorem 8 is pointed out in [9] .

Corollary 9. *Let $\ell\geq 1$ and $m\geq 1$ be the vanishing multiplicities of $\mathcal{F}(r,\theta)$ and $V(r,\theta)$ at $r=0$, respectively. If $m<\ell$ then the origin of system* (1) *is a center.*

We will prove a sufficient condition to compute the value of the vanishing multiplicity m of $V(r,\theta)$ at $r=0$, when the origin of the analytic differential system (1) is a focus and we do not know the explicit expression of $V(r,\theta)$.

Proof of Theorem 1. When the origin is a focus, we know that $m\geq\ell\geq 1$, see [9]. Introducing the Taylor series (5) and (7) into the partial differential equation (8) we have

$$(19)\qquad \left[v'_m(\theta)r^m + \cdots\right] + \left[m v_m(\theta) r^{m-1} + \cdots\right]\left[\mathcal{F}_\ell(\theta) r^\ell + \cdots\right] = \left[\ell \mathcal{F}_\ell(\theta) r^{\ell-1} + \cdots\right]\left[v_m(\theta) r^m + \cdots\right],$$

where the dots denote higher order terms. In order to obtain the minimum exponent of the powers of r in (19) we must compare the integer numbers m and $m+\ell-1$, which are equal only in the case $\ell = 1$. Therefore we split the proof into two cases, namely $\ell = 1$ and $\ell \geq 2$.

Case $\ell = 1$. If $\ell = 1$, then equating the coefficients of the power r^m in (19) we get that $v'_m(\theta) = (1-m)\,\mathcal{F}_1(\theta)\, v_m(\theta)$. Therefore

$$v_m(\theta) = v_m(0) \exp\left(\int_0^\theta (1-m)\mathcal{F}_1(\alpha)\right) d\alpha.$$

Now, using the T–periodicity of $v_m(\theta)$ and the fact that $\mathcal{F}_1(\theta) \not\equiv 0$ and $v_m(\theta) \not\equiv 0$, we obtain that $m = 1$ if and only if $v_1(\theta)$ is constant for all $\theta \in [0, T)$, and that if $\int_0^T \mathcal{F}_1(\theta)\, d\theta \neq 0$ then $m = 1$.

In short, we get both statements (a) with $m = \ell = 1$, and (b) with $k = 0$ and $m = \ell = 1$.

Case $\ell \geq 2$. In this case, equating in (19) the coefficients of the powers r^k for $k = m, \ldots, m+\ell-2$, we get that $v_k(\theta) = C_k \in \mathbb{R}$ are constants with $C_m \neq 0$. Now, comparing the coefficients in (19) of the next power $r^{m+\ell-1}$ we obtain

$$(20)\qquad v'_{m+\ell-1}(\theta) = C_m(\ell - m)\mathcal{F}_\ell(\theta)\ .$$

Since $\mathcal{F}_\ell(\theta) \neq 0$, we get that

$$v_{m+\ell-1}(\theta) = v_{m+\ell-1}(0) \exp\left(\int_0^\theta (\ell-m)\mathcal{F}_1(\alpha)\right) d\alpha.$$

From this expression we see that $m = \ell$ if and only if $v_{m+\ell-1}(\theta) = C_{m+\ell-1} = v_{m+\ell-1}(0)$. This proves statement (a) of the theorem.

Since by hypothesis $\ell \geq 2$, equating again in (19) the coefficients of all the powers r^s for $s \geq m+\ell$, we get that

$$(21)\qquad v'_{m+\ell+j}(\theta) = \sum_{i=0}^{j+1} (\ell - m + j + 1 - 2i)\, v_{m+i}(\theta)\, \mathcal{F}_{\ell+j+1-i}(\theta)\ ,$$

for any integer index $j \geq 0$. It is clear that, in order to preserve the T–periodicity of the functions $v_{m+\ell+j}(\theta)$, we must impose that the righthand side of (21) be a function with zero average.

Since $\ell \geq 2+k$, from a previous analysis we know that $v_i(\theta) = C_i \in \mathbb{R}$ are constants for $i = m, \ldots, m+k$ with $C_m \neq 0$. Now, consider the expression of $v'_{m+\ell+k-1}(\theta)$ given by (21) with $j = k-1$, that is,

$$v'_{m+\ell+k-1}(\theta) = \sum_{i=0}^{k} (\ell - m + k - 2i)\, C_{m+i}\, \mathcal{F}_{\ell+k-i}(\theta). \tag{22}$$

Assume now the hypotheses of the statement (b) of the theorem. If $k = 0$, we have $\int_0^T \mathcal{F}_\ell(\theta)\, d\theta \neq 0$. Therefore imposing zero average in the righthand side of (20) we get $m = \ell$.

If $k \geq 1$, then taking zero average in (22) we have that

$$(\ell - m + k)\, C_m \int_0^T \mathcal{F}_{\ell+k}(\theta)\, d\theta = 0,$$

or equivalently $m = \ell + k$. This proves statement (b). □

Remark 10. We will sketch that using the Bautin's method for computing the Poincaré–Liapunov constants we obtain the same conclusion than statement (b) of Theorem 1.

Let $\Psi(\theta; r_0) = \sum_{i \geq 1} \Psi_i(\theta) r_0^i$ be the flow associated to equation (5) such that $\Psi(0; r_0) = r_0$. The Poincaré map $\Pi : \Sigma \subseteq \mathbb{R} \to \mathbb{R}$ is defined as $\Pi(r_0) = \Psi(T; r_0) = \sum_{i \geq 1} c_i r_0^i$ where the $c_i := \Psi_i(T)$ are called Poincaré–Liapunov constants. The values of the Poincaré–Liapunov constants c_i can be determined in a recursive way, although many computations are involved. The standard Bautin's method consists in imposing that $\Psi(\theta; r_0)$ is a solution of (5) and next equating the same powers of r_0. In this way one has a set of recursive linear differential equations for each $\Psi_i(\theta)$ which are uniquely determined from the initial condition $\Psi(0; r_0) = r_0$ which implies that $\Psi_1(0) = 1$ and $\Psi_i(0) = 0$ for $i \geq 2$.

We have $\partial\Psi/\partial\theta = \mathcal{F}(\Psi, \theta)$, that is,

$$\Psi'_1(\theta) r_0 + \cdots = \mathcal{F}_\ell(\theta)\, [\Psi_1(\theta) r_0 + \cdots]^\ell + \mathcal{O}(r_0^{\ell+1}). \tag{23}$$

Equating in (23) the coefficients of the powers r_0^k for $k = 1, \ldots, \ell - 1$ gives $\Psi'_k(\theta) = 0$. Therefore, $\Psi_1(\theta) = 1$ and $\Psi_i(\theta) = 0$ for $i = 2, \ldots, \ell-1$. Thus we have $c_1 = 1$ and $c_i = 0$ for $i = 2, \ldots, \ell - 1$. Equating now the coefficient of r_0^ℓ, the next equation is $\Psi'_\ell(\theta) = \mathcal{F}_\ell(\theta)$ or equivalently $\Psi_\ell(\theta) = \int_0^\theta \mathcal{F}_\ell(\theta)\, d\theta$. Therefore $c_\ell = \int_0^T \mathcal{F}_\ell(\theta)\, d\theta$.

Now we are in position to show statement (b) of Theorem 1 with $k = 0$, that is, if $\int_0^T \mathcal{F}_\ell(\theta)\, d\theta \neq 0$ then $c_\ell \neq 0$ and hence $m = \ell$. Statement (b) of Theorem 1 with $k \geq 1$ is proved in the same way, going even further in the performed analysis.

Remark 11. Since in statement (b) of Theorem 1 we have $\ell \geq 2+k$ (thus $k \leq \ell - 2$) and $m = \ell + k$, the possible values of m allowed by Theorem 1 are bounded by

$$2(k+1) \leq m \leq 2(\ell-1) \ .$$

Actually, since m must be odd in the nondegenerate focus case, the improved bound in this case is $2k+3 \leq m \leq 2\ell - 3$. Unfortunately, the maximum value $2\ell-3$ that m can reach is far from an optimal upper bound for m as the following example shows.

We reproduce using our notation the example of Theorem 1 in [16]. For any $\ell \geq 4$ such that $\ell+1$ is either a prime number or an integer power of a prime number, the equation

$$\frac{dr}{d\theta} = \mathcal{F}(r,\theta) = \frac{r^\ell a(\theta)}{1 + r^{\ell-1} b(\theta)} = \sum_{j\geq 0} (-1)^j \, a(\theta) \, b^j(\theta) \, r^{(j+1)\ell - j},$$

with $a(\theta) = R\sin((\ell+1)\theta)$, $b(\theta) = R\cos((\ell+1)\theta) + 2\sin((\ell-1)\theta)$ and R a nonzero real parameter has $m = 2\ell(\ell-1)+1$.

4. Case $a(\theta)$ constant

We shall need the following result which is a partial result of Theorem 1 of [6].

Theorem 12. *Let* (7) *be an inverse integrating factor of the polynomial differential system* (10) *having a weak focus at the origin. Then* $m \geq 1$ *is odd and the cyclicity of this focus is* $(n-1)/2$.

Proof of Theorem 2. First we claim that system (12) is the more general form of a system (10) whose associated homogeneous trigonometric polynomial $a(\theta)$ is constant. To prove the claim, note that if $a(\theta) = a$ is a constant, we must have

$$xP_n(x,y) + yQ_n(x,y) = a\,(x^2+y^2)^{(n+1)/2}. \tag{24}$$

Define now the polynomial

$$\Delta(x,y) = xQ_n(x,y) - yP_n(x,y). \tag{25}$$

Using the Cramer's rule for solving system (24) and (25) with respect to the variables P_n and Q_n we obtain

$$\begin{aligned} P_n(x,y) &= \frac{\begin{vmatrix} a\,(x^2+y^2)^{(n+1)/2} & y \\ \Delta(x,y) & x \end{vmatrix}}{\begin{vmatrix} x & y \\ -y & x \end{vmatrix}}, \\ Q_n(x,y) &= \frac{\begin{vmatrix} x & a\,(x^2+y^2)^{(n+1)/2} \\ -y & \Delta(x,y) \end{vmatrix}}{\begin{vmatrix} x & y \\ -y & x \end{vmatrix}}. \end{aligned} \tag{26}$$

We have $(x^2+y^2)P_n(x,y) = a\,x(x^2+y^2)^{(n+1)/2} - y\Delta(x,y)$. Thus, we get that x^2+y^2 divides $\Delta(x,y)$ and, therefore $P_n(x,y) = ax(x^2+y^2)^{\frac{n-1}{2}} -$

$y\Lambda_{n-1}(x,y)$, with Λ_{n-1} an arbitrary homogeneous polynomial of degree $n-1$. Moreover, the second equation of (26) gives $Q_n(x,y) = x\Lambda_{n-1}(x,y)) + ay(x^2+y^2)^{\frac{n-1}{2}}$, proving thus the claim.

In polar coordinates system (12) becomes

$$\dot r = ar^n, \quad \dot\theta = 1 + b(\theta)\, r^{n-1}, \tag{27}$$

where $b(\theta) = \Lambda_{n-1}(\cos\theta, \sin\theta)$ is an arbitrary homogeneous trigonometric polynomial of degree $n-1$. If $a>0$ (resp. $a<0$) then in $\mathbb{R}^2\backslash\{(0,0)\}$ we have that $\dot r > 0$ (resp. $\dot r < 0$) and system (12) has no periodic orbit. In particular, the origin is the unique singularity of (12) which is a global repeller or attractor according with $a>0$ or $a<0$, respectively. Note that if $a=0$, then system (12) becomes orbitally equivalent to the linear center, that is, $\dot x = -y(1+\Lambda_{n-1}(x,y))$, $\dot y = x(1+\Lambda_{n-1}(x,y))$.

Near the origin the equation of the orbits of (27) is

$$\frac{dr}{d\theta} = \mathcal{F}(r,\theta) = \frac{ar^n}{1+b(\theta)\, r^{n-1}} = a\, r^n \; + \, \mathcal{O}\left(r^{2n-1}\right). \tag{28}$$

Hence, the vanishing multiplicity of $\mathcal{F}(r,\theta)$ at $r=0$ is n. Since

$$\int_0^{2\pi} \mathcal{F}_n(\theta)\, d\theta = \int_0^{2\pi} a\, d\theta \neq 0,$$

taking into account statement (b) of Theorem 1 with $k=0$, the vanishing multiplicity of $V(r,\theta)$ at $r=0$ is $m=n$. From Theorem 12 the cyclicity of the origin of system (12) is $(n-1)/2$. □

5. Case $b(\theta)$ constant

Proof of Theorem 3. Using an analogous proof to that of Theorem 2 we would see that (13) is the more general form of a system (10) whose homogeneous trigonometric polynomial $b(\theta)$ is constant. If $b(\theta)=b$ is constant, we must have

$$xQ_n(x,y) - yP_n(x,y) = b\,(x^2+y^2)^{(n+1)/2}. \tag{29}$$

Define now the polynomial

$$\Psi(x,y) = xP_n(x,y) + yQ_n(x,y). \tag{30}$$

Applying the Cramer's rule for solving system (29) and (30) with respect to the variables P_n and Q_n we obtain that x^2+y^2 divides $\Psi(x,y)$. Therefore $P_n(x,y) = x\,\Omega_{n-1}(x,y) - b\,y(x^2+y^2)^{\frac{n-1}{2}}$ with Ω_{n-1} an arbitrary homogeneous polynomial of degree $n-1$. Moreover, we obtain that $Q_n(x,y) = y\,\Omega_{n-1}(x,y) + b\,x(x^2+y^2)^{\frac{n-1}{2}}$, and system (10) becomes system (13).

In polar coordinates system (13) becomes

$$\dot r = r^n\, a(\theta), \quad \dot\theta = 1 + b\, r^{n-1}, \tag{31}$$

where $a(\theta) = \Omega_{n-1}(\cos\theta, \sin\theta)$. On the other hand, the differential equation of the orbits of system (31) is

$$\frac{dr}{d\theta} = \mathcal{F}(\theta, r) = \frac{r^n\, a(\theta)}{1 + b\, r^{n-1}} = a(\theta) r^{n+1} + O(r^{2n-1}), \tag{32}$$

which has the inverse integrating factor

$$V(\theta, r) = \frac{r^n}{1 + b\, r^{n-1}} = r^n + O(r^{2n-1}).$$

Therefore, by Theorem 8, the origin is a center of (13) if and only if

$$\int_0^{2\pi} \frac{\mathcal{F}(\theta, r)}{V(\theta, r)}\, d\theta = 0,$$

which implies that the unique center condition is

$$\int_0^{2\pi} a(\theta)\, d\theta = 0,$$

equivalently to (14).

Besides, since n is the vanishing multiplicity of $V(r, \theta)$ on $r = 0$, taking again into account Theorem 12, the cyclicity of the focus at the origin of system (12) is $(n-1)/2$. □

Corollary 13. *System* (13) *has in polar coordinates the first integral*

$$H(\theta, r) = \frac{r^{1-n}}{1-n} + b\, \ln r - A(\theta),$$

being $A(\theta)$ *a primitive of* $a(\theta)$.

Proof. The variables in (32) can be separated as

$$\frac{1 + b\, r^{n-1}}{r^n}\, dr = a(\theta)\, d\theta.$$

Integrating we get the first integral

$$H(\theta, r) = \frac{r^{1-n}}{1-n} + b\, \ln r - A(\theta),$$

being $A(\theta)$ a primitive of $a(\theta)$, that is $A'(\theta) = a(\theta)$. □

Remark 14. System (13) possesses the following dynamic behavior according with the sign of the parameter $b \in \mathbb{R}$.

- If $b = 0$ then $\dot{\theta} = 1$ and therefore the origin is an isochronous monodromic singular point of system (13). The origin is the only finite singularity of (13). Moreover, (13) has a degenerate infinity (i.e. the equator of the Poincaré disc is filled of singular points).
- If $b > 0$ then $\dot{\theta} > 0$. Thus the origin is the only finite singularity of (13) which is monodromic.

- If $b < 0$ then, besides the origin, system (13) can have other singularities. These singular points are located on a circle of radius $R^* = \sqrt[n-1]{-1/b}$. More precisely, the polar coordinates of them are $(r, \theta) = (R^*, \theta^*)$ with $a(\theta^*) = 0$.

6. The cyclicity of some nilpotent focus

The following theorem is one of the main results of [6] and will be strongly used in the proofs of Theorem 5 and Proposition 6 given in this section.

Theorem 15 ([6])**.** *We assume that the origin of system* (17) *is monodromic with Andreev number n. Let $V(r, \theta)$ be an inverse integrating factor of the corresponding equation* (5) *which has a Laurent expansion in a neighborhood of $r = 0$ of the form $V(r, \theta) = v_m(\theta)\, r^m + \mathcal{O}(r^{m+1})$, with $v_m(\theta) \not\equiv 0$ and $m \in \mathbb{Z}$.*

(a) *If the origin of system* (17) *is a focus, then $m \geq 1$, $m + n$ is even and its cyclicity* $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0)$ *satisfies* $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0) \geq (m + n)/2 - 1$. *In this case, m is the vanishing multiplicity of $V(r, \theta)$ on $r = 0$.*

(b) *If the origin of system* (18) *is a focus and if only analytic perturbations of $(1, n)$–quasihomogeneous weighted subdegrees (w_x, w_y) with $w_x \geq n$ and $w_y \geq 2n - 1$ are taken into account, then the maximum number of limit cycles which bifurcate from the origin is $\lfloor (m-1)/2 \rfloor$, that is,* $\mathrm{Cycl}(\mathcal{X}_\varepsilon^{[n,2n-1]}, 0) = \lfloor (m-1)/2 \rfloor$.

Proof of Theorem 5. First of all we recall that the fact of having the Andreev number n odd is a necessary condition for the existence of an analytic inverse integrating factor $V(x, y)$ around any nilpotent focus at the origin for system (17), see [6].

We will denote by $\mathcal{P}_k^{(1,n)} \subset \mathbb{R}[x, y]$ the set of $(1, n)$–quasihomogeneous polynomials of weighted degree k. That is, $p_k(x, y) \in \mathcal{P}_k^{(1,n)}$ if $p_k(\lambda x, \lambda^n y) = \lambda^k p_k(x, y)$ for all $\lambda \in \mathbb{R}$. Moreover, a vector field $\mathcal{X}_i = p_{i+1}\partial_x + q_{i+n}\partial_y$ is a $(1, n)$–quasihomogeneous polynomial vector field of weighted degree i if $p_{i+1} \in \mathcal{P}_{i+1}^{(1,n)}$ and $q_{i+n} \in \mathcal{P}_{i+n}^{(1,n)}$.

With the former definitions, we write the analytic normal form (15) as

$$\dot{x} = \sum_{i \geq n} p_i(x, y)\ ,\ \ \dot{y} = \sum_{i \geq 2n-1} q_i(x, y)\ , \tag{33}$$

where p_i and q_i are in $\mathcal{P}_i^{(1,n)}$. It is clear that

$$p_n(x, y) = -y\ ,\ \ q_{2n-1}(x, y) = \begin{cases} x^{2n-1} & \text{if} \quad \beta > n - 1\ , \\ x^{2n-1} + b_{n-1} y x^{n-1} & \text{if} \quad \beta = n - 1, \end{cases}$$

and $p_j(x, y) \equiv 0$ for all $j > n$. In other words, if $\mathcal{X}$ denotes the associated vector field to system (15), then $\mathcal{X} = \sum_{i \geq n-1} \mathcal{X}_i$ where $\mathcal{X}_i$ denotes a $(1, n)$–quasihomogeneous polynomial vector field of weighted degree i.

Let $V(x,y)$ be an analytic inverse integrating factor of system (15). Then we can assume the following Taylor expansion $V(x,y) = \sum_{i\geq s} V_i(x,y)$ with V_i a given $(1,n)$–quasihomogeneous polynomial of weighted degree i. As usual we denote by $\mathrm{div}\mathcal{X}$ the diveregence of the vector field $\mathcal{X}$. Imposing $\mathcal{X}V = V\,\mathrm{div}\mathcal{X}$ and taking its lower $(1,n)$–quasihomogeneous terms (in fact of weighted degree $n+s-1$) we get

$$\mathcal{X}_{n-1}V_s = V_s\,\mathrm{div}\mathcal{X}_{n-1}\ . \tag{34}$$

That is, $V_s(x,y)$ is an inverse integrating factor of $\mathcal{X}_{n-1}$. Due to the quasi-homogeneity of $\mathcal{X}_{n-1}$, it follows that $V_{2n}(x,y) = xq_{2n-1} - nyp_n$ is an inverse integrating factor of $\mathcal{X}_{n-1}$ and thus a polynomial solution of the partial differential equation (34). In order to have another polynomial solution of (34) linearly independent with V_{2n} it is necessary that $\mathcal{X}_{n-1}$ possesses a polynomial first integral. But this option is only possible when $\beta > n-1$, and therefore $\mathcal{X}_{n-1} = -y\partial_x + x^{2n-1}\partial_y$ is Hamiltonian, see [1].

In short we have proved that, when $\beta = n-1$ any analytic local inverse integrating factor of (15) has a Taylor expansion around the origin of the form $V(x,y) = \sum_{i\geq 2n} V_i(x,y)$ with $V_{2n}(x,y) = x^{2n} + b_{n-1}yx^n + ny^2$. In this case, after taking generalized polar coordinates $x = r\,\mathrm{Cs}\,\theta$, $y = r^n\,\mathrm{Sn}\,\theta$ and using that $x^{2n} + ny^2 = r^{2n}$, system (15) becomes

$$\dot r \;=\; \frac{x^{2n-1}\dot x + y\dot y}{r^{2n-1}} = \Xi(r,\theta), \qquad \dot\theta \;=\; \frac{x\dot y - ny\dot x}{r^{n+1}} = \Theta(r,\theta).$$

Finally, from here we get an ordinary analytic differential equation (5) over a cylinder. Using that $\mathrm{Cs}^{2n}\theta \,+\, n\,\mathrm{Sn}^2\theta \;=\; 1$, we have that the Jacobian determinant of the polar blow-up is r^n and therefore the associated equation (5) has the inverse integrating factor given by

$$\tilde V(r,\theta) \;=\; \frac{V(r\,\mathrm{Cs}\,\theta, r^n\mathrm{Sn}\,\theta)}{r^n\,\Theta(r,\theta)},$$

where $\Theta(r,\theta) \;=\; \Theta_{n-1}(\theta)\,r^{n-1} + \mathcal{O}(r^n)$ with $\Theta_{n-1}(\theta) \;=\; 1 + b_{n-1}\,\mathrm{Cs}^n\theta\,\mathrm{Sn}\,\theta$. We emphasize that $\Theta_{n-1}(\theta) > 0$ as it is proved in [6]. Using quasi-homogeneity leads $V(r\,\mathrm{Cs}\,\theta, r^n\mathrm{Sn}\,\theta) = \sum_{i\geq 2n} w_i(\theta)r^i$ with $w_i(\theta) = V_i(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta)$ and hence $w_{2n}(\theta) = \Theta_{n-1}(\theta)$. Putting all together the result is $\tilde V(r,\theta) = r + \cdots$, that is, the vanishing multiplicity m of $\tilde V$ at $r=0$ is $m=1$. Now, using the results of [6] we obtain the desired result. More precisely, our statements (a) and (b) follow from statements (a) and (b) of Theorem 15 with $m=1$, respectively. □

Proof of Proposition 6. The statements about the cyclicity of the nilpotent focus follow from Theorem 15 once we know the value of m.

First we prove statement (a). Notice that, since β is even we have a focus at the origin of system (15). In short, we have $\dot x = -y$, $\dot y = x^{2n-1} + y\sum_{i\geq\beta} b_i x^i$ which is expressed in generalized polar coordinates $x = r\,\mathrm{Cs}\,\theta$, $y = r^n\,\mathrm{Sn}\,\theta$ as $\dot r = \sum_{i\geq\beta} a_i(\theta)r^{i+1}$, $\dot\theta = r^{n-1} + \sum_{i\geq\beta}\hat b_i(\theta)r^i$

with $a_i(\theta) = b_i \mathrm{Cs}^i\,\theta\,\mathrm{Sn}^2\,\theta$ and $\hat{b}_i(\theta) = b_i \mathrm{Cs}^{i+1}\,\theta\,\mathrm{Sn}\,\theta$ for $i \geq \beta$. Hence we obtain an equation (5) of the form

$$\frac{dr}{d\theta} = \mathcal{F}(r,\theta) = \frac{\sum_{i\geq\beta} a_i(\theta) r^{i-n+2}}{1+\sum_{i\geq\beta}\hat{b}_i(\theta) r^{i-n+1}} = \sum_{i\geq\ell} \mathcal{F}_i(\theta) r^i,$$

with $\ell = \beta - n + 2$. Observe that $\ell \geq 2$ due to the assumption $\beta > n-1$.

We recall briefly at this point two properties of integrals along one period T of the generalized trigonometric functions that we shall use along this proof, see [12] for more details:

$$\int_0^T \mathrm{Cs}^q\,\theta\,\mathrm{Sn}^p\,\theta\,d\theta = \begin{cases} 0 & \text{if} \quad p \text{ or } q \text{ are odd;} \\ \dfrac{2}{\sqrt{n^{p+1}}} \dfrac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2n})}{\Gamma(\frac{p+1}{2}+\frac{q+1}{2n})} & \text{if} \quad \text{both } p \text{ and } q \text{ are even.} \end{cases}$$

Actually we get

$$\int_0^T \mathcal{F}_\ell(\theta)\,d\theta = \int_0^T a(\theta)\,d\theta = b_\beta \int_0^T \mathrm{Cs}^\beta\,\theta\,\mathrm{Sn}^2\,\theta\,d\theta \neq 0,$$

because $b_\beta \neq 0$ and β is even by hypothesis.

Using now statement (b) of Theorem 1 with $k=0$, the vanishing multiplicity of the inverse integrating factor $V(r,\theta)$ at $r=0$ is $m=\ell$.

Now, we shall prove statement (b). We remark that, since β is odd and γ is even we have a focus at the origin for system (15). In short, taking again generalized polar coordinates as before we obtain an equation (5) of the form

$$\frac{dr}{d\theta} = \mathcal{F}(r,\theta) = \frac{a_\beta(\theta) r^\ell + a_\gamma(\theta) r^j}{1+\hat{b}_\beta(\theta) r^{\ell-1} + \hat{b}_\gamma(\theta) r^{j-1}} = \sum_{i\geq\ell} \mathcal{F}_i(\theta) r^i,$$

with $a_i(\theta) = b_i \mathrm{Cs}^i\,\theta\,\mathrm{Sn}^2\,\theta$ and $\hat{b}_i(\theta) = b_i \mathrm{Cs}^{i+1}\,\theta\,\mathrm{Sn}\,\theta$ for $i \in \{\alpha,\beta\}$. Here $\ell = \beta - n + 2 \geq 2$ and $j = \gamma - n + 2 > \ell$. Actually we obtain that

$$\int_0^T \mathcal{F}_\ell(\theta)\,d\theta = \int_0^T a_\beta(\theta)\,d\theta = b_\beta \int_0^T \mathrm{Cs}^\beta\,\theta\,\mathrm{Sn}^2\,\theta\,d\theta = 0,$$

because β is odd. Now we split the proof of the subcases (b.1) and (b.2):

(b.1) Take $\gamma = \beta + 1$, hence $j = \ell + 1$. Now we have the expansion $\mathcal{F}(r,\theta) = a_\beta(\theta) r^\ell + a_{\beta+1}(\theta) r^{\ell+1} + \cdots$ because $\ell \geq 3$ since $\beta \geq n+1$. Therefore

$$\int_0^T \mathcal{F}_{\ell+1}(\theta)\,d\theta = \int_0^T a_{\beta+1}(\theta)\,d\theta = b_{\beta+1} \int_0^T \mathrm{Cs}^{\beta+1}\,\theta\,\mathrm{Sn}^2\,\theta\,d\theta \neq 0,$$

because β is odd. Using now statement (b) of Theorem 1 with $k=1$, the vanishing multiplicity of the inverse integrating factor $V(r,\theta)$ at $r=0$ is $m = \ell+1 = \beta - n + 3$. Recall that we can use Theorem 1 because of the extra hypothesis $\beta \geq n+1$.

(b.2) Take $\gamma = \beta + 3$, hence $j = \ell + 3$. Now we have the expansion $\mathcal{F}(r, \theta) = a_\beta(\theta) r^\ell + a_{\beta+3}(\theta) r^{\ell+3} + \cdots$ because $\ell \geq 5$ since $\beta \geq n + 3$. Hence $\mathcal{F}_\ell(\theta) = a_\beta(\theta)$, $\mathcal{F}_{\ell+1}(\theta) = \mathcal{F}_{\ell+2}(\theta) \equiv 0$ and $\mathcal{F}_{\ell+3}(\theta) = a_{\beta+3}(\theta)$. Therefore

$$\int_0^T \mathcal{F}_\ell(\theta)\, d\theta = \int_0^T \mathcal{F}_{\ell+1}(\theta)\, d\theta = \int_0^T \mathcal{F}_{\ell+2}(\theta)\, d\theta = 0,$$

but

$$\int_0^T \mathcal{F}_{\ell+3}(\theta)\, d\theta = \int_0^T a_{\beta+3}(\theta)\, d\theta = b_{\beta+3} \int_0^T \mathrm{Cs}^{\beta+3}\, \theta \, \mathrm{Sn}^2\, \theta \, d\theta \neq 0,$$

because β is odd. From statement (b) of Theorem 1 with $k = 3$, we have $m = \ell + 3 = \beta - n + 5$.

The proof is finished. □

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