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ON THE NUMBER OF LIMIT CYCLES FOR DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS IN $\mathbb{R}^{2n}$ WITH TWO ZONES

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Abstract. We study the number of limit cycles of the discontinuous piecewise linear differential systems in $\mathbb{R}^{2n}$ with two zones separated by a hyperplane. Our main result shows that at most $(8n-6)^{n-1}$ limit cycles can bifurcate up to first–order expansion of the displacement function with respect to a small parameter. For proving this result we use the averaging theory in a form where the differentiability of the system is not necessary.

1. Introduction and statement of the main result

For a given differential system a *limit cycle* is a periodic orbit isolated in the set of all its periodic orbits. Inside the qualitative theory of differential systems the study of their limit cycles is one of the main topics.

Many questions are considered on the limit cycles of the differential systems in $\mathbb{R}^2$. Thus, one of the main lines of research for such systems is the study of how many limit cycles emerge from the periodic orbits of a centre when we perturb it inside a given class of differential equations, see for example the book [6] and the references there in. More precisely, the problem of considering the planar linear differential centre

$$\dot{x} = -y, \quad \dot{y} = x$$

and perturbing it

$$\dot{x} = -y + \varepsilon f(x, y), \quad \dot{y} = x + \varepsilon g(x, y),$$

inside a given class of differential equations for studying the limit cycles which bifurcate from the periodic orbits of the linear centre. Of course, ε is a small parameter. Here our main concerning is to bring this problem to higher dimension when the perturbation is discontinuous and piecewise linear.

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In [16] Lum and Chua conjectured that a continuous piecewise linear differential system in the plane with two zones separated by an straight line has at most one limit cycle. This conjecture was proved by Freire, Ponce, Rodrigo and Torres [9]. The number of limit cycles for continuous piecewise linear differential system with three zones separated by two parallel hyperplanes in $\mathbb{R}^{2n}$ has been studied in [5] by Cardin, De Carvalho and Llibre.

The purpose of this paper is to study the problem of Lum and Chua extended to discontinuous piecewise linear differential systems in $\mathbb{R}^{2n}$ with two zones separated by a hyperplane. More precisely, we want to study the number of limit cycles of the discontinuous piecewise linear vector fields with two zones in $\mathbb{R}^{2n}$ separated by a hyperplane. This problem in $\mathbb{R}^2$ has been studied by several authors, see for instance [10, 12, 13, 14, 15]. The same problem in $\mathbb{R}^4$ has been studied in [4]. Here we shall study this problem in $\mathbb{R}^{2n}$ for $n \geq 3$.

The study of continuous and discontinuous piecewise linear differential systems goes back to Andronov and coworkers [1], and nowadays they are also studied by many researchers. Moreover, these systems can exhibit complicated dynamical phenomena such as those exhibited by general nonlinear differential systems. Thus, these last years a big interest from the mathematical community takes place trying to understand their dynamical richness, because such systems are widely used to model many real processes and different modern devices, see for more details the book [8] and the references therein. More recently, these systems become also important as idealized models of cell activity, see [7, 18, 19].

We shall study the number of limit cycles of the discontinuous piecewise linear differential systems in $\mathbb{R}^{2n}$ of the form

$$\dot{x} = A_0 x + \varepsilon F(x), \tag{1}$$

where $\varepsilon \neq 0$ is a small parameter,

$$A_0 = \operatorname{diag}\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\},$$

and

$$F(x) = Ax + \varphi(k^T x) b,$$

with $A \in \mathcal{M}_{2n}(\mathbb{R})$, $k, b \in \mathbb{R}^{2n} \backslash \{0\}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is the discontinuous function

$$\varphi(s) = \begin{cases} 0 & s < a \ (a > 0), \\ ms & s \geq a. \end{cases} \tag{2}$$

Our main result is the following *generic* upper bound for the maximum number of limit cycles of the discontinuous piecewise differential linear systems (1) in $\mathbb{R}^{2n}$, which can bifurcate up to first–order expansion of the displacement function, from the periodic orbits of the $2n$–dimensional center (1) with $\varepsilon = 0$. Here by "generic" we mean that the first–order expansion of the displacement function of (1) is not identically zero.

Theorem 1. *For $n \geq 3$ system* (1) *generically has at most* $(8n-6)^{n-1}$ *limit cycles bifurcating from the periodic orbits of the $2n$-dimensional global isochronous center.*

We recall the basic results from averaging theory that we shall need for proving Theorem 1 in section 2, and we shall prove Theorem 1 in section 3.

We note that the bound obtained in Theorem 1 for the discontinuous piecewise differential linear systems (1) in $\mathbb{R}^{2n}$ with two zones separated by a hyperplane, coincides with the bound obtained in [5] for the continuous piecewise differential linear systems in $\mathbb{R}^{2n}$ with three pieces separated by two parallel hyperplanes.

2. Averaging method

The averaging method is a powerful tool in the qualitative theory of differential equations. We present below the first-order averaging method as obtained in [3]. For a systematic treatment of the averaging theory, see [17].

Consider the following differential system

$$\dot{x} = G(t, x, \varepsilon) = \varepsilon H(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{3}$$

where $H : \mathbb{R} \times D \to \mathbb{R}^m$, $R : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^m$ are continuous functions, T–periodic in the first variable, locally Lipschitz in the second variable and D is an open subset of $\mathbb{R}^m$.

For each $z \in D$, we denote by $x(\cdot, z, \varepsilon) : [0, t_z) \to \mathbb{R}^m$ the solution of (3) with $x(0, z, \varepsilon) = z$. We assume that $t_z > T$ for all $z \in D$. Define the *displacement map* $g : D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^m$ associated to system (3) as

$$g(z, \varepsilon) = \int_0^T G(t, x(t, z, \varepsilon), \varepsilon) dt. \tag{4}$$

Then, the zeros of the displacement map g correspond to periodic solutions of the system (3), while isolated zeros of g correspond to limit cycles of (3).

The first–order averaging method for studying the periodic orbits of the differential system (3) is stated in the next result.

Theorem 2. [3, Theorem 1.1] *For the differential system* (3) *we define* $h : D \to \mathbb{R}^m$ *as*

$$h(z) = \int_0^T H(t, z)dt. \tag{5}$$

Assume that for $a \in D$ *with* $h(a) = 0$*, there exists a neighborhood* V *of* a *such that* $h(z) \neq 0$ *for all* $z \in \overline{V}\backslash\{a\}$*, and the Brouwder degree* $d_B(h, V, a) \neq 0$*. Then, for* $|\varepsilon| > 0$ *sufficiently small, there exists an isolated* T*–periodic solution* $\psi(\cdot, \varepsilon)$ *of system* (3) *such that* $\psi(0, \varepsilon) \to a$ *as* $\varepsilon \to 0$*.*

In this paper, we want to use the averaging method to study discontinuous differential systems. Let

$$\dot{x} = G_0(t, x, \varepsilon) = \varepsilon H_0(t, x) + \varepsilon^2 R_0(t, x, \varepsilon) \tag{6}$$

be the differential system (3) satisfying the same assumptions with the exception that G_0 is discontinuous in the second variable at $x = 0$. Clearly, our discontinuous piecewise differential linear systems (1) satisfy the assumptions of the differential systems (6). Moreover, the displacement function of systems (1) are smooth. We have the following result.

Proposition 3. *Assume that the displacement map* g_0 *associated to system* (6) *is smooth. If there exists a family of* T*–periodic smooth maps* $G_\delta(t, x, \varepsilon)$ *such that* $\lim\limits_{\delta \to 0} G_\delta(t, x, \varepsilon) = G_0(t, x, \varepsilon)$*, then we can apply the first–order averaging method directly to system* (6)*.*

Proof. Denote by g_δ the displacement map associated to

$$\dot{x} = G_\delta(t, x, \varepsilon) = \varepsilon H_\delta(t, x) + \varepsilon^2 R_\delta(t, x, \varepsilon). \tag{7}$$

Then g_δ is smooth and $\lim_{\delta \to 0} g_\delta = g_0$. Write

$$h_0(z) = \int_0^T H_0(t, z)dt, \quad h_\delta(z) = \int_0^T H_\delta(t, z)dt.$$

Then since g_0 and g_δ are smooth, we have $g_0(z, \varepsilon) = \varepsilon h_0(z) + O(\varepsilon^2)$ and $g_\delta(z, \varepsilon) = \varepsilon h_\delta(z) + O(\varepsilon^2)$ (cf. [3]). Therefore, we have $\lim_{\delta \to 0} h_\delta = h_0$. Since we can apply the first–order averaging method to systems (7), by passing to the limit, we see that it is equivalent to applying the first–order averaging method directly to system (6). $\square$

3. On the number of limit cycles

For applying the averaging method of section 2 to our discontinuous piecewise differential linear systems (1), we must write system (1) into the "normal" form of the averaging method, i.e. into the form (3), for doing this we need the following lemma of [5].

Lemma 4. [5, Lemma 2] *By a linear change of variables system* (1) *can be transformed into*

$$\dot{x} = A_0 x + \varepsilon \bar{A} x + \varepsilon \varphi(x_1)\bar{b}, \tag{8}$$

where $\bar{A} = (a_{ij})_{1\le i,j\le 2n} \in \mathcal{M}_{2n}(\mathbb{R})$ *and* $\bar{b} = (b_1, \cdots, b_n)^T = e_l$ *for some odd* l*, with* e_l *being the* l*–th vector of the canonical basis of* $\mathbb{R}^{2n}$.

Now we consider the following change of variables

$$x_{2k+1} = r_k \cos\theta_k, \quad x_{2k+2} = r_k \sin\theta_k, \quad k = 0, \cdots, n-1, \tag{9}$$

and write $r = r_0$, $\theta = \theta_0$ and $\varphi_k = \theta_k - \theta$.

Lemma 5. *Doing the change of variables* (9) *system* (8) *is transformed into*

$$\begin{aligned} \frac{dr_k}{d\theta} &= \varepsilon H_{2k+1} + O(\varepsilon^2), \qquad 0 \le k \le n-1, \\ \frac{d\varphi_k}{d\theta} &= \varepsilon H_{2k+2} + O(\varepsilon^2), \qquad 1 \le k \le n-1, \end{aligned} \tag{10}$$

where

$$H_{2k+1} = F_{2k+1}\cos\theta_k + F_{2k+2}\sin\theta_k,$$

$$H_{2k+2} = -\frac{1}{r}(F_2\cos\theta - F_1\sin\theta) + \frac{1}{r_k}(F_{2k+2}\cos\theta_k - F_{2k+1}\sin\theta_k),$$

with

$$F_i = \sum_{j=0}^{n-1}(a_{i(2j+1)} r_j \cos\theta_j + a_{i(2j+2)} r_j \sin\theta_j) + \varphi(r\cos\theta) b_i.$$

Proof. Under the change of variables (9), we have

$$\dot{r}_k = \dot{x}_{2k+1}\cos\theta_k + \dot{x}_{2k+2}\sin\theta_k, \quad \dot{\theta}_k = \frac{1}{r}(\dot{x}_{2k+2}\cos\theta_k - \dot{x}_{2k+1}\sin\theta_k).$$

The lemma then follows from a straightforward computation. □

Note that the displacement map for system (10) is smooth because the Poincaré map is a composition of two smooth maps. It is also easy to find smooth functions approximating the function φ in (2). Therefore, by Proposition 3, we can apply the averaging method directly to the discontinuous system (10).

Write

$$h_j = \int_0^{2\pi} H_j d\theta, \quad j = 1, 3, 4, \cdots, 2n,$$

and

$$I_1(r) = \int_0^{2\pi} \varphi(r\cos\theta)\cos\theta d\theta, \quad I_2(r) = \int_0^{2\pi} \varphi(r\cos\theta)\sin\theta d\theta, \quad r > 0.$$

Lemma 6. *We have $I_2(r) = 0$ for all $r > 0$ and*

$$I_1(r) = \begin{cases} 0 & \text{if } r < a, \\ m\left(\dfrac{a\sqrt{r^2-a^2}}{r} + r\arctan\dfrac{\sqrt{r^2-a^2}}{a}\right) & \text{if } r \geq a. \end{cases} \tag{11}$$

Proof. For $r < a$, we have $\varphi(r\cos\theta) \equiv 0$. Thus assume $r \geq a$, and let θ^* be such that $\cos\theta^* = a/r$. Then

$$I_1(r) = \int_0^{\theta^*} \varphi(r\cos\theta)\cos\theta d\theta + \int_{2\pi-\theta^*}^{2\pi} \varphi(r\cos\theta)\cos\theta d\theta,$$
$$I_2(r) = \int_0^{\theta^*} \varphi(r\cos\theta)\sin\theta d\theta + \int_{2\pi-\theta^*}^{2\pi} \varphi(r\cos\theta)\sin\theta d\theta.$$

The lemma then follows from a straightforward computation. □

Lemma 7. *We have*

(12)

$$h_{2k+1} = \sum_{j=0}^{n-1}(c_{kj}\cos(\varphi_j - \varphi_k) + d_{kj}\sin(\varphi_j - \varphi_k))r_j + b_{2k+1}\cos\varphi_k I_1(r),$$

$$h_{2k+2} = d_{00} + \sum_{j=1}^{n-1}(d_{0j}\cos\varphi_j - c_{0j}\sin\varphi_j)\frac{r_j}{r} -$$
$$\sum_{j=0}^{n-1}(d_{kj}\cos(\varphi_j - \varphi_k) - c_{kj}\sin(\varphi_j - \varphi_k))\frac{r_j}{r_k} - \frac{b_{2k+1}}{r_k}\sin\varphi_k I_1(r),$$

where for $0 \leq k, j \leq n-1$,

$$c_{kj} = \pi(a_{(2k+1)(2j+1)} + a_{(2k+2)(2j+2)}),$$
$$d_{kj} = \pi(a_{(2k+1)(2j+2)} - a_{(2k+2)(2j+1)}).$$

Proof. It is easy to check that we have

$$\int_0^{2\pi} \cos\theta_k \cos\theta_j d\theta = \int_0^{2\pi} \sin\theta_k \sin\theta_j d\theta = \pi\cos(\varphi_j - \varphi_k),$$
$$\int_0^{2\pi} \cos\theta_k \sin\theta_j d\theta = -\int_0^{2\pi} \cos\theta_j \sin\theta_k d\theta = \pi\sin(\varphi_j - \varphi_k),$$

$$\int_0^{2\pi} \varphi(r\cos\theta)\cos\theta_k d\theta = \cos\varphi_k I_1(r),$$
$$\int_0^{2\pi} \varphi(r\cos\theta)\sin\theta_k d\theta = \sin\varphi_k I_1(r).$$

Note also that by Lemma 4, we have $b_i = 0$ for all i even. The lemma then follows from a straightforward computation. □

We also need the following result for proving our main result.

Lemma 8. *The equation $I_1(r) = cr$ $(c \neq 0)$ has a unique solution if $0 < c/m < \pi/2$, and no solutions otherwise.*

Proof. Set $u = (\sqrt{r^2 - a^2})/a$. Then, by Lemma 6, $I_1(r) = cr$ is equivalent to

$$\arctan u = \frac{c}{m} - \frac{u}{1+u^2}.$$

It is easy to check that the above equation has a (unique) solution for $u > 0$ if and only if $0 < c/m < \pi/2$. □

We are now ready to prove our main result.

Proof of Theorem 1. First consider the case $\bar{b} = e_1$. For $0 \leq k, j \leq n-1$, set

$$\alpha_{kj} = c_{kj}\cos(\varphi_j - \varphi_k) + d_{kj}\sin(\varphi_j - \varphi_k).$$

Write $B = (\alpha_{kj})_{1\leq k,j\leq n-1}$ and $\alpha = -(\alpha_{10}, \cdots, \alpha_{(n-1)0})^T$. Then the system of equations $\{h_{2k+1} = 0\}_{1\leq k\leq n-1}$ can be written as

$$B\gamma = r\alpha, \qquad \gamma = (r_1, \cdots, r_{n-1})^T. \tag{13}$$

Write $u_k = \cos\varphi_k$ and $v_k = \sin\varphi_k$. Then α_{kj} are polynomials of u_k and v_k with degα_{kj} equal to zero if $k = j$, one if $k = 0$ or $j = 0$, and two otherwise. From (13) we get

$$r_i = r\frac{\Delta_i}{\Delta}, \qquad 1 \leq i \leq n-1, \tag{14}$$

where Δ_i and Δ are polynomials of u_k and v_k with deg$\Delta_i = 2n-3$ and deg$\Delta = 2n-2$.

Substituting (14) into the system of equations $\{h_{2k+2} = 0\}_{1\leq k\leq n-1}$ and multiplying the k–th equation by $r_k\Delta^2/r$, we get

$$\sum_{j=0}^{n-1}(\mu_j\Delta_k + \nu_{kj}\Delta)\Delta_j = 0, \qquad 1 \leq k \leq n-1, \tag{15}$$

where μ_j and ν_{kj} are polynomials of u_k and v_k with deg$\mu_j = 1$ and deg$\nu_{kj} = 2$. Thus each equation of (15) is of degree $4n-3$.

Note that u_k and v_k also satisfy the set of equations

$$u_k^2 + v_k^2 = 1, \qquad 1 \le k \le n-1. \tag{16}$$

Therefore, by Bezout's Theorem (see [11]), the system (15) and (16) has at most $(8n-6)^{n-1}$ solutions (counted with their multiplicities).

Substituting each solution of (15) and (16) into $h_1 = 0$, we get an equation of the form $I_1(r) = cr$. By genericity we have $c \neq 0$. And by Lemma 8, we get at most one solution of r for each solution of $(u_k, v_k)_{1\le k\le n-1}$.

The cases $\bar{b} = e_l$, $l = 3, 5, \cdots, 2n-1$, can be treated similarly. We first solve the system of equations $\{h_{2k+1} = 0\}_{2k+1\neq l}$ and then substitute the solutions into the system of equations $\{h_{2k+2} = 0\}_{1\le k\le n-1}$. We get a system of equations similar to (15), but with degree less than $4n-3$. Thus the number of solutions in each of these cases does not exceed $(8n-6)^{n-1}$. $\square$

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