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GENERALIZED WEIERSTRASS INTEGRABILITY OF THE ABEL DIFFERENTIAL EQUATIONS

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Abstract. We study the Abel differential equations that admits either a generalized Weierstrass first integral or a generalized Weierstrass inverse integrating factor.

1. Introduction and statement of the main results

Let x and y be complex variables. In this paper we study the differential equations of the form

$$\frac{dy}{dx} = a(x)y^3 + b(x)y^2 + c(x)y + d(x), \tag{1}$$

where a, b, c, d are meromorphic functions of x. In fact, the differential equation (1) is called *Abel differential equation* when $a(x) \not\equiv 0$, is called *Riccati differential equation* when $a(x) \equiv 0$ and $b(x) \not\equiv 0$, and is called *linear differential equation* when $a(x) = b(x) \equiv 0$ and $c(x) \not\equiv 0$.

Equations (1) where studied by the first time by Abel in his analysis on the elliptic functions (see [1]). Abel equations appear in the reduction of order of many second and higher order families, and hence are frequently found in the modeling of real problems in several areas. Thus, for instance Abel differential equations appear in cosmology (see [10]), in control theory of electrical circuits (see [6]), in ecology (see [5]), ...

In what follows instead of working with the Abel differential equation (1) we shall work with the equivalent differential system

$$\dot{x} = 1, \quad \dot{y} = a(x)y^3 + b(x)y^2 + c(x)y + d(x), \tag{2}$$

where the dot denotes derivative with respect to the time t, real or complex.

Let $U \subset \mathbb{C}^2$ be the domain of definition of system (2). Let W be an open and dense subset of U. If this system has a non-constant function $H : W \to \mathbb{C}$ which is constant over its solutions, then H is a *first integral* of it.

The objective of this paper is to study the integrability of the Abel differential equations restricted to a special kind of first integrals. For such systems the notion of integrability is based on the existence of a first integral, and we want to characterize

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when the differential equations (1) have either a Weierstrass first integral or a Weierstrass inverse integrating factor. The integrability of the Abel differential equations has been studied by several authors, see [3, 4, 8, 11, 8] to cite just a few. For instance in [3], [4] and [8], the authors provide a list of the known integrable Abel differential equations with a, b, c and d rational functions.

As usual $\mathbb{C}[[x]]$ is the ring of formal power series in the variable x with coefficients in $\mathbb{C}$, and $\mathbb{C}[y]$ is the ring of polynomials in the variable y with coefficients in $\mathbb{C}$. A polynomial of the form

$$\sum_{i=0}^{n} a_i(x)y^i \in \mathbb{C}[[x]][y], \tag{3}$$

is called a *formal Weierstrass polynomial* in y of degree n if and only if $a_n(x) = 1$ and $a_i(0) = 0$ for $i < n$. A formal polynomial whose coefficients are convergent is called *Weierstrass polynomial*, see [2].

A polynomial of the form (3) is called a *formal generalized Weierstrass polynomial* in y of degree n if and only if $a_n(x) \not\equiv 0$. A formal polynomial whose coefficients are convergent is called *generalized Weierstrass polynomial.*

Let $V \colon W \to \mathbb{C}$ be a function satisfying

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}(a(x)y^3 + b(x)y^2 + c(x)y + d(x)) = (3a(x)y^2 + 2b(x)y + c(x))V.$$

Then V is an *inverse integrating factor*, and it is known that there exists a first integral H such that

$$\frac{1}{V} = \frac{\partial H}{\partial y}, \qquad \frac{a(x)y^3 + b(x)y^2 + c(x)y + d(x)}{V} = -\frac{\partial H}{\partial x}. \tag{4}$$

We say that a differential system (2) is *Weierstrass integrable* if it admits a first integral or an inverse integrating factor which is a Weierstrass polynomial. In [7] this definition is given in a more general context. We say that a differential system (2) is *generalized Weierstrass integrable* if it admits a first integral or an inverse integrating factor which is a generalized Weierstrass polynomial.

The main objective of this paper is to provide the Abel differential equations that have (generalized) Weierstrass first integrals, or (generalized) Weierstrass integrating factors. More precisely: *How to recognize functions $a(x)$, $b(x)$, $c(x)$ and $d(x)$ for which the Abel differential equation* (1) *is (generalized) Weierstrass integrable*?

Our main results are the following.

Theorem 1. *System* (2) *admits a generalized Weierstrass first integral of the form*

$$H = H_s(x)y^s + H_{s-1}(x)y^{s-1} + \cdots + H_1(x)y + H_0(x) = \sum_{i=0}^{s} H_i(x)y^i. \tag{5}$$

if and only if $a(x) = b(x) \equiv 0$ and in this case the first integral is

$$H = H_1(x)y + H_0(x) = e^{-\int c(x)\,dx}y - \int d(x)e^{-\int c(u)\,du}\,dx.$$

Corollary 2. *System* (2) *admits a Weierstrass first integral of the form* (5) *if and only if* $a(x) = b(x) = c(x) \equiv 0$. *In this case the first integral is*

$$H = H_1(x)y + H_0(x) = y - \int_0^x d(s)\, ds.$$

Theorem 1 and Corollary 2 are proved in section 2. From Theorem 1 and Corollary 2 it follows that the Abel and Riccati differential equations have neither generalized Weierstrass first integrals, nor Weierstrass first integrals.

We look for inverse integrating factors of the form

$$V = V_s(x)y^s + V_{s-1}(x)y^{s-1} + \cdots + V_1(x)y + V_0(x) = \sum_{i=0}^{s} V_i(x)y^i \tag{6}$$

with $V_s(x) \not\equiv 0$.

Theorem 3. *The following holds for system* (2).

(a) *If* $a(x) \not\equiv 0$ *it admits a generalized Weierstrass inverse integrating factor of the form* $V_0(x) + V_1(x)y + V_2(x)y^2 + V_3(x)y^3$ *if and only if*
  (i) *either* $d(x) \neq -\big(2b(x)^3 - 9b(x)(a(x)c(x) + a'(x)) + 9a(x)b'(x)\big)/(27a(x)^2)$ *and* $K_1 = 0$ *and* $K_2 = 0$,
  (ii) *or* $d(x) = -\big(2b(x)^3 - 9b(x)(a(x)c(x) + a'(x)) + 9a(x)b'(x)\big)/(27a(x)^2)$ *and*

$$c(x) = \frac{b(x)^2}{3a(x)} + \frac{a(x)c_1}{V_3(x)} - \frac{V_3'(x)}{2V_3(x)}, \quad c_1 \in \mathbb{C}.$$

(b) *If* $a(x) \equiv 0$ *and* $b(x) \not\equiv 0$ *it admits a generalized Weierstrass inverse integrating factor of the form* $V_0(x) + V_1(x)y + V_2(x)y^2$ *if and only if*

$$d(x) = \frac{1}{4b(x)^2V_2(x)^2}\big(4b(x)^3d_1 + 2V_2(x)b'(x)(c(x)V_2(x) + V_2'(x)) + b(x)(c(x)^2V_2(x)^2 + V_2'(x)^2 - 2V_2(x)(V_2(x)c'(x) + V_2''(x)))\big), \quad d_1 \in \mathbb{C}.$$

(c) *If* $a(x) = b(x) \equiv 0$, *then* V *admits a generalized Weierstrass inverse integrating factor* $V = e^{\int c(x)\, dx}$.

The conditions $K_1 = 0$, $K_2 = 0$ *are given in the appendix.*

The explicit expressions for the generalized Weierstrass inverse integrating factors are given along the proof of Theorem 3.

Theorem 4. *The following holds for system* (2).

(a) *If* $a(x) \not\equiv 0$ *it admits a Weierstrass inverse integrating factor of the form* (6) *if and only if* $a(0) \neq 0$, $b(0) = 0$, $c(x) = b(x)^2/(3a(x))$ *and*

$$d(x) = \frac{1}{27a(x)^2}\big(b(x)^3 + 9b(x)a'(x) - 9b'(x)a(x)\big).$$

In this case $V = V_0(x) + V_1(x)y + V_2(x)y^2 + y^3$ *where* $V_2(x) = b(x)/a(x)$, $V_1(x) = c(x)/a(x)$ *and*

$$V_0(x) = \frac{1}{3a(x)^3}(-b(x)a'(x) + a(x)(3a(x)d(x) + b'(x))).$$

(b) *If* $a(x) \equiv 0$ *and* $b(x) \not\equiv 0$ *it admits a Weierstrass inverse integrating factor if and only if* $b(0) \neq 0$, $c(0) = 0$ *and*

$$d(x) = \frac{1}{4b(x)^2}(c(x)^2 b(x) + 2c(x)b'(x) - 2b(x)c'(x)).$$

In this case $V = V_0(x) + V_1(x)y + y^2$ *where* $V_1(x) = c(x)/b(x)$ *and*

$$V_0(x) = \frac{1}{2b(x)^3}(2b(x)^2 d(x) - c(x)b'(x) + b(x)c'(x)).$$

(c) *If* $a(x) = b(x) \equiv 0$ *then admits a Weierstrass inverse integrating factor if and only if* $c(x) \equiv 0$ *and in this case the inverse integrating factor is* $V = 1$.

The proof of Theorems 3 and 4 is given in section 3. Note that, from Theorems 3 and 4, there are Abel, Riccati and linear differential equations having either generalized Weierstrass inverse integrating factors, or Weierstrass integrating factors.

The analytic conditions for the existence of either generalized Weierstrass inverse integrating factors, or Weierstrass inverse integrating factors have been computed with the help of the algebraic manipulator mathematica.

2. Proof of Theorems 1 and 2

To prove Theorems 1 and 2 we first state and prove an auxiliary result.

Proposition 5. *If* $a(x) \not\equiv 0$ *or* $b(x) \not\equiv 0$, *system* (2) *does not admit a generalized Weierstrass first integral of the form* (5).

Proof. Imposing that system (2) has a first integral (5) (with $H_s(x) \neq 0$) we obtain a polynomial in y whose coefficients must be zero. Hence we get that

$$\sum_{i=0}^{s} H_i'(x)y^i + \sum_{i=0}^{s} iH_i(x)y^{i-1}(a(x)y^3 + b(x)y^2 + c(x)y + d(x)) = 0. \tag{7}$$

We first assume that $a(x) \not\equiv 0$. Hence we obtain a recursive differential system to determine the functions $H_i(x)$ for $i = 0, \ldots, s$. The highest power is y^{s+2} and its coefficient is $sH_s(x)a(x) = 0$. Since we are assuming that $a(x) \neq 0$ we get $H_s(x) = 0$ or $s = 0$. So $H = H_0(x)$. Then, from (7), it follows that $H_0'(x) = 0$, that is $H = H_0$, a constant in contradiction with the fact that H is a first integral.

Now we assume that $a(x) \equiv 0$ and $b(x) \not\equiv 0$. Again we obtain a recursive differential system to determine the functions $H_i(x)$ for $i = 0, \ldots, s$. The highest power is y^{s+1} and its coefficient is $sH_s(x)b(x) = 0$. Since we are assuming that $b(x) \neq 0$ we get $H_s(x) = 0$ or $s = 0$. So $H = H_0(x)$, and the proof follows as in the case $a(x) \neq 0$. □

Proof of Theorem 1. In view of Proposition 5 in order that system (2) has a generalized Weierstrass first integral of the form (5) we must have that $a(x) \equiv b(x) \equiv 0$. Therefore, system (2) becomes

$$\dot{x} = 1, \quad \dot{y} = c(x)y + d(x). \tag{8}$$

This is a linear differential system having the first integral

$$H = e^{-\int c(s)\,ds} y - \int d(s) e^{-\int c(u)\,du}\,ds. \tag{9}$$

This completes the proof. □

Proof of Corollary 2. It follows immediately from the definition of Weierstrass first integral and Theorem 1. □

3. Proof of Theorems 3 and 4

To prove Theorems 3 and 4 we will first state and prove some auxiliary results.

Proposition 6. *If $a(x) \not\equiv 0$ then system (2) admits a generalized Weierstrass inverse integrating factor of the form (6) with $s = 3$. Moreover, if $a(x) \equiv 0$ and $b(x) \not\equiv 0$, system (2) admits a generalized Weierstrass inverse integrating factor of the form (6) with $s = 2$.*

Proof. Imposing that system (2) has an inverse integrating factor of the form (6) we obtain a polynomial in y whose coefficients must be zero. Hence we get that

$$\begin{aligned}&\sum_{i=0}^{s} V_i'(x) y^i + \sum_{i=0}^{s} i V_i(x) y^{i-1}(a(x)y^3 + b(x)y^2 + c(x)y + d)\\ &= (3a(x)y^2 + 2b(x)y + c(x))\Big(\sum_{i=0}^{s} V_i(x) y^i\Big).\end{aligned} \tag{10}$$

We first assume that $a(x) \not\equiv 0$. In this case, computing the terms in (10) with y^{s+2} we get

$$sV_s(x)a(x) = 3V_s(x)a(x), \quad \text{i. e.} \quad s = 3.$$

Now assume $a(x) \equiv 0$ and $b(x) \not\equiv 0$. Computing the terms in (10) with y^{s+1} we get

$$sV_s(x)b(x) = 2V_s(x)b(x), \quad \text{i. e.} \quad s = 2.$$

□

Proposition 7. *Assume that $a(x) \not\equiv 0$ in system (2). Then this system admits a generalized Weierstrass inverse integrating factor of the form $V = V_0(x) + V_1(x)y + V_2(x)y^2 + V_3(x)y^3$ if and only if*

(i) $d(x) \neq -\big(2b(x)^3 - 9b(x)(a(x)c(x) + a'(x)) + 9a(x)b'(x)\big)/(27a(x)^2)$ *and* $K_1 = 0$ *and* $K_2 = 0$,

(ii) $d(x) = -\big(2b(x)^3 - 9b(x)(a(x)c(x) + a'(x)) + 9a(x)b'(x)\big)/(27a(x)^2)$ *and*

$$c(x) = \frac{b(x)^2}{3a(x)} + \frac{a(x)c_1}{V_3(x)} - \frac{V_3'(x)}{2V_3(x)}, \quad c_1 \in \mathbb{C}.$$

The conditions $K_1 = 0$ and $K_2 = 0$ are given in the Appendix.

Proof. By Proposition 6 if V is an inverse integrating factor of the form (10) it must have $s = 3$. Hence, computing in (10) with $s = 3$ the coefficients of y^k for $k = 4, 3, 2, 1, 0$ are

$$\begin{aligned} e_4 &= -a(x)V_2(x) + b(x)V_3(x), \\ e_3 &= -2a(x)V_1(x) + 2c(x)V_3(x) + V_3'(x), \\ e_2 &= -3a(x)V_0(x) - b(x)V_1(x) + c(x)V_2(x) + 3d(x)V_3(x) + V_2'(x), \\ e_1 &= -2b(x)V_0(x) + 2d(x)V_2(x) + V_1'(x), \\ e_0 &= -c(x)V_0(x) + d(x)V_1(x) + V_0'(x). \end{aligned} \tag{11}$$

Solving $e_4 = 0$ we get that

$$V_2(x) = \frac{b(x)V_3(x)}{a(x)}. \tag{12}$$

Then, from $e_3 = 0$ we obtain

$$V_1(x) = \frac{2c(x)V_3(x) + V_3'(x)}{2a(x)}. \tag{13}$$

Now from $e_2 = 0$ we have that

$$V_0(x) = \frac{2V_3(x)(-b(x)a'(x) + a(x)(3a(x)d(x) + b'(x))) + a(x)b(x)V_3'(x)}{6a(x)^3}. \tag{14}$$

Case 1: $2b(x)^3 - 9b(x)(a(x)c(x) + a'(x)) + 9a(x)(3a(x)d(x) + b'(x)) \neq 0$. Substituting $V_k(x)$ for $k = 0, 1, 2$ into $e_1 = 0$ and $e_0 = 0$ we obtain two expressions where appear $V_3''(x)$. Isolating from both expressions $V_3''(x)$ and equating them we have a differential equation for $V_3'(x)$. Solving this differential equation we get

$$V_3(x) = \frac{a(x)^2}{(2b(x)^3 - 9b(x)(a(x)c(x) + a'(x)) + 9a(x)(3a(x)d(x) + b'(x)))^{2/3}}.$$

Substituting $V_3(x)$ in $e_1 = 0$ and $e_0 = 0$ we obtain the two conditions $K_1 = 0$ and $K_2 = 0$ that the coefficients of the Abel equation must satisfy in order that the equation has a generalized Weierstrass inverse integrating factor.

Case 2: $2b(x)^3 - 9b(x)(a(x)c(x) + a'(x)) + 9a(x)(3a(x)d(x) + b'(x)) = 0$, i.e.

$$d(x) = -\frac{1}{27a(x)^2}\big(2b(x)^3 - 9b(x)(a(x)c(x) + a'(x)) + 9a(x)b'(x)\big).$$

Substituting $d(x)$ and $V_k(x)$ for $k = 0, 1, 2$ into e_1 and e_0 we obtain that $e_0 = b(x)e_1/(3a(x))$. Therefore, we just need to make $e_1 = 0$, or in other words,

$$\begin{aligned} &-4a(x)b(x)V_3(x)b'(x) + b(x)^2(4V_3(x)\,a'(x) - 2a(x)V_3'(x)) + \\ &3a(x)\big(-(a'(x)V_3'(x)) + c(x)(-2V_3(x)a'(x) + 2a(x)V_3'(x)) + \\ &a(x)(2\,V_3(x)c'(x) + V_3''(x))\big) = 0. \end{aligned}$$

Therefore, for any function $V_3(x) \neq 0$ we must have

$$c(x) = \frac{b(x)^2}{3a(x)} + \frac{a(x)c_1}{V_3(x)} - \frac{V_3'(x)}{2V_3(x)}, \quad c_1 \in \mathbb{C}.$$

This completes the proof of the theorem. □

Proposition 8. *Assume that $a(x) \equiv 0$ and $b(x) \not\equiv 0$ in system (2). Then this system admits a generalized Weierstrass inverse integrating factor of the form $V = V_0(x) + V_1(x)y + V_2(x)y^2$ if and only if*

$$d(x) = \frac{1}{4b(x)^2V_2(x)^2}\big(4b(x)^3d_1 + 2V_2(x)b'(x)(c(x)V_2(x) + V_2'(x)) + b(x)(c(x)^2V_2(x)^2 + V_2'(x)^2 - 2V_2(x)(V_2(x)c'(x) + V_2''(x)))\big), \quad d_1 \in \mathbb{C}.$$

Proof. By Proposition 6 if V is an inverse integrating factor of the form (10) it must have $s = 2$. Hence, computing in (10) with $s = 2$ the coefficients of y^k for $k = 2, 1, 0$ are

$$\begin{aligned} e_2 &= -b(x)V_1(x) + c(x)V_2(x) + V_2'(x), \\ e_1 &= -2b(x)V_0(x) + 2d(x)V_2(x) + V_1'(x), \\ e_0 &= -c(x)V_0(x) + d(x)V_1(x) + V_0'(x). \end{aligned} \tag{15}$$

Solving $e_2 = 0$ we get that

$$V_1(x) = \frac{c(x)V_2(x) + V_2'(x)}{b(x)}. \tag{16}$$

Now from $e_1 = 0$ we have that

$$\begin{aligned} V_0(x) = {} & \frac{1}{2b(x)^3}\big(2b(x)^2d(x)V_2(x) - c(x)V_2(x)b'(x) + b(x)V_2(x)c'(x) \\ & + b(x)c(x)V_2'(x) - b'(x)V_2'(x) + b(x)V_2''(x)\big). \end{aligned} \tag{17}$$

Now the condition $e_0 = 0$ becomes

$$\begin{aligned} & \frac{1}{2b(x)^4}\big(3b'(x)^2(c(x)V_2(x) + V_2'(x)) + 2b(x)^3(V_2(x)d'(x) + 2d(x)V_2'(x)) \\ & + b(x)(c(x)^2V_2(x)b'(x) - V_2'(x)b''(x) - c(x)(2b'(x)V_2'(x) + V_2(x)b''(x)) \\ & - 3b'(x)(V_2(x)c'(x) + V_2''(x))) + b(x)^2(-2d(x)V_2(x)b'(x) - c(x)V_2(x)c'(x) \\ & - c(x)^2V_2'(x) + 2c'(x)V_2'(x) + V_2(x)c''(x) + V_2(x)^3\big) = 0, \end{aligned}$$

or equivalently

$$\begin{aligned} d(x) = {} & \frac{1}{4b(x)^2V_2(x)^2}\big(4b(x)^3d_1 + 2V_2(x)b'(x)(c(x)V_2(x) + V_2'(x)) + b(x)(c(x)^2V_2(x)^2 \\ & + V_2'(x)^2 - 2V_2(x)(V_2(x)c'(x) + V_2''(x)))\big), \quad d_1 \in \mathbb{C}. \end{aligned} \tag{18}$$

This completes the proof. □

Proposition 9. *Assume that $a(x) = b(x) \equiv 0$ and $c(x) \not\equiv 0$ in system (2). Then it admits an inverse integrating factor of the form $V = e^{\int c(x)\,dx}$.*

Proof. When $a(x) = b(x) \equiv 0$ the Abel system (2) becomes a linear system whose first integral is $H = e^{-\int c(x)\,dx}y - \int d(x)e^{-\int c(s)\,ds}\,dx$. Now the proposition follows from equation (4). □

Proof of Theorem 3. The proof of Theorem 3 follows directly from Propositions 7, 8 and 9. □

Example. We provide an example of the existence of a generalized Weierstrass inverse integrating factor of the Abel differential system (2) under condition (i) in Theorem 3. Consider the Abel differential system (2) with $a(x) \not\equiv 0$, $b(x) \equiv 0$, $d(x) \not\equiv 0$ and

$$c(x) = \frac{1}{9}a(x)^{1/3}d(x)^{2/3}\int \frac{q(x,y)}{a(x)^{7/3}d(x)^{8/3}}\,dx,$$

where

$$\begin{aligned} q(x,y) = \ & 4d(x)^2a'(x)^2 + a(x)d(x)a'(x)d'(x) - 5a(x)^2d'(x)^2 - 3a(x)d(x)^2a''(x) \\ & +3a(x)^2d(x)d''(x). \end{aligned}$$

The generalized inverse integrating factor is of the form $V(x,y) = V_0(x) + V_1(x)y + V_3(x)y^3$ with

$$V_0(x) = \frac{(a(x)^2d(x))^{1/3}}{9a(x)}, \quad V_3(x) = \frac{(a(x)^2d(x))^{1/3}}{9d(x)}$$

and

$$\begin{aligned} V_1(x) = \ & \frac{a(x)^2}{81(a(x)^2d(x))^{5/3}}\Bigg(a(x)^{4/3}d(x)^{5/3}\int \frac{q(x,y)}{a(x)^{7/3}d(x)^{8/3}}\,dx + 3d(x)a'(x) \\ & -3a(x)d'(x)\Bigg). \end{aligned}$$

Proof of Theorem 4. We first assume $a(x) \not\equiv 0$. By Proposition 6 if V is an inverse integrating factor of the form (10) it must have $s = 3$. The coefficients of y^k for $k = 4,3,2,1,0$ are given in (11) with $V_3 = 1$. Then proceeding as in the proof of Proposition 7 it follows from (12), (13) and (14) with $V_3 = 1$ that

$$V_2(x) = \frac{b(x)}{a(x)}, \quad V_1(x) = \frac{c(x)}{a(x)}$$

and

$$V_0(x) = \frac{-b(x)a'(x) + a(x)(3a(x)d(x) + b'(x))}{3a(x)^3}.$$

Since $V_2(0) = 0$ we must have $a(0) \neq 0$ and $b(0) = 0$. Moreover, since $V_1(0) = 0$ we must have $c(0) = 0$, and since $V_0(0) = 0$ we must have $3a(0)d(0) = -b'(0)$. Now it follows from $e_1 = 0$ that

$$c(x) = \frac{b(x)^2}{3a(x)} + a(x)c_0, \quad c_0 \in \mathbb{C}.$$

Since $c(0) = 0$ then $c_0 = 0$, and thus $c(x) = b(x)^2/(3a(x))$. Now it follows from $e_0 = 0$ that

$$d(x) = a(x)d_0 + \frac{1}{27a(x)^2}\big(b(x)^3 + 9b(x)a'(x) - 9b'(x)a(x)\big), \quad d_0 \in \mathbb{C}.$$

Using that $3a(0)d(0) = -b'(0)$ we get $d_0 = 0$, and thus $d(x) = \big(b(x)^3 + 9b(x)a'(x) - 9b'(x)a(x)\big)/(27a(x)^2)$.

Now we assume that $a(x) \equiv 0$ and $b(x) \not\equiv 0$. By Proposition 6 if V is an inverse integrating factor of the form (10) it must have $s = 2$. The coefficients of y^k for $k = 2, 1, 0$ are given in (15) with $V_2 = 1$. Then proceeding as in the proof of Proposition 8 it follows from (16), (17) and (18) with $V_2 = 1$ that

$$V_1(x) = \frac{c(x)}{b(x)}, \quad V_0(x) = \frac{2b(x)^2 d(x) - c(x)b'(x) + b(x)c'(x)}{2b(x)^3}$$

and

$$d(x) = b(x)d_0 + \frac{1}{4b(x)^2}\big(c(x)^2 b(x) + 2c(x)b'(x) - 2c'(x)b(x)\big), \quad d_0 \in \mathbb{C}.$$

Since $V_1(0) = 0$ we must have $b(0) \neq 0$ and $c(0) = 0$. Moreover, since $V_0(0) = 0$ we must have $2b(0)d(0) = -b(0)c'(0)$ and thus $d_0 = 0$. Hence $d(x) = \big(c(x)^2 b(x) + 2c(x)b'(x) - 2c'(x)b(x)\big)/(4b(x)^2)$.

The case $a(x) = b(x) \equiv 0$ follows easily from Theorem 3(c). □

Appendix

The two conditions K_1 and K_2 are:

$$\begin{aligned}
K_1 = \;& -4(2d(x)a'(x) + a(x)d'(x))b(x)^5 + 2(a'(x)c(x)^2 + a(x)c'(x)c(x) + 6a(x)d(x)b'(x) - \\
& 2a'(x)c'(x) - a(x)c''(x) - a(x)^3)b(x)^4 + 2(3(-3d(x)c'(x) + 5c(x)d'(x) + d''(x))a(x)^2 \\
& +(-2b'(x)c(x)^2 + (21d(x)a'(x) + b''(x))c(x) + 5b'(x)c'(x) + 30a'(x)d'(x) - 3d(x)a''(x) \\
& +b(x)^3)a(x) + 42d(x)a'(x)^2 + 2b'(x)(c(x)a'(x) + 3a''(x)) + 4a'(x)b''(x))b(x)^3 - \\
& (54d(x)d'(x)a(x)^3 + 3(3c'(x)c(x)^2 + 18d(x)b'(x)c(x) - 3c''(x)c(x) + 5c'(x)^2 + \\
& 36d(x)^2a'(x) + 26b'(x)d'(x))a(x)^2 + (9a'(x)c(x)^3 + (10b'(x)^2 + 21a'(x)c'(x) - \\
& 9a(x)^3)c(x) + 30c'(x)a''(x) + 10b'(x)(21d(x)a'(x) + 2b''(x)) - 9a'(x)c''(x))a(x) + \\
& 3(8c(x)^2a'(x)^2 + 10c(x)a''(x)a'(x) + (10b'(x)^2 - 3(2a'(x)c'(x) + a(x)^3))a'(x) + \\
& 5a''(x)^2))b(x)^2 - 3(3(-18b'(x)d(x)^2 + 3(c''(x) - 4c(x)c'(x))d(x) + \\
& 2(3c(x)^2 - 5c'(x))d'(x) + 3c(x)d''(x))a(x)^3 + (-6b'(x)c(x)^3 + 3b''(x)c(x)^2 + \\
& (-8b'(x)c'(x) + 51a'(x)d'(x) - 9d(x)a''(x) + 3b(x)^3)c(x) - 30d'(x)a''(x) - \\
& 10c'(x)b''(x) + 3b'(x)c''(x) + 9a'(x)d''(x) + d(x)(-48b'(x)^2 - 69a'(x)c'(x) + \\
& 9a(x)^3))a(x)^2 + (-10b'(x)^3 - 23c(x)^2a'(x)b'(x) + 3(a(x)^3 - 3a'(x)c'(x))b'(x) + \\
& 63a'(x)^2d'(x) - 10a''(x)b''(x) + c(x)(21d(x)a'(x)^2 + 5b''(x)a'(x) - 2b'(x)a''(x)) + \\
& 3a'(x)(b(x)^3 - 23d(x)a''(x)))a(x) + 12a'(x)(6d(x)a'(x)^2 + b''(x)a'(x) - \\
& b'(x)(2c(x)a'(x) + a''(x))))b(x) + 3a(x)(9(-9c'(x)d(x)^2 + 3(2c(x)d'(x) + d''(x))d(x) - \\
& 5d'(x)^2)a(x)^3 + 3(-6d(x)b'(x)c(x)^2 + (16b'(x)d'(x) + 3d(x)(3d(x)a'(x) + b''(x)))c(x) - \\
& 9d(x)^2a''(x) - 10d'(x)b''(x) + 3b'(x)d''(x) + 3d(x)(-8b'(x)c'(x) + a'(x)d'(x) + b(x)^3))a(x)^2 \\
& +(36d(x)^2a'(x)^2 - 11c(x)^2b'(x)^2 - 5b''(x)^2 + 13c(x)b'(x)b''(x) - 3d(x)(b'(x)(c(x)a'(x) \\
& +15a''(x)) + 8a'(x)b''(x)) + 3b'(x)(-5b'(x)c'(x) + 21a'(x)d'(x) + b(x)^3))a(x) - \\
& 12b'(x)(-6d(x)a'(x)^2 - b''(x)a'(x) + b'(x)(2c(x)a'(x) + a''(x)))),
\end{aligned}$$

$$\begin{aligned}
K_2 = \;& b(x)(4(2d(x)a'(x) + a(x)d'(x))b(x)^5 + 2(-a'(x)c(x)^2 - a(x)c'(x)c(x) + 2a'(x)c'(x) + \\
& a(x)(c''(x) - 6d(x)b'(x)) + a^{(3)}(x))b(x)^4 - 2(3(-3d(x)c'(x) + 5c(x)d'(x) + d''(x))a(x)^2 + \\
& (-2b'(x)c(x)^2 + (21d(x)a'(x) + b''(x))c(x) + 5b'(x)c'(x) + 30a'(x)d'(x) - 3d(x)a''(x) + \\
& b(x)^3)a(x) + 42d(x)a'(x)^2 + 2b'(x)(c(x)a'(x) + 3a''(x)) + 4a'(x)b''(x))b(x)^3 + \\
& (54d(x)d'(x)a(x)^3 + 3(3c'(x)c(x)^2 + 18d(x)b'(x)c(x) - 3c''(x)c(x) + 5c'(x)^2 + 36d(x)^2a'(x) \\
& +26b'(x)d'(x))a(x)^2 + (9a'(x)c(x)^3 + (10b'(x)^2 + 21a'(x)c'(x) - 9a(x)^3)c(x) + 30c'(x)a''(x) \\
& +10b'(x)(21d(x)a'(x) + 2b''(x)) - 9a'(x)c''(x))a(x) + 3(8c(x)^2a'(x)^2 + 10c(x)a''(x)a'(x) + \\
& (10b'(x)^2 - 3(2a'(x)c'(x) + a(x)^3))a'(x) + 5a''(x)^2))b(x)^2 + 3(3(-18b'(x)d(x)^2 + 3(c''(x) - \\
& 4c(x)c'(x))d(x) + 2(3c(x)^2 - 5c'(x))d'(x) + 3c(x)d''(x))a(x)^3 + (-6b'(x)c(x)^3 + 3b''(x)c(x)^2
\end{aligned}$$

$+(-8b'(x)c'(x)+51a'(x)d'(x)-9d(x)a''(x)+3b(x)^3)c(x)-30d'(x)a''(x)-10c'(x)b''(x)+$
$3b'(x)c''(x)+9a'(x)d''(x)+d(x)(-48b'(x)^2-69a'(x)c'(x)+9a(x)^3))a(x)^2+(-10b'(x)^3$
$-23c(x)^2a'(x)b'(x)+3(a(x)^3-3a'(x)c'(x))b'(x)+63a'(x)^2d'(x)-10a''(x)b''(x)+$
$c(x)(21d(x)a'(x)^2+5b''(x)a'(x)-2b'(x)a''(x))+3a'(x)(b(x)^3-23d(x)a''(x)))a(x)+$
$12a'(x)(6d(x)a'(x)^2+b''(x)a'(x)-b'(x)(2c(x)a'(x)+a''(x))))b(x)-3a(x)(9(-9c'(x)d(x)^2$
$+3(2c(x)d'(x)+d''(x))d(x)-5d'(x)^2)a(x)^3+3(-6d(x)b'(x)c(x)^2+(16b'(x)d'(x)+$
$3d(x)(3d(x)a'(x)+b''(x)))c(x)-9d(x)^2a''(x)-10d'(x)b''(x)+3b'(x)d''(x)+$
$3d(x)(-8b'(x)c'(x)+a'(x)d'(x)+b(x)^3))a(x)^2+(36d(x)^2a'(x)^2-11c(x)^2b'(x)^2-$
$5b''(x)^2+13c(x)b'(x)b''(x)-3d(x)(b'(x)(c(x)a'(x)+15a''(x))+8a'(x)b''(x))+$
$3b'(x)(-5b'(x)c'(x)+21a'(x)d'(x)+b(x)^3))a(x)-12b'(x)(-6d(x)a'(x)^2-b''(x)a'(x)+$
$b'(x)(2c(x)a'(x)+a''(x))))).$

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