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On periodic solutions of 2–periodic Lyness difference equations*

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Abstract

We study the existence of periodic solutions of the *non–autonomous periodic Lyness' recurrence* $u_{n+2} = (a_n + u_{n+1})/u_n$, where $\{a_n\}_n$ is a cycle with positive values a,b and with positive initial conditions. It is known that for $a = b = 1$ all the sequences generated by this recurrence are 5–periodic. We prove that for each pair $(a, b) \neq (1, 1)$ there are infinitely many initial conditions giving rise to periodic sequences, and that the family of recurrences have almost all the even periods. If $a \neq b$, then any odd period, except 1, appears.

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1 Introduction and main results

The dynamics of the *autonomous Lyness' difference equation*

$$u_{n+2} = \frac{a + u_{n+1}}{u_n} \text{ with } a > 0, \text{ and } u_1, u_2 > 0. \tag{1}$$

is completely understood after the research done in [1, 3] and [25] (see also [15]). In summary, the dynamics of equation (1) can be studied through the dynamics of the map $F_a(x, y) = (y, (a + y)/x)$. This map has a first integral V_a such that their level sets in $\mathcal{Q}^+ := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$, except the one corresponding to the unique fixed point, are the ovals of some *elliptic curves* of the form

$$\{V_a(x, y) = h\} = \{(x + 1)(y + 1)(x + y + a) - hxy = 0\}. \tag{2}$$

The action of the map on each of the above curves can be described in terms of the *group law* of them (the Lyness' one is, in fact, a particular case of the well known family of QRT maps, [11]). In particular all possible periods of the recurrences generated by (1) are known, and for any $a \notin \{0, 1\}$ infinitely many different prime periods appear (the cases $a = 0$ and 1 are globally periodic with periods 6 and 5 respectively).

Recently, there has been some progress concerning the study of the *non–autonomous* periodic Lyness' equations

$$u_{n+2} = \frac{a_n + u_{n+1}}{u_n}, \tag{3}$$

when $\{a_n\}_n$ is a k-periodic sequence taking positive values, and the initial conditions u_1, u_2 are, as well, positive (see [6, 7, 8, 17, 20], and also [16, 21]). These works focus on some qualitative aspects of the dynamics like persistence, stability, etc, as well as some integrability issues. In this paper we will focus on the characterization of periodic solutions of the 2–periodic case given by

$$a_n = \begin{cases} a & \text{for} \quad n = 2\ell + 1, \\ b & \text{for} \quad n = 2\ell, \end{cases} \tag{4}$$

where $\ell \in \mathbb{N}$ and $a > 0, b > 0$. In this sense, in the recent years, and mainly driven by some conjectures in mathematical biology, there has been some attention to the study of periodic orbits of periodic non–autonomous difference equations, see [4, 9, 10, 12, 13, 22] for instance.

The solutions of equation (3) can be studied through the dynamics given by the *composition map*

$$F_{b,a}(x, y) := (F_b \circ F_a)(x, y) = \left(\frac{a + y}{x}, \frac{a + bx + y}{xy} \right), \tag{5}$$

since the relation between the terms of recurrence (3)–(4) and the iterates of the composition map is given by

$$(u_{2n+1}, u_{2n+2}) = F_{b,a}(u_{2n-1}, u_{2n}), \text{ and } (u_{2n+2}, u_{2n+3}) = F_{a,b}(u_{2n}, u_{2n+1}), \tag{6}$$

where $(u_1, u_2) \in \mathcal{Q}^+$ and $n \geq 1$.

The map $F_{b,a}$ is *integrable* in the sense that it has a first integral given by

$$V_{b,a}(x, y) = \frac{(bx + a)(ay + b)(ax + by + ab)}{xy}. \tag{7}$$

This means that each map $F_{b,a}$ preserves a foliation of the plane given by cubic curves which are, generically, elliptic. The composition maps $F_{b,a}$ associated to the 2–periodic Lyness' equations are also particular cases of *QRT maps* (see again [11]). It is interesting, however, to notice that the 2–periodic case is one of the few *rationally integrable* (thus giving rise to QRT maps) or *meromorphically integrable* k–periodic Lyness' equations (see [7, 8]). The maps $F_{b,a}$ are also a generalization of the map considered in [14]. The main goal of this paper is to study the global periodic structure of these maps.

It is known that a map $F_{b,a}$ has a unique fixed point $(x_c, y_c) \in \mathcal{Q}^+$ given by the solution of the system

$$\left\{ \begin{array}{l} x^2 = a + y, \\ y^2 = b + x. \end{array} \right. \tag{8}$$

which corresponds to the unique global minimum of $V_{b,a}$ in $\mathcal{Q}^+$. Furthermore, setting $h_c := \{V_{b,a}(x_c, y_c)\}$, the level sets $\mathcal{C}_h^+ := \{\{V_{b,a} = h\} \cap \mathcal{Q}^+$ for $h > h_c\}$ are closed curves and the dynamics of $F_{b,a}$ restricted to these sets is *conjugate to a rotation on the unit circle* with associated *rotation number* $\theta_{b,a}(h)$, [6]. In Section 2 we give an alternative proof of this fact (see Corollary 6). In this paper we study some properties of the rotation number in order to characterize the possible periods that can appear in the family of recurrences (3).

The main results of the paper are the following.

Theorem 1. *Set*

$$I(a, b) := \left\langle \sigma(a, b), \frac{2}{5} \right\rangle,$$

where $\langle c, d\rangle = (\min(c, d), \max(c, d))$*, and*

$$\sigma(a, b) = \frac{1}{2\pi} \arccos \left(\frac{1}{2} \left[-1 - \frac{a + b\, x_c}{x_c\, (b + x_c)} \right] \right) = \frac{1}{2\pi} \arccos \left(\frac{1}{2} \left[-2 + \frac{1}{x_c\, y_c} \right] \right).$$

For any fixed $a, b > 0$, and any value $\theta \in I(a, b)$, there exist at least an oval of the form $\mathcal{C}_h^+$ such that the map $F_{b,a}$ restricted to the this oval is conjugate to a rotation, with a rotation number $\theta_{b,a}(h) = \theta$.

Notice that $F_{1,1}$ is the doubling of the well–known globally autonomous Lyness' map F_1 which is globally 5–periodic and whose rotation number function takes the constant value

$1/5^*$, and therefore $I(1,1)=\{2/5\}$. In Proposition 18 we give a more precise description of $I(a,b)$.

Theorem 2. *Consider the family of maps $F_{b,a}$ given in (5) for $a,b>0$.*

(i) If $(a,b)\neq(1,1)$, then there exists a computable value $p_0(a,b)\in\mathbb{N}$ such that for any $p>p_0(a,b)$ there exist at least a continuum of initial conditions in $\mathcal{Q}^+$ (an oval $\mathcal{C}_h^+$) giving rise to p–periodic orbits of $F_{b,a}$.

(ii) For each number θ in $(1/3,1/2)$ there exists some $a>0$ and $b>0$ and at least an oval $\mathcal{C}_h^+$, such that the action of $F_{b,a}$ restricted to this oval is conjugate to a rotation with rotation number $\theta_{b,a}(h)=\theta$. In particular, for all the irreducible rational numbers $q/p\in(1/3,1/2)$, there exist periodic orbits of $F_{b,a}$ of prime period p.

(iii) The set of periods arising in the family $\{F_{b,a},\ a>0,b>0\}$ restricted to $\mathcal{Q}^+$ contains all prime periods except 2, 3, 4, 6 and 10.

In fact, as we will see in Section 4, the prime periods 2 and 3 do not appear for any a and b in the whole domain of definition of the dynamical system defined by $F_{b,a}$, but periods 4, 6, and 10 appear for some $a,b>0$ and some initial conditions in $\mathcal{G}\setminus\mathcal{Q}^+$.

Corollary 3. *Consider the 2–periodic Lyness' recurrence (3)-(4) for $a>0$, $b>0$ and positive initial conditions u_1 and u_2.*

(i) If $(a,b)\neq(1,1)$, then there exists a computable value $p_0(a,b)\in\mathbb{N}$ such that for any $p>p_0(a,b)$ there exist continua of initial conditions giving rise to $2p$–periodic sequences.

(ii) The set of prime periods arising when $(a,b)\in(0,\infty)^2$ and positive initial conditions are considered contains all the even numbers except 4, 6, 8, 12 and 20. If $a\neq b$, then it does not appear any odd period, except 1.

Observe that Theorem 2 characterizes all the prime periods that can appear for the iterations of the maps $F_{b,a}$ in $\mathcal{Q}^+$. In fact it characterizes the set of periods of any planar system of first order difference equations associated to any map conjugate to $F_{b,a}$, like the two families of first order systems of difference equations

$$\left\{\begin{array}{l} u_{n+1}u_n=a+v_n,\\ v_{n+1}v_n=b+u_{n+1}, \end{array}\right. \quad\text{and}\quad \left\{\begin{array}{l} u_{n+1}u_n=\alpha(1+v_n),\\ v_{n+1}v_n=\beta(1+u_{n+1}). \end{array}\right.$$

*Here we adopt the notation in [1] and [25] concerning the determination of the rotation number. Notice that in [6] it is adopted the determination given by $1-\theta_{b,a}(h)$.

for some choices of $a, b, \alpha, \beta > 0$, and $u_1 > 0$, $v_1 > 0$. The second one is associated to the map G, defined in (11) below.

The paper is structured as follows. In Section 2 we see that on the level sets $\mathcal{C}_h^+$, the action of the map $F_{b,a}$ can be seen as a linear action of a birational map on an elliptic curve, and thus we reobtain that on these level sets it is conjugate to a rotation (these also follows from the fact of being a QRT map, but the notation and elements introduced here will be useful in our further analysis of the rotation number function). The main part of this section is devoted to proof the elliptic nature of the invariant curves on $\mathbb{R}^2$, $\mathcal{C}_h := \{V_{b,a} = h\}$, as well as to derive a *Weierstrass normal form* representation of both the invariant curves and the map. This one, will be a key step in our approach to the asymptotic behavior of the rotation number function when h tends to infinity. This is done in Section 3, which is devoted to study the limit of the rotation function. In particular, we obtain that the asymptotic behavior of this function at the energy level corresponding to the fixed point and at the infinity do not coincide. It is worth noticing that further tools to study the rotation intervals have to be developed to face the problem of finding the set of periods when these limits coincide. The main results, as well as some other ones concerning periodic orbits are proved in Section 4.

2 $F_{b,a}$ as a linear action in terms of the group law of a cubic

2.1 An overview from an algebraic geometric viewpoint

In this section we will see $F_{b,a}$ as a linear action in terms of the group law of the cubic. The level sets $\{V_{b,a} = h\}$ are given by the cubic curves in $\mathbb{R}^2$:

$$\mathcal{C}_h = \{(bx + a)(ay + b)(ax + by + ab) - hxy = 0\}.$$

These curves $\mathcal{C}_h$, in homogeneous coordinates $[x : y : t] \in \mathbb{C}P^2$, write as

$$\widetilde{\mathcal{C}}_h = \{(bx + at)(ay + bt)(ax + by + abt) - hxyt = 0\}.$$

Observe that there are three infinite points at infinity which are common to all the above curves

$$H = [1 : 0 : 0]; \;\; V = [0 : 1 : 0]; \;\; D = [b : -a : 0].$$

Notice that none of these points is an inflection point.

The map $F_{b,a}$ extends naturally to $\mathbb{C}P^2$ as

$$\widetilde{F}_{b,a}\left([x : y : t]\right) = \left[ayt + y^2 : at^2 + bxt + yt : xy\right],$$

and leaves the curves $\widetilde{\mathcal{C}}_h$ invariant.

Notice that only a finite number curves $\widetilde{\mathcal{C}}_h$ are singular, and thus correspond to non–elliptic curves $\mathcal{C}_h$, since the discriminant is a polynomial in h (see Section 2.2.3). In particular, we will prove that for all the energy levels $h > h_c$, the curves $\widetilde{\mathcal{C}}_h$ are non-singular.

Proposition 4. *If $a > 0$ and $b > 0$, and for all $h > h_c$, the curves $\widetilde{\mathcal{C}}_h$ are elliptic.*

The above result will be a direct consequence of Proposition 9 (proved in Section 2.2), and it implies that in $\mathcal{Q}^+ \setminus \{(x_c, y_c)\}$ the map $F_{b,a}$ is a *birational transformation on an elliptic curve*, and therefore it can be expressed as a linear action in terms of the group law of the curve ([18, Theorem 3]). Indeed, for those cases such that $\widetilde{\mathcal{C}}_h$ is elliptic, if we consider $P_0 = [x_0 : y_0 : 1] \in \widetilde{\mathcal{C}}_h$, and taking the infinite point V as the zero element of $\widetilde{\mathcal{C}}_h$, then $P_0 + H$ can be computed as follows: take the horizontal line passing through P_0 and H, it cuts $\widetilde{\mathcal{C}}_h$ in a second point $(P_0 * H) = (x_1, y_0)$ (where $M * N$ denotes the point where the line $\overline{MN}$ cuts again the cubic). A computation shows that $x_1 = (a + y_0)/x_0$. The vertical line passing through $P_0 * H$ cuts $\widetilde{\mathcal{C}}_h$ at a new point $P_1 = [x_1 : y_1 : 1] = P_0 + H$. Again a computation shows that $y_1 = (a + bx_0 + y_0)/(x_0 y_0)$. Hence $(x_1, y_1) = F_{b,a}(x_0, y_0)$, and we get the following result:

Proposition 5. *For each value of h such that $\widetilde{\mathcal{C}}_h$ is an elliptic curve, $\widetilde{F}_{b,a|_{\widetilde{\mathcal{C}}_h}}(P) = P + H$, where $+$ is the addition of the group law of $\widetilde{\mathcal{C}}_h$ taking the infinite point V as the zero element.*

So we have the following immediate corollary

Corollary 6. *On each level set $\mathcal{C}_h^+$ the map $F_{b,a}$ is conjugate to a rotation.*

Remark 7. *Notice that the zero element V of the group law is not an inflection point of the cubic. So the relation of collinearity for three points A, B, C does not mean that $A+B+C = V$, but $A+B+C = V*V$ (an inflection point U satisfies $U*U = U$). Another useful relation is $-P = P*(V*V)$, where $V*V = R := [-a/b : 0 : 1]$.*

The above results are a consequence of the ones proved in the next section. Now we are able to give a first result about admissible periodic orbits of $F_{b,a}$.

Proposition 8. *For any admissible period p of $F_{b,a}$, there is only a finite number of level curves of the admissible invariant levels sets $\mathcal{C}_h$, which may have this period.*

Proof. As we have noticed, there is only a finite number of energy levels $\{V_{b,a} = h\}$, which correspond to non–elliptic curves. For the rest of values (those such that $\widetilde{\mathcal{C}}_h$ is an elliptic curve) we have the following: if there is a periodic orbit with minimal period $2n$ in the curve $\widetilde{\mathcal{C}}_h \cap \mathcal{Q}^+$ with $h > h_c$, since $F_{b,a}$ is conjugate to a rotation, then $\widetilde{\mathcal{C}}_h$ must be filled by with periodic orbits of minimal period $2n$, and also, taking into account the natural extension $\widetilde{F}_{b,a}$ (defined in Proposition 5) we have $2nH = V$ or, in other words, if and only if $nH = -nH$. Using Remark 7 we have that $-nH = nH * R$, where $R = (-a/b, 0, 1)$. So there exists $2n$–periodic orbits if and only if $nH = nH * R$, that is, if and only if R is on the tangent line to the curve $\widetilde{\mathcal{C}}_h$ at the point nH. Computing this tangent line and imposing this condition, we obtain that the energy level of corresponding to this curve must be a zero of a polynomial $P_{2n}(h)$, so it has a finite number of solutions.

Suppose that $\mathcal{C}_h$ is filled by periodic orbits of period $p = 2n+1$, then

$$(2n+1)H = V \Leftrightarrow nH + H = -nH \Leftrightarrow (nH * H) * V = nH * R,$$

which is again an algebraic relation on h giving rise to a finite number of solutions. ∎

2.2 Proof of the ellipticity of the curves $\widetilde{\mathcal{C}}_h$ for $h > h_c$ and a Weierstrass normal form representation for $F_{b,a}$

In this section we prove Proposition 4. In order to simplify the computations we introduce a preliminary change of variables and parameters. It is easy to check from (7) that the integral $V_{b,a}$ admits the following form

$$V_{b,a}(x,y)/(a^2b^2) = \Big(\frac{x}{b} + \frac{a}{b^2}\Big)\Big(\frac{y}{a} + \frac{b}{a^2}\Big)\Big(1 + \frac{x}{b} + \frac{y}{a}\Big)/\Big[\Big(\frac{x}{b}\Big)\Big(\frac{y}{a}\Big)\Big].$$

This means that taking the new variables

$$\Big\{X := \frac{x}{b},\ Y := \frac{y}{a},\ \alpha := \frac{a}{b^2},\ \beta := \frac{b}{a^2}, \tag{9}$$

the first integral is now given by

$$W(X,Y) = \frac{(X+\alpha)(Y+\beta)(1+X+Y)}{XY}; \tag{10}$$

the map $F_{b,a}$ becomes

$$G(X,Y) = \Big(\frac{\alpha(1+Y)}{X}, \frac{\beta(X+\alpha Y+\alpha)}{XY}\Big); \tag{11}$$

and the curves $\widetilde{\mathcal{C}}_h$ are brought into

$$\mathcal{D}_L = \{(X+\alpha T)(Y+\beta T)(T+X+Y) - LXYT = 0\},$$

where $L = h/(a^2b^2)$. Observe that all the curves $\mathcal{D}_L$ have three common infinite points given by $H = [1:0:0]$; $V = [0:1:0]$; and $D = [1:-1:0]$, and again, none of these points is an inflection point. The map G extends to $\mathbb{C}P^2$ as

$$\widetilde{G}\,([X:Y:T]) = \big[\alpha(YT+Y^2) : \beta(XT+\alpha YT+\alpha T^2) : XY\big]$$

The level $L_c = h_c/(a^2b^2)$ corresponds to the level of the global minimum of the invariant W in $\mathcal{Q}^+$. Proposition 4 is, then, a direct consequence of the following result, that will be proved in Subsection 2.2.3:

Proposition 9. *If $\alpha > 0$ and $\beta > 0$, for all $L > L_c = h_c/(a^2b^2)$, the curves $\mathcal{D}_L$ are elliptic.*

Reasoning as in the case of $F_{b,a}$ we obtain the following result.

Proposition 10. *For each L such that $\mathcal{D}_L$ is an elliptic curve, $\widetilde{G}_{|\mathcal{D}_L}(P) = P + H$, where $+$ is the addition of the group law of $\mathcal{D}_L$ taking the infinite point V as the zero element*[†].

As a consequence of the Propositions 9 and 10 we obtain that the action of G on each level $\mathcal{D}_L \cap \mathcal{Q}^+$, with $L > L_c$ is conjugate to a rotation.

In order to prove Proposition 9 as well as to significatively simplify some of the proofs of the results concerning the behavior of $\theta_{b,a}(h)$ done in Section 3, we first present a new conjugation between the linear action of $\widetilde{G}$ on $\mathcal{D}_L$ (and therefore for $\widetilde{F}_{b,a}$ on $\widetilde{\mathcal{C}}_h$) and a linear action on a standard Weierstrass normal form of the invariant curves. We prove:

Proposition 11. *(i) For any fixed L, the curves $\mathcal{D}_L$ have an associated Weierstass form given by*

$$\mathcal{E}_L = \{[x : y : t],\, y^2t = 4\,x^3 - g_2\,xt^2 - g_3\,t^3\}, \tag{12}$$

being

$$g_2 = \frac{1}{192}\left(L^8 + \sum_{i=4}^{7} p_i(\alpha,\beta)L^i\right) \quad \text{and} \quad g_3 = \frac{1}{13824}\left(-L^{12} + \sum_{i=6}^{11} q_i(\alpha,\beta)L^i\right),$$

where

$$\begin{aligned}
p_7(a,b) &= \; -4\,(\alpha+\beta+1)\,,\\
p_6(a,b) &= \; 2\left(3(\alpha-\beta)^2 + 2(\alpha+\beta) + 3\right),\\
p_5(a,b) &= \; -4\,(\alpha+\beta-1)\left(\alpha^2 - 4\beta\alpha + \beta^2 - 1\right),\\
p_4(a,b) &= \; (\alpha+\beta-1)^4\,.
\end{aligned}$$

and

$$\begin{aligned}
q_{11}(a,b) &= \; 6\,(\alpha+\beta+1)\,,\\
q_{10}(a,b) &= \; 3\left(-5\alpha^2 + 2\alpha\beta - 5\beta^2 - 6\alpha - 6\beta - 5\right)\\
q_9(a,b) &= \; 4\left(5\alpha^3 - 12\alpha^2\beta - 12\alpha\beta^2 + 5\beta^3 + 3\alpha^2 - 3\alpha\beta + 3\beta^2 + 3\alpha + 3\beta + 5\right)\\
q_8(a,b) &= \; 3\left(-5\alpha^4 + 16\alpha^3\beta - 30\alpha^2\beta^2 + 16\alpha\beta^3 - 5\beta^4 + 4\alpha^3\right.\\
&\quad \left. -12\alpha^2\beta - 12\alpha\beta^2 + 4\beta^3 + 2\alpha^2 - 8\alpha\beta + 2\beta^2 + 4\alpha + 4\beta - 5\right)\\
q_7(a,b) &= \; 6\left(\alpha^2 - 4\alpha\beta + \beta^2 - 1\right)(\alpha+\beta-1)^3\\
q_6(a,b) &= \; -\,(\alpha+\beta-1)^6
\end{aligned}$$

[†] Notice that, again, the zero element V of the group law is not an inflection point of the cubic (see Remark 7).

(ii) *For any* $L > L_c$*, the curve* $\mathcal{E}_L$ *is an elliptic curve.*

(iii) *For any* $L > L_c$*,* $\widetilde{G}_{|\mathcal{D}_L}$ *is conjugate to the linear action*

$$\widehat{G}_{|\mathcal{E}_L} : P \mapsto P \star \widehat{H}, \tag{13}$$

where

$$\widehat{H} = \left[\frac{1}{48}\left(L^2 - 2\left(\alpha+\beta+1\right)L + \left(\alpha+\beta-1\right)^2\right)L^2 \,:\, -\frac{1}{8}\alpha\beta L^4 \,:\, 1\right] \tag{14}$$

and $\star$ *denotes the sum operation of the cubic* $\mathcal{E}_L$*, taking the infinity point* $\widehat{V} = [0 : 1 : 0]$ *as the zero of the group law.*

In the next subsections we give the proof of the above result.

2.2.1 Computing the Weierstrass normal form of $\mathcal{D}_L$

In this section we explicit the transformations which pass $\mathcal{D}_L$ into their Weierstrass normal form. Although it is a standard calculation, we prefer to include it because in the proof of Proposition 11, we need the explicit expressions of some of the intermediate transformations below.

First transformation: Following [24, Section I.3], we take the following reference projective system: Let $t = 0$ be the tangent line to $\mathcal{D}_L$ at V. This tangent line intersects $\mathcal{D}_L$ at $R = [-\alpha : 0 : 1]$, so we will take the reference $x = 0$ to be the tangent line at R. Finally $y = 0$ will be the Y axis. The new coordinates x, y, t are, then, obtained via

$$x = \beta(1-\alpha)X + \alpha LY + \alpha\beta(1-\alpha)T;\; y = X; t = X + \alpha T, \tag{15}$$

and the equation of the cubic becomes

$$xy^2 + y(a_1xt + a_2t^2) = a_3x^2t + a_4xt^2 + a_5t^3, \tag{16}$$

where

$$\left\{\begin{array}{l} a_1 = (\alpha-\beta-1-L)/L, \quad a_2 = \beta\left((1-a)^2 + \beta(1-\alpha) - L\right)/L, \quad a_3 = -1/L^2, \\ a_4 = \left(2\beta(1-\alpha) - L(1+\beta)\right)/L^2, \quad a_5 = \beta(\beta(1-\alpha) - L)(L-1+\alpha)/L^2. \end{array}\right. \tag{17}$$

Second transformation: We multiply both members of (16) by x, and we take the new variables (denoted again X, Y, T):

$$X = x; Y = xy/t; T = t, \tag{18}$$

obtaining the new equation

$$T^2Y^2 + (a_1X + a_2T)YT^2 = a_3X^3T + a_4X^2T^2 + a_5XT^3. \tag{19}$$

Third transformation: Removing the factor T in (19) and taking the new variables x, y, t

$$x = X;\ y = Y + (a_1X + a_2T)/2;\ t = T, \tag{20}$$

we obtain the new equation

$$ty^2 = \lambda x^3 + \mu x^2 t + \nu x t^2 + \gamma t^3,$$

where

$$\lambda = a_3, \mu = a_4 + a_1^2/4, \nu = a_5 + a_1a_2/2, \gamma = a_2^2/4. \tag{21}$$

The last two transformations Let X, Y and T be the new variables given by

$$X = x/(4\lambda),\ \ Y = y/(4\lambda^2),\ \ T = t \tag{22}$$

In these new variables the curve writes as

$$TY^2 = 4X^3 + \frac{\mu}{\lambda^2}X^2T + \frac{\nu}{4\lambda^3}XT^2 + \frac{\gamma}{4^2\lambda^4}T^3.$$

Finally, with the last change

$$x = \frac{1}{12}\frac{12X\lambda^2 + \mu t}{\lambda^2},\ \ y = Y,\ \ t = T, \tag{23}$$

the curve is written in the Weierstrass' normal form

$$y^2t = 4x^3 - g_2xt^2 - g_3t^3,$$

where

$$g_2 = \frac{\mu^2}{12\lambda^4} - \frac{\nu}{4\lambda^3}, g_3 = -\frac{\mu^3}{216\lambda^6} + \frac{\mu\nu}{48\lambda^5} - \frac{\gamma}{16\lambda^4}. \tag{24}$$

2.2.2 Proof of Proposition 11

Proof of Proposition 11. (i) The expression of the equation (12) comes from the transformations leading $\mathcal{D}_L$ to its Weierstrass form given in Section 2.2.1. In particular the expressions of the coefficients of g_2 and g_3 in terms of α, β and L come from formulae (17), (21) and (24).

(ii) From Proposition 9, the curves $\mathcal{D}_L$, with $L > L_c$ are elliptic. This means that for $L > L_c$ the curves $\mathcal{E}_L$ will be also elliptic.

Statement (iii) is a direct consequence of Proposition 10. But it is still necessary to prove that the zero point V is transformed into $\widehat{V} = [0 : 1 : 0]$. To do this we should obtain the image of V under the same transformations as above, but the image of V under the first transformation, say $V_1 := [1:0:0]$, has no image for the second transformation. So we use a continuity argument: Set $V_1(x) = [x : y(x) : 1]$, where $y(x)$ is choose in sort that $V_1(x)$ is on the cubic (16), that is, $y(x)$ can be either $y_+(x)$ or $y_-(x)$, where

$$y_\pm(x) = \frac{-a_1x - a_2 \pm \sqrt{4\,a_3x^3 + 4\,(a_4 + a_1{}^2)x^2 + 2\,(a_1a_2 + 4\,a_5)\,x + a_2{}^2}}{2x}$$

The leading term of $y_\pm(x)$ when $x \to +\infty$ is $\pm\sqrt{a_3\,x}$ (observe that these two numbers are imaginary, because $a_3 = -1/L^2$). So we have two complex points of the cubic $[x : y_\pm(x) : 1]$ where $y_\pm = \pm\sqrt{a_3\,x} + O(1)$:

$$V_1(x) = [x : \pm\sqrt{a_3\,x} + O(1) : 1] = [1 : \pm\sqrt{a_3/x} + O(1/x) : 1/x]$$

which tends to $V_1 = [1:0:0]$ when x tends to $+\infty$.

But if we take the images of these two complex points, in their first form, by the transformation (18), we obtain

$$V_2(x) = [1/\sqrt{x} : \pm\sqrt{a_3} + O(1/\sqrt{x}) : 1/\sqrt{x^3}],$$

which tends to $[0 : \pm\sqrt{a_3} : 0] = [0:1:0]$. So this point is V_2, the image of V_1 obtained by continuity. Now the successive transformations can be done without any problem and we obtain $\widehat{V} = [0:1:0]$.

The point $\widehat{H}$ is the image of $H = [1:0:0]$ under the successive transformations (15), (18), (20), (22) and (23). ∎

2.2.3 Proof of the ellipticity of the curves $\mathcal{D}_L$ for $L > L_c$

In this section we prove Proposition 9, but first we notice the following facts:

Fact 1. The curves $\mathcal{D}_L$ have no singular points at the infinity of $\mathbb{C}P^2$. Indeed, setting $D := (X + \alpha T)(Y + \beta T)(T + X + Y) - LXYT$, we have that

$$\frac{\partial D}{\partial X}(X, Y, 0) = Y(Y + 2X),\ \frac{\partial D}{\partial Y}(X, Y, 0) = X(2Y + X),$$

so an straightforward computation shows that on the line $T = 0$ the only singular point must satisfy $X = 0$ and $Y = 0$, thus not in $\mathbb{C}P^2$.

Fact 2. An straightforward computation shows that the set of affine singular points of $\mathcal{D}_L$ coincides with the set of singular points of the invariant function W given in (10). On the other hand, a computation shows that

$$\begin{aligned} W'_x &= (Y + \beta)(X^2 - \alpha(1 + Y))/(X^2Y) \text{ and} \\ W'_y &= (X + \alpha)(Y^2 - \beta(1 + X))/(XY^2). \end{aligned} \tag{25}$$

So for $L > L_c$, the other possible singular point of a curve $\mathcal{D}_L$ is one of the point intersection of the two parabolas $\mathcal{P}_1 := \{X^2 = \alpha(1+Y)\}$ and $\mathcal{P}_2 := \{Y^2 = \beta(1+X)\}$.

Fact 3. The equilibrium point $P = (p, q)$ which is in the positive quadrant, is one of the (at most) four real points on $\mathcal{P}_1 \cap \mathcal{P}_2$, and it is on $\mathcal{D}_L$ and singular in this curve only for $L = L_c$ (in this case, it is a real isolated point).

Now we introduce the some new parameters (p, q) which correspond to the coordinates of the fixed point in $\mathcal{Q}^+$, thus given by

$$\begin{cases} p^2 = \alpha(1+q), \\ q^2 = \beta(1+p). \end{cases} \tag{26}$$

With these new parameters it is easy to obtain an explicit form for L_c:

$$L_c = \frac{(1+p+q)^3}{(1+p)(1+q)}. \tag{27}$$

In fact this explicit expression allows to obtain an straightforward proof that $L_c > 1$ since

$$L_c = (p+q+1)\left(1 + \frac{q}{p+1}\right)\left(1 + \frac{p}{q+1}\right) > 1.$$

Furthermore, after some computations (skipped here) it is possible to prove that

$$\lim_{(\alpha,\beta)\to(0,0)} (p, q) = (0, 0).$$

This implies that the above one is the sharpest lower bound for L_c.

Fact 4. Any singular point of $\mathcal{D}_L$ with $L > L_c$ must be real. If a singular point is not real, the conjugate point is also singular and different, and so the (real) line which support these two points is in the curve, which splits in a right line and a conic. It is easy to see that, after the identification of the coefficients the equation of the cubic must be written as

$$D = (X + Y + (1-L)T)\left(XY + \beta XT + \alpha YT + \frac{\alpha\beta}{1-L}T^2\right) = 0.$$

The point $[0 : -1 : 1]$ is on the cubic, hence $-L\left(-\alpha + \alpha\beta/(1-L)\right) = 0$, which is impossible because the above number is positive, since we have just seen that $L > L_c > 1$.

Taking into account all the above information, to prove Proposition 9, we only need to show that if $\mathcal{D}_L$ is singular then $L \le L_c$.

Lemma 12. *If the curve $\mathcal{D}_L$ is singular, then $L \le L_c$.*

Proof. Using the Weierstrass form given by (12) and the expressions of g_2 and g_3 given by from formulae (17), (21) and (24). We obtain that the discriminant of $\mathcal{D}_L$ is given by

$$\Delta := g_2^3 - 27g_3^2 = \frac{1}{4096}\alpha^2\beta^2 L^{15}\Delta_1(L; \alpha, \beta),$$

where $\Delta_1(L) = L^4 + \sum_{i=0}^{3} \delta_i(\alpha,\beta)\, L^i$, and being

$$\begin{aligned}
\delta_3(\alpha,\beta) = &\ \alpha\beta - 4\alpha - 4\beta - 4,\\
\delta_2(\alpha,\beta) = &\ -3\alpha^2\beta - 3\alpha\beta^2 + 6\alpha^2 - 21\alpha\beta + 6\beta^2 + 4\alpha + 4\beta + 6,\\
\delta_1(\alpha,\beta) = &\ 3\alpha^3\beta - 21\alpha^2\beta^2 + 3\alpha\beta^3 - 4\alpha^3 + 18\alpha^2\beta + 18\alpha\beta^2 - 4\beta^3 + 4\alpha^2\\
&-25\alpha\beta + 4\beta^2 + 4\alpha + 4\beta - 4,\\
\delta_0(\alpha,\beta) = &\ -(\alpha-1)(\beta-1)(\alpha+\beta-1)^3.
\end{aligned}$$

Obviously $L = 0$ gives a singular curve (a union of three lines). To see that all the real roots of Δ_1 are lower than L_c, we take into account the explicit expression of L_c given by (27), we use the change

$$\left\{ \alpha = \frac{p^2}{1+q}, \quad \beta = \frac{q^2}{1+p} \right. ,$$

and also the translation given by $L = \widetilde{L} + L_c$, obtaining that $\Delta_1(L;\alpha,\beta) = 0$ if and only if the numerator of $\Delta_1(\widetilde{L} + L_c; p^2/(1+q), q^2/(1+p))$ vanishes, which happens if and only if $\widetilde{\Delta}_1(\widetilde{L}) := \widetilde{L}\left[\sum_{i=0}^{3} d_i(p,q)\widetilde{L}^i\right] = 0$, where

$$\begin{aligned}
d_3(p,q) = &\ (1+p)^3(1+q)^3,\\
d_2(p,q) = &\ (1+p)^2(1+q)^2\left(p^2q^2 + 12p^2q + 12pq^2 + 8p^2 + 20pq + 8q^2 + 8p + 8q\right),\\
d_1(p,q) = &\ (1+p)^2(1+q)^2\left(9p^4q^3 + 9p^3q^4 + 60p^4q^2 + 93p^3q^3 + 60p^2q^4 + 64p^4q + 228p^3q^2\right.\\
&+228p^2q^3 + 64pq^4 + 16p^4 + 176p^3q + 312p^2q^2 + 176pq^3 + 16q^4 + 32p^3 + 160p^2q\\
&\left.+160pq^2 + 32q^3 + 16p^2 + 48pq + 16q^2\right),\\
d_0(p,q) = &\ pq\left(p^2 + pq + q^2 + p + q\right)(3pq + 4p + 4q + 4)^3.
\end{aligned}$$

Since $\widetilde{\Delta}_1$ is a polynomial with positive coefficients, its real roots are negative or zero, and thus, the real roots of Δ_1 are less or equal than L_c. ∎

3 Behavior of $\theta_{b,a}(h)$.

To prove Theorem 1, as well as to study the periods appearing in the family $F_{b,a}$ we need to analyze the rotation number function $\theta_{b,a}(h)$, with special emphasis on its asymptotic behavior when $h \to h_c^+$, and when $h \to +\infty$. Regarding this last question, in [6], there were pointed out strong numerical evidences that $\lim\limits_{h\to+\infty} \theta_{b,a}(h) = 2/5$. Notice that these evidences agree with the fact, proved in [1], that for the autonomous case the rotation number associated to the invariants curves (2), say $\theta_a(h)$, satisfies $\lim\limits_{h\to+\infty} \theta_a(h) = 1/5$ so it is independent of the value of the parameter a. In this section we prove

Proposition 13. *The following statements hold*

(i) The rotation number map $\theta_{b,a}(h)$ is analytic on $(h_c, +\infty)$.

(ii) $\lim_{h\to+\infty} \theta_{b,a}(h) = \frac{2}{5}$.

(iii) The rotation number map $\theta_{b,a}(h)$ is continuous in $[h_c, +\infty)$. Furthermore, setting

$$\sigma(a,b) := \lim_{h\to h_c^+} \theta_{b,a}(h), \tag{28}$$

it holds that

$$\sigma(a,b) = \frac{1}{2\pi} \arccos\left(\frac{1}{2}\left[-1 - \frac{a + b\,x_c}{x_c\,(b+x_c)}\right]\right) = \tag{29}$$

$$= \frac{1}{2\pi} \arccos\left(\frac{1}{2}\left[-2 + \frac{1}{x_c\,y_c}\right]\right). \tag{30}$$

To prove the statements (i) and (ii) of Proposition 13, it will be helpful to use the conjugation of $F_{b,a}$ given by the map $\widehat{G}$ defined in (13), for which its invariant level sets are described by the Weierstrass normal form $\mathcal{E}_L$ defined in (12), and then to apply a similar machinery as in [1]. First, we recall that given the elliptic curve $\mathcal{E}_L$, for $L > L_c$, there exist two positive numbers ω_1 and ω_2 depending on α, β and L and a lattice in $\mathbb{C}$

$$\Lambda = \{2n\omega_1 + 2m\, i\omega_2 \text{ such that } (n,m) \in \mathbb{Z}^2\} \subset \mathbb{C},$$

such that the *Weierstrass $\wp$ function relative to* Λ

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right]$$

gives a parametrization of $\mathcal{E}_L$. This is because the map

$$\begin{array}{rccl} \phi: & \mathbb{C}/\Lambda & \longrightarrow & \mathcal{E}_L \\ & z & \longrightarrow & \left\{ \begin{array}{ll} [\wp(z) : \wp'(z) : 1] & \text{if } \; z \notin \Lambda, \\ \ [0:1:0] = \widehat{V} & \text{if } \; z \in \Lambda, \end{array} \right. \end{array} \tag{31}$$

is an holomorphic homeomorphism, and therefore

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \tag{32}$$

(see [19, Lemma 5.17] and [23, Proposition 3.6]). Another interesting property of the parametrization of $\mathcal{E}_L$ given by ϕ is that it is injective for the real interval $(0, \omega_1)$ onto the real unbounded branch of $\mathcal{E}_L$ whose points have *negative y–coordinates*, and that $\phi(\omega_1) = [e_1, 0, 1]$ or, in other words:

$$e_1 = \wp(\omega_1) \tag{33}$$

(see again [19, 23]). Finally, observe that $\phi(0) = \phi(2\omega_1) = \widehat{V}$, and $\lim_{u\to 0} \wp(u) = +\infty$. Taking all the preceding considerations into account, and by direct integration of the differential equation (32) on $[0, u)$, we have that in real variables

$$u = \int_{\wp(u)}^{+\infty} \frac{\mathrm{d}s}{\sqrt{4s^3 - g_2 s - g_3}}. \tag{34}$$

Proof of Proposition 13 (i). Observe that equation (12) can be written as

$$ty^2 = 4(x - te_1)(x - te_2)(x - te_3),$$

where

$$e_1 + e_2 + e_3 = 0, \;\; e_1e_2 + e_2e_3 + e_3e_1 = -g_2/4, \;\; e_1e_2e_3 = g_3/4,$$

and $e_3 < e_2 < e_1$. Then, equation (34) writes:

$$u = \int_{\wp(u)}^{+\infty} \frac{\mathrm{d}s}{\sqrt{4(s - e_1)(s - e_2)(s - e_3)}}, \tag{35}$$

and from the above relation and equation (33) we get:

$$\omega_1 = \int_{e_1}^{+\infty} \frac{\mathrm{d}s}{\sqrt{4(s - e_1)(s - e_2)(s - e_3)}},$$

From Proposition 11, the action of $\widehat{G}$ on each level $\mathcal{E}_L$, with $L > L_c$ is conjugate to a rotation of angle $2\pi\Theta(L)$, where Θ is the rotation number associated to $\widehat{G}_{|\mathcal{E}_L}$. On the other hand from the expression of $\widehat{H}$ given in (14) we know that its y–coordinate is negative, so from the above considerations on the parametrization ϕ we know that there exists $u \in (0, \omega_1)$ such that it has the following representation

$$\exp\left(\frac{2\pi i}{2\omega_1} u\right) = \exp\left(2\pi i \Theta(L)\right) \in [0, 2\omega_1]/\Lambda \cong \mathbb{S}^1,$$

so $u = 2\omega_1\Theta(L)$, and since $\widehat{H}$ is the image of $\widehat{V}$ under the transformation $\widehat{G}_{|\mathcal{E}_L}$, it is given by $\phi(2\omega_1\Theta(L))$, and therefore

$$\wp(2\omega_1\Theta(L)) = X(L),$$

where $X(L)$ is the abscissa of point $\widehat{H}$ given in (14). Applying (35) we obtain that

$$2\omega_1\Theta(L) = \int_{X(L)}^{+\infty} \frac{\mathrm{d}s}{\sqrt{4(s - e_1)(s - e_2)(s - e_3)}},$$

hence

$$2\Theta(L) = \frac{\displaystyle\int_{X(L)}^{+\infty} \frac{\mathrm{d}s}{\sqrt{(s - e_1)(s - e_2)(s - e_3)}}}{\displaystyle\int_{e_1}^{+\infty} \frac{\mathrm{d}s}{\sqrt{(s - e_1)(s - e_2)(s - e_3)}}}. \tag{36}$$

By using the change of variables $s = e_1 + 1/r^2$ and $r\sqrt{e_1 - e_3} = u$, it can be proved that equation (36) becomes

$$2\Theta(L) = \frac{\displaystyle\int_0^{\sqrt{\frac{e_1-e_3}{\nu}}} \frac{\mathrm{d}u}{\sqrt{(1+u^2)(1+\varepsilon u^2)}}}{\displaystyle\int_0^{+\infty} \frac{\mathrm{d}u}{\sqrt{(1+u^2)(1+\varepsilon u^2)}}}, \tag{37}$$

where $\nu = X(L) - e_1$, and $\varepsilon = (e_1 - e_2)/(e_1 - e_3)$.

The analyticity of $\Theta(L)$ in $(L_c, +\infty)$ can be derived now from expression (37), by taking into account that when $L > L_c$, e_1, e_2, e_3 are different, and that from the geometry of the problem we have $e_3 < e_2 < e_1 < X(L)$, so all the elements in the left hand side of (37) are holomorphic in a neighborhood of $(L_c, +\infty)$. Hence $\Theta(L)$ is analytic on $(L_c, +\infty)$. ∎

Observe that $\lim_{h\to+\infty} \theta_{b,a}(h) = \lim_{L\to+\infty} \Theta(L)$, hence our main objective now is to prove that $\lim_{L\to+\infty} \Theta(L) = 2/5$. To do this we will need the following auxiliary result

Lemma 14. *([1, Lemma 4]) Let $\lambda, \varepsilon, \gamma$ be positive numbers. For any map $\phi(\varepsilon)$ such that $\lim_{\varepsilon\to 0} \phi(\varepsilon) = 0$, and $\lambda + \phi(\varepsilon) > 0$, set*

$$N(\varepsilon, \lambda, \gamma) = \int_0^{\frac{\lambda+\phi(\varepsilon)}{\varepsilon^\gamma}} \frac{\mathrm{d}u}{\sqrt{(1+u^2)(1+\varepsilon u^2)}}, \ \textit{and}\ D(\varepsilon) = \int_0^{+\infty} \frac{\mathrm{d}u}{\sqrt{(1+u^2)(1+\varepsilon u^2)}}.$$

Then $D(\varepsilon) \sim (1/2)\ln(1/\varepsilon)$, and if $\gamma < 1/2$ we have $N(\varepsilon, \lambda, \gamma) \sim \gamma \ln(1/\varepsilon)$, where $\sim$ denotes the equivalence with the leading term of the asymptotic development at zero.

Proof of Proposition 13 (ii). Following the steps in [1], we will compute $\lim_{L\to+\infty} \Theta(L)$ by using the asymptotic developments of the elements involved in equation (37).

Solving the cubic equation $4\,s^3 - g_2\,s - g_3 = 0$ by leaving the solutions in terms of g_2 and g_3; using the expressions of these coefficients in terms of α, β and L given by Proposition 11; and after some computations done with a symbolic algebra system (Maple 12), we obtain that

$$e_1 \sim L^4/48,\ e_2 \sim L^4/48 \ \text{and}\ e_3 \sim -L^4/24,$$

when L tends to infinity, and

$$e_1 - e_2 \sim \alpha\beta\, L^{3/2} \ \text{and}\ e_1 - e_3 \sim L^4/16.$$

Setting $\varepsilon = (e_1 - e_2)/(e_1 - e_3)$, we obtain $\varepsilon \sim 16\,\alpha\beta/L^{5/2}$ (when L tends to infinity), so

$$L \sim \frac{2\,2^{3/5}\,(\alpha\beta)^{2/5}}{\varepsilon^{2/5}}. \tag{38}$$

(when ε tends to zero). From the expression of $\widehat{H}$ given in (14) we have

$$\nu = X(L) - e_1 \sim \alpha\beta\, L^2/4,$$

and therefore, using (38), we get

$$\sqrt{\frac{e_1 - e_3}{\nu}} \sim \frac{L}{2\sqrt{\alpha\beta}} \sim A\,\varepsilon^{-2/5},$$

where $A = 2^{3/5}\,(\alpha\beta)^{-1/(10)}$. So from equation (37), and Lemma 14 we finally obtain

$$2\Theta(L) = \frac{N(\varepsilon, A, 2/5)}{D(\varepsilon)} \sim \frac{\frac{2}{5}\ln(1/\varepsilon)}{\frac{1}{2}\ln(1/\varepsilon)} = \frac{4}{5},$$

and therefore $\lim\limits_{L\to\infty} \Theta(L) = 2/5$. ∎

It is well known (see [2] for instance, and also [5, 25]) that under some regularity conditions the rotation number function of a differentiable planar map F near an elliptic fixed point, can be extended continuously to the fixed point. In this sense, the proof of statement (ii) follows by applying the following version of [2, Proposition 8].

Lemma 15. *Let U be an open set of $\mathbb{R}^2$ and a $\mathcal{C}^1$ map $F : U \to U$. Suppose that F has a unique elliptic fixed point P in U (that is $DF(P)$ has two distinct non real eigenvalues with modulus 1). Suppose that F has a first integral V in U, attaining a strict minimum $h_p := V(P)$ at the point P, and such that*

- *(i) For $h > h_p$, the invariant level sets $\{V = h\}$ (except the level set $\{V(P) = h_P\}$) are closed curves, surrounding P.*
- *(ii) For $h > h_p$, the sets $\overline{\mathrm{Int}\,(\{V = h\})}$ are starlike with respect to P, and each half line from P cuts $\{V = h\} \cap U$ exactly in one point.*
- *(iii) On each curve $\{V = h\}$ for $h > h_p$, the map $F_{|\{V=h\}}$ is conjugate to a rotation of angle $2\pi\theta(h)$.*

Then,

$$\theta(P) := \lim_{h\to h_P} \theta(h) = \frac{1}{2\pi} \arccos\left(\frac{\mathrm{Trace}(DF(P))}{2}\right). \tag{39}$$

Proof of Proposition 13 (iii). As a direct consequence of Lemma 15, the continuous extension of $\theta_{b,a}(h)$ to h_c^+, is

$$\sigma(a, b) = \frac{1}{2\pi} \arccos\left(\frac{1}{2}\mathrm{Trace}(DF_{b,a}(x_c, y_c))\right).$$

Then, the expressions (29–30) are then easily derived from (8) after some easy manipulations. ∎

4 Periodic solutions

4.1 The rotation interval $I(a,b)$. Proof of Theorem 1

In this Section we prove Theorem 1, and we give a more precise characterization of the rotation number interval $I(a,b)$.

Proof of Theorem 1. In proposition 13, it is proved that $\theta_{b,a}(h)$ is an analytic function on $(h_c,+\infty)$ and continuous in $[h_c,+\infty]$; that $\lim_{h\to h_c^+}\theta_{b,a}(h)=\sigma(a,b)$; and $\lim_{h\to+\infty}\theta_{b,a}(h)=2/5$.

It is clear then that for any θ in the interval $I(a,b)$ defined by

$$I(a,b):=\left\langle \lim_{h\to h_c^+}\theta_{b,a}(h),\lim_{h\to+\infty}\theta_{b,a}(h)\right\rangle=\left\langle \sigma(a,b),\frac{2}{5}\right\rangle,$$

where $\langle c,d\rangle=(\min(c,d),\max(c,d))$, there is a value of $h\in[h_c,+\infty)$ such that the action of $F_{b,a}$ restricted to $\mathcal{C}_h^+$ is a rotation with rotation number θ, since

$$I(a,b)\subseteq \mathrm{Image}\left(\theta_{b,a}((h_c,+\infty))\right).$$

∎

To have a better description of the interval $I(a,b)$, it would be interesting to know the relative position of $\sigma(a,b)$ with respect to $2/5$ for given values of a and b. This is issue is considered in Proposition 18 and Corollary 19 (see also Figure 1), but before proving them we need a preliminary result.

Set $J(a,b):=\mathrm{Image}\left(\theta_{b,a}\left((h_c,+\infty)\right)\right)\subset[0,\frac{1}{2}]$. As as stated above, it holds that $I(a,b)\subseteq J(a,b)$. Observe that both intervals are the same when the function $\theta_{b,a}(h)$ is monotonic, but notice that in [6] it is proved that there are values of a and b for which the map is non–monotonic. With respect the interval $J(a,b)$, we prove:

Lemma 16. *The following statements hold.*

(i) For any $a,b>0$, $\sigma(a,b)>\dfrac{1}{3}$.

(ii) $\lim_{b\to+\infty}\sigma(b^2,b)=\dfrac{1}{2}$. Therefore, when b is sufficiently large $\sigma(b^2,b)>2/5$, and then $I(b^2,b)=\big(2/5,\sigma(b^2,b)\big)$.

(iii) $\lim_{b\to0^+}\sigma(b^2,b)=\dfrac{1}{3}$. Therefore, when b is small enough $\sigma(b^2,b)<2/5$, and then $I(b^2,b)=\big(\sigma(b^2,b),2/5\big)$.

As a first direct consequence of the above result, we have

Proposition 17. *It holds that*

$$\left(\frac{1}{3}, \frac{1}{2}\right) \subset \bigcup_{a>0,\, b>0} J(a,b).$$

Furthermore

$$\left(\frac{1}{3}, \frac{1}{2}\right) \subset \bigcup_{b>0} J(b^2,b).$$

Proof of Lemma 16. (i) Using that

$$\sigma(a,b) = \frac{1}{2\pi} \arccos\left(\frac{1}{2}\mathrm{Trace}(DF_{b,a}(x_c, y_c))\right),$$

we have that $\sigma(a,b) > 1/3$ if and only if $\mathrm{Trace}(DF_{b,a}(x_c,y_c))/2 < \cos(2\pi/3) = -1/2$ or, equivalently using equation (29),

$$-1 - \frac{a + b\,x_c}{x_c\,(b + x_c)} < -1,$$

which is true, since a, b and x_c are positive.
(ii) From equation (8) we have that $y_c > \sqrt{b}$. Suppose that $a = b^2$, then $x_c = \sqrt{a + y_c} > \sqrt{a} = b$. So $\lim\limits_{b\to+\infty} x_c = \lim\limits_{b\to+\infty} y_c = +\infty$ and then, using equation (30)

$$\lim_{b\to+\infty} \theta(b^2,b) = \lim_{b\to+\infty} \frac{1}{2\pi} \arccos\left(\frac{1}{2}\left[-2 + \frac{1}{x_c\,y_c}\right]\right) = \frac{1}{2\pi}\arccos(-1) = \frac{1}{2}.$$

(iii) When $a = b^2$, we consider the polynomials $R(x,y) := x^2 - y - b^2$ and $S(x,y) := y^2 - x - b$ whose common zeros are the fixed points of $F_{b,a}$. Observe that y_c must be a root of the resultant

$$\mathrm{Resultant}(R, S; x) = -y(-y^3 + 2by + 1),$$

so $y_c^3 = 2by_c + 1$, thus $y_c > 1$ and $y_c < y_c^3 = 2by_c + 1$, hence

$$y_c < \frac{1}{1-2b} < 2 \ \text{ if } b < \frac{1}{4}.$$

So if $b < 1/4$ we have

$$1 < y_c^3 = 1 + 2by_c < 1 + 4b;$$

since the right hand side of the above inequality tends to 1 when b tends to 0, we obtain that y_c tends to 1. Since $x_c = y_c^2 + b$, we obtain that $\lim\limits_{b\to0^+} x_c = \lim\limits_{b\to0^+} y_c = 1$, hence

$$\lim_{b\to0^+} \theta(b^2,b) = \lim_{b\to0^+} \frac{1}{2\pi} \arccos\left(\frac{1}{2}\left[-2 + \frac{1}{x_c\,y_c}\right]\right) = \frac{1}{2\pi}\arccos(-1) = \frac{1}{2}.$$

∎

The next result characterizes the relative position of $\sigma(a,b)$ with respect $2/5$.

Proposition 18. *Let (x_c, y_c) be the fixed point of the map $F_{b,a}$ given by (8). Set $R(x,y) := \phi\, y - b\, x - a$. Then, $\sigma(a,b) < 2/5$ if and only if $R(x_c,y_c) > 0$; $\sigma(a,b) = 2/5$ if and only if $R(x_c,y_c) = 0$; and $\sigma(a,b) > 2/5$ if and only if $R(x_c,y_c) < 0$.*

Proof. Observe that using expression (29), the condition $\sigma(a,b) \leq 2/5$ gives

$$\frac{1}{2}\left[-1 - \frac{a + b\,x_c}{x_c\,(b + x_c)}\right] \geq \cos\left(\frac{4\pi}{5}\right) = -\frac{1+\sqrt{5}}{4},$$

thus

$$\frac{a + b\,x_c}{x_c\,(b + x_c)} \leq \phi - 1 = \frac{1}{\phi},$$

where $\phi = (1+\sqrt{5})/2$. By using (8) this last inequality turns to

$$\phi\,(a + b\,x_c) \leq x_c\,(x_c + b) = x_c^2 + b\,x_c = a + y_c + b\,x_c.$$

Using again $\phi - 1 = 1/\phi$, we obtain that $\phi y_c - b x_c - a \geq 0$. ∎

In order to obtain a characterization of the condition $\sigma(a,b) = 2/5$ in terms of the parameters a and b instead of x_c and y_c, we isolate a and b from the equations (8) and plug them in the equation $R(x_c,y_c) = 0$, obtaining

$$y_c(\phi - x_c y_c + 1) = 0.$$

Using that $\phi + 1 = \phi^2$, we obtain that $R(x_c,y_c) = 0$ if and only if $y_c = \phi^2/x_c$, thus taking the parameter $t := x_c$ we have that

$$a = \frac{t^3 - \phi^2}{t} \quad \text{and} \quad b = \frac{\phi^4 - t^3}{t^2}.$$

A simple computation shows that the above formulae describe a curve with a single branch in the parameter's space $\mathcal{P} := \{(a,b),\, a,b > 0\}$ when $\phi^{\frac{2}{3}} < t < \phi^{\frac{4}{3}}$. In summary, we have proved the following corollary:

Corollary 19. *The curve $\sigma(a,b) = 2/5$ for $a,b > 0$ is given by*

$$\Gamma := \{\sigma(a,b) = 2/5,\, a,b > 0\} = \left\{(a,b) = \left(\frac{t^3 - \phi^2}{t}, \frac{\phi^4 - t^3}{t^2}\right),\, t \in (\phi^{\frac{2}{3}}, \phi^{\frac{4}{3}})\right\} \subset \mathcal{P}.$$

It is easy to check that this curve cuts the axes at the points $(0,\sigma_*)$ and $(\sigma_*,0)$ where $\sigma_* = \phi^{\frac{5}{3}} \simeq 2.2300$, and it splits the parameter space into two connected components. By using Lemma 16 (ii) and (iii), it is easy to obtain that one of them corresponds to the set $\{\sigma(a,b) < 2/5\}$ and the other to $\{\sigma(a,b) > 2/5\}$, and that they have the geometry depicted in Figure 1.

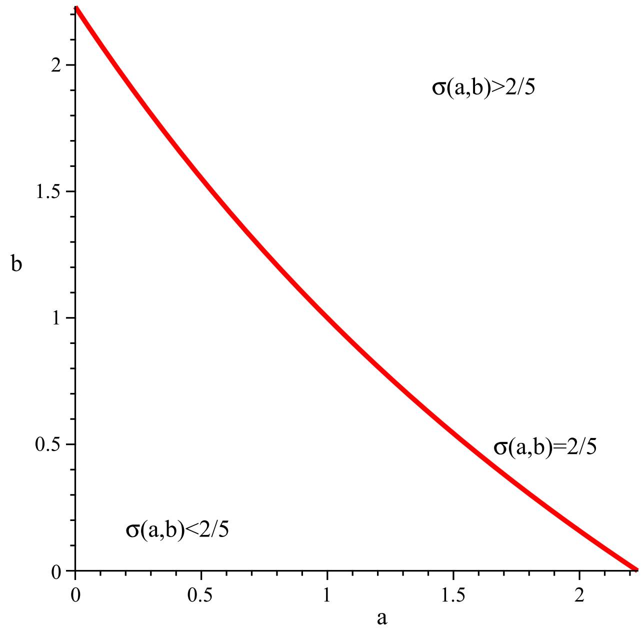

Figure 1: The sets $\{\sigma(a,b) < 2/5\}$ and $\{\sigma(a,b) > 2/5\}$, and the curve $\sigma(a,b) = 2/5$.

4.2 Forbidden periods and periods not appearing in $\mathcal{Q}^+$

Proposition 20. *The map $F_{b,a}$ do not have periodic orbits of prime periods 2 and 3.*

Proof. Observe that the orbits of the dynamical system defined by the iterations of $F_{b,a}$ are also given by the planar first order system of difference equations given by

$$\begin{cases} u_{n+1}u_n = a + v_n, \\ v_{n+1}v_n = b + u_{n+1}. \end{cases}$$

If the above system has a periodic solution of prime period 2, then we have that $u_{n+1}u_n$ is constant, so it also must be $a + v_n$, and therefore v_n and u_n are constant.

If there is a periodic solution of prime period 3 of the above system we have

$$u_{n+2}u_{n+1}u_n = c = u_{n+2}(a + v_n), \tag{40}$$

where c is a constant; and also we have

$$v_{n+2}v_{n+1}v_n = d = (b + u_{n+2})v_n,$$

where d is a constant. Hence

$$u_{n+2} = \frac{d}{v_n} - b.$$

Using equation (40) we have $c = (a + v_n)\left(\frac{d}{v_n} - b\right)$, so

$$b\, v_n^2 \;+\; (ab + c - d)\, v_n \;-\; ad \;=\; 0,$$

and therefore v_n is constant. ∎

Proposition 21. *The map $F_{b,a}$ do not have periodic orbits of prime periods 4, 6 and 10 in $\mathcal{Q}^+$.*

Observe, however, that there can exist periodic orbits of minimal period 4, 6 and 10 in $\mathbb{R}^2 \setminus \mathcal{Q}^+$ for some values of a and b. For instance, if $a = 1, b = 2$, then the curves $\mathcal{C}_h$ with $h = 3$; $h = 4 + \sqrt{13}$; and $h \simeq 9.8309187775$ gives rise to continua of initial conditions with minimal period 4, 6 and 10 respectively. These curves are located in $\mathbb{R}^2 \setminus \mathcal{Q}^+$. Of course, the period 10 appears when considering the map $F_{1,1}$, but not as a prime period.

Proof of Proposition 21. We consider again the coordinates X, Y and T introduced in (9), and the curves $\mathcal{D}_L$. Observe that, there exist periodic orbits with minimal period $2n$ in the curve $\mathcal{D}_L$ if and only if $2nH = V$, or in other words, if and only if $nH = -nH$. Using that $V * V = R := [-\alpha : 0 : 1]$, we have

$$nH = V \Longleftrightarrow nH = -nH = nH * (V * V) \Longleftrightarrow nH = nH * R.$$

The last of the above condition means that there exist a $2n$ periodic orbit if and only if R belongs to the tangent line to $\mathcal{D}_L$ in nH. In the following we will see that for periods 4, 6 and 10 this condition implies $L < L_c$, thus proving that these periods do not appear in $\mathcal{Q}^+$.

Period 4. Since $2H = (H * H) * V = [0 : -1 : 1] =: Q$, there exist a 4−periodic orbit if and only if R belongs to the tangent line to $\mathcal{D}_h$ in Q. An straightforward computation allows us to find the explicit expression of the tangent line to $2H$: $Y = m(L; \alpha, \beta)\, X \,+\, n(L; \alpha, \beta)$, where

$$m(L; \alpha, \beta) = -\frac{L + \alpha(\beta - 1)}{\alpha(\beta - 1)} \;\text{ and }\; n(L; \alpha, \beta) = -1.$$

This tangent line passes through R if and only if

$$-\,\alpha\, m(L; \alpha, \beta) + n(L; \alpha, \beta) = 0, \tag{41}$$

or equivalently, after some manipulations, if and only if $L + (\beta - 1)\,(\alpha - 1) = 0$. That is, when

$$L = L_4 := -\,(\beta - 1)\,(\alpha - 1)\,.$$

To see that $L_4 < L_c$, we consider again the parameters p and q given by the coordinates of the fixed point in $\mathcal{Q}^+$, given by (26), that is, we use the change

$$\left\{\alpha = \frac{p^2}{1+q}, \quad \beta = \frac{q^2}{1+p}\right. , \tag{42}$$

obtaining that

$$L_4 = \frac{\left(-q^2+p+1\right)\left(p^2-q-1\right)}{(1+p)\,(1+q)} < \frac{(1+p+q)^3}{(1+p)(1+q)} = L_c.$$

The last inequality can be obtained by subtraction of the numerators, for instance.

Period 6. A computation shows that

$$3H = [-L-\alpha\,(\beta-1) : L\beta+\beta\,(\beta-1)\,(\alpha-1) : \beta-1]$$

Proceeding as in the period 4 case, we compute the tangent line to $\mathcal{D}_L$ at $3H$, and we impose condition (41), obtaining that there exists a continuum of periodic orbits characterized by the energy level L if and only if either $L = 0 < L_c$ or if L is a root of

$$P_6(L) := L^2 + (3\alpha\beta-2\alpha-2\beta+1)\,L + (\beta-1)\,(\alpha-1)\,(2\alpha\beta-\alpha-\beta+1)\,.$$

Using the change (42) and the translation $L = \widetilde{L} + L_c$ we obtain that $P_6(L) = 0$ if and only if $\widetilde{L}$ is not a root of

$$\begin{aligned}\widetilde{P}_6(\widetilde{L}) := \;& (1+q)\,(1+p)\,\widetilde{L}^2 + \left(3p^2q^2+6p^2q+6pq^2+4p^2+13pq+4q^2+7p+7q+3\right)\widetilde{L}\\ & + (1+q)^2\,(1+p)^2\,(2pq+3p+3q+3)\,.\end{aligned}$$

Observe that all the coefficients of $\widetilde{P}_6(\widetilde{L})$ are positive when p and q are positive, so all its real roots are negative. Therefore, all the real roots of P_6 are less than L_c.

Period 10. A computation done with the aid of a computer algebra system gives $5H := [A : B : C]$, where

$$\begin{aligned}A :=\;& -\left[(\alpha\,\beta-1)\,L+\alpha\,\beta\,(\beta-1)\,(\alpha-1)\right]\left[L^2+\alpha\,(\beta-1)\,(\beta-2)\,L-\alpha\,(\beta-1)^3\,(\alpha-1)\right],\\ B :=\;& -\beta\,[L+(\beta-1)\,(\alpha-1)]\,[L+\alpha\,(\beta-1)]\,\big[L^2+(-2\alpha+1-2\beta+3\alpha\beta)\,L+\\ & +(\beta-1)\,(\alpha-1)\,(2\alpha\beta-\alpha-\beta+1)\big]\,,\\ C :=\;& [\beta L+(\beta-1)\,(\beta(\alpha-1)+1)]\,[L+(\beta-1)\,(\alpha-1)]\,[(\alpha\beta-1)\,L+\alpha\beta\,(\beta-1)\,(\alpha-1)]\,.\end{aligned}$$

We compute the tangent line to $\mathcal{D}_L$ at $5H$ and we impose condition (41), obtaining that there exists a continuum of periodic orbits characterized by the energy level L if and only if

$$P_{10,a}(L;\alpha,\beta)\,P_{10,b}(L;\alpha,\beta)\,P_{10,c}(L;\alpha,\beta) = 0,$$

where

$$P_{10,a}(L;\alpha,\beta) = (\alpha\,\beta-1)\,L+\alpha\,\beta\,(\beta-1)\,(-1+\alpha)\,,\; P_{10,b}(L,\alpha,\beta) = (L+\alpha(\beta-1))^2\,;$$

and $P_{10,c}(L;\alpha,\beta) := \sum_{i=0}^{5} p_i(\alpha,\beta)\,L^i$, being

$$\begin{aligned}
p_5(\alpha,\beta) =\ & \alpha\beta+1,\\
p_4(\alpha,\beta) =\ & 5\alpha^2\beta^2-5\alpha^2\beta-5\alpha\beta^2+12\alpha\beta-4\alpha-4\beta+1,\\
p_3(\alpha,\beta) =\ & 10\alpha^3\beta^3-20\alpha^3\beta^2-20\alpha^2\beta^3+10\alpha^3\beta+56\alpha^2\beta^2+10\alpha\beta^3-41\alpha^2\beta-41\alpha\beta^2\\
& +6\alpha^2+35\alpha\beta+6\beta^2-6\alpha-6\beta+1,\\
p_2(\alpha,\beta) =\ & (\beta-1)\,(\alpha-1)\,\big(10\alpha^3\beta^3-20\alpha^3\beta^2-20\alpha^2\beta^3+10\alpha^3\beta+55\alpha^2\beta^2+10\alpha\beta^3\\
& -37\alpha^2\beta-37\alpha\beta^2+4\alpha^2+30\alpha\beta+4\beta^2-5\alpha-5\beta+1\big)\,,\\
p_1(\alpha,\beta) =\ & (\beta\,(\alpha-1)-1)^4\,\big(5\alpha^3\beta^3-10\alpha^3\beta^2-10\alpha^2\beta^3+5\alpha^3\beta+25\alpha^2\beta^2+5\alpha\beta^3\\
& -15\alpha^2\beta-15\alpha\beta^2+\alpha^2+12\alpha\beta+\beta^2-2\alpha-2\beta+1\big)\,,\\
p_0(\alpha,\beta) =\ & \alpha\beta\,(\beta-1)^5\,(\alpha-1)^5\,.
\end{aligned}$$

It is easy to check, using the expression of $5H$ given above, that when $\alpha\beta-1\neq 0$ the value of L corresponding to the root of $P_{10,a}$ give $5H=V$, thus characterizing orbits of prime period 5, and so 10 is not a prime period for the orbits in this level set. If $\alpha\beta-1=0$, then $P_{10,a}$ does not depends on L, and it is zero only if $\alpha=\beta=1$, which corresponds to the well–known globally 10 periodic case $a=b=1$.

Observe that, by using (42), the root of $P_{10,b}$ is

$$L=\alpha(1-\beta)=\frac{p^2(1+p-q^2)}{(p+1)(q+1)},$$

which is obviously less than L_c.

To prove that the real zeros of $P_{10,c}$ are lower than L_c, we proceed as in the period 6 case, by using the change (42) and the translation $L=\widetilde{L}+L_c$, obtaining that $P_{10,c}(L)=0$ if and only if the degree 5 polynomial $\widetilde{P}_{10,c}(\widetilde{L})$ obtained when considering the equation $P_{10,c}(\widetilde{L}+L_c)=0$ with parameters p and q, vanishes. Once again it is easy to check that this polynomial has positive coefficients when p and q are positive, so its real roots are all negative, and therefore the roots of $P_{10,c}$ are all lower than L_c. ∎

4.3 Possible periods. Proof of Theorem 2

To prove Theorem 2, we need the following auxiliary result which allow us to characterize, constructively, which periods arise when the rotation number takes values in a given interval.

Lemma 22. *([5, Theorem 25 and Corollary 26]) Consider an open interval (c,d); denote by $p_1=2,p_2=3,p_3,\ldots,p_n,\ldots$ the set of all the prime numbers, ordered following the usual order. Also consider the following natural numbers:*

- *Let p_{m+1} be the smallest prime number satisfying that $p_{m+1}>\max(3/(d-c),2)$,*

- *Given any prime number p_n, $1 \leq n \leq m$, let s_n be the smallest natural number such that $p_n^{s_n} > 4/(d-c)$.*

- *Set $p := p_1^{s_1-1} p_2^{s_2-1} \cdots p_m^{s_m-1}$.*

Then, for any $r > p$ there exists an irreducible fraction q/r such that $q/r \in (c,d)$.

Proof of Theorem 2. Statement (i) is a direct consequence of Theorem 1 and Lemma 22.

Statement (ii) is a direct consequence of Proposition 17, which implies that for each number in $(1/3, 1/2)$ there exists some $a, b > 0$ and some initial condition in $\mathcal{Q}^+$ with this associated rotation number for $F_{b,a}$. In particular, for all the irreducible rational numbers $q/p \in (1/3, 1/2)$ there are positive values of a and b such that $F_{b,a}$ has continuum of periodic orbits of period p.

(iii) Setting $c = 1/3$ and $d = 1/2$, and using the notation introduced in Lemma 22, we have that, $m = 7$; $p_1 = 2$ (with $s_1 = 5$); $p_2 = 3$ (with $s_2 = 3$); $p_3 = 5$; $p_4 = 7$; $p_5 = 11$; $p_6 = 13$ and $p_7 = 17$ (with $s_i = 1$ for $i = 3, \ldots, 7$). From Lemma 22, for all $p \in \mathbb{N}$, such that $p > p_0 := 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = 12\,252\,240$ there exists an irreducible fraction $q/p \in (1/3, 1/2)$. Hence by Proposition 17 there exists some $a, b > 0$ such that there exists a continuum of initial conditions with rotation number $\theta_{b,a}(h) = q/p$, thus giving rise to p–periodic orbits of $F_{b,a}$. Now, using a finite algorithm we can determine which irreducible fractions q/p with $p \leq p_0$ are in $(1/2, 1/3)$, resulting that there appear irreducible fractions with all the denominators except $2, 3, 4, 6$ and 10.

Finally, the periods 2, 3, 4, 6 and 10 do not appear as a consequence of Propositions 20 and 21. ∎

Proof of Corollary 3. First observe that if $a \neq b$, then the recurrence (3)-(4) do not have periodic solutions of odd period $p \neq 1$. Indeed, suppose that for $u_1 > 0$,$u_2 > 0$ the sequence $\{u_n\}$ is a periodic solution with period $p = 2k+1$. Then, by using the relations in (6), we have that

$$(u_1, u_2) = (u_{2k+2}, u_{2k+3}) = F_a \left(F_{b,a}\right)^k (u_1, u_2). \tag{43}$$

But on the other hand, from (6) and (43) we have:

$$\begin{aligned} F_a(u_1, u_2) &= (u_2, u_3) = (u_{2k+3}, u_{2k+4}) = F_b \left(F_{a,b}\right)^k (u_2, u_3) \\ &= F_b \left(F_{a,b}\right)^k F_a(u_1, u_2) = \left(F_{b,a}\right)^{k+1} (u_1, u_2) \\ &= F_b \Big(F_a \left(F_{b,a}\right)^k (u_1, u_2) \Big) \\ &= F_b \left(u_1, u_2\right). \end{aligned}$$

Hence $F_a(u_1, u_2) = F_b\left(u_1, u_2\right)$, which implies that $a = b$.

Finally, observe that any p–periodic orbit of $F_{b,a}$ with initial conditions (u_1, u_2) will give rise to a $2p$–periodic solution of the recurrence (3). Now, both statements (i) and (ii) follow as a straightforward application of Theorem 2. ∎

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