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ANALYTIC INTEGRABILITY OF THE BIANCHI CLASS A COSMOLOGICAL MODELS WITH $0 \leq k < 1$

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Abstract. There are many works studying the integrability of the Bianchi class A cosmologies with $k = 1$. Here we characterize the analytic integrability of the Bianchi class A cosmological models when $0 \leq k < 1$.

1. Introduction

Bianchi models describe space-times which are foliated by homogeneous (and so we have three dimensional Lie algebras) hypersurfaces of constant time. Bianchi [2, 3] was the first to classify three dimensional Lie algebras which are nonisomorphic. There are nine types of models according to the dimension n of the algebra:

(a) $n = 0$: type I;
(b) $n = 1$: types II, III;
(c) $n = 2$: types IV, V, VI, VII;
(d) $n = 3$: types VIII, IX.

If we consider X_1, X_2, X_3 an appropriate basis of the 3-dimensional Lie Algebra, then the classification depends on a scalar $a \in \mathbb{R}$ and a vector (n_1, n_2, n_3), with $n_i \in \{+1, -1, 0\}$ such that

$$[X_1, X_2] = n_3 X_3, \quad [X_2, X_3] = n_1 X_1 - a X_2, \quad [X_3, X_1] = n_2 X_2 + a X_1,$$

where $[,]$ is the Lie bracket. In particular for $a = 0$ we obtain models of class A and for $a \neq 0$ we obtain models of class B. A good reference for the Bianchi models is Bogoyavlensky [4].

In a cosmological model Einstein's equations connect the geometry of the space-time with the properties of the matter. The matter occupying the space-time is determined by the stress energy tensor of the matter. In our study we follow [4] and we consider the hydrodynamical tensor of the matter. We will work with an equation of state of matter of the form $p = k\varepsilon$, where ε is the energy density of the matter, p is the pressure and $0 \leq k \leq 1$.

Following [4] the Einstein equations for the homogenous cosmologies of class A without motion of matter can be formalized as a Hamiltonian system in the phase space p_i, q_i for

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$i = 1, 2, 3$ with the Hamiltonian function

$$H = \frac{1}{(q_1q_2q_3)^{\frac{1-k}{2}}}\left(T(p_iq_i) + \frac{1}{4}V_G(q_i)\right).$$

Here T is the kinetic energy (not positive defined) and V_G is the potential. According to [4] (Section 4 of Chapter II) the kinetic and the potential energy are given, respectively, by

$$\begin{array}{rcl} T(p_iq_i) & = & 2\sum_{i<j}^{3} p_ip_jq_iq_j - \sum_{i=1}^{3} p_i^2q_i^2, \\ V_G(q_i) & = & 2\sum_{i<j}^{3} n_in_jq_iq_j - \sum_{i=1}^{3} n_i^2q_i^2, \end{array}$$

for $i, j \in \{1, 2, 3\}$.

Consider the time τ defined by $d\tau = (q_1q_2q_3)^{-k/2}dt$, where t is the synchronous time. The Hamiltonian system in the new time τ is written as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

or equivalently as

$$\begin{aligned}
\dot{q}_1 =& 2q_1(q_1q_2q_3)^{\frac{k-1}{2}}(-p_1q_1 + p_2q_2 + p_3q_3),\\
\dot{q}_2 =& 2q_2(q_1q_2q_3)^{\frac{k-1}{2}}(p_1q_1 - p_2q_2 + p_3q_3),\\
\dot{q}_3 =& 2q_3(q_1q_2q_3)^{\frac{k-1}{2}}(p_1q_1 + p_2q_2 - p_3q_3),\\
\dot{p}_1 =& -(q_1q_2q_3)^{\frac{k-1}{2}}\left(2p_1(-p_1q_1 + p_2q_2 + p_3q_3) + \frac{1}{2}n_1(-n_1q_1 + n_2q_2 + n_3q_3)\right) + \frac{1-k}{2q_1}H,\\
\dot{p}_2 =& -(q_1q_2q_3)^{\frac{k-1}{2}}\left(2p_2(p_1q_1 - p_2q_2 + p_3q_3) + \frac{1}{2}n_2(n_1q_1 - n_2q_2 + n_3q_3)\right) + \frac{1-k}{2q_2}H,\\
\dot{p}_3 =& -(q_1q_2q_3)^{\frac{k-1}{2}}\left(2p_3(p_1q_1 + p_2q_2 - p_3q_3) + \frac{1}{2}n_3(n_1q_1 + n_2q_2 - n_3q_3)\right) + \frac{1-k}{2q_2}H.
\end{aligned}$$

Note that the constants n_1, n_2, n_3 determine the type of the model according to Table 1. After the change of coordinates $ds = (q_1q_2q_3)^{\frac{1-k}{2}}d\tau$, $q_i = x_i$, $p_i = x_{i+3}/(2x_i)$, $i = 1, 2, 3$, we obtain the quadratic homogeneous polynomial differential system

$$\begin{array}{rcl}
\dot{x}_1 & = & x_1(-x_4 + x_5 + x_6),\\
\dot{x}_2 & = & x_2(x_4 - x_5 + x_6),\\
\dot{x}_3 & = & x_3(x_4 + x_5 - x_6),\\
\dot{x}_4 & = & n_1x_1(n_1x_1 - n_2x_2 - n_3x_3) + \frac{k-1}{4}F,\\
\dot{x}_5 & = & n_2x_2(-n_1x_1 + n_2x_2 - n_3x_3) + \frac{k-1}{4}F,\\
\dot{x}_6 & = & n_3x_3(-n_1x_1 - n_2x_2 + n_3x_3) + \frac{k-1}{4}F,
\end{array} \tag{1}$$

Type	I	II	VI_0	VII_0	VIII	IX
a	0	0	0	0	0	0
n_1	0	1	1	1	1	1
n_2	0	0	-1	1	1	1
n_3	0	0	0	0	-1	1

Table 1. The classification of Bianchi class A cosmologies.

where

$$
\begin{aligned}
F =& n_1^2x_1^2 + n_2^2x_2^2 + n_3^2x_3^2 - 2n_1n_2x_1x_2 - 2n_1n_3x_1x_3 - 2n_2n_3x_2x_3 \\
&+ x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_5x_6 - 2x_4x_6.
\end{aligned} \tag{2}
$$

Note that system (1) is a homogeneous polynomial differential system of degree 2. The Hamiltonian H becomes after the changes of variables the first integral

$$
\begin{aligned}
\mathcal{H} =& (x_1x_2x_3)^{\frac{k-1}{2}} F \\
=& (x_1x_2x_3)^{\frac{k-1}{2}} (n_1^2x_1^2 + n_2^2x_2^2 + n_3^2x_3^2 - 2n_1n_2x_1x_2 - 2n_1n_3x_1x_3 \\
&- 2n_2n_3x_2x_3 + x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_5x_6 - 2x_4x_6)
\end{aligned} \tag{3}
$$

of system (1).

Let U be an open and dense subset of $\mathbb{R}^6$. Then we recall that system (1) has a first integral $\mathcal{H} : U \to \mathbb{R}$ if $\mathcal{H}$ is a non-constant $\mathcal{C}^1$-function such that

$$\dot{x}_1 \frac{\partial \mathcal{H}}{\partial x_1} + \cdots + \dot{x}_6 \frac{\partial \mathcal{H}}{\partial x_6} = 0.$$

Many authors have studied some models of Class A for the case $k = 1$ considering different types of integrability, see for exemple [5, 6, 7, 8, 9, 11, 12]. In this work we study the analytic integrability of all Bianchi models of class A in the variables $(x_1, x_2, x_3, x_4, x_5, x_6)$ for $0 \le k < 1$. The following result is well known, see for instance [10].

Proposition 1. *Let F be an analytic function and let $F = \sum_i F_i$ be its decomposition into homogeneous polynomials of degree i. Then F is an analytic first integral of the homogeneous differential system* (1) *if and only if F_i is a homogeneous polynomial first integral of system* (1) *for all i.*

A Hamiltonian system with n degrees of freedom is *completely integrable* if it admits n independent first integrals in involution, see for more details [1]. A differential system of n variables is *completely integrable* if it admits $n-1$ independent first integrals.

According to Proposition 1 the study of the analytic first integrals of the homogeneous system (1) is reduced to the study of its polynomial homogeneous first integrals. The main result of this paper is the characterization of the polynomial first integrals of the Bianchi models of class A. Section 2 provides three technical lemmas that we will use in Section 3 to prove the following theorem.

Theorem 2. *For $0 \le k < 1$ the following statements hold.*

(a) *The Bianchi type* I *model is completely integrable.*
(b) *The Bianchi type* II *model has the polynomial first integral $K = x_5 - x_6$. This model does not admit any additional polynomial first integral independent from $\mathcal{H}$ and K.*
(c) *The Bianchi type* VI_0 *and* VII_0 *models have no polynomial first integrals.*
(d) *The Bianchi type* VIII *and* IX *models have no polynomial first integrals.*

2. Some auxiliary lemmas

In order to prove Theorem 2 we shall use the following three lemmas.

Lemma 3 (see [9])**.** *Let x_k be a one-dimensional variable, $k \in \{1, \ldots, n\}$, $n > 1$ and let $f = f(x_1, \ldots, x_n)$ be a polynomial. For $l \in \{1, \cdots, n\}$ and c_0 a constant let $f_l = f(x_1, \ldots, x_n)|_{x_l = c_0}$. Then there exists a polynomial $g = g(x_1, \ldots, x_n)$ such that $f = f_l + (x_l - c_0)g$.*

Lemma 4. *Let $g = g(x_4, x_5, x_6)$ be a homogeneous polynomial solution of the homogeneous partial differential equation*

$$(a_1 x_4 + a_2 x_5 + a_3 x_6) g + \frac{k-1}{4} F_{123} \left(\frac{\partial g}{\partial x_4} + \frac{\partial g}{\partial x_5} + \frac{\partial g}{\partial x_6} \right) = 0, \qquad (4)$$

where $F_{123} = x_4^2 + x_5^2 + x_6^2 - 2(x_4 x_5 + x_4 x_6 + x_5 x_6)$ and $a_1, a_2, a_3 \in \mathbb{R}$ are such that $(a_1 - a_2)^2 + (a_1 - a_3)^2 \neq 0$. Then $g \equiv 0$.

Proof. The general solution of equation (4) is

$$\begin{aligned} g(x_4, x_5, x_6) = & f(x_4 - x_5, x_4 - x_6) \\ & \left(-x_4 - x_5 - x_6 + 2\sqrt{\Delta}\right)^{\Delta_1 + \Delta_2} \left(x_4 + x_5 + x_6 + 2\sqrt{\Delta}\right)^{\Delta_1 - \Delta_2}, \end{aligned}$$

where $\Delta_1 = 2(a_1 + a_2 + a_3)/(3(k-1))$, $\Delta_2 = ((2a_1 - a_2 - a_3)x_4 + (-a_1 + 2a_2 - a_3)x_5 + (-a_1 - a_2 + 2a_3)x_6)/(3(k-1)\sqrt{\Delta})$, $\Delta = x_4^2 + x_5^2 + x_6^2 - x_4 x_5 - x_4 x_6 - x_5 x_6$ and f is an arbitrary function. We note that g is a polynomial if and only if $\Delta_1 \in \mathbb{N}$, $\Delta_2 = 0$ and f is a polynomial. In particular, the relation $\Delta_2 = 0$ is equivalent to the linear system

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The solution of this system is $a_1 = a_2 = a_3$. This cannot happen by assumption. Therefore g is not a polynomial unless $g \equiv 0$. ☐

Lemma 5. *Let $g = g(x_4, x_5, x_6)$ and $h = h(x_4 - x_5, x_4 - x_6)$ be homogeneous polynomials of respective degrees $n-2$ and n such that*

$$2(x_4 - x_5 + x_6) g + \frac{k-1}{4} F_{123} \left(\frac{\partial g}{\partial x_4} + \frac{\partial g}{\partial x_5} + \frac{\partial g}{\partial x_6} \right) + \frac{\partial h}{\partial x_5} = 0, \qquad (5)$$

where $F_{123} = x_4^2 + x_5^2 + x_6^2 - 2(x_4 x_5 + x_4 x_6 + x_5 x_6)$. Then $h = h(x_4 - x_6)$ and $g \equiv 0$.

Proof. Let $h = \sum_{i=0}^{n} a_i (x_4 - x_5)^i (x_4 - x_6)^{n-i}$ and $g = \sum_{i+j=0}^{n-2} b_{ij} x_4^i x_5^j x_6^{n-2-i-j}$. Suppose that $g \not\equiv 0$. Forcing that the solution of (5) be a polynomial, Mathematica (see [13]) shows that g is of the form

$$g(x_4, x_5, x_6) = \frac{4}{1-k} \int \frac{h_5}{F_{123}} \, dx_4 + f(x_4 - x_5, x_4 - x_6),$$

where f is a homogeneous polynomial, $h_5 = \dfrac{\partial h}{\partial x_5}$ and the integral is to be a polynomial.

Let $A_1 = \sqrt{x_5} - \sqrt{x_6}$ and $A_2 = \sqrt{x_5} + \sqrt{x_6}$. Under this notation $F_{123} = (x_4 - A_1^2)(x_4 - A_2^2)$. The fraction inside the above integral can be written as

$$\frac{h_5}{F_{123}} = X_0 + \frac{1}{A_1^2 - A_2^2} \left(\frac{X_1}{x_4 - A_1^2} - \frac{X_2}{x_4 - A_2^2} \right),$$

where $X_0 = X_0(x_4, A_1, A_2)$, $X_1 = X_1(A_1, A_2)$ and $X_2 = X_2(A_1, A_2)$ are homogeneous polynomials. The integrals of the fractions in the right hand side with respect to x_4 are $X_i \log(x_4 - A_i^2)$, $i = 1, 2$; hence X_1 and X_2 must be identically zero. $X_1 = 0$ and $X_2 = 0$

have the same solutions $a_1, \dots, a_n$ because of symmetry. Indeed $X_1 = 0$ (or $X_2 = 0$) reduces to $S_n = 0$, where

$$S_n = \sum_{i=1}^{n} (3A_1 - A_2)^{n-i}(A_1 + A_2)^{n-i}(3A_1 + A_2)^{i-1}(A_1 - A_2)^{i-1} i\, a_i.$$

We note that we have the recursive equality

$$S_n = (3A_1 - A_2)(A_1 + A_2)S_{n-1} + (3A_1 + A_2)^{n-1}(A_1 - A_2)^{n-1} n\, a_n.$$

On $A_1 = -A_2$ (or equivalently on $x_5 = 0$) we have $n4^{n-1}A_2^{2n-2}a_n = 0$, and hence we have $a_n = 0$. Induction arguments prove that $S_n = 0$ implies $a_1 = \cdots = a_n = 0$. Therefore $h_5 = 0$, which means that equation (5) is a particular case of equation (4) and then by Lemma 4 we get $g \equiv 0$ and then we are finished. □

3. Proof of Theorem 2

In this section we prove the four statements of Theorem 2.

3.1. **Proof of statement (a) of Theorem 2.** According to Table 1 the Bianchi cosmological model I is obtained for $n_1 = n_2 = n_3 = 0$. System (1) becomes

$$\begin{aligned}
\dot{x}_1 &= x_1(-x_4 + x_5 + x_6),\\
\dot{x}_2 &= x_2(x_4 - x_5 + x_6),\\
\dot{x}_3 &= x_3(x_4 + x_5 - x_6),\\
\dot{x}_4 &= \frac{k-1}{4}F,\\
\dot{x}_5 &= \frac{k-1}{4}F,\\
\dot{x}_6 &= \frac{k-1}{4}F,
\end{aligned} \tag{6}$$

where $F = (x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_5x_6 - 2x_4x_6)$. Straightforward computations show that system (6) has the five first integrals $x_4 - x_5$, $x_4 - x_6$, $\mathcal{H}$ defined in (3),

$$\left(\frac{x_1}{x_2}\right)^{\frac{1-k}{2}} \left(\frac{x_4 + x_5 + x_6 - 2\sqrt{\Delta}}{x_4 + x_5 + x_6 + 2\sqrt{\Delta}}\right)^{\frac{x_4 - x_5}{\sqrt{\Delta}}},$$

and

$$\left(\frac{x_2}{x_3}\right)^{\frac{1-k}{2}} \left(\frac{x_4 + x_5 + x_6 - 2\sqrt{\Delta}}{x_4 + x_5 + x_6 + 2\sqrt{\Delta}}\right)^{\frac{x_5 - x_6}{\sqrt{\Delta}}},$$

with $\Delta = x_4^2 + x_5^2 + x_6^2 - x_4x_5 - x_5x_6 - x_4x_6$. Note that the five first integrals are independent. Statement (a) is proved.

3.2. **Prove of statement (b) of Theorem 2.** The Bianchi cosmological model II is obtained for $n_1 = 1$ and $n_2 = n_3 = 0$. System (1) writes as

$$\begin{aligned} \dot{x}_1 =& x_1(-x_4 + x_5 + x_6),\\ \dot{x}_2 =& x_2(x_4 - x_5 + x_6),\\ \dot{x}_3 =& x_3(x_4 + x_5 - x_6),\\ \dot{x}_4 =& x_1^2 + \frac{k-1}{4}F,\\ \dot{x}_5 =& \frac{k-1}{4}F,\\ \dot{x}_6 =& \frac{k-1}{4}F, \end{aligned} \tag{7}$$

where $F = x_1^2 + x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_5x_6 - 2x_4x_6$.

Let $h = h(x_1, x_2, x_3, x_4, x_5, x_6)$ be a homogeneous polynomial first integral of (7). Using Lemma 3 we can write $h = h_1(x_2, x_3, x_4, x_5, x_6) + x_1^j g_1(x_1, x_2, x_3, x_4, x_5, x_6)$, with $j \in \mathbb{N}$ and h_1 and g_1 homogeneous polynomials such that $x_1 \nmid g_1$. On $x_1 = 0$ system (7) becomes

$$\begin{aligned} \dot{x}_2 =& x_2(x_4 - x_5 + x_6),\\ \dot{x}_3 =& x_3(x_4 + x_5 - x_6),\\ \dot{x}_4 =& \frac{k-1}{4}F_1,\\ \dot{x}_5 =& \frac{k-1}{4}F_1,\\ \dot{x}_6 =& \frac{k-1}{4}F_1 \end{aligned} \tag{8}$$

where $F_1 = F|_{x_1=0}$. System (8) admits the two polynomial first integrals x_4-x_5 and x_5-x_6 and the two non-polynomial first integrals

$$x_2^{\frac{3}{2}(k-1)} F_1 \left(\frac{x_4 + x_5 + x_6 - 2\sqrt{\Delta}}{x_4 + x_5 + x_6 + 2\sqrt{\Delta}} \right)^{\frac{x_4-2x_5+x_6}{\sqrt{\Delta}}}$$

and

$$x_3^{\frac{3}{2}(k-1)} F_1 \left(\frac{x_4 + x_5 + x_6 - 2\sqrt{\Delta}}{x_4 + x_5 + x_6 + 2\sqrt{\Delta}} \right)^{\frac{x_4+x_5-2x_6}{\sqrt{\Delta}}},$$

where $\Delta = x_4^2 + x_5^2 + x_6^2 - x_4x_5 - x_4x_6 - x_5x_6$. As these four first integrals of system (8) are independent and h_1 is a polynomial first integral of (8), we have $h_1 = h_1(x_4 - x_5, x_5 - x_6)$.

The following lemma ends the proof of statement (b) of Theorem 2.

Lemma 6. *For system* (7) *we have that* $h_1 = h_1(x_5 - x_6)$ *and* $g_1 \equiv 0$.

Proof. Suppose that $g_1 \not\equiv 0$. As h is a first integral of (7), we have

$$\begin{aligned} &x_1^j \left[j(-x_4 + x_5 + x_6)g_1 + x_1(-x_4 + x_5 + x_6)\frac{\partial g_1}{\partial x_1} + x_2(x_4 - x_5 + x_6)\frac{\partial g_1}{\partial x_2} \right.\\ &\left. + x_3(x_4 + x_5 - x_6)\frac{\partial g_1}{\partial x_3} + x_1^2\frac{\partial g_1}{\partial x_4} + \frac{k-1}{4}F\left(\frac{\partial g_1}{\partial x_4} + \frac{\partial g_1}{\partial x_5} + \frac{\partial g_1}{\partial x_6}\right)\right] + x_1^2\frac{\partial h_1}{\partial x_4} = 0. \end{aligned} \tag{9}$$

We distinguish three cases depending on the value of j. If $j = 1$ then equation (9) becomes

$$(-x_4+x_5+x_6)g_1 + x_1(-x_4+x_5+x_6)\frac{\partial g_1}{\partial x_1} + x_2(x_4-x_5+x_6)\frac{\partial g_1}{\partial x_2} + x_3(x_4+x_5-x_6)\frac{\partial g_1}{\partial x_3} + x_1^2\frac{\partial g_1}{\partial x_4} + \frac{k-1}{4}F\left(\frac{\partial g_1}{\partial x_4}+\frac{\partial g_1}{\partial x_5}+\frac{\partial g_1}{\partial x_6}\right) + x_1\frac{\partial h_1}{\partial x_4} = 0.$$

Let $\bar{g}_1 = g_1|_{x_1=0} \not\equiv 0$. Equation (9) on $x_1 = 0$ can be written as

$$(-x_4+x_5+x_6)\bar{g}_1 + x_2(x_4-x_5+x_6)\frac{\partial \bar{g}_1}{\partial x_2} + x_3(x_4+x_5-x_6)\frac{\partial \bar{g}_1}{\partial x_3} + \frac{k-1}{4}F_1\left(\frac{\partial \bar{g}_1}{\partial x_4}+\frac{\partial \bar{g}_1}{\partial x_5}+\frac{\partial \bar{g}_1}{\partial x_6}\right) = 0.$$

Write $\bar{g}_1 = x_2^l g_2 \not\equiv 0$, with $l \in \mathbb{N}\cup\{0\}$ and $x_2 \nmid g_2$. We get

$$((-x_4+x_5+x_6)+l(x_4-x_5+x_6))g_2 + x_2(x_4-x_5+x_6)\frac{\partial g_2}{\partial x_2} + x_3(x_4+x_5-x_6)\frac{\partial g_2}{\partial x_3} + \frac{k-1}{4}F_1\left(\frac{\partial g_2}{\partial x_4}+\frac{\partial g_2}{\partial x_5}+\frac{\partial g_2}{\partial x_6}\right) = 0.$$

Let $\bar{g}_2 = g_2|_{x_2=0} \not\equiv 0$. On $x_2 = 0$ we have

$$((-x_4+x_5+x_6)+l(x_4-x_5+x_6))\bar{g}_2 + x_3(x_4+x_5-x_6)\frac{\partial \bar{g}_2}{\partial x_3} + \frac{k-1}{4}F_{12}\left(\frac{\partial \bar{g}_2}{\partial x_4}+\frac{\partial \bar{g}_2}{\partial x_5}+\frac{\partial \bar{g}_2}{\partial x_6}\right) = 0,$$

where $F_{12} = F_1|_{x_2=0}$. Now we write $\bar{g}_2 = x_3^m g_3 \not\equiv 0$, with $m \in \mathbb{N}\cup\{0\}$ and $x_3 \nmid g_3$. We obtain

$$((-x_4+x_5+x_6)+l(x_4-x_5+x_6)+m(x_4+x_5-x_6))g_3 + x_3(x_4+x_5-x_6)\frac{\partial g_3}{\partial x_3} + \frac{k-1}{4}F_{12}\left(\frac{\partial g_3}{\partial x_4}+\frac{\partial g_3}{\partial x_5}+\frac{\partial g_3}{\partial x_6}\right) = 0.$$

Let $\bar{g}_3 = g_3|_{x_3=0} \not\equiv 0$. On $x_3 = 0$ we have

$$((-x_4+x_5+x_6)+l(x_4-x_5+x_6)+m(x_4+x_5-x_6))\bar{g}_3 + \frac{k-1}{4}F_{123}\left(\frac{\partial \bar{g}_3}{\partial x_4}+\frac{\partial \bar{g}_3}{\partial x_5}+\frac{\partial \bar{g}_3}{\partial x_6}\right) = 0,$$

where $F_{123} = F_{12}|_{x_3=0}$. Applying Lemma 4 we get $\bar{g}_3 \equiv 0$, which is a contradiction. Hence we have $g_1 \equiv 0$ and therefore $\dfrac{\partial h_1}{\partial x_4} \equiv 0$. The lemma follows in this case.

If $j > 2$ then $x_1 \left| \dfrac{\partial h_1}{\partial x_4}\right.$, and then $\dfrac{\partial h_1}{\partial x_4} \equiv 0$. Now we can proceed in a similar way as in the case $j = 1$ to prove that $g_1 \equiv 0$ by using Lemma 4.

If $j = 2$ then equation (9) becomes

$$2(-x_4+x_5+x_6)g_1 + x_1(-x_4+x_5+x_6)\frac{\partial g_1}{\partial x_1} + x_2(x_4-x_5+x_6)\frac{\partial g_1}{\partial x_2} + x_3(x_4+x_5-x_6)\frac{\partial g_1}{\partial x_3} + x_1^2\frac{\partial g_1}{\partial x_4} + \frac{k-1}{4}F\left(\frac{\partial g_1}{\partial x_4}+\frac{\partial g_1}{\partial x_5}+\frac{\partial g_1}{\partial x_6}\right) + \frac{\partial h_1}{\partial x_4} = 0.$$

Let $\bar{g}_1 = g_1|_{x_1=0} \not\equiv 0$. Equation (9) on $x_1 = 0$ can be written as

$$2(-x_4+x_5+x_6)\bar{g}_1 + x_2(x_4-x_5+x_6)\frac{\partial \bar{g}_1}{\partial x_2} + x_3(x_4+x_5-x_6)\frac{\partial \bar{g}_1}{\partial x_3} + \frac{k-1}{4}F_1\left(\frac{\partial \bar{g}_1}{\partial x_4}+\frac{\partial \bar{g}_1}{\partial x_5}+\frac{\partial \bar{g}_1}{\partial x_6}\right) + \frac{\partial h_1}{\partial x_4} = 0.$$

Write $\bar{g}_1 = x_2^l g_2 \not\equiv 0$, with $l \in \mathbb{N}\cup\{0\}$ and $x_2 \nmid g_2$. We get

$$(2(-x_4+x_5+x_6)+l(x_4-x_5+x_6))g_2 + x_2(x_4-x_5+x_6)\frac{\partial g_2}{\partial x_2} + x_3(x_4+x_5-x_6)\frac{\partial g_2}{\partial x_3} + \frac{k-1}{4}F_1\left(\frac{\partial g_2}{\partial x_4}+\frac{\partial g_2}{\partial x_5}+\frac{\partial g_2}{\partial x_6}\right) + \frac{\partial h_1}{\partial x_4} = 0.$$

If $l > 0$ then $\dfrac{\partial h_1}{\partial x_4} \equiv 0$. Similar arguments to those used above lead to the desired result after applying Lemma 4. If $l = 0$, let $\bar{g}_2 = g_2|_{x_2=0} \not\equiv 0$. On $x_2 = 0$ we have

$$2(-x_4+x_5+x_6)\bar{g}_2 + x_3(x_4+x_5-x_6)\frac{\partial \bar{g}_2}{\partial x_3} + \frac{k-1}{4}F_{12}\left(\frac{\partial \bar{g}_2}{\partial x_4}+\frac{\partial \bar{g}_2}{\partial x_5}+\frac{\partial \bar{g}_2}{\partial x_6}\right) + \frac{\partial h_1}{\partial x_4} = 0,$$

where $F_{12} = F_1|_{x_2=0}$. Now we write $\bar{g}_2 = x_3^m g_3 \not\equiv 0$, with $m \in \mathbb{N}\cup\{0\}$ and $x_3 \nmid g_3$. We obtain

$$(2(-x_4+x_5+x_6)+m(x_4+x_5-x_6))g_3 + x_3(x_4+x_5-x_6)\frac{\partial g_3}{\partial x_3} + \frac{k-1}{4}F_{12}\left(\frac{\partial g_3}{\partial x_4}+\frac{\partial g_3}{\partial x_5}+\frac{\partial g_3}{\partial x_6}\right) + \frac{\partial h_1}{\partial x_4} = 0.$$

If $m > 0$ then $\dfrac{\partial h_1}{\partial x_4} \equiv 0$. Again the usual arguments lead to the desired result after applying Lemma 4. If $m = 0$, let $\bar{g}_3 = g_3|_{x_3=0} \not\equiv 0$. On $x_3 = 0$ we have

$$2(-x_4+x_5+x_6)\bar{g}_3 + \frac{k-1}{4}F_{123}\left(\frac{\partial \bar{g}_3}{\partial x_4}+\frac{\partial \bar{g}_3}{\partial x_5}+\frac{\partial \bar{g}_3}{\partial x_6}\right) + \frac{\partial h_1}{\partial x_4} = 0,$$

where $F_{123} = F_{12}|_{x_3=0}$. Applying Lemma 5 swapping x_4 and x_5 we get $\bar{g}_3 \equiv 0$, which is a contradiction. Hence we have $g_1 \equiv 0$ and therefore $\dfrac{\partial h_1}{\partial x_4} \equiv 0$. The lemma follows also in this case.

□

After Lemma 6, $h = h(x_5 - x_6)$. Hence statement (b) of Theorem 2 follows.

3.3. **Proof of statement (c) of Theorem 2.** According to Table 1, system (1) in cases VI_0 and VII_0 can be written as

$$\begin{aligned}
\dot{x}_1 &= x_1(-x_4+x_5+x_6),\\
\dot{x}_2 &= x_2(x_4-x_5+x_6),\\
\dot{x}_3 &= x_3(x_4+x_5-x_6),\\
\dot{x}_4 &= x_1(x_1-n_2x_2) + \frac{k-1}{4}F,\\
\dot{x}_5 &= n_2x_2(-x_1+n_2x_2) + \frac{k-1}{4}F,\\
\dot{x}_6 &= \frac{k-1}{4}F,
\end{aligned} \tag{10}$$

where $F = (x_1 - n_2 x_2)^2 + x_4^2 + x_5^2 + x_6^2 - 2x_4x_5 - 2x_5x_6 - 2x_4x_6$ and $n_2^2 = 1$. Suppose that system (10) has a homogeneous polynomial first integral $h(x_1, \ldots, x_6)$. From Lemma 3 we can write $h = h_1(x_2, \ldots, x_6) + x_1^j g_1(x_1, \ldots, x_6)$, with $j \in \mathbb{N}$ and h_1 and g_1 homogeneous polynomials such that $x_1 \nmid g_1$. System (10) on $x_1 = 0$ is

$$\begin{aligned}
\dot{x}_2 &= x_2(x_4 - x_5 + x_6),\\
\dot{x}_3 &= x_3(x_4 + x_5 - x_6),\\
\dot{x}_4 &= \frac{k-1}{4}F_1,\\
\dot{x}_5 &= x_2^2 + \frac{k-1}{4}F_1,\\
\dot{x}_6 &= \frac{k-1}{4}F_1,
\end{aligned} \tag{11}$$

where $F_1 = F|_{x_1=0}$. We note that h_1 is a first integral of system (11). From Lemma 3 we can write $h_1 = h_2(x_3, \ldots, x_6) + x_2^l g_2(x_2, \ldots, x_6)$, with $l \in \mathbb{N}$ and h_2 and g_2 homogeneous polynomials such that $x_2 \nmid g_2$. System (11) on $x_2 = 0$ writes

$$\begin{aligned}
\dot{x}_3 &= x_3(x_4 + x_5 - x_6),\\
\dot{x}_4 &= \frac{k-1}{4}F_{12},\\
\dot{x}_5 &= \frac{k-1}{4}F_{12},\\
\dot{x}_6 &= \frac{k-1}{4}F_{12},
\end{aligned} \tag{12}$$

where $F_{12} = F_1|_{x_2=0}$. We note that h_2 is a first integral of system (12). Straightforward computations show that system (12) has the three independent first integrals $x_4 - x_5$, $x_5 - x_6$ and

$$x_3^{\frac{3}{2}(k-1)} F_{12} \left(\frac{x_4 + x_5 + x_6 - 2\sqrt{\Delta}}{x_4 + x_5 + x_6 + 2\sqrt{\Delta}} \right)^{\frac{x_4 + x_5 - 2x_6}{\sqrt{\Delta}}},$$

where $\Delta = x_4^2 + x_5^2 + x_6^2 - x_4x_5 - x_4x_6 - x_5x_6$. Therefore $h_2 = h_2(x_4 - x_5, x_4 - x_6)$.

Lemma 7. *For system* (11) *we have that* $h_2 = h_2(x_4 - x_6)$ *and* $g_2 \equiv 0$.

Proof. Suppose that $g_2 \not\equiv 0$. As $h_1 = h_2 + x_2^l g_2$ is a first integral of system (11), we can write

$$\begin{aligned}
x_2^l \Bigg[& l(x_4 - x_5 + x_6)g_2 + x_2(x_4 - x_5 + x_6)\frac{\partial g_2}{\partial x_2} + x_3(x_4 + x_5 - x_6)\frac{\partial g_2}{\partial x_3} \\
& + x_2^2 \frac{\partial g_2}{\partial x_5} + \frac{k-1}{4}F_1 \left(\frac{\partial g_2}{\partial x_4} + \frac{\partial g_2}{\partial x_5} + \frac{\partial g_2}{\partial x_6} \right) \Bigg] + x_2^2 \frac{\partial h_2}{\partial x_5} = 0.
\end{aligned} \tag{13}$$

We distinguish three cases depending on the value of l. If $l = 1$ then equation (13) becomes

$$\begin{aligned}
& (x_4 - x_5 + x_6)g_2 + x_2(x_4 - x_5 + x_6)\frac{\partial g_2}{\partial x_2} + x_3(x_4 + x_5 - x_6)\frac{\partial g_2}{\partial x_3} \\
& + \frac{k-1}{4}F_1 \left(\frac{\partial g_2}{\partial x_4} + \frac{\partial g_2}{\partial x_5} + \frac{\partial g_2}{\partial x_6} \right) + x_2 \frac{\partial h_2}{\partial x_5} + x_2^2 \frac{\partial g_2}{\partial x_5} = 0.
\end{aligned}$$

Let $\bar{g}_2 = g_2|_{x_2=0} \not\equiv 0$. On $x_2 = 0$ we have

$$(x_4 - x_5 + x_6)\bar{g}_2 + x_3(x_4 + x_5 - x_6)\frac{\partial \bar{g}_2}{\partial x_3} + \frac{k-1}{4}F_{12}\left(\frac{\partial \bar{g}_2}{\partial x_4} + \frac{\partial \bar{g}_2}{\partial x_5} + \frac{\partial \bar{g}_2}{\partial x_6}\right) = 0.$$

Write $\bar{g}_2 = x_3^m g_3 \not\equiv 0$, with $m \in \mathbb{N} \cup \{0\}$ and $x_3 \nmid g_3$. Then

$$\begin{aligned}((x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6))g_3 + x_3(x_4 + x_5 - x_6)\frac{\partial g_3}{\partial x_3}& \\ + \frac{k-1}{4}F_{12}\left(\frac{\partial g_3}{\partial x_4} + \frac{\partial g_3}{\partial x_5} + \frac{\partial g_3}{\partial x_6}\right) = 0.&\end{aligned}$$

Let $\bar{g}_3 = g_3|_{x_3=0} \not\equiv 0$. On $x_3 = 0$ we get

$$((x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6))\bar{g}_3 + \frac{k-1}{4}F_{123}\left(\frac{\partial \bar{g}_3}{\partial x_4} + \frac{\partial \bar{g}_3}{\partial x_5} + \frac{\partial \bar{g}_3}{\partial x_6}\right) = 0,$$

where $F_{123} = F_{12}|_{x_3=0}$. Applying Lemma 4 we obtain $\bar{g}_3 \equiv 0$, which is a contradiction. Hence $g_2 \equiv 0$. Back to equation (13) we have $\dfrac{\partial h_2}{\partial x_5} \equiv 0$. Then the lemma follows.

If $l > 2$, then from equation (13) we have that $x_2 \left| \dfrac{\partial h_2}{\partial x_5}\right.$ and thus $\dfrac{\partial h_2}{\partial x_5} \equiv 0$. Therefore $h_2 = h_2(x_4 - x_6)$. Now we can proceed in a similar way as in the case $l = 1$ to obtain the equation

$$(l(x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6))\bar{g}_3 + \frac{k-1}{4}F_{123}\left(\frac{\partial \bar{g}_3}{\partial x_4} + \frac{\partial \bar{g}_3}{\partial x_5} + \frac{\partial \bar{g}_3}{\partial x_6}\right) = 0.$$

Applying again Lemma 4 we arrive to contradiction and hence $g_2 \equiv 0$.

If $l = 2$, then equation (13) writes as

$$\begin{aligned}2(x_4 - x_5 + x_6)g_2 + x_2(x_4 - x_5 + x_6)\frac{\partial g_2}{\partial x_2} + x_3(x_4 + x_5 - x_6)\frac{\partial g_2}{\partial x_3}& \\ + \frac{k-1}{4}F_1\left(\frac{\partial g_2}{\partial x_4} + \frac{\partial g_2}{\partial x_5} + \frac{\partial g_2}{\partial x_6}\right) + x_2^2\frac{\partial g_2}{\partial x_5} + \frac{\partial h_2}{\partial x_5} = 0.&\end{aligned}$$

Let $\bar{g}_2 = g_2|_{x_2=0} \not\equiv 0$. On $x_2 = 0$ we have

$$2(x_4 - x_5 + x_6)\bar{g}_2 + x_3(x_4 + x_5 - x_6)\frac{\partial \bar{g}_2}{\partial x_3} + \frac{k-1}{4}F_{12}\left(\frac{\partial \bar{g}_2}{\partial x_4} + \frac{\partial \bar{g}_2}{\partial x_5} + \frac{\partial \bar{g}_2}{\partial x_6}\right) + \frac{\partial h_2}{\partial x_5} = 0.$$

Write $\bar{g}_2 = x_3^m g_3 \not\equiv 0$, with $m \in \mathbb{N} \cup \{0\}$ and $x_3 \nmid g_3$. Then

$$\begin{aligned}x_3^m\Bigg[(2(x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6))g_3 + x_3(x_4 + x_5 - x_6)\frac{\partial g_3}{\partial x_3}& \\ + \frac{k-1}{4}F_{12}\left(\frac{\partial g_3}{\partial x_4} + \frac{\partial g_3}{\partial x_5} + \ pdg_3x_6\right)\Bigg] + \frac{\partial h_2}{\partial x_5} = 0.&\end{aligned}$$

We distinguish two cases depending on the value of m. If $m > 0$ then $x_3 \left| \dfrac{\partial h_2}{\partial x_5}\right.$. Hence $\dfrac{\partial h_2}{\partial x_5} \equiv 0$ and $h_2 = h_2(x_4 - x_6)$. Now let $\bar{g}_3 = g_3|_{x_3=0} \not\equiv 0$. On $x_3 = 0$ we obtain

$$(2(x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6))\bar{g}_3 + \frac{k-1}{4}F_{123}\left(\frac{\partial \bar{g}_3}{\partial x_4} + \frac{\partial \bar{g}_3}{\partial x_5} + \frac{\partial \bar{g}_3}{\partial x_6}\right) = 0.$$

Applying Lemma 4 we get a contradiction and hence $g_2 \equiv 0$.

If $m=0$, let $\bar g_3 = g_3|_{x_3=0} \not\equiv 0$. On $x_3=0$ we obtain

$$2(x_4-x_5+x_6)\bar g_3 + \frac{k-1}{4}F_{123}\left(\frac{\partial \bar g_3}{\partial x_4}+\frac{\partial \bar g_3}{\partial x_5}+\frac{\partial \bar g_3}{\partial x_6}\right)+\frac{\partial h_2}{\partial x_5}=0.$$

Applying Lemma 5 we get $\dfrac{\partial h_2}{\partial x_5}\equiv 0$ and $\bar g_3\equiv 0$, hence the lemma follows.

All the subcases are finished and therefore the lemma is proved. □

After Lemma 7 we have that $h = h_2(x_4-x_6)+x_1^j g_1(x_1,\ldots,x_6)$, with $j\in\mathbb{N}$ and $x_1 \nmid g_1$. We recall that h is a first integral of system (10). Thus

$$\begin{aligned} x_1^j \Bigg[& j(-x_4+x_5+x_6)g_1 + x_1(-x_4+x_5+x_6)\frac{\partial g_1}{\partial x_1} \\ & + x_2(x_4-x_5+x_6)\frac{\partial g_1}{\partial x_2} + x_3(x_4+x_5-x_6)\frac{\partial g_1}{\partial x_3} + x_1(x_1-n_2x_2)\frac{\partial g_1}{\partial x_4} \\ & -n_2x_2(x_1-n_2x_2)\frac{\partial g_1}{\partial x_5} + \frac{k-1}{4}F\left(\frac{\partial g_1}{\partial x_4}+\frac{\partial g_1}{\partial x_5}+\frac{\partial g_1}{\partial x_6}\right)\Bigg] + x_1(x_1-n_2x_2)\frac{\partial h_2}{\partial x_4}=0. \end{aligned} \tag{14}$$

Lemma 8. *For system* (10) *we have that* $h_2\equiv 0$ *and* $g_1\equiv 0$.

Proof. Suppose that $g_1\not\equiv 0$. We distinguish two cases depending on the value of j. If $j>1$ then from equation (14) we have that $x_1 \,\Big|\, \dfrac{\partial h_2}{\partial x_4}$, and hence $h_2\equiv 0$. Therefore equation (14) can be written as

$$\begin{aligned} & j(-x_4+x_5+x_6)g_1 + x_1(-x_4+x_5+x_6)\frac{\partial g_1}{\partial x_1} + x_2(x_4-x_5+x_6)\frac{\partial g_1}{\partial x_2} \\ & \quad + x_3(x_4+x_5-x_6)\frac{\partial g_1}{\partial x_3} + x_1(x_1-n_2x_2)\frac{\partial g_1}{\partial x_4} - n_2x_2(x_1-n_2x_2)\frac{\partial g_1}{\partial x_5} \\ & \quad + \frac{k-1}{4}F\left(\frac{\partial g_1}{\partial x_4}+\frac{\partial g_1}{\partial x_5}+\frac{\partial g_1}{\partial x_6}\right)=0. \end{aligned}$$

Let $\bar g_1 = g_1|_{x_1=0}\not\equiv 0$. Equation (14) on $x_1=0$ becomes

$$\begin{aligned} & j(-x_4+x_5+x_6)\bar g_1 + x_2(x_4-x_5+x_6)\frac{\partial \bar g_1}{\partial x_2} + x_3(x_4+x_5-x_6)\frac{\partial \bar g_1}{\partial x_3} \\ & \quad + x_2^2\frac{\partial \bar g_1}{\partial x_5} + \frac{k-1}{4}F_1\left(\frac{\partial \bar g_1}{\partial x_4}+\frac{\partial \bar g_1}{\partial x_5}+\frac{\partial \bar g_1}{\partial x_6}\right)=0. \end{aligned}$$

Write $\bar g_1 = x_2^l g_2\not\equiv 0$, with $l\in\mathbb{N}\cup\{0\}$ and $x_2\nmid g_2$. We get

$$\begin{aligned} & (j(-x_4+x_5+x_6)+l(x_4-x_5+x_6))g_2 + x_2(x_4-x_5+x_6)\frac{\partial g_2}{\partial x_2} \\ & \quad + x_3(x_4+x_5-x_6)\frac{\partial g_2}{\partial x_3} + x_2^2\frac{\partial g_2}{\partial x_5} + \frac{k-1}{4}F_1\left(\frac{\partial g_2}{\partial x_4}+\frac{\partial g_2}{\partial x_5}+\frac{\partial g_2}{\partial x_6}\right)=0. \end{aligned}$$

Let $\bar g_2 = g_2|_{x_2=0}\not\equiv 0$. Then, on $x_2=0$ we have

$$\begin{aligned} & (j(-x_4+x_5+x_6)+l(x_4-x_5+x_6))\bar g_2 + x_3(x_4+x_5-x_6)\frac{\partial \bar g_2}{\partial x_3} \\ & \quad + \frac{k-1}{4}F_{12}\left(\frac{\partial \bar g_2}{\partial x_4}+\frac{\partial \bar g_2}{\partial x_5}+\frac{\partial \bar g_2}{\partial x_6}\right)=0. \end{aligned}$$

Now write $\bar{g}_2 = x_3^m g_3 \not\equiv 0$, with $m \in \mathbb{N} \cup \{0\}$ and $x_3 \nmid g_3$. We get

$$(j(-x_4+x_5+x_6)+l(x_4-x_5+x_6)+m(x_4+x_5-x_6))g_3 \\ + x_3(x_4+x_5-x_6)\frac{\partial g_3}{\partial x_3} + \frac{k-1}{4}F_{12}\left(\frac{\partial g_3}{\partial x_4}+\frac{\partial g_3}{\partial x_5}+\frac{\partial g_3}{\partial x_6}\right) = 0.$$

Let $\bar{g}_3 = g_3|_{x_3=0} \not\equiv 0$. Then, on $x_3 = 0$ we have

$$(j(-x_4+x_5+x_6)+l(x_4-x_5+x_6)+m(x_4+x_5-x_6))\bar{g}_3 \\ + \frac{k-1}{4}F_{123}\left(\frac{\partial \bar{g}_3}{\partial x_4}+\frac{\partial \bar{g}_3}{\partial x_5}+\frac{\partial \bar{g}_3}{\partial x_6}\right) = 0.$$

Applying Lemma 4 we obtain $\bar{g}_3 \equiv 0$, a contradiction. Hence $g_1 \equiv 0$ and the lemma follows in this case.

If $j = 1$ then equation (14) becomes

$$(-x_4+x_5+x_6)g_1 + x_1(-x_4+x_5+x_6)\frac{\partial g_1}{\partial x_1} + x_2(x_4-x_5+x_6)\frac{\partial g_1}{\partial x_2} \\ + x_3(x_4+x_5-x_6)\frac{\partial g_1}{\partial x_3} + x_1(x_1-n_2x_2)\frac{\partial g_1}{\partial x_4} \\ - n_2x_2(x_1-n_2x_2)\frac{\partial g_1}{\partial x_5} + \frac{k-1}{4}F\left(\frac{\partial g_1}{\partial x_4}+\frac{\partial g_1}{\partial x_5}+\frac{\partial g_1}{\partial x_6}\right) + (x_1-n_2x_2)\frac{\partial h_2}{\partial x_4} = 0.$$

Let $\bar{g}_1 = g_1|_{x_1=0} \not\equiv 0$. On $x_1 = 0$ we have

$$(-x_4+x_5+x_6)\bar{g}_1 + x_2(x_4-x_5+x_6)\frac{\partial \bar{g}_1}{\partial x_2} + x_3(x_4+x_5-x_6)\frac{\partial \bar{g}_1}{\partial x_3} \\ + x_2^2\frac{\partial \bar{g}_1}{\partial x_5} + \frac{k-1}{4}F_1\left(\frac{\partial \bar{g}_1}{\partial x_4}+\frac{\partial \bar{g}_1}{\partial x_5}+\frac{\partial \bar{g}_1}{\partial x_6}\right) - n_2x_2\frac{\partial h_2}{\partial x_4} = 0.$$

Write $\bar{g}_1 = x_2^l g_2 \not\equiv 0$, with $l \in \mathbb{N} \cup \{0\}$ and $x_2 \nmid g_2$. We get

$$\begin{aligned} &x_2^l\left[((-x_4+x_5+x_6)+l(x_4-x_5+x_6))g_2 + x_2(x_4-x_5+x_6)\frac{\partial g_2}{\partial x_2}\right. \\ &\left. + x_3(x_4+x_5-x_6)\frac{\partial g_2}{\partial x_3} + x_2^2\frac{\partial g_2}{\partial x_5} + \frac{k-1}{4}F_1\left(\frac{\partial g_2}{\partial x_4}+\frac{\partial g_2}{\partial x_5}+\frac{\partial g_2}{\partial x_6}\right)\right] \\ &- n_2x_2\frac{\partial h_2}{\partial x_4} = 0. \end{aligned} \tag{15}$$

We distinguish three cases depending on the value of l. If $l > 1$ then $x_2 \left| \frac{\partial h_2}{\partial x_4}\right.$. Hence $h_2 \equiv 0$. Thus, from (15),

$$((-x_4+x_5+x_6)+l(x_4-x_5+x_6))g_2 + x_2(x_4-x_5+x_6)\frac{\partial g_2}{\partial x_2} \\ + x_3(x_4+x_5-x_6)\frac{\partial g_2}{\partial x_3} + x_2^2\frac{\partial g_2}{\partial x_5} + \frac{k-1}{4}F_1\left(\frac{\partial g_2}{\partial x_4}+\frac{\partial g_2}{\partial x_5}+\frac{\partial g_2}{\partial x_6}\right) = 0.$$

Similar arguments to those used before lead to an equation of type (4) and hence applying Lemma 4 we get a contradiction. Therefore $g_1 \equiv 0$ and the lemma follows.

If $l = 0$ then we can use the same arguments to arrive from equation (15) to an equation of type (4), and hence applying Lemma 4 we get a contradiction. Therefore $g_1 \equiv 0$ and $h_2 \equiv 0$, so the lemma follows.

It only remains to consider the case $l=1$. Let $\bar{g}_2 = g_2|_{x_2=0} \not\equiv 0$. From equation (15) on $x_2=0$ we have

$$2x_6\bar{g}_2 + x_3(x_4+x_5-x_6)\frac{\partial \bar{g}_2}{\partial x_3} + \frac{k-1}{4}F_{12}\left(\frac{\partial \bar{g}_2}{\partial x_4}+\frac{\partial \bar{g}_2}{\partial x_5}+\frac{\partial \bar{g}_2}{\partial x_6}\right) - n_2\frac{\partial h_2}{\partial x_4} = 0.$$

Write $\bar{g}_2 = x_3^m g_3 \not\equiv 0$, with $m \in \mathbb{N}\cup\{0\}$ and $x_3 \nmid g_3$. We get:

$$\begin{aligned} x_3^m \Bigg[&(2x_6 + m(x_4+x_5-x_6))g_3 + x_3(x_4+x_5-x_6)\frac{\partial g_3}{\partial x_3} \\ &+\frac{k-1}{4}F_{12}\left(\frac{\partial g_3}{\partial x_4}+\frac{\partial g_3}{\partial x_5}+\frac{\partial g_3}{\partial x_6}\right)\Bigg] - n_2\frac{\partial h_2}{\partial x_4} = 0. \end{aligned}$$

If $m>0$ then $x_3 \left| \dfrac{\partial h_2}{\partial x_4} \right.$, and hence $h_2 \equiv 0$. Therefore we obtain an equation of type (4) and hence by Lemma 4 we get $g_1 \equiv 0$. If $m=0$, let $\bar{g}_3 = g_3|_{x_3=0} \not\equiv 0$. On $x_3=0$, we have

$$2x_6\bar{g}_3 + \frac{k-1}{4}F_{123}\left(\frac{\partial \bar{g}_3}{\partial x_4}+\frac{\partial \bar{g}_3}{\partial x_5}+\frac{\partial \bar{g}_3}{\partial x_6}\right) - n_2\frac{\partial h_2}{\partial x_4} = 0. \tag{16}$$

As $h_2 = h_2(x_4-x_6)$ is a homogeneous polynomial of degree n, we have $h_2 = a_0(x_4-x_6)^n$. Thus $\dfrac{\partial h_2}{\partial x_4} = a_0 n(x_4-x_6)^{n-1}$. On $x_6=0$ equation (16) writes

$$\frac{k-1}{4}(x_4-x_5)^2\left(\frac{\partial \bar{g}_3}{\partial x_4}+\frac{\partial \bar{g}_3}{\partial x_5}+\frac{\partial \bar{g}_3}{\partial x_6}\right)\Bigg|_{x_6=0} - n_2 a_0 n x_4^{n-1} = 0.$$

Therefore we must take $a_0=0$ and hence $h_2\equiv 0$. Now equation (16) is of type (4) and hence by Lemma 4 we get $g_1 \equiv 0$ and the lemma follows. □

After Lemma 8 statement (c) of Theorem 2 is proved, as it follows that $h \equiv 0$.

3.4. **Proof of statement (d) of Theorem 2.** According to Table 1, Bianchi cases VIII and IX correspond to $n_1 = n_2 = n_3^2 = 1$ and can be written into the form

$$\begin{aligned} \dot{x}_1 &= x_1(-x_4+x_5+x_6), \\ \dot{x}_2 &= x_2(x_4-x_5+x_6), \\ \dot{x}_3 &= x_3(x_4+x_5-x_6), \\ \dot{x}_4 &= x_1(x_1-x_2-n_3x_3) + \frac{k-1}{4}F, \\ \dot{x}_5 &= x_2(-x_1+x_2-n_3x_3) + \frac{k-1}{4}F, \\ \dot{x}_6 &= n_3x_3(-x_1-x_2+n_3x_3) + \frac{k-1}{4}F, \end{aligned} \tag{17}$$

where $F = x_1^2+x_2^2+x_3^2-2x_1x_2-2n_3x_1x_3-2n_3x_2x_3+x_4^2+x_5^2+x_6^2-2x_4x_5-2x_5x_6-2x_4x_6$ and $n_3^2=1$. Let $h = h(x_1,\cdots,x_6)$ be a homogeneous polynomial first integral of degree n of system (17). Write $h = h_1(x_2,\cdots,x_6) + x_1^j g_1(x_1,\cdots,x_6)$, with $j\in\mathbb{N}$, h_1 and g_1

homogeneous polynomials and $x_1 \nmid g_1$. System (17) on $x_1 = 0$ becomes

$$\begin{aligned}
\dot{x}_2 &= x_2(x_4 - x_5 + x_6),\\
\dot{x}_3 &= x_3(x_4 + x_5 - x_6),\\
\dot{x}_4 &= \frac{k-1}{4}F_1,\\
\dot{x}_5 &= x_2(x_2 - n_3x_3) + \frac{k-1}{4}F_1,\\
\dot{x}_6 &= -n_3x_3(x_2 - n_3x_3) + \frac{k-1}{4}F_1,
\end{aligned} \tag{18}$$

where $F_1 = F|_{x_1=0}$. System (18) admits $h_1 = h_1(x_2, \cdots, x_6)$ as first integral. Write $h_1 = h_2(x_3, \cdots, x_6) + x_2^l g_2(x_2, \cdots, x_6)$, with $l \in \mathbb{N}$, h_2 and g_2 homogeneous polynomials and $x_2 \nmid g_2$. System (18) on $x_2 = 0$ becomes

$$\begin{aligned}
\dot{x}_3 &= x_3(x_4 + x_5 - x_6),\\
\dot{x}_4 &= \frac{k-1}{4}F_{12},\\
\dot{x}_5 &= \frac{k-1}{4}F_{12},\\
\dot{x}_6 &= x_3^2 + \frac{k-1}{4}F_{12},
\end{aligned} \tag{19}$$

where $F_{12} = F_1|_{x_2=0}$. We note that $h_2 = h_2(x_3, \cdots, x_6)$ is a first integral of system (19). Write $h_2 = h_3(x_4, x_5, x_6) + x_3^m g_3(x_3, x_4, x_5, x_6)$, with $m \in \mathbb{N}$, h_3 and g_3 homogeneous polynomials and $x_3 \nmid g_3$. System (19) on $x_3 = 0$ is

$$\begin{aligned}
\dot{x}_4 &= \frac{k-1}{4}F_{123},\\
\dot{x}_5 &= \frac{k-1}{4}F_{123},\\
\dot{x}_6 &= \frac{k-1}{4}F_{123},
\end{aligned} \tag{20}$$

where $F_{123} = F_{12}|_{x_3=0}$. Note that h_3 is a polynomial first integral of system (20). Since system (20) admits the two independent first integrals $x_4 - x_5$ and $x_5 - x_6$, any polynomial first integral of (20) must be a polynomial in the variables $x_4 - x_5$ and $x_5 - x_6$. Therefore $h_3 = h_3(x_4 - x_5, x_5 - x_6)$.

The next three lemmas end the proof of statement (d) of Theorem 2. The first one shows that $h_2 = h_2(x_4 - x_5)$.

Lemma 9. *For system* (19)*we have that* $g_3 \equiv 0$ *and* $h_3 = h_3(x_4 - x_5)$.

Proof. Suppose that $g_3 \not\equiv 0$. We recall that $h_2 = h_3(x_4 - x_5, x_5 - x_6) + x_3^m g_3(x_3, x_4, x_5, x_6)$, where $m \in \mathbb{N}$ and $x_3 \nmid g_3$. As h_2 is a first integral of system (19), we have

$$\begin{aligned}
x_3^m \Bigg[& m(x_4 + x_5 - x_6)g_3 + x_3(x_4 + x_5 - x_6)\frac{\partial g_3}{\partial x_3} + x_3^2\frac{\partial g_3}{\partial x_6}\\
&+ \frac{k-1}{4}F_{12}\left(\frac{\partial g_3}{\partial x_4} + \frac{\partial g_3}{\partial x_5} + \frac{\partial g_3}{\partial x_6}\right)\Bigg] + x_3^2\frac{\partial h_3}{\partial x_6} = 0.
\end{aligned} \tag{21}$$

We distinguish three cases depending on the value of m.

If $m=1$ then

$$(x_4+x_5-x_6)g_3+x_3(x_4+x_5-x_6)\frac{\partial g_3}{\partial x_3}+x_3^2\frac{\partial g_3}{\partial x_6} + \frac{k-1}{4}F_{12}\left(\frac{\partial g_3}{\partial x_4}+\frac{\partial g_3}{\partial x_5}+\frac{\partial g_3}{\partial x_6}\right)+x_3\frac{\partial h_3}{\partial x_6}=0.$$

Let $\bar{g}_3=g_3|_{x_3=0}\not\equiv 0$. On $x_3=0$ we have

$$(x_4+x_5-x_6)\bar{g}_3+\frac{k-1}{4}F_{123}\left(\frac{\partial \bar{g}_3}{\partial x_4}+\frac{\partial \bar{g}_3}{\partial x_5}+\frac{\partial \bar{g}_3}{\partial x_6}\right)=0.$$

Applying Lemma 4 we get $\bar{g}_3\equiv 0$ and hence $g_3\equiv 0$. Consequently $\dfrac{\partial h_3}{\partial x_6}=0$ and the lemma follows in this case.

If $m>2$ then from (21) we have $x_3\left|\dfrac{\partial h_3}{\partial x_6}\right.$ and so $h_3=h_3(x_4-x_5)$. Now from equation (21) on $x_3=0$ we get an equation of type (4), hence applying Lemma 4 we get $g_3\equiv 0$ and the lemma follows in this case.

If $m=2$ then from (21) we have

$$2(x_4+x_5-x_6)g_3+x_3(x_4+x_5-x_6)\frac{\partial g_3}{\partial x_3}+x_3^2\frac{\partial g_3}{\partial x_6} + \frac{k-1}{4}F_{12}\left(\frac{\partial g_3}{\partial x_4}+\frac{\partial g_3}{\partial x_5}+\frac{\partial g_3}{\partial x_6}\right)+\frac{\partial h_3}{\partial x_6}=0.$$

Let $\bar{g}_3=g_3|_{x_3=0}\not\equiv 0$. On $x_3=0$ we obtain

$$2(x_4+x_5-x_6)\bar{g}_3+\frac{k-1}{4}F_{123}\left(\frac{\partial \bar{g}_3}{\partial x_4}+\frac{\partial \bar{g}_3}{\partial x_5}+\frac{\partial \bar{g}_3}{\partial x_6}\right)+\frac{\partial h_3}{\partial x_6}=0.$$

Applying Lemma 5 swapping x_5 and x_6 we get $\dfrac{\partial h_3}{\partial x_6}=0$ and $\bar{g}_3\equiv 0$. Hence $h_3=h_3(x_4-x_5)$, $g_3\equiv 0$ and the lemma follows in this case. □

The second lemma shows that $h_1\equiv 0$.

Lemma 10. *For system* (18) *we have that* $g_2\equiv 0$ *and* $h_2\equiv 0$.

Proof. Suppose that $g_2\not\equiv 0$. We recall that $h_1=h_2(x_4-x_5)+x_2^l g_2$, with $l\in\mathbb{N}$ and $x_2\nmid g_2$. As h_1 is a first integral of system (18), we have

$$x_2^l\left[l(x_4-x_5+x_6)g_2+x_2(x_4-x_5+x_6)\frac{\partial g_2}{\partial x_2}+x_3(x_4+x_5-x_6)\frac{\partial g_2}{\partial x_3}+x_2(x_2-n_3x_3)\frac{\partial g_2}{\partial x_5} - n_3x_3(x_2-n_3x_3)\frac{\partial g_2}{\partial x_6}+\frac{k-1}{4}F_1\left(\frac{\partial g_2}{\partial x_4}+\frac{\partial g_2}{\partial x_5}+\frac{\partial g_2}{\partial x_6}\right)\right]+x_2(x_2-n_3x_3)\frac{\partial h_2}{\partial x_5}=0.$$

We distinguish two cases depending on the value of l. If $l>1$ then $x_2\left|\dfrac{\partial h_2}{\partial x_5}\right.$ and hence $\dfrac{\partial h_2}{\partial x_5}\equiv 0$, which means that $h_2\equiv 0$. Substituting in the equation above we have

$$l(x_4-x_5+x_6)g_2+x_2(x_4-x_5+x_6)\frac{\partial g_2}{\partial x_2}+x_3(x_4+x_5-x_6)\frac{\partial g_2}{\partial x_3} + x_2(x_2-n_3x_3)\frac{\partial g_2}{\partial x_5}-n_3x_3(x_2-n_3x_3)\frac{\partial g_2}{\partial x_6}+\frac{k-1}{4}F_1\left(\frac{\partial g_2}{\partial x_4}+\frac{\partial g_2}{\partial x_5}+\frac{\partial g_2}{\partial x_6}\right)=0.$$

The usual arguments lead to equation (4), hence we obtain $g_2 \equiv 0$ by Lemma 4.

If $l = 1$ then we have

$$(x_4 - x_5 + x_6)g_2 + x_2(x_4 - x_5 + x_6)\frac{\partial g_2}{\partial x_2} + x_3(x_4 + x_5 - x_6)\frac{\partial g_2}{\partial x_3} + x_2(x_2 - n_3x_3)\frac{\partial g_2}{\partial x_5}$$
$$- n_3x_3(x_2 - n_3x_3)\frac{\partial g_2}{\partial x_6} + \frac{k-1}{4}F_1\left(\frac{\partial g_2}{\partial x_4} + \frac{\partial g_2}{\partial x_5} + \frac{\partial g_2}{\partial x_6}\right) + (x_2 - n_3x_3)\frac{\partial h_2}{\partial x_5} = 0.$$

Let $\bar{g}_2 = g_2|_{x_2=0} \not\equiv 0$. On $x_2 = 0$ we have

$$\begin{aligned}(x_4 - x_5 + x_6)\bar{g}_2 + x_3(x_4 + x_5 - x_6)\frac{\partial \bar{g}_2}{\partial x_3} + x_3^2\frac{\partial \bar{g}_2}{\partial x_6}\\ + \frac{k-1}{4}F_{12}\left(\frac{\partial \bar{g}_2}{\partial x_4} + \frac{\partial \bar{g}_2}{\partial x_5} + \frac{\partial \bar{g}_2}{\partial x_6}\right) - n_3x_3\frac{\partial h_2}{\partial x_5} = 0.\end{aligned} \tag{22}$$

Write $\bar{g}_2 = x_3^m g_3 \not\equiv 0$, with $m \in \mathbb{N} \cup \{0\}$ and $x_3 \nmid g_3$. Then

$$x_3^m\left[((x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6))g_3 + x_3(x_4 + x_5 - x_6)\frac{\partial g_3}{\partial x_3}\right.$$
$$\left.+x_3^2\frac{\partial g_3}{\partial x_6} + \frac{k-1}{4}F_{12}\left(\frac{\partial g_3}{\partial x_4} + \frac{\partial g_3}{\partial x_5} + \frac{\partial g_3}{\partial x_6}\right)\right] - n_3x_3\frac{\partial h_2}{\partial x_5} = 0.$$

Now we distinguish three cases depending on the value of m. If $m = 0$ then we are in (22) again and the usual arguments lead to $g_2 \equiv 0$ and $h_2 \equiv 0$.

If $m > 1$ then $x_3 \left| \dfrac{\partial h_2}{\partial x_5}\right.$ and hence $\dfrac{\partial h_2}{\partial x_5} \equiv 0$, which means that $h_2 \equiv 0$. Then we have

$$((x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6))g_3 + x_3(x_4 + x_5 - x_6)\frac{\partial g_3}{\partial x_3}$$
$$+ x_3^2\frac{\partial g_3}{\partial x_6} + \frac{k-1}{4}F_{12}\left(\frac{\partial g_3}{\partial x_4} + \frac{\partial g_3}{\partial x_5} + \frac{\partial g_3}{\partial x_6}\right) = 0.$$

The usual arguments finish the proof in this case.

Finally if $m = 1$ then we have

$$2x_4g_3 + x_3(x_4 + x_5 - x_6)\frac{\partial g_3}{\partial x_3} + x_3^2\frac{\partial g_3}{\partial x_6} + \frac{k-1}{4}F_{12}\left(\frac{\partial g_3}{\partial x_4} + \frac{\partial g_3}{\partial x_5} + \frac{\partial g_3}{\partial x_6}\right) - n_3\frac{\partial h_2}{\partial x_5} = 0.$$

Let $\bar{g}_3 = g_3|_{x_3=0} \not\equiv 0$. On $x_3 = 0$ we have

$$2x_4\bar{g}_3 + \frac{k-1}{4}F_{123}\left(\frac{\partial \bar{g}_3}{\partial x_4} + \frac{\partial \bar{g}_3}{\partial x_5} + \frac{\partial \bar{g}_3}{\partial x_6}\right) - n_3\frac{\partial h_2}{\partial x_5} = 0.$$

As $h_2 = h_2(x_4 - x_5)$ is a homogeneous polynomial of degree n, we have $h_2 = a_0(x_4 - x_5)^n$. Hence

$$2x_4\bar{g}_3 + \frac{k-1}{4}F_{123}\left(\frac{\partial \bar{g}_3}{\partial x_4} + \frac{\partial \bar{g}_3}{\partial x_5} + \frac{\partial \bar{g}_3}{\partial x_6}\right) + n_3a_0n(x_4 - x_5)^{n-1} = 0.$$

On $x_4 = 0$ we have

$$\frac{k-1}{4}(x_5 - x_6)^2\left.\left(\frac{\partial \bar{g}_3}{\partial x_4} + \frac{\partial \bar{g}_3}{\partial x_5} + \frac{\partial \bar{g}_3}{\partial x_6}\right)\right|_{x_4=0} + n_3a_0n(-x_5)^{n-1} = 0,$$

which means that $a_0 = 0$. Therefore $h_2 \equiv 0$. The equation is now of type (4) and leads to $g_2 \equiv 0$ by Lemma 4.

All the subcases are considered and the proof of the lemma is finished. $\square$

The last lemma shows that $g_1 \equiv 0$ and therefore that $h \equiv 0$.

Lemma 11. *For system* (17) *we have that* $g_1 \equiv 0$.

Proof. Suppose that $g_1 \not\equiv 0$. We recall that $h = x_1^j g_1$, with $j \in \mathbb{N}$ and $x_1 \nmid g_1$, is a first integral of system (17). Then

$$\begin{aligned} &j(-x_4+x_5+x_6)g_1 + x_1(-x_4+x_5+x_6)\frac{\partial g_1}{\partial x_1} + x_2(x_4-x_5+x_6)\frac{\partial g_1}{\partial x_2} \\ &\quad + x_3(x_4+x_5-x_6)\frac{\partial g_1}{\partial x_3} + x_1(x_1-x_2-n_3x_3)\frac{\partial g_1}{\partial x_4} \\ &\quad + x_2(-x_1+x_2-n_3x_3)\frac{\partial g_1}{\partial x_5} + n_3x_3(-x_1-x_2+n_3x_3)\frac{\partial g_1}{\partial x_6} \\ &\quad + \frac{k-1}{4}F\left(\frac{\partial g_1}{\partial x_4}+\frac{\partial g_1}{\partial x_5}+\frac{\partial g_1}{\partial x_6}\right) = 0. \end{aligned}$$

Let $\bar{g}_1 = g_1|_{x_1=0} \not\equiv 0$. On $x_1 = 0$ we have

$$\begin{aligned} &j(-x_4+x_5+x_6)\bar{g}_1 + x_2(x_4-x_5+x_6)\frac{\partial \bar{g}_1}{\partial x_2} + x_3(x_4+x_5-x_6)\frac{\partial \bar{g}_1}{\partial x_3} \\ &\quad + x_2(x_2-n_3x_3)\frac{\partial \bar{g}_1}{\partial x_5} + n_3x_3(-x_2+n_3x_3)\frac{\partial \bar{g}_1}{\partial x_6} + \frac{k-1}{4}F_1\left(\frac{\partial \bar{g}_1}{\partial x_4}+\frac{\partial \bar{g}_1}{\partial x_5}+\frac{\partial \bar{g}_1}{\partial x_6}\right) = 0. \end{aligned}$$

Write $\bar{g}_1 = x_2^l g_2 \not\equiv 0$, with $l \in \mathbb{N}\cup\{0\}$ and $x_2 \nmid g_2$. We get

$$\begin{aligned} &(j(-x_4+x_5+x_6)+l(x_4-x_5+x_6))g_2 + x_2(x_4-x_5+x_6)\frac{\partial g_2}{\partial x_2} \\ &\quad + x_3(x_4+x_5-x_6)\frac{\partial g_2}{\partial x_3} + x_2(x_2-n_3x_3)\frac{\partial g_2}{\partial x_5} \\ &\quad + n_3x_3(-x_2+n_3x_3)\frac{\partial g_2}{\partial x_6} + \frac{k-1}{4}F_1\left(\frac{\partial g_2}{\partial x_4}+\frac{\partial g_2}{\partial x_5}+\frac{\partial g_2}{\partial x_6}\right) = 0. \end{aligned}$$

Let $\bar{g}_2 = g_2|_{x_2=0} \not\equiv 0$. On $x_2 = 0$ we have

$$\begin{aligned} &(j(-x_4+x_5+x_6)+l(x_4-x_5+x_6))\bar{g}_2 + x_3(x_4+x_5-x_6)\frac{\partial \bar{g}_2}{\partial x_3} \\ &\quad + x_3^2\frac{\partial \bar{g}_2}{\partial x_6} + \frac{k-1}{4}F_{12}\left(\frac{\partial \bar{g}_2}{\partial x_4}+\frac{\partial \bar{g}_2}{\partial x_5}+\frac{\partial \bar{g}_2}{\partial x_6}\right) = 0. \end{aligned}$$

Write $\bar{g}_2 = x_3^m g_3 \not\equiv 0$, with $m \in \mathbb{N}\cup\{0\}$ and $x_3 \nmid g_3$. We get

$$\begin{aligned} &(j(-x_4+x_5+x_6)+l(x_4-x_5+x_6)+m(x_4+x_5-x_6))g_3 \\ &\quad + x_3(x_4+x_5-x_6)\frac{\partial g_3}{\partial x_3} + x_3^2\frac{\partial g_3}{\partial x_6} + \frac{k-1}{4}F_{12}\left(\frac{\partial g_3}{\partial x_4}+\frac{\partial g_3}{\partial x_5}+\frac{\partial g_3}{\partial x_6}\right) = 0. \end{aligned}$$

Let $\bar{g}_3 = g_3|_{x_3=0} \not\equiv 0$. On $x_3 = 0$ we have

$$\begin{aligned} &(j(-x_4+x_5+x_6)+l(x_4-x_5+x_6)+m(x_4+x_5-x_6))\bar{g}_3 \\ &\quad + \frac{k-1}{4}F_{123}\left(\frac{\partial \bar{g}_3}{\partial x_4}+\frac{\partial \bar{g}_3}{\partial x_5}+\frac{\partial \bar{g}_3}{\partial x_6}\right) = 0. \end{aligned}$$

We can apply Lemma 4. Hence the lemma follows. □

After Lemma 11, we get $h \equiv 0$. Thus the proof of statement (d) of Theorem 2 is finished.

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