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MAXIMIZING ENTROPY OF CYCLES ON TREES

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Abstract. In this paper we give a partial characterization of the periodic tree patterns of maximum entropy. More precisely, we prove that each periodic pattern with maximal entropy is irreducible and simplicial. Moreover, it is also maximodal in the sense that for every monotone representative of the pattern every periodic point is a "turning point".

1. Introduction

A *pattern* is a classical and well studied object in the theory of one-dimensional combinatorial dynamics. Given a topological space X and a continuous map $f\colon X \longrightarrow X$ which is known to exhibit a finite invariant set P, the *pattern* of P is a combinatorial object that encodes information about both the relative positions of the points of P inside the space X and the way these positions are permuted under the action of $f\big|_P$.

When X is an interval, the pattern of P can be identified with a permutation π in a natural way: set $P = \{p_1, p_2, \ldots, p_n\}$ with $p_1 < p_2 < \ldots < p_n$ and define $\pi\colon \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ as $\pi(i) = j$ if and only if $f(p_i) = p_j$. If P is a periodic orbit then π is a cyclic permutation and the pattern is called *cyclic* or *periodic*. The notion of pattern for maps of the interval has its roots in the well known Sharkovskii's Theorem [21, 23], but it was formalized and developed by Misiurewicz and Nitecki [19] in the early 1990s building on a previous work by Baldwin [9].

On the other hand, the *topological entropy* of a continuous map $f\colon X \longrightarrow X$ of a compact metric space is a non-negative real number that measures the dynamical complexity of f. This well known topological invariant was first introduced in 1965 [1]. This notion can also be used to define the *topological entropy of a pattern* π, which is the infimum of the topological entropies of all self-maps of X having an invariant set with pattern π.

Although computing the entropy of a map of the interval is difficult in general, the computation of the entropy of a pattern π can be easily done by using some algebraic tools. Indeed, it turns out that in this case the entropy of π is equal to the entropy of a particularly simple map $f_\pi\colon [1, n] \longrightarrow [1, n]$ which satisfies:

(1) $f_\pi(i) = \pi(i)$, for $i \in \{1, \ldots, n\}$,
(2) f_π is monotone on each interval $I_i = \{x \in [1, n] : i \le x \le i + 1\}$ for each $i \in \{1, \ldots, n-1\}$.

The map f_π is essentially unique and it is known in the literature as the *monotone representative of* π or the "connect-the-dots" map. Its entropy is minimum in the set of all maps exhibiting an invariant set with pattern π and it is equal (see [11]) to the logarithm of the spectral radius of the so-called *Markov matrix* $(m_{i,j})_{i,j=1}^{n-1}$

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of π, whose entries are given by

$$m_{i,j} = \begin{cases} 1 & \text{if } f_\pi(I_i) \supset I_j \\ 0 & \text{otherwise,} \end{cases}$$

for $i, j \in \{1, \dots, n-1\}$.

Besides the classical case of maps of the interval, recently there has been a growing interest in extending the notion of *pattern* to more general one-dimensional spaces such as graphs (see [6], where the notion of pattern is termed *action*, and [2]) or trees (see [3]).

In this paper we are interested in patterns of maps defined on trees. In [3], the authors introduce a notion of *pattern* of a finite invariant set of a continuous map from a tree into itself (from now on such a map will be called a *tree map*). Next we informally explain this notion. In Section 2 we will give the precise version of this concept.

A *model* will be a triplet (T, P, f) such that $f: T \longrightarrow T$ is a tree map and P is a finite invariant subset of T. Two points x, y of P will be called *consecutive* if the interior of the unique interval in T whose endpoints are x and y contains no points of P. Any maximal subset of P consisting only of pairwise consecutive points will be called a *discrete component*.

A *pattern* is an object which can be identified with the conjugacy class of all models with a fixed distribution of discrete components and images of points in the distinguished finite invariant set P. For instance, in Figure 1 we can see two models which represent the same cyclic pattern. Observe that two points p_i, p_j of P are consecutive in T if and only if the corresponding points p'_i, p'_j of P' are consecutive in T'. However, note that the trees T and T' need not be homeomorphic.

While for a pattern of the interval it is trivial to construct a monotone model, this is not the case for tree maps. However, Theorem A of [3] states that for any tree pattern π there exists a monotone representative (T, P, f) satisfying properties very similar to those of the "connect-the-dots" interval maps. In particular, the entropy of π equals the entropy of its monotone representative f, which can be easily computed as the logarithm of the spectral radius of a certain non-negative matrix (the *path transition matrix* defined later).

Given $n \in \mathbb{N}$, the set of all n-periodic patterns is finite. Hence, the set of entropies of all the n-periodic patterns has a maximum.

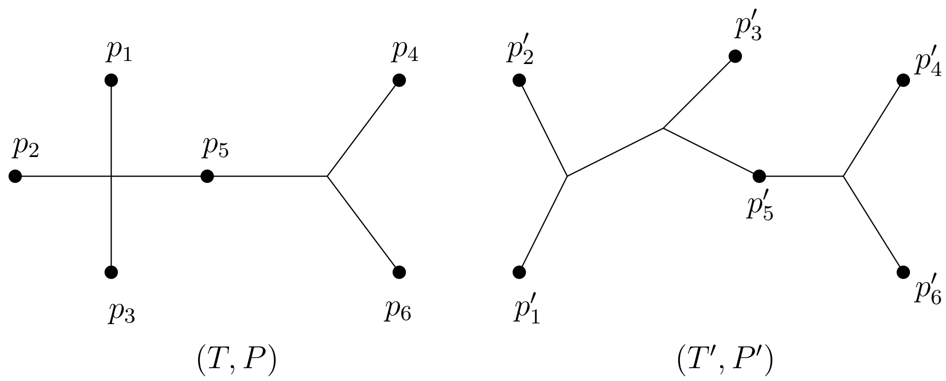

FIGURE 1. Set $P = \{p_i\}_{i=1}^6$ and $P' = \{p'_i\}_{i=1}^6$. If $f: T \longrightarrow T$ and $f': T' \longrightarrow T'$ are tree maps such that $f(p_i) = p_{i+1}$ and $f'(p'_i) = p'_{p+1}$ for $1 \le i \le 5$, $f(p_6) = p_1$ and $f'(p'_6) = p'_1$, then the models (T, P, f) and (T', P', f') represent the same cyclic pattern π.

Once we have depicted the idea of tree pattern and established that for each tree pattern one can compute its topological entropy, we are ready to explain the aim of this paper. It comes from one of the natural questions arising in this setting: given any $n \in \mathbb{N}$, can we identify the tree patterns with maximum entropy in the set of all patterns (and/or cyclic patterns) of cardinality n? Answering this question is a formidable challenge that at this moment remains unsolved even when one restricts to the interval case.

For interval patterns, the answer to the above question is known for any $n \in \mathbb{N}$ in the case of n-permutations, and for $n \neq 4k+2$ in the case of n-cycles. For $n = 4k+1$ Misiurewicz and Nitecki, in [19], constructed a family of entropy-maximal n-periodic orbits. Geller and Tolosa [12] extended this definition to a family of periodic orbits of period $n = 4k + 3$ and proved that this family in fact has maximum entropy among all n-permutations. In [13] it was shown that this family is unique. So, the characterization of the entropy-maximal n-cycles and n-permutations is complete for n odd. For the case n even the situation is more complicated, since the entropy-maximal n-permutations are not cyclic. All entropy-maximal n-permutations for n even were described by King [15, 16] and independently by Geller and Zhang [14]. Finally, two families of entropy-maximal n-periodic orbits for $n = 4k$ have been recently described in [17], and they have been shown to be unique up to a reversal of orientation. The characterization of the entropy-maximal interval patterns is still unknown in the case $n = 4k + 2$. However, a very recent paper [4] studies this case from a computational point of view and proposes a family of $(4k + 2)$-periodic orbits which the authors conjecture are entropy-maximal.

As far as we know, there is no literature about this problem in the setting of tree patterns. In this paper we give a partial characterization of the entropy-maximal n-periodic tree patterns for any $n \in \mathbb{N}$. We start by restricting our study to the setting of *simplicial* patterns. A pattern is said to be *simplicial* if all its discrete components have cardinality 2. First we show that, for each n, the maximum entropy is attained in the class of simplicial n-periodic patterns and that these patterns have to be *irreducible*. This means that the periodic orbit cannot be partitioned according to what we call a *block structure*. Equivalently, the Markov matrix of its monotone representative is irreducible in the usual algebraic sense. Then we show that any simplicial periodic pattern with maximum entropy has to be *maximodal* in the sense that for every monotone representative (T, P, f) of the pattern and for every $x \in P$ there exists a small neighbourhood U of x such that $\mathrm{Cl}(f(U))$ is an interval and $f(x)$ is an endpoint of $\mathrm{Cl}(f(U))$. Finally, for each n-periodic pattern which is not simplicial we construct an n-periodic simplicial pattern with strictly more entropy, showing that in fact every periodic pattern with maximum entropy has to be simplicial. Putting all these results together we get the main result of this paper, which states that, for any $n \in \mathbb{N}$, each n-periodic pattern with maximum entropy has to be simplicial, irreducible and maximodal. In Section 2 we give the precise definitions of these notions.

This paper is dedicated to the memory of our friend Pere Mumbrú, deceased on July 28, 2005. Some of the underlying ideas of the manuscript were born when he started to work on this problem together with Lluís Alsedà and Francesc Mañosas.

2. Definitions and statement of the main results

A *tree* is a compact uniquely arcwise connected space which is a point or a union of a finite number of intervals (by an *interval* we mean any space homeomorphic to $[0, 1]$). Any continuous map $f : T \longrightarrow T$ from a tree T into itself will be called a *tree map*. A set $X \subset T$ will be called f*-invariant* if $f(X) \subset X$. For each $x \in T$, we define the *valence* of x, denoted by $\mathrm{Val}_T(x)$ or simply $\mathrm{Val}(x)$, to be the number of connected components of $T \setminus \{x\}$. A point of valence different from 2 will be called a *vertex* of T and the set of vertices of T will be denoted by $\mathrm{V}(T)$. Each point of valence 1 will be called an *endpoint* of T. The set of such points will be denoted by $\mathrm{En}(T)$. The points in $\mathrm{V}(T) \setminus \mathrm{En}(T)$ have valence greater than or equal to 3. They will be called the *branching points* of T and the set of such points will be denoted by $\mathrm{Br}(T)$. Also, the closure of a connected component of $T \setminus \mathrm{V}(T)$ will be called an *edge of* T.

Given any subset X of a topological space, we will denote by $\mathrm{Int}(X)$ and $\mathrm{Cl}(X)$ the interior and the closure of X, respectively. For a finite set P we will denote its cardinality by $|P|$.

A triplet (T, P, f) will be called a *model* if $f \colon T \longrightarrow T$ is a tree map and P is a finite f-invariant set such that $\mathrm{En}(T) \subset P$. In particular, if P is a periodic orbit of f and $|P| = n$ then (T, P, f) will be called an *n-periodic model*. Given $X \subset T$ we will define the *convex hull* of X, denoted by $\langle X \rangle_T$ or simply by $\langle X \rangle$, as the smallest closed connected subset of T containing X. We will write $\langle x, y \rangle$ to denote $\langle \{x, y\} \rangle$ and (x, y) to denote $\langle \{x, y\} \rangle \setminus \{x, y\}$.

Let T be a tree and let $P \subset T$ be a finite subset of T. The pair (T, P) will be called a *pointed tree*. A set $Q \subset P$ is said to be a *discrete component* of (T, P) if either $|Q| > 1$ and there is a connected component C of $T \setminus P$ such that $Q = \mathrm{Cl}(C) \cap P$, or $|Q| = 1$ and $Q = P$. We say that two pointed trees (T, P) and (T', P') are *equivalent* if there exists a bijection $\phi \colon P \longrightarrow P'$ which preserves discrete components. The equivalence class of a pointed tree (T, P) will be denoted by $[T, P]$.

Let (T, P) and (T', P') be equivalent pointed trees, and let $\theta \colon P \longrightarrow P$ and $\theta' \colon P' \longrightarrow P'$ be maps. We will say that θ and θ' are *equivalent* if $\theta' = \varphi \circ \theta \circ \varphi^{-1}$ for a bijection $\varphi \colon P \longrightarrow P'$ which preserves discrete components. The equivalence class of θ by this relation will be denoted by $[\theta]$. If $[T, P]$ is an equivalence class of pointed trees and $[\theta]$ is an equivalence class of maps then the pair $([T, P], [\theta])$ will be called a *pattern*. We say that a model (T, P, f) *exhibits* a pattern $(\mathcal{T}, \Theta)$ if $\mathcal{T} = [T, P]$ and $\Theta = [f\big|_P]$. This pattern will be denoted by $[T, P, f]$. Alternatively, we will say that the model (T, P, f) is a *representative* of the pattern $(\mathcal{T}, \Theta)$.

The topological entropy [1] is a well known quantitative measure of the dynamical complexity of a model. It is an important topological invariant which is defined for continuous maps on compact metric spaces. The topological entropy of a map $f \colon T \longrightarrow T$ will be denoted by $h(f)$. Given a pattern $\mathcal{P}$, the topological entropy of $\mathcal{P}$ is defined to be

$$h(\mathcal{P}) := \inf\{h(f) : (T, P, f) \text{ is a model exhibiting } \mathcal{P}\}.$$

The simplest models exhibiting a given pattern are the monotone ones, according to the following definition. Let S and T be trees and let $f \colon T \longrightarrow S$ be a map. Given $a, b \in T$ we say that $f\big|_{\langle a,b \rangle}$ is *monotone* if either $f(\langle a, b \rangle)$ is a point or it is an interval and, given two homeomorphisms $\phi \colon [0, 1] \longrightarrow \langle a, b \rangle$ and $\varphi \colon g(\langle a, b \rangle) \longrightarrow [0, 1]$, then $\varphi \circ f \circ \phi \colon [0, 1] \longrightarrow [0, 1]$ is monotone (as a real function). Let (T, P, f) be a model. A pair $\{a, b\} \subset P$ will be called a *basic path of* (T, P) if $\langle a, b \rangle \cap P = \{a, b\}$. We will say that f is *P-monotone* if $f(\langle a, b \rangle) = \langle f(a), f(b) \rangle$ and $f\big|_{\langle a,b \rangle}$ is monotone for any basic path $\{a, b\}$. The model (T, P, f) will be called *monotone*.

Theorem 2.1 (Theorem A of [3])**.** *Let $\mathcal{P}$ be a pattern. Then the following statements hold.*

(a) There exists a monotone model (T, P, f) exhibiting the pattern $\mathcal{P}$.
(b) The topological entropy of f is the minimum within the class of models which exhibit $\mathcal{P}$.

Remark 2.2. From the proof of Theorem A of [3] it follows that if (T, P, f) is a monotone model then the set $P \cup \mathrm{V}(T)$ is f-invariant. It easily follows that the map f, which is P-monotone, is also $(P \cup \mathrm{V}(T))$-monotone.

The monotone models from Theorem 2.1 are essentially unique in the following sense. Let (T, P, f) be a monotone model and let S be a non-empty union of edges disjoint from P. We will say that S is an *invariant forest of* (T, P, f) if either $f^i(S) \cap P = \emptyset$ for every $i \geq 0$ or there exists $n > 0$ such that $f^i(S) \cap P = \emptyset$ for every $i = 0, 1, \ldots, n-1$ and $f^n(S)$ degenerates to a point of P. Then, we will say that (T, P, f) is a *canonical model* of the pattern $[T, P, f]$ if it has no invariant forests. From [3, Theorem B] it follows that every pattern has a canonical model. Moreover, given two canonical models (T, P, f) and (T', P', f') of the same pattern there exists a homeomorphism $\varphi\colon T \longrightarrow T'$ such that $\varphi(P) = P'$, and $f' \circ \varphi\big|_P = \varphi \circ f\big|_P$. Hence, a canonical model of a pattern is essentially unique.

The topological entropy of a pattern $\mathcal{P}$ can be easily computed as the logarithm of the spectral radius of a certain non-negative matrix called a *path transition matrix*, which depends only on the combinatorial data of $\mathcal{P}$. This notion will be introduced (and frequently used) in Section 7.

To establish the main result of this paper we need to introduce some definitions on periodic patterns.

The first notion we need has to do with the spatial distribution of the points of the invariant set. A pattern $([T, P], [\theta])$ will be called *simplicial* if each discrete component of (T, P) has two points. Observe that, in this case, for each pointed tree $(S, Q) \in [T, P]$ we have that $\mathrm{V}(S) \subset Q$ and, for each discrete component π of (S, Q), $\langle \pi \rangle_S$ is an interval. Hence, if (S, Q) and (S', Q') belong to $[T, P]$ then S and S' are homeomorphic. It follows that if a pattern is simplicial then it has essentially a unique monotone representative. Conversely, given a monotone model (T, P, f) such that $\mathrm{V}(T) \subset P$, the pattern $[T, P, f]$ is simplicial. We will also say that the model (T, P, f) is *simplicial*. In particular, if (T, P, f) is a monotone model then, by Remark 2.2, $(T, P \cup \mathrm{V}(T), f)$ is monotone and simplicial.

The *Markov graph* and the *Markov matrix* associated to a simplicial model (T, Q, f) are standard combinatorial objects which codify the dynamical behaviour of f. In particular, the topological entropy of f can be computed by means of the Markov matrix. Let us recall the definitions. An interval of T will be called *Q-basic* if it is the closure of a connected component of $T \setminus Q$. Observe that two different Q-basic intervals have pairwise disjoint interiors. Given $K, L \subset T$, we will say that K *f-covers* L if $f(K) \supset L$. Consider a labelling $I_1, I_2, \ldots I_n$ of all Q-basic intervals. The *Markov graph of* (T, Q, f) associated to this labelling is a combinatorial directed graph whose vertices are the Q-basic intervals and there is an arrow from I_i to I_j if and only if I_i f-covers I_j. On the other hand, the *Markov matrix of* (T, Q, f) associated to this labelling is an $n \times n$ matrix $(m_{i,j})_{i,j=1}^n$ such that $m_{i,j} = 1$ if and only if I_i f-covers I_j, and $m_{i,j} = 0$ otherwise. Given two different labellings of the set of Q-basic intervals and their associated Markov matrices M and N, there exists a permutation matrix P such that $M = P^T N P$ (where P^T denotes the transpose of P), and the corresponding Markov graphs are isomorphic.

The *spectral radius* of the Markov matrix M of a simplicial model (T, P, f) is a very useful algebraic notion which has powerful implications for the dynamics of f. We shall denote it by $\sigma(M)$. Recall that, by definition, it is the maximum of the moduli of the eigenvalues of M. In a similar way to [5, Theorem 4.4.5] it follows that $h(f) = \max\{0, \log(\sigma(M))\}$.

We recall [22] that an $n \times n$ matrix is called *reducible* if there exists a permutation matrix P such that

$$P^T M P = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix} \tag{1}$$

where M_{11} and M_{22} are square matrices of sizes $i \times i$ and $j \times j$ ($i, j \geq 1$) respectively and 0 stands for the $i \times j$ matrix whose entries are all 0. If there does not exist such P then the matrix M is called *irreducible*. In this spirit, a simplicial model (T, Q, f) will be called *reducible* if there exists a particular labelling of the set of Q-basic intervals such that the associated Markov matrix of (T, Q, f) reads as the right hand side of (1). The corresponding simplicial pattern $[T, Q, f]$ will be also called *reducible*. Observe that, in this case, the Markov matrix of (T, Q, f) associated to any labelling of the set of Q-basic intervals is reducible. If a simplicial model (or a simplicial pattern) is not reducible then we will call it *irreducible*.

Finally we introduce a notion of *maximodality* for simplicial patterns. We borrow this terminology from the usual and well known notion for interval maps (see for instance [5]): the *modality* of a P-monotone interval map f is the number of points of P at which f has a local extremum, and the map f is called *maximodal* if its modality is $|P|$.

The extension to trees of the notion of maximodality is based on extending to this setting the notion of local extremum. Indeed, if (T, P, f) is a monotone model and $x \in T$ then we say that x is a local extremum of (T, P, f) if there exists a small neighbourhood U of x such that $\mathrm{Cl}(f(U))$ is an interval and $f(x)$ is an endpoint of $\mathrm{Cl}(f(U))$. Then, a simplicial model (T, P, f) will be called maximodal when every point from P is a local extremum. Taking these comments into account we formalize the notion of maximodality in the following way.

Let (T, P, f) be a simplicial model. A triplet a, x, b of pairwise different points from P will be called an *ordered triplet with midpoint* x, denoted by $[a; x; b]$ (or $[b; x; a]$), if and only if $x \in \langle a, b\rangle$. Observe that a, x, b can fail being an ordered triplet with midpoint x whenever $\langle\{a, x, b\}\rangle_T$ is a tree with 3 endpoints or when $\langle\{a, x, b\}\rangle_T$ is an interval but $x \notin \langle a, b\rangle$. An ordered triplet $[a; x; b]$ will be called a *basic triplet* if $\langle a, b\rangle \cap P = \{a, x, b\}$. We say that a basic triplet $[a; x; b]$ is *monotone* if and only if $f\big|_{\langle a,b\rangle}$ is monotone. Finally, a simplicial model (T, P, f) (and the corresponding pattern $[T, P, f]$) will be called *maximodal* if it has no monotone basic triplets.

The set of tree patterns of a fixed period is finite and hence the set of entropies of all n-periodic tree patterns is finite for every $n \in \mathbb{N}$. Any n-periodic pattern whose entropy is maximal in the set of entropies of all n-periodic patterns will be called *maximal*.

Now we are ready to state the main result of this paper.

Theorem A. *The maximal periodic patterns are simplicial, irreducible and maximodal.*

This paper is organized as follows. In the next section we define the notion of a combinatorial oriented generalized graph and we recall and prove some algebraic properties of the spectral radius of their transition matrices. In Section 4 we show that there exist simplicial patterns with maximum entropy. In Section 5 we prove that a simplicial n-periodic pattern which is reducible cannot attain the maximum entropy in the set of all n-periodic patterns. Analogously, in Section 6 we show that a simplicial irreducible n-periodic pattern with monotone triplets cannot attain the maximum entropy. Finally, in Section 7 we show that any maximal periodic pattern is simplicial and prove Theorem A.

3. A technical lemma

In this section we define the notion of a combinatorial oriented generalized graph and we prove a lemma which states some useful algebraic properties of the spectral radius of transition matrices of combinatorial oriented generalized graphs.

A *combinatorial oriented generalized graph* is a pair $\mathcal{G} = (V, U)$ where $V = \{v_1, v_2, \dots, v_n\}$ is a finite set, $U \subset V \times V \times \mathbb{N}$, and for every $i, j \in \{1, 2, \dots, n\}$ there exists $t_{i,j} \geq 0$ such that $\{k \in \mathbb{N} : (v_i, v_j, k) \in U\} = \{1, 2, \dots, t_{i,j}\}$. The elements of V are called the *vertices of* $\mathcal{G}$ and each element (v_i, v_j, k) in U is called a *(labelled) arrow of* $\mathcal{G}$. When the graph has a unique arrow from v_i to v_j (i.e. $t_{i,j} = 1$) we will omit the label for simplicity.

The notions of path and loop of a combinatorial oriented generalized graph are defined in the standard way by using labelled arrows instead of arrows. Also the length of a path is defined as the number of arrows in the path.

Observe that a Markov graph is, in particular, a combinatorial oriented generalized graph such that $t_{i,j} \leq 1$ for every i, j.

The *transition matrix* of a combinatorial oriented generalized graph is defined as the matrix $T = (t_{i,j})$. Clearly T is an $n \times n$ non-negative integer matrix.

Given two matrices $A = (a_{i,j})$ and $B = (b_{i,j})$, we will write $A \geq B$ if and only if $a_{i,j} \geq b_{i,j}$ for each pair i, j. Also we will denote by $a^k_{i,j}$ the entry i, j of A^k (that is, $A^k = (a^k_{i,j})$).

Lemma 3.1. *Let $M = (m_{i,j})_{i,j=1}^n$ be an $n \times n$ non-negative integer matrix. Then, the following statements hold.*

(a) Assume that M is the transition matrix of a combinatorial oriented generalized graph $\mathcal{G}$. Then, for every $k \geq 1$, $m^k_{i,j}$ is the number of paths from vertex i to vertex j of length k in $\mathcal{G}$.

(b) The matrix M is irreducible if and only if for every i, j there exists a $k = k(i, j) \geq 1$ such that $m^k_{i,j} > 0$.

(c) Assume that M is irreducible. Then,

$$\sigma(M) = \limsup_{k\to\infty} \sqrt[k]{m^k_{i,i}}$$

for every $i \in \{1, 2, \dots, n\}$.

(d) Assume that $A \neq M$ is a matrix such that $A \geq M$. Then, $\sigma(A) \geq \sigma(M)$. Moreover, if one of the two matrices is irreducible then the inequality is strict.

Remark 3.2. Statements (a) and (b) of the above lemma imply that the transition matrix of a combinatorial oriented generalized graph $\mathcal{G}$ is irreducible if and only if $\mathcal{G}$ is *transitive*. This means that for each pair of vertices i, j there exists a path from i to j in $\mathcal{G}$.

Proof of Lemma 3.1. The proof of Statement (a) follows directly from the definitions and the proof of (b) follows from [24, Page 6].

Now we prove (d). From [20] it follows that $\sigma(M)$ is equal to the limit as k goes to infinity of the k-th root of the sum of all entries of M^k. So, it follows by induction that $\sigma(A) \geq \sigma(M)$ whenever $A \geq M$. On the other hand, if $A \neq M$ and A is irreducible, then we get $\sigma(A) > \sigma(M)$ from [22, Lemma 5.3.3]. Finally, if $A \geq M$ and M is irreducible, it follows from (b) that A is irreducible. Thus, again, $\sigma(A) > \sigma(M)$ when $A \neq M$.

Finally, we prove (c). From [5, Lemma 4.4.2] it follows that

$$\sigma(M) = \limsup_{k\to\infty} \sqrt[k]{|\operatorname{tr}(M^k)|}, \tag{2}$$

where $\operatorname{tr}(\cdot)$ denotes the trace function. Consequently, there exists a strictly increasing sequence $\{k_l\}_{l=1}^\infty$ such that

$$\sigma(M) = \lim_{l\to\infty} \sqrt[k_l]{|\operatorname{tr}(M^{k_l})|}. \tag{3}$$

On the other hand, there exists $p \in \{1, 2, \dots, n\}$ and a partial sequence of $\{k_l\}$, denoted also by $\{k_l\}$ for simplicity, such that $m_{p,p}^{k_l} = \max\{m_{1,1}^{k_l}, m_{2,2}^{k_l}, \dots, m_{n,n}^{k_l}\}$ for every $l \geq 1$. For every k_l we have

$$\tfrac{1}{n}|\operatorname{tr}(M^{k_l})| \leq m_{p,p}^{k_l} \leq |\operatorname{tr}(M^{k_l})|.$$

Consequently, from (3) we get, $\sigma(M) = \lim_{l\to\infty} \sqrt[k_l]{m_{p,p}^{k_l}}$.

Observe also that, by (a) and (b), for any $i \in \{1, 2, \dots, n\}$, there exist paths α and β in $\mathcal{G}$ from i to p and from p to i, respectively. We denote by $s = s(i, p)$ the sum of the lengths of these paths (clearly, we can take $s = 0$ when $i = p$). Then, any loop of length k in $\mathcal{G}$ from p to p gives a loop of length $k + s$ in $\mathcal{G}$ from i to i. Thus, $m_{p,p}^k \leq m_{i,i}^{k+s}$ for every $k \geq 1$.

Since $\sqrt[k_l]{m_{p,p}^{k_l}} = \left(\left(m_{p,p}^{k_l}\right)^{\frac{1}{k_l+s}}\right)^{\frac{k_l+s}{k_l}}$, it follows that

$$\begin{aligned}\sigma(M) = \lim_{l\to\infty}\left(\left(m_{p,p}^{k_l}\right)^{\frac{1}{k_l+s}}\right)^{\frac{k_l+s}{k_l}} &= \lim_{l\to\infty}\left(m_{p,p}^{k_l}\right)^{\frac{1}{k_l+s}} \leq \limsup_{l\to\infty}\left(m_{i,i}^{k_l+s}\right)^{\frac{1}{k_l+s}} \\ &\leq \limsup_{k\to\infty}\sqrt[k]{m_{i,i}^k} \leq \limsup_{k\to\infty}\sqrt[k]{|\operatorname{tr}(M^k)|} = \sigma(M).\end{aligned}$$

This ends the proof of the lemma. □

4. On the existence of maximal simplicial patterns

In this section we show (Corollary 4.2) that for each n there exist simplicial maximal n-periodic patterns. To do it, given a periodic pattern we define a procedure to construct another periodic pattern of the same period without decreasing the entropy which is "simpler" in the sense that it has less vertices not contained in the invariant set. The new pattern will be called a *collapse*. After performing this operation finitely many times we will obtain an n-periodic simplicial monotone model without decreasing the entropy. So, starting the process with a maximal n-periodic pattern we get a maximal simplicial pattern.

Now let us define the notion of a *collapse*. Given a pattern $\mathcal{P}$ we define the number of *free vertices of* $\mathcal{P}$ as the cardinality of $\operatorname{Br}(T) \setminus P$ where (T, P, f) is a canonical model of $\mathcal{P}$ (recall that $\operatorname{En}(T) \subset P$). Observe that, since two different canonical models of the same pattern are conjugated by a homeomorphism (and, hence, essentially unique) the number of free vertices of a pattern is well defined. The number of free vertices of $\mathcal{P}$ will be denoted by $\nu(\mathcal{P})$.

Let $\mathcal{P}$ be a periodic pattern such that $\nu(\mathcal{P}) > 0$ and let (T, P, f) be a canonical model of $\mathcal{P}$. Clearly, there exists a pair (v, x) such that $v \in \operatorname{Br}(T) \setminus P$, $x \in P$ and $\langle x, v\rangle \cap (P \cup \operatorname{V}(T)) = \{x, v\}$. Let T' be the tree obtained from T by collapsing the interval $\langle x, v\rangle$ to one point and let κ be the standard projection from T to T'. Observe that $\kappa\big|_P$ is bijective. Then, clearly $\mathcal{P}' = \left([T', \kappa(P)], [\kappa \circ f \circ \kappa^{-1}\big|_{\kappa(P)}]\right)$ is a periodic pattern of the same period as $\mathcal{P}$ such that $\nu(\mathcal{P}') < \nu(\mathcal{P})$. The pattern $\mathcal{P}'$ will be called a *collapse of* $\mathcal{P}$.

The fact that canonical models of the same pattern are conjugated imply that the collapses of a pattern are independent of the chosen canonical representative. They only depend on the original pattern and the collapsing pair x, v.

The next result states the properties of collapses that we need.

Proposition 4.1. *Let $\mathcal{P}$ be a periodic pattern such that $\nu(\mathcal{P}) > 0$ and let $\mathcal{P}'$ be a collapse of $\mathcal{P}$. Then $h(\mathcal{P}) \leq h(\mathcal{P}')$.*

From the iterative use of Proposition 4.1 we see that given any periodic pattern $\mathcal{P}$ there exists a simplicial periodic pattern $\mathcal{P}'$ of the same period such that $h(\mathcal{P}') \geq h(\mathcal{P})$. Thus, starting with a maximal pattern $\mathcal{P}$, we obtain the following

Corollary 4.2. *For each n there exist simplicial maximal n-periodic patterns.*

Now we prove the proposition.

Proof of Proposition 4.1. Let (T, P, f) be a canonical model of $\mathcal{P}$ and let $v \in \operatorname{Br}(T) \setminus P$ and $x \in P$ such that $\langle x, v\rangle \cap (P \cup \operatorname{V}(T)) = \{x, v\}$. By Remark 2.2 it follows that f is Q-monotone, where $Q = P \cup \operatorname{V}(T)$.

Let C_v be the largest connected subset of T such that $C_v \cap (P \cup \operatorname{V}(T)) = \{v\}$. Clearly, $\operatorname{Cl}(C_v)$ is a star with v as a branching point and $\operatorname{Cl}(C_v) \supset \langle x, v\rangle$.

Let $g\colon T \longrightarrow T$ be a Q-monotone map such that $g\big|_{T\setminus C_v} = f\big|_{T\setminus C_v}$ and $g(\langle v, x\rangle) = f(x)$. In particular, $f\big|_P = g\big|_P$ and hence the patterns $\mathcal{P} = [T, P, f]$ and $[T, P, g]$ coincide. Therefore, $h(g) \geq h(f) = h(\mathcal{P})$ because f is P-monotone (Theorem 2.1).

Now, let T' be the tree obtained from T by collapsing the edge $\langle v, x\rangle$ to a point and let κ be the standard projection from T to T'. Set $P' = \kappa(P)$ which has the same cardinality as P since $\kappa\big|_P$ is bijective. Moreover, $\kappa(Q) = P' \cup \operatorname{V}(T')$ and $|\operatorname{Br}(T') \setminus P'| = |\operatorname{Br}(T) \setminus P| - 1 = \nu(\mathcal{P}) - 1$.

Next we define a tree map $f'\colon T' \longrightarrow T'$ as follows. For each $t \in T'$, we set $f'(t) := \kappa \circ g \circ \kappa^{-1}(t)$. Note that $\kappa^{-1}(t)$ is a single point except when $t = \kappa(x)$ and $\kappa^{-1}(t) = \langle v, x\rangle$. But since $g(\langle v, x\rangle)$ degenerates to $f(x)$, it follows that the map f' is well defined and continuous on T'. Moreover, $f'\big|_{P'} = \kappa \circ f \circ \kappa^{-1}\big|_{P'}$. So clearly $[T', P', f'] = \Big([T', \kappa(P)], [\kappa \circ f \circ \kappa^{-1}\big|_{\kappa(P)}]\Big)$ is a collapse of $\mathcal{P}$.

We will show that (T', P', f') is a monotone model of $[T', P', f']$ and $h(f') = h(g)$. Hence, $h([T', P', f']) = h(f') = h(g) \geq h(\mathcal{P})$.

To see that f' is P'-monotone take any basic path $\{a', b'\}$ of (T', P'). We have to see that f' maps monotonically $\langle a', b'\rangle_{T'}$ onto $\langle f'(a'), f'(b')\rangle_{T'}$. Since κ sends P bijectively to P' there exist unique points $a, b \in P$ such that $\kappa(a) = a'$ and $\kappa(b) = b'$. Moreover, from the definition of κ it follows that, for every $u, w \in T$, κ sends $\langle u, w\rangle_T$ monotonically onto $\langle \kappa(u), \kappa(w)\rangle_{T'}$ and, conversely, κ^{-1} sends $\langle \kappa(u), \kappa(w)\rangle_{T'}$ monotonically onto $\langle u, w\rangle_T$. Therefore, κ^{-1} sends $\langle a', b'\rangle_{T'}$ monotonically onto $\langle a, b\rangle_T$ and $|\langle a, b\rangle_T \cap P| = |\langle a', b'\rangle_{T'} \cap P'| = 2$. Hence, $\{a, b\}$ is a basic path of (T, P).

To prove the monotonicity of $f' = \kappa \circ g \circ \kappa^{-1}$, we only have to see that g maps monotonically $\langle a, b\rangle_T$ onto $\langle g(a), g(b)\rangle_T$. Since $g\big|_{T\setminus C_v} = f\big|_{T\setminus C_v}$ and f is P-monotone, the statement holds trivially when $\langle a, b\rangle \cap C_v = \emptyset$.

Assume now that $\langle a, b\rangle \cap C_v \neq \emptyset$. Since $v \notin P$ and C_v is a star that has v as a branching point, $v \in \langle a, b\rangle_T$. Hence, $\kappa(v) = \kappa(x) \in \langle a', b'\rangle_{T'}$. Thus, $\kappa(x) \in \{a', b'\}$ and hence, $x \in \{a, b\}$. Assume for definiteness that $x = b$. Since f is P-monotone we have $f(v) \in \langle f(a), f(x)\rangle$. By definition (and the P-monotonicity of f) g maps monotonically $\langle a, v\rangle$ onto $\langle g(a), g(v)\rangle = \langle f(a), f(x)\rangle$ and $g(\langle v, x\rangle) = f(x)$. Summarizing, g is monotone on $\langle a, x\rangle$, which proves the claim.

Next we will show that $h(f') = h(g)$. Let $I_1, I_2, \ldots, I_k$ be a labelling of the set of all Q-basic intervals of (T, Q, g) such that $I_k = \langle v, x\rangle_T$. Let $M = (m_{i,j})_{i,j=1}^k$ be the associated Markov matrix. Since $g(I_k)$ reduces to a point, $m_{k,j} = 0$ for $j = 1, 2, \ldots, k$. Hence, $h(g) = \max\{0, \log(\sigma(M))\} = \max\{0, \log(\sigma(N))\}$, where $N = (m_{i,j})_{i,j=1}^{k-1}$. Set $I_i = \langle a_i, b_i\rangle_T$ and define $J_i = \langle \kappa(a_i), \kappa(b_i)\rangle_{T'}$ for $j = 1, 2, \ldots, k-1$. From the definitions of g and f' we have that I_i g-covers I_j if and only if J_i f'-covers J_j for every $j = 1, 2, \ldots, k-1$. In other words, the Markov matrix of $(T', \kappa(Q), f') = (T', P' \cup \operatorname{V}(T'), f')$ is N. Since f' is P'-monotone, it is also $P' \cup \operatorname{V}(T')$-monotone by Remark 2.2. Consequently, $h(f') = \max\{0, \log(\sigma(N))\} = h(g)$. This ends the proof of the proposition. $\square$

In Section 7 we will need the following slight improvement of Proposition 4.1.

Lemma 4.3. *In the situation of Proposition 4.1 we have $\nu(\mathcal{P}') > 0$ whenever $\nu(\mathcal{P}) > 1$.*

Proof. We use the notation of Proposition 4.1 and its proof. Assume that $|\operatorname{Br}(T) \setminus P| = \nu(\mathcal{P}) > 1$. If (T', P', f') is already a canonical model, then

$$\nu([T', P', f']) = |\operatorname{Br}(T') \setminus P'| = |\operatorname{Br}(T) \setminus P| - 1 > 0,$$

and the proposition holds. Otherwise, let $(\widetilde{T}, \widetilde{P}, \widetilde{f})$ be the canonical model of $[T', P', f']$ obtained from (T', P', f') by collapsing every connected component of an invariant forest of (T', P', f') to a point. We have to prove that $\operatorname{Br}(\widetilde{T}) \setminus \widetilde{P} \neq \emptyset$.

By assumption there exists $w \in \operatorname{Br}(T') \setminus P'$. Denote by $\widetilde{\kappa}$ the standard projection from T' to $\widetilde{T}$. Clearly, $\operatorname{Val}_{\widetilde{T}}(\widetilde{\kappa}(w)) \geq \operatorname{Val}_{T'}(w) \geq 3$ and hence, $\widetilde{\kappa}(w) \in \operatorname{Br}(\widetilde{T})$. Assume now that $\widetilde{\kappa}(w) = \widetilde{\kappa}(p)$ with $p \in P'$. Since $w \notin P'$ it follows that $w \neq p$ and hence $\widetilde{\kappa}$ is not one-to-one in $\{w, p\}$. Therefore, w and p belong to the same connected component of an invariant forest; a contradiction since every invariant forest of (T', P', f') is disjoint from P' by definition. Consequently, $\widetilde{\kappa}(w) \notin \widetilde{\kappa}(P') = \widetilde{P}$. □

5. The maximal simplicial patterns are irreducible

In this Section we prove (Corollary 5.4) that a simplicial n-periodic pattern which is reducible cannot attain the maximum entropy in the set of all n-periodic patterns. To do it, first we show (Proposition 5.1) that the reducible patterns have a very particular structure (what we call a *block structure*). Then we find (Proposition 5.2) a strict upper bound for the entropy of any n-periodic model having a block structure. This bound is $\log(\lfloor \frac{n}{2} \rfloor - 1)$. Finally (Theorem 5.3) we construct an n-periodic interval model whose entropy is larger than or equal to $\log(\lfloor \frac{n}{2} \rfloor)$. This result has interest in itself since, in particular, it proves that the maximum entropy of all n-cycles in the interval is larger than or equal to $\log(\lfloor \frac{n}{2} \rfloor)$ thus giving a lower bound for this maximum entropy.

Let (T, P, f) be a simplicial model. We will say that (T, P, f) and its corresponding pattern $[T, P, f]$ have a *p-block structure* (or simply a *block structure*) if $p \geq 2$ and there exists a partition $P = P_1 \cup P_2 \cup \cdots \cup P_p$ such that $|P_1| = |P_2| = \cdots = |P_p|$, $\langle P_i \rangle \cap \langle P_j \rangle = \emptyset$ for $i \neq j$, and $f(P_i) = P_{i+1}$ for $1 \leq i < p$ and $f(P_p) = P_1$. Observe that $|P_i| = |P|/p > 1$ for $1 \leq i \leq p$ and $T \setminus (\cup_{i=1}^{p} \langle P_i \rangle)$ is a finite union of pairwise disjoint open intervals because $\operatorname{V}(T) \subset P$. Moreover, $f(\langle P_i \rangle) = \langle P_{i+1} \rangle$ for $1 \leq i < p$ and $f(\langle P_p \rangle) = \langle P_1 \rangle$ since a simplicial model (T, P, f) is monotone by assumption.

In the literature one can find several kinds of block structures and related notions for periodic orbits. In the interval case, Sharkovskii's *square root construction* (see [21, 23] or [5]) is an earlier example of a block structure. The notion of *extension*, which first appeared in [10], gives rise to some particular cases of block structures for interval periodic orbits. The notion of *division*, introduced in [18] for interval periodic orbits and generalized in [7] and [8] to several kinds of trees, is a special case of block structure which has been used in a number of papers to study the topological entropy of tree maps.

The next result tells us that each reducible model has a block structure. In fact, the proof of Proposition 5.2 will tell us that the converse is also true, so that the periodic simplicial models having a block structure are precisely the reducible ones.

Proposition 5.1. *Let (T, P, f) be a periodic simplicial model. If (T, P, f) is reducible then it has a block structure.*

Proof. Since (T,P,f) is reducible, there exists a particular labelling $I_1, I_2, \dots, I_n$ of the set of P-basic intervals such that the associated Markov matrix M of (T,P,f) reads

$$\begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix},$$

where M_{11} and M_{22} are square matrices of sizes $k\times k$ and $l\times l$ $(k,l\geq 1)$ respectively and 0 stands for the $k\times l$ matrix whose entries are all 0. In particular, the set $X := \cup_{i=1}^{k} I_i$ is f-invariant.

Note that $\mathrm{En}(I_i)\subset P$ for $1\leq i\leq n$ and that $\cup_{i=1}^{k}\mathrm{En}(I_i)$ is f-invariant. Therefore, since P is a periodic orbit, $\cup_{i=1}^{k}\mathrm{En}(I_i)=P$. Note that $X\neq T$ because $k<n$. Since $X\supset P\supset \mathrm{En}(T)$, it follows that X is not connected. Since f maps any connected component of X onto a connected component of X and P is a periodic orbit of f, it easily follows that f acts as a cyclic permutation of the set of connected components of X. So, there exists a divisor $p\geq 2$ of $|P|$ and a labelling $X_1, X_2\dots, X_p$ of the set of connected components of X such that $f(X_i)=X_{i+1}$ for $1\leq i<p$ and $f(X_p)=X_1$. We set $P_i=X_i\cap P$ and note that $X_i=\langle P_i\rangle$ for $1\leq i\leq p$ because each X_i is a union of P-basic intervals. Moreover, $f(P_i)=P_{i+1}$ for $1\leq i<p$ and $f(P_p)=P_1$. Therefore, (T,P,f) has a p-block structure. □

Now we aim at obtaining an upper bound for the topological entropy of a model (T,P,f) with a block structure, in terms of the period of P. In what follows $\lfloor\cdot\rfloor$ will denote the integer part function.

Proposition 5.2. *Let (T,P,f) be an n-periodic simplicial model with a block structure. Then, $n\geq 4$ and $h(f)\leq \log(\lfloor\frac{n}{2}\rfloor-1)$.*

Proof. Let $p\geq 2$ and let $P_1\cup P_2\cup\ldots\cup P_p$ be a partition of P such that $\langle P_i\rangle\cap\langle P_j\rangle=\emptyset$ for $i\neq j$, $f(P_i)=P_{i+1}$ for $1\leq i<p$ and $f(P_p)=P_1$. Then $n=pq$, where $|P_i|=q\geq 2$ for $1\leq i\leq p$. Hence, $n\geq 4$.

Since each $\langle P_i\rangle$ is a tree with all its vertices contained in P_i, it follows that the number of P-basic intervals contained in $\langle P_i\rangle$ is $q-1$, for each $1\leq i\leq p$. Therefore, since the total number of P-basic intervals is $n-1$, the number of P-basic intervals whose interior does not intersect $\cup_{i=1}^{p}\langle P_i\rangle$ is $n-1-p(q-1)=p-1$. Let $I_1, I_2, \dots, I_{n-1}$ be a labelling of the P-basic intervals such that, $I_{(i-1)(q-1)+j}\subset\langle P_i\rangle$ for every $1\leq i\leq p$ and $1\leq j\leq q-1$, and $I_{p(q-1)+1}, I_{p(q-1)+2},\dots, I_{n-1}\subset T\setminus(\cup_{i=1}^{p}\langle P_i\rangle)$.

Let M be the Markov matrix of (T,P,f) associated to this labelling. From Lemma 3.1(a) it follows that the i,j entry of M^p is the number of paths of length p in the Markov graph starting at I_i and ending at I_j. Therefore,

$$M^p=\begin{pmatrix} A & 0 \\ B & C \end{pmatrix},$$

where A and C are square matrices of sizes $p(q-1)\times p(q-1)$ and $(p-1)\times(p-1)$ respectively, 0 stands for the $p(q-1)\times(p-1)$ matrix whose entries are all 0, and A has the form

$$\begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & A_p \end{pmatrix},$$

where each A_i is a matrix of size $(q-1)\times(q-1)$ and 0 stands for the $(q-1)\times(q-1)$ matrix whose entries are all 0. Moreover, for each i, every entry of A_i is bounded above by $(q-1)^{p-1}$ and every entry of C is bounded above by $(p-1)^{p-1}$. To prove this consider an entry of A_i. It is an entry $m_{l,k}^{(p)}$ of M^p with $(i-1)(q-1)+1\leq$

$l, k \leq i(q-1)$ (i.e. I_l and I_k are P-basic intervals contained in $\langle P_i \rangle$). Hence, $m_{l,k}^{(p)}$ is the number of paths of length p in the Markov graph starting at I_l and ending at I_k. Such paths are of the form $I_l \longrightarrow J_{i+1} \longrightarrow \cdots \longrightarrow J_p \longrightarrow J_1 \longrightarrow \cdots \longrightarrow J_{i-1} \longrightarrow I_k$, where $J_j \subset \langle P_j \rangle$ for $j \in \{1, 2, \ldots, p\} \setminus \{i\}$. Since there are at most $q-1$ choices for every J_k it follows that there are at most $(q-1)^{p-1}$ loops of this form and hence, $m_{l,k}^{(p)} \leq (q-1)^{p-1}$. The same argument can be used to show that every entry of the matrix C is upper bounded by $(p-1)^{p-1}$. Consequently, since the spectral radius of a matrix is bounded above by any norm, $\sigma(A_i) \leq ||A_i||_\infty \leq (q-1)^p$ for $1 \leq i \leq p$ and $\sigma(C) \leq ||C||_\infty \leq (p-1)^p$.

Since $h(f) = \max\{0, \log(\sigma(M))\}$ and $\sigma(M^p) = \sigma(M)^p$, we have that $h(f) = \max\{0, \frac{1}{p}\log(\sigma(M^p))\}$. Now observe that $\sigma(M^p)$ is the maximum of the spectral radius of the matrices $A_1, A_2, \ldots, A_p, C$. Taking into account the above estimates of $\sigma(A_i)$ and $\sigma(C)$ we get $h(f) \leq \max\{0, \log(q-1), \log(p-1)\}$. Therefore, since $n = pq$ with $p, q \geq 2$, it follows that $p, q \leq n/2$ and hence,

$$h(f) \leq \log\left(\tfrac{n}{2} - 1\right).$$

So, if n is even the proposition holds. If n is odd, then $n \geq 9$ and $p, q \leq n/3$. Thus, $h(f) \leq \log\left(\frac{n}{3} - 1\right)$. The inequality $n \geq 9$ implies $2n < 3(n-1)$ which is equivalent to $\frac{n}{3} < \frac{n-1}{2} = \left\lfloor \frac{n}{2} \right\rfloor$. This ends the proof of the proposition. □

Next we will show that, for each $n \geq 4$, there exists an n-periodic interval model whose entropy is larger than or equal to $\log(\left\lfloor \frac{n}{2} \right\rfloor)$. This construction is based on the well known notion of a *horseshoe*. If $f\colon I \longrightarrow I$ is a continuous interval map and $s \geq 2$, then an *s-horseshoe* for f is an interval $J \subset I$ and a partition $\mathcal{D}$ of J into s subintervals such that the closure of each element of $\mathcal{D}$ f-covers J. It is well known (see for instance [5, Proposition 4.3.2]) that if f has an s-horseshoe then $h(f) \geq \log(s)$.

As we have already said, this result has interest in itself since, in particular, it proves in a constructive way that the maximum entropy of all n-cycles in the interval is larger than or equal to $\log(\left\lfloor \frac{n}{2} \right\rfloor)$.

Theorem 5.3. *Let $n \in \mathbb{N}$ be such that $n \geq 4$. Then, there exists a monotone n-periodic interval model $([1, n], \{1, 2, \ldots, n\}, f)$ such that f has an $\left\lfloor \frac{n}{2} \right\rfloor$-horseshoe and, hence, $h(f) \geq \log(\left\lfloor \frac{n}{2} \right\rfloor)$.*

Corollary 5.4. *The maximal periodic simplicial patterns are irreducible.*

Proof. Let $\mathcal{P}$ be a reducible n-periodic simplicial pattern. By Proposition 5.1 it has a block structure. In particular, $n = pq$ for some $p, q \geq 2$. Hence, $n \geq 4$.

By Proposition 5.2 and Theorem 5.3 we get that $\mathcal{P}$ is not maximal. □

Notice that the proof of the above corollary works in the same manner restricted to maximal *interval* patterns. So we also get

Corollary 5.5. *The maximal periodic interval patterns are irreducible.*

The rest of this section is devoted to prove Theorem 5.3. To this end we will introduce some notation concerning permutations and we will prove a simple, but useful, technical lemma.

A permutation of order n will be written as a bijective map from the set $Z_n := \{1, 2, \ldots, n\}$ to itself. Given a permutation θ, as usual, we denote by θ^k the k-th iterate of θ. Observe that θ is cyclic if and only if $\theta^k(i) \neq i$ for every $i \in Z_n$ and $1 \leq k < n$.

In a similar way, a one-to-one map $\varphi\colon A \longrightarrow Z_n$, where A is a proper non-empty subset of Z_n, will be called a *partial permutation of order n*. For such a map we also denote by φ^k the k-th iterate of φ. Observe that, given $k \geq 2$ and $i \in A$, $\varphi^k(i)$

will not be defined whenever $\varphi^l(i) \notin A$ for some $l < k$ (and $l \leq |A|$). We will say that the partial permutation φ is *periodic* if there exists $i \in A$ and $1 \leq k < n$ such that $\varphi^k(i) = i$, otherwise φ is non-periodic.

Lemma 5.6. *Let $\varphi\colon A \longrightarrow Z_n$ be a non-periodic partial permutation of order n. Then, there exists a cyclic permutation $\theta\colon Z_n \longrightarrow Z_n$ such that $\theta\big|_A = \varphi$.*

Proof. Since $A \neq Z_n$, $\varphi(A) \neq Z_n$. Let $x_1, x_2, \dots, x_l$ be an enumeration of the elements of $Z_n \setminus \varphi(A)$. Clearly, for every x_i there exists a sequence $x_0^i = x_i, x_1^i, \dots, x_{j_i}^i$ in Z_n with $j_i \geq 0$ such that $x_p^i \in A$ and $x_{p+1}^i = \varphi(x_p^i)$ for $p = 0, 1, \dots, j_i - 1$; and $x_{j_i}^i \notin A$. Hence, $A = \bigcup_{i=1}^l \{x_0^i, x_1^i, \dots, x_{j_i-1}^i\}$ and $Z_n \setminus A = \{x_{j_1}^1, x_{j_2}^2, \dots, x_{j_l}^l\}$.

Now we define $\theta(w) = \varphi(w)$ for every $w \in A$, $\theta(x_{j_i}^i) = x_0^{i+1}$ for $i = 1, 2, \dots, l-1$ and $\theta(x_{j_l}^l) = x_0^1$. This defines θ on the whole Z_n in such a way that it is one-to-one and non-periodic. So, θ is a cyclic permutation of order n. □

Now we are ready to prove Theorem 5.3.

Proof of Theorem 5.3. Set $k = \left\lfloor \frac{n}{2} \right\rfloor$. We claim that there exists a non-periodic partial permutation $\varphi\colon A \longrightarrow Z_n$, where $A = \{l, l+1, \dots, l+k\} \subset Z_n$ for a certain $l \in Z_n$ that will be defined later, such that

$$\langle \varphi(i), \varphi(i+1) \rangle \supset [l, l+k] \quad \text{for } i = l, l+1, \dots, l+k-1. \tag{4}$$

Assume that the claim holds and let us prove the theorem. By Lemma 5.6, we know that there exists a cyclic permutation $\theta\colon Z_n \longrightarrow Z_n$ of order n such that $\theta\big|_A = \varphi$. Now we define $f\colon [1,n] \longrightarrow [1,n]$ as the unique continuous map such that $f(i) = \theta(i)$ for every $i \in Z_n$ and f is affine on each interval of the form $[i, i+1]$ for $i = 1, 2, \dots, n-1$. This completely specifies the n-periodic model $([1,n], \{1, 2, \dots, n\}, f)$. Clearly, for $i \in A \setminus \{l+k\}$,

$$f([i, i+1]) = \langle \varphi(i), \varphi(i+1) \rangle \supset [l, l+k].$$

Hence, the model $([1,n], \{1, 2, \dots, n\}, f)$ has a k-horseshoe and, by [5, Proposition 4.3.2]), $h(f) \geq \log(k)$.

Now we will prove the claim. To this end we set $u = \left\lfloor \frac{n-1}{4} \right\rfloor$ (that is, $n = 4u + r$ with $r \in \{1, 2, 3, 4\}$). Observe that $u \geq 0$ because $n \geq 4$.

We define $A = \{x_0, x_1, \dots, x_k\}$ by setting $l = u + 1$ and $x_i = l + i = u + 1 + i$ for $i = 0, 1, \dots, k$. Observe that $x_k = l + k = \left\lfloor \frac{n-1}{4} \right\rfloor + 1 + \left\lfloor \frac{n}{2} \right\rfloor < n$ for every $n \geq 4$. Hence, A is a proper non-empty subset of Z_n. We also set

$$\begin{aligned}
A_e &= \{x_i \in A : i \text{ is even}\},\\
A_o &= \{x_i \in A : i \text{ is odd}\},\\
L &= \{i \in Z_n : i \leq x_0\} = \{1, 2, \dots, x_0\},\\
R &= \{i \in Z_n : i \geq x_k\} = \{x_k, x_k + 1, \dots, n\}.
\end{aligned}$$

Observe that, since $k \geq 2$, $L \cap R = \emptyset$. Also, $A_e \cap A_o = \emptyset$ by definition.

We define φ on A_o by

$$\varphi(x_{2i-1}) = i \quad \text{for } i = 1, 2, \dots, |A_o|$$

and we define φ on A_e by

$$\varphi(x_{2i}) = \begin{cases} x_k + i & \text{when } k \text{ is even} \\ n - i & \text{when } k \text{ is odd} \end{cases} \qquad \text{for } i = 0, 1, \dots, |A_e| - 1.$$

Clearly, $\varphi(A_o) = \{1, 2, \dots, |A_o|\}$ and

$$\varphi(A_e) = \begin{cases} \{x_k, x_k + 1, \dots, x_k + |A_e| - 1\} & \text{when } k \text{ is even,} \\ \{n - |A_e| + 1, n - |A_e| + 2, \dots, n\} & \text{when } k \text{ is odd.} \end{cases}$$

We will prove that

$$|R| \geq |A_e| \quad \text{and} \quad |L| \geq |A_o|. \tag{5}$$

This clearly implies

$$\varphi(A_e) \subset R \quad \text{and} \quad \varphi(A_o) \subset L \tag{6}$$

and, hence, $\varphi(A) \subset Z_n$ and φ is one-to-one because $L \cap R = \emptyset$. So, φ is a partial permutation. Moreover, for $i \in A \setminus \{l+k\}$, $\langle \varphi(i), \varphi(i+1) \rangle \supset [x_0, x_k] = [l, l+k]$. Hence, (4) holds.

To prove (5), a simple computation shows that

$$k = \left\lfloor \tfrac{4u+r}{2} \right\rfloor = 2u + \left\lfloor \tfrac{r}{2} \right\rfloor = 2u + (r-1) - s, \text{ with}$$
$$s = \begin{cases} 0 & \text{when } n - 4u \in \{1, 2\}, \\ 1 & \text{when } n - 4u \in \{3, 4\}. \end{cases}$$

Observe that k is odd if and only if $\left\lfloor \frac{r}{2} \right\rfloor = 1$; which is equivalent to $n - 4u \in \{2, 3\}$ and $k = 2u + 1$. Otherwise, $n - 4u \in \{1, 4\}$ and $k = 2(u+s)$. We have

$$|L| = u + 1, \text{ and}$$
$$|R| = n - (x_k - 1) = n - (u + k) = 4u + r - (3u + (r-1) - s) = u + 1 + s.$$

On the other hand, when k is odd (that is, when $n - 4u \in \{2, 3\}$),

$$|A_o| = |A_e| = \frac{k+1}{2} = \frac{2u+2}{2} = u + 1.$$

and, when k is even ($n - 4u \in \{1, 4\}$),

$$|A_o| = |A_e| - 1 = \frac{k}{2} = u + s.$$

Therefore, since $s \in \{0, 1\}$, we have

$$|A_o| \leq u + 1 = |L| \qquad \text{and} \qquad |A_e| \leq u + 1 + s = |R|$$

and (5) holds.

Now, to end the proof of the theorem, we have to show that φ is non-periodic. From (6) it follows that the only possible periodic points of φ have to be contained in $\{x_0, x_k\}$.

When k is even $x_0, x_k \in A_e$. Hence, $\varphi(x_0) = x_k$ and $\varphi(x_k) = x_k + \frac{k}{2}$. Since $k \geq 2$, $\varphi(x_0) \neq x_0$ and $\varphi^2(x_0) = \varphi(x_k) > x_k$. Hence, $\varphi^2(x_0) = \varphi(x_k) \notin A$. So, φ is non-periodic.

When k is odd, $\varphi(x_0) = n \notin A$. On the other hand, $x_k \in A_o$ and, hence, $\varphi(x_k) \in \varphi(A_o) \subset L$. So, either $\varphi(x_k) \notin A$ or $\varphi(x_k) = x_0$. In any case, φ is non-periodic as above. □

6. The maximal simplicial patterns are maximodal

In this Section we prove that an n-periodic irreducible simplicial pattern $\mathcal{P}$ which has monotone triplets cannot attain the maximum entropy in the set of all n-periodic patterns (Corollary 6.2). This basically follows from the following theorem.

Theorem 6.1. *Let $\mathcal{P}$ be an n-periodic irreducible simplicial pattern. If $\mathcal{P}$ has monotone triplets, then there exists an n-periodic simplicial pattern $\mathcal{Q}$ such that $h(\mathcal{Q}) > h(\mathcal{P})$.*

Then from Corollary 5.4 and Theorem 6.1 we immediately get:

Corollary 6.2. *The maximal periodic simplicial patterns are irreducible and maximodal.*

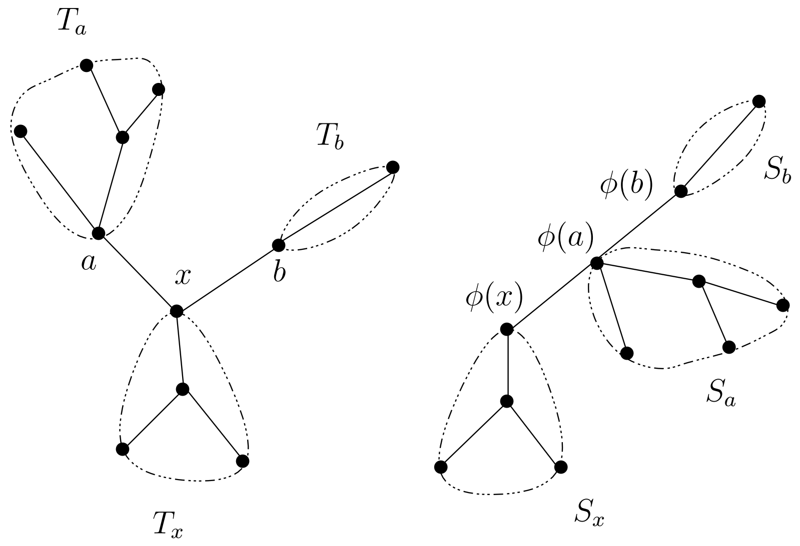

FIGURE 2. On the left picture, a model (T, P, f) and the trees T_a, T_b and T_x as defined in the proof of Theorem 6.1. On the right picture, the model (S, Q, g) as constructed in the proof of Theorem 6.1.

Proof of Theorem 6.1. If $h(\mathcal{P}) = 0$ the result follows trivially since, for every $n \geq 3$, there exist patterns with positive topological entropy (for instance in the interval; for $n \geq 4$ this follows also from Theorem 5.3). So, in what follows we assume that $h(\mathcal{P}) > 0$.

Let (T, P, f) be a monotone model of $\mathcal{P}$ and let $[a; x; b]$ be a monotone triplet of (T, P, f). We define T_a (respectively, T_b) as the connected component of $T \setminus (a, b)$ which contains a (respectively b). We also define T_x as the connected component of $T \setminus \{x\}$ which does not contain a neither b (see the left part of Figure 2). Observe that T_a, T_b and T_x are trees (maybe reduced to one point). Clearly, by construction, $T_a \cup T_b \cup T_x = T \setminus ((a, x) \cup (x, b))$.

We start by claiming that there exists a P-basic interval which f-covers either $\langle a, x\rangle$ or $\langle x, b\rangle$, but not both intervals. To prove the claim, let x^{-1} (respectively b^{-1}) denote the only point in $f^{-1}(x) \cap P$ (respectively $f^{-1}(b) \cap P$). Since $x \neq b$, $x^{-1} \neq b^{-1}$. Take a linear ordering $\prec$ in $\langle x^{-1}, b^{-1}\rangle$ such that $x^{-1} \prec b^{-1}$. Let $y_1, y_2, \ldots, y_r$ denote all points in $P \cap \langle x^{-1}, b^{-1}\rangle$, labelled in such a way that $x^{-1} = y_1 \prec y_2 \prec \cdots \prec y_r = b^{-1}$. Note that $f(y_1) = x \in T_x$, $f(y_r) = b \in T_b$ and each $f(y_i)$ belongs either to T_x or T_a or T_b. It follows that there exists a P-basic interval I of the form $\langle y_i, y_{i+1}\rangle$ for some $1 \leq i < r$ such that $f(y_i) \in T_x$ and $f(y_{i+1}) \notin T_x$. Then I f-covers $\langle a, x\rangle$ when $f(y_{i+1}) \in T_a$ and $\langle x, b\rangle$ when $f(y_{i+1}) \in T_b$. So the claim is proved.

Without loss of generality, from now on, we assume that the basic interval I f-covers $\langle x, b\rangle$ and does not f-cover $\langle a, x\rangle$.

Next we are going to construct a new model (S, Q, g) in the following way: we remove the interval (a, x) from T and attach T_a at some point v in the interval (x, b) by gluing together a and v (see Figure 2 for an example). More precisely, there exists a map $\phi\colon T \longrightarrow S$ such that $\phi\big|_{T_x \cup T_b \cup \langle x, b\rangle}$ and $\phi\big|_{T_a \cup \langle a, x\rangle}$ are homeomorphisms, and $\phi(a) = v \in (x, b)$. Observe that $\phi\big|_{T_a \cup T_b \cup T_x}$ is bijective. Then we set $Q = \phi(P)$ and $g\big|_{S_a \cup S_b \cup S_x} = \phi \circ f \circ \phi^{-1}\big|_{S_a \cup S_b \cup S_x}$, where $S_\alpha = \phi(T_\alpha)$ for every $\alpha \in \{a, x, b\}$. Since $P \subset T_a \cup T_b \cup T_x$, Q is a periodic orbit of g of period n. Next we extend g to the whole S in such a way that it is continuous and (S,Q,g) is a monotone model. Clearly, since (T,P,f) is simplicial, (S,Q,g) is simplicial.

Let $\mathcal{Q}$ be the pattern $[S,Q,g]$. To end the proof of the theorem we have to show that $h(\mathcal{Q}) > h(\mathcal{P})$.

Since (S,Q,g) is n-periodic and simplicial, it has $n-1$ Q-basic intervals. Observe that if $\langle y,z\rangle$ is a P-basic interval then $\langle \phi(y),\phi(z)\rangle$ is a Q-basic interval if and only if $\langle y,z\rangle \neq \langle x,b\rangle$. Moreover, if $\langle y,z\rangle$ is a Q-basic interval then $\langle \phi^{-1}(y),\phi^{-1}(z)\rangle$ is a P-basic interval if and only if $\langle y,z\rangle \neq \langle \phi(a),\phi(b)\rangle$.

Now consider a labelling $\{I_i\}_{i=1}^{n-1}$ of the set of P-basic intervals of (T,P,f) such that $I_{n-2} = \langle a,x\rangle$ and $I_{n-1} = \langle x,b\rangle$, and let $M = (m_{i,j})_{i,j=1}^{n-1}$ be the associated Markov matrix. Since $\mathcal{P}$ is irreducible, M is irreducible. For every P-basic interval $I_i = \langle y,z\rangle$ of (T,P,f) set $\phi(I_i) := \langle \phi(y),\phi(z)\rangle$. Consider a labelling $\{J_i\}_{i=1}^{n-1}$ of the set of Q-basic intervals of (S,Q,g) such that $J_i = \phi(I_i)$ for $1 \le i \le n-2$ and $J_{n-1} = \langle \phi(a),\phi(b)\rangle$. Let $R = (r_{i,j})_{i,j=1}^{n-1}$ be the associated Markov matrix. The matrix R can be obtained from the matrix M in the following way:

(1) $m_{i,j} = r_{i,j}$ for $1 \le i \le n-2$ and $1 \le j \le n-3$.
(2) For any $1 \le i \le n-3$, the ordered pair $(m_{i,n-2}, m_{i,n-1})$ changes to the pair $(r_{i,n-2}, r_{i,n-1})$ according to the following rules: $(0,0) \to (0,0)$, $(0,1) \to (1,1)$, $(1,0) \to (1,0)$ and $(1,1) \to (0,1)$.
(3) $r_{n-1,j} = m_{n-1,j} + m_{n-2,j}$ for each $1 \le j \le n-3$. Observe that, since $[a;x;b]$ is monotone, $m_{n-1,j}$ and $m_{n-2,j}$ cannot simultaneously be equal to 1.
(4) The 2×2 submatrix

$$\begin{pmatrix} m_{n-2,n-2} & m_{n-2,n-1} \\ m_{n-1,n-2} & m_{n-1,n-1} \end{pmatrix} \quad \text{changes to} \quad \begin{pmatrix} r_{n-2,n-2} & r_{n-2,n-1} \\ r_{n-1,n-2} & r_{n-1,n-1} \end{pmatrix}$$

according to the following rules:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and}$$
$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

No other configurations are possible for this 2×2 submatrix of M.

The above Statements (1–4) follow almost directly from the definition of the model (S,Q,g) and the labelling of the P-basic and Q-basic intervals. As an example let us prove Statement (3). We have to show that if I_j is f-covered either by $I_{n-2} = \langle a,x\rangle$ or by $I_{n-1} = \langle x,b\rangle$ for some $1 \le j \le n-3$, then $J_{n-1} = \langle \phi(a),\phi(b)\rangle$ g-covers J_j. Assume that $I_j \subset \langle f(a), f(x)\rangle$ (the proof is analogous if I_j is f-covered by $\langle x,b\rangle$). Since $1 \le j \le n-3$, I_j is contained in T_a, T_b or T_x. Then, from the definition of (S,Q,g) it follows that $J_j = \phi(I_j)$ is contained respectively in S_a, S_b or S_x, and $J_j \subset \langle \phi(f(a)), \phi(f(x))\rangle = \langle g(\phi(a)), g(\phi(x))\rangle$. Now observe that, since $[a;x;b]$ is a monotone triplet then $[f(a);f(x);f(b)]$ is an ordered triplet in T. Note also that, if $[x_1;x_2;x_3]$ is any ordered triplet in T, then $[\phi(x_1);\phi(x_2);\phi(x_3)]$ is not an ordered triplet in S if and only if $x_1 \in T_a$, $x_2 = x$ and $x_3 \in T_b$. Since $f(x) \neq x$, it follows that $[\phi(f(a));\phi(f(x));\phi(f(b))]$, which can be rewritten as $[g(\phi(a));g(\phi(x));g(\phi(b))]$, is an ordered triplet in S. So, $\langle g(\phi(a)), g(\phi(b))\rangle \supset \langle g(\phi(a)), g(\phi(x))\rangle \supset J_j$, which proves (3).

Let L be the matrix of the linear transformation of $\mathbb{R}^{n-1}$ given by

$$(x_1, x_2, \ldots, x_{n-1}) \longrightarrow (x_1, x_2, \ldots, x_{n-2}, x_{n-1} + x_{n-2}).$$

We have

$$h(\mathcal{Q}) = h(g) = \max\{0, \log(\sigma(R))\} = \max\{0, \log(\sigma(L^{-1}RL))\} \text{ and}$$
$$h(\mathcal{P}) = h(f) = \max\{0, \log(\sigma(M))\}.$$

Since $h(\mathcal{P}) > 0$ it follows that $0 < h(\mathcal{P}) = \log(\sigma(M))$. So, to end the proof of the theorem it is enough to show that $\sigma(L^{-1}RL) > \sigma(M)$.

Observe that L coincides with the $(n-1) \times (n-1)$ identity matrix except for the element placed at row $n-1$, column $n-2$, which is 1. Moreover, all entries of L^{-1} coincide with those of L except the element placed at row $n-1$, column $n-2$, which is -1. We also note that, given any $(n-1) \times (n-1)$ matrix X:

- The product XL gives a matrix whose $(n-2)$-th column is the sum of the $(n-2)$-th and $(n-1)$-th columns of X (the rest of columns of X remain intact).
- The product $L^{-1}X$ gives a matrix whose $(n-1)$-th row is the $(n-1)$-th row of X minus the $(n-2)$-th row of X (the rest of rows of X remain intact).

Collecting the two statements above together with Statements (1–4) it is straightforward to check that $L^{-1}RL$ is a non-negative matrix whose entries coincide with those of M except when the last two elements of a row of M are 0 and 1, in which case the corresponding elements of $L^{-1}RL$ are respectively 2 and 1. In other words, if $L^{-1}RL = (s_{i,j})_{i,j=1}^{n-1}$, then $s_{i,j} \neq m_{i,j}$ if and only if $j = n-2$, $m_{i,j} = 0$ and $m_{i,j+1} = 1$, and in this case $s_{i,j} = 2$ and $s_{i,j+1} = 1$. Hence, $L^{-1}RL \geq M$.

Recovering the claim at the beginning of the proof, recall that there exists a P-basic interval I_i which f-covers I_{n-1} and does not f-cover I_{n-2}. In terms of the matrix M this amounts to $m_{i,n-2} = 0$ and $m_{i,n-1} = 1$. Therefore, from above, $s_{i,n-2} = 2 > m_{i,n-2}$ and, hence, $L^{-1}RL \neq M$.

On the other hand, M is irreducible by hypothesis. Thus, $\sigma(L^{-1}RL) > \sigma(M)$ by Lemma 3.1(d). □

7. The maximal patterns are simplicial. Proof of Theorem A

Theorem A follows directly from Corollary 6.2 and the following theorem which is the main result of this section.

Theorem 7.1. *Any maximal periodic pattern is simplicial.*

As it has been said in Section 2, to prove the above theorem we need to introduce some new tools. In particular the notion of *path transition matrix*.

Notation and tools. Let $\mathcal{P} = [T, P, f]$ be a pattern where (T, P, f) is a monotone model of $\mathcal{P}$. Let $\{\pi_1, \pi_2, \dots, \pi_k\}$ be the set of basic paths of the pointed tree (T, P). We will say that the basic path π_i *f-covers* the basic path π_j whenever $\pi_j \subset \langle f(\pi_i) \rangle$. The fact that π_i *f-covers* π_j will be denoted by $\pi_i \longrightarrow \pi_j$.

The *$\mathcal{P}$-path graph* is the combinatorial oriented graph whose vertices are in one-to-one correspondence with the basic paths of (T, P), and there is an oriented edge (or arrow) from the vertex i to the vertex j if and only if π_i f-covers π_j.

The $k \times k$ matrix $M_P(\mathcal{P}) = (m_{ij})$ defined by

$$m_{ij} = \begin{cases} 1 & \text{if } \pi_i \ f\text{-covers } \pi_j, \\ 0 & \text{otherwise} \end{cases}$$

(i.e. the transition matrix of the $\mathcal{P}$-path graph) will be called the *path transition matrix of $\mathcal{P}$*.

It can be seen that the definitions of $\mathcal{P}$-path graph and $M_P(\mathcal{P})$ are independent of the particular choice of the model (T, P, f). Thus, they are well-defined pattern invariants.

A crucial fact about the path transition matrix of a pattern $\mathcal{P}$ is the following (see [3])

$$h(\mathcal{P}) = \max\{0, \log \sigma(M_P(\mathcal{P}))\}. \tag{7}$$

Remark 7.2. In view of Theorem 5.3 and the fact that every 3-periodic interval model has positive topological entropy (see for instance [5, Theorem 4.4.20]) it follows that if $\mathcal{P}$ is a maximal n-periodic pattern with $n \geq 3$ then, by (7), $\sigma(M_P(\mathcal{P})) > 1$ and $h(\mathcal{P}) = \log \sigma(M_P(\mathcal{P}))$.

There is also a *converse* of the operation just described (i.e. going from the $\mathcal{P}$-path graph to the path transition matrix). Indeed, let $M = (m_{ij})$ be a $k \times k$ matrix whose entries are non-negative integers. To M we can associate the combinatorial oriented generalized graph whose vertices are $1, 2, \ldots, k$ and there are m_{ij} labelled arrows from the vertex i to the vertex j. Such a graph will be called the *M-induced graph*. Clearly, M is the transition matrix of the M-induced graph. In particular, the $\mathcal{P}$-path graph is the $M_P(\mathcal{P})$-induced graph.

Now we are ready to start the

Proof of Theorem 7.1. The strategy of the proof of Theorem 7.1 is as follows. Assume that $\mathcal{P}$ is a maximal periodic pattern which is not simplicial (i.e. $\nu(\mathcal{P}) > 0$). By the iterative use of Proposition 4.1 and Lemma 4.3 we may assume that $\mathcal{P}$ has a unique free vertex. In the rest of this section we will prove that any maximal periodic pattern $\mathcal{P}$ with a unique free vertex admits a (special) collapse $\mathcal{P}'$ such that $h(\mathcal{P}') > h(\mathcal{P})$, contradicting the maximality of $\mathcal{P}$. To this end we fix the notation for the rest of this section as follows.

Set $\mathcal{P} = [T, P, f]$ where (T, P, f) is a canonical monotone n-periodic model and assume that $\mathrm{Br}(T) \setminus P = \{v\}$. Notice that with these assumptions, $n \geq 3$.

By Remark 2.2 it follows that $f(v) \in P \cup \{v\}$ and f is $(P \cup \{v\})$-monotone.

Let $C_v = \{p_1, p_2, \ldots, p_k\} \subset P$ denote the discrete component of (T, P) such that $v \in \langle C_v \rangle$. Observe that v has valence k and that $\langle C_v \rangle$ is a star that has v as branching point. Since $v \in \mathrm{Br}(T)$ it follows that $k \geq 3$. Note also that $|P| > k$ since otherwise $\mathcal{P}$ has a unique discrete component $C_v = P$ and f is a permutation over P. In such case $h(\mathcal{P}) = 0$, contradicting Remark 7.2.

For $i \in \{1, 2, \ldots, k\}$ we will denote by C_i the connected component of $T \setminus \mathrm{Int}(\langle C_v \rangle)$ such that $p_i \in C_i$. Observe that $P \subset C_1 \cup C_2 \cup \ldots C_k$.

The basic path $\{p_i, p_j\}$ with $i, j \in \{1, 2, \ldots, k\}$ will be denoted by $p_{i,j}$ (or $p_{j,i}$) and be called an *interior basic path of the $\mathcal{P}$-path graph*. Every other basic path will be called an *exterior basic path of the $\mathcal{P}$-path graph*. The set of exterior basic paths will be denoted by $\mathcal{E}$. Observe that every exterior basic path is contained in $\langle C_i \rangle$ for some i.

We will separate the proof of Theorem 7.1 into two cases according to the fact that v is fixed by f or not.

Case $f(v) \neq v$.

Since $f(v) \in P \cup \{v\}$ it follows that $f(v) \in P$.

We claim that the valence of $f(v)$ is k. To prove the claim assume that the valence of $f(v)$ is smaller than k. Then, since f is $P \cup \{v\}-$monotone, the valence of $f(v)$ is $k-1$ and there exists $i \in \{1, 2, \ldots, k\}$ such that $f(\langle v, p_i \rangle) = f(p_i)$.

We consider the pattern $\mathcal{P}'$ obtained by collapsing the edge $\langle v, p_i \rangle$ to the point $\{p_i\}$. Clearly this pattern is simplicial. Moreover, since $f\big|_{\langle v, p_i \rangle}$ is constant, the entropy of $\mathcal{P}'$ coincides with the entropy of $\mathcal{P}$ in a similar way to the proof of Proposition 4.1. Hence, $\mathcal{P}'$ is maximal by assumption. On the other hand, the fact that $f(v)$ has valence $k-1$ implies that in the pattern $\mathcal{P}'$, $[p_l; p_i; p_m]$ with $i \notin \{l, m\}$

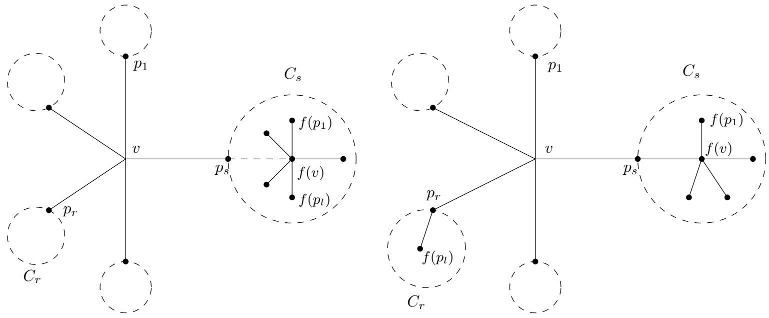

FIGURE 3. The two subcases of the proof of Theorem 7.1 in the case $f(v) \neq v$. On the left the situation of Subcase 1 and on the right the situation of Subcase 2.

is a monotone ordered triplet and hence, $\mathcal{P}'$ is not maximodal. This contradicts Corollary 6.2 and ends the proof of the claim.

Let $s \in \{1, 2, \dots, k\}$ be such that $f(v) \in C_s$. Since the valence of $f(v)$ is k (and, hence, f is a local homeomorphism at v) it follows that we are in one of the following two cases (see Figure 3):

Subcase 1. $f(p_i) \in C_s$ for all $i \in \{1, \dots, k\}$.

Observe that if $f(C_i) \subset C_s$ for every $i \in \{1, \dots, k\}$ then P cannot be a periodic orbit. Hence, $f(C_i) \not\subset C_s$ for some $i \in \{1, \dots, k\}$ and, since $f(p_i) \in C_s$, some exterior basic path must $f-$cover some interior basic path $p_{r,s}$.

Subcase 2. There exists a unique $l \in \{1, \dots, k\}$ such that $f(p_l) \notin C_s$.

Let $r \in \{1, \dots, k\} \setminus \{s\}$ be such that $f(p_l) \in C_r$.

In both subcases, since $k \geq 3$, we may assume by relabelling the points p_i if necessary that $1 \notin \{r, s\}$.

We consider the pattern obtained from $\mathcal{P}$ by collapsing the edge $\langle v, p_1 \rangle$ to a point (see the definition of a collapse from Section 4). In this way we obtain a pattern $\mathcal{P}' = \big([T', P'], [\kappa \circ f \circ \kappa^{-1}\big|_{P'}]\big)$ (where κ is the standard projection from T to the tree T' obtained by collapsing the edge $\langle v, p_1 \rangle$ of T and $P' = \kappa(P)$). Clearly $\mathcal{P}$ is simplicial and from Proposition 4.1 it follows that $h(\mathcal{P}') \geq h(\mathcal{P})$. Consequently, $\mathcal{P}'$ is maximal. Since $\mathcal{P}'$ is simplicial, the path-transition matrix of $\mathcal{P}'$, $M_P(\mathcal{P}')$, coincides with the Markov matrix of $\mathcal{P}'$. Therefore, $M_P(\mathcal{P}')$ is irreducible by Corollary 6.2.

We have to show that $h(\mathcal{P}) < h(\mathcal{P}')$. To do this we will compare the spectral radius of $M_P(\mathcal{P})$ and $M_P(\mathcal{P}')$ (Remark 7.2). Unfortunately these matrices cannot be compared directly and we need the help of an auxiliary intermediate matrix M obtained from $M_P(\mathcal{P})$ as follows. The fact that f is $P \cup \{v\}-$monotone and $f(v)$ has valence k implies that $\langle f(v), f(p_i) \rangle \cap \langle f(v), f(p_1) \rangle = \{f(v)\}$ for every $i \in \{2, 3, \dots, k\}$. Therefore, the matrix $M_P(\mathcal{P})$ has a zero in the entries lying in a row corresponding to every interior basic path $p_{i,j}$ with $1 < i < j$ and in a column corresponding to every basic path contained in $\langle f(v), f(p_1) \rangle$. The matrix M has the same size as $M_P(\mathcal{P})$ and coincides with $M_P(\mathcal{P})$ except for the entries lying in a row corresponding to every interior basic path $p_{i,j}$ with $1 < i < j$ and column corresponding to every basic path contained in $\langle f(v), f(p_1) \rangle$, where we replace the zero in $M_P(\mathcal{P})$ by a 2 in M. Observe that all basic paths contained in $\langle f(v), f(p_1) \rangle$ are exterior.

We will prove that

$$h(\mathcal{P}) < \log \sigma(M) = h(\mathcal{P}'). \tag{8}$$

This contradicts the maximality of $\mathcal{P}$ and ends the proof of the theorem in the case $f(v) \neq v$.

The above inequalities will be proved by using Lemma 3.1. To this end we need to extend the notions of interior and exterior basic path, defined for the $\mathcal{P}$-path graph, to the $\mathcal{P}'$-path graph and to the M-induced graph.

Let f' be a map from T' to itself such that $\mathcal{P}' = [T', P', f']$ and (T', P', f') is a monotone model. Set $p'_i = \kappa(p_i)$ for $i = 1, 2, \dots, k$. The basic path $\{p'_1, p'_j\}$ of (T', P') with $j \in \{2, 3, \dots, k\}$ will be denoted by $p'_{1,j}$ and be called an *interior basic path of the $\mathcal{P}'$-path graph* (observe that $\{p'_i, p'_j\}$ with $i, j \in \{2, 3, \dots, k\}$ is not a basic path of $\mathcal{P}'$). Every other basic path of the $\mathcal{P}'$-path graph will be called an *exterior basic path*. The set of exterior basic paths of the $\mathcal{P}'$-path graph will be denoted by $\mathcal{E}'$.

By the construction of the matrix M there is a one-to-one correspondence between the rows of $M_P(\mathcal{P})$ and the rows of M. Thus, we can identify the basic path which determines a given row of the matrix $M_P(\mathcal{P})$ with the corresponding row index of the matrix M and, hence, with the vertices of the M-induced graph. In the rest of this section we will use this notation for the vertices of the M-induced graph (i.e. if a node i of the M-induced graph is identified with the basic path π_m of the $\mathcal{P}$-path graph, in what follows it will be denoted by π_m). In this setting the interior and exterior basic paths of the $\mathcal{P}$-path graph and the M-induced graph coincide. Moreover, the $\mathcal{P}$-path graph is a subgraph of the M-induced graph.

Observe that in the collapsing of the edge $\langle v, p_1 \rangle$ to obtain T' from T we did not modify any exterior basic path of the $\mathcal{P}$-path graph. Hence, the exterior basic paths of the $\mathcal{P}'$-path graph are in one-to-one correspondence with those of the $\mathcal{P}$-path graph which, in turn, are identified with the exterior basic paths of the M-induced graph. So, if $E \in \mathcal{E}$ we will denote by E' the corresponding element of $\mathcal{E}'$, and conversely. Also, the coverings between the elements of $\mathcal{E}$ in the $\mathcal{P}$-path graph and the M-induced graph and the coverings between the elements of $\mathcal{E}'$ in the $\mathcal{P}'$-path graph are in one-to-one correspondence because when constructing the pattern $\mathcal{P}'$ and the matrix M we only have modified the interior basic paths and the coverings of interior basic paths, respectively.

The next lemma describes all f-coverings between interior paths of the $\mathcal{P}$-path, the $\mathcal{P}'$-path and the M-induced graphs. It follows directly from the definitions.

Lemma 7.3. *With the previous notation the $\mathcal{P}$-path, the $\mathcal{P}'$-path and the M-induced graphs have exactly the following covers between interior basic paths.*

Subcase 1. *No interior basic path covers another interior basic path in the the $\mathcal{P}$-path, the $\mathcal{P}'$-path and the M-induced graphs.*

Subcase 2. *In the cases*

- $l \neq 1$ *we have $f(p_1) \in C_s$ and the $\mathcal{P}$-path and the M-induced graphs contain the arrows $p_{i,l} \longrightarrow p_{r,s}$ for all $i \neq l$. The $\mathcal{P}'$-path graph contains the arrows $p'_{1,l} \longrightarrow p'_{1,r}$ and $p'_{1,l} \longrightarrow p'_{1,s}$.*
- $l = 1$ *the $\mathcal{P}$-path graph contains the arrows $p_{i,1} \longrightarrow p_{r,s}$ for all $i \neq 1$; the M-induced graph contains the arrows $p_{i,1} \longrightarrow p_{r,s}$ for all $i \neq 1$ and $p_{i,j} \xrightarrow{1} p_{r,s}$ and $p_{i,j} \xrightarrow{2} p_{r,s}$ for all $1 < i < j \leq k$ and the $\mathcal{P}'$-path graph contains the arrows $p'_{i,1} \longrightarrow p'_{r,1}$ and $p'_{i,1} \longrightarrow p'_{s,1}$ for all $i \neq 1$.*

The proof of (8) will be split into two lemmas. Before stating and proving them we will recall the notion of a *concatenation* of paths. Let $\alpha = \alpha_0 \xrightarrow{l_0} \alpha_1 \xrightarrow{l_1} \cdots \xrightarrow{l_{n-2}} \alpha_{n-1} \xrightarrow{l_{n-1}} \alpha_n$ and $\beta = \beta_0 \xrightarrow{r_0} \beta_1 \xrightarrow{r_1} \cdots \xrightarrow{r_{m-2}} \beta_{m-1} \xrightarrow{r_{m-1}} \beta_m$ be two paths in an oriented generalized graph. If $\alpha_n = \beta_0$ then we can concatenate α and

β to get the path $\alpha_0 \xrightarrow{l_0} \alpha_1 \xrightarrow{l_1} \cdots \xrightarrow{l_{n-2}} \alpha_{n-1} \xrightarrow{l_{n-1}} \alpha_n \xrightarrow{r_0} \beta_1 \xrightarrow{r_1} \cdots \xrightarrow{r_{m-2}} \beta_{m-1} \xrightarrow{r_{m-1}} \beta_m$. Such a path will be denoted by $\alpha\beta$ and be called the *concatenation of* α *and* β.

Lemma 7.4. *With the above notations* $h(\mathcal{P}) < \log \sigma(M)$.

Proof. From Remark 7.2 it follows that $h(\mathcal{P}) = \log \sigma(M_P(\mathcal{P}))$. So, we have to show that $\sigma(M_P(\mathcal{P})) < \sigma(M)$. As previously stated, to compare the entries of these matrices we will use the interpretation in terms of loops of the corresponding graphs given by Lemma 3.1(a).

The proof of the lemma will be achieved through a series of 3 claims which follow.

Claim 1. Any pair of exterior basic paths of the $\mathcal{P}$-path graph (and, hence, of the M-induced graph) can be joined by a path in the M-induced graph. To prove the claim take $E, \widetilde{E} \in \mathcal{E}$ and let $E', \widetilde{E}' \in \mathcal{E}'$ denote the corresponding exterior basic paths in the $\mathcal{P}'$-path graph (recall that the elements of $\mathcal{E}$ are in one-to-one correspondence with those of $\mathcal{E}'$). Since $M_P(\mathcal{P}')$ is irreducible, it follows from Lemma 3.1(a,b) that there exists a path from E' to $\widetilde{E}'$ in the $\mathcal{P}'$-path graph. Among these paths, let us take one with a minimal length. If such a path does not contain interior basic paths, a corresponding path also exists in the M-induced graph (recall that the coverings between the elements of $\mathcal{E}'$ in the $\mathcal{P}'$-path graph and the coverings between the elements of $\mathcal{E}$ in the M-induced graph are in one-to-one correspondence). This ends the proof of the claim in this case.

Assume now that the above minimal path contains interior basic paths. Such a path can be written as a concatenation of paths $\gamma'_1\gamma'_2\cdots\gamma'_m$ such that each γ'_j begins and ends with an external basic path (γ'_1 begins with E' and γ'_m ends with $\widetilde{E}'$), contains interior basic paths and all interior basic paths contained in each γ'_j are consecutive.

We will show that for each $\gamma'_j = E'^j_1 \longrightarrow E'^j_2 \longrightarrow \cdots \longrightarrow E'^j_{n_j}$ there exists a path γ_j from E^j_1 to $E^j_{n_j}$ in the M-induced graph, where E^j_1 (respectively $E^j_{n_j}$) is the external basic path of the M-induced graph that corresponds to E'^j_1 (respectively $E'^j_{n_j}$). In particular, γ_1 begins with E and γ_m ends with $\widetilde{E}$. Thus, we can concatenate the paths γ_j to get a path $\gamma_1\gamma_2\cdots\gamma_m$ from E to $\widetilde{E}$ and the claim follows.

Now we will show that for each path $\gamma'_j = E'^j_1 \longrightarrow E'^j_2 \longrightarrow \cdots \longrightarrow E'^j_{n_j}$ with the above properties there exists a path γ_j from E^j_1 to $E^j_{n_j}$ in the M-induced graph. According to Lemma 7.3 and in view of the minimality of the length of the path $\gamma'_1\gamma'_2\cdots\gamma'_m$, each path γ'_j must be of one of the following forms with $i \in \{2, 3, \ldots, k\}$:

- $E'_1 \longrightarrow \cdots \longrightarrow E'_{m_1} \longrightarrow p'_{1,i} \longrightarrow \widetilde{E}'_1 \longrightarrow \cdots \longrightarrow \widetilde{E}'_{m_2}$, or
- $E'_1 \longrightarrow \cdots \longrightarrow E'_{m_1} \longrightarrow p'_{1,l} \longrightarrow p'_{1,r} \longrightarrow \widetilde{E}'_1 \longrightarrow \cdots \longrightarrow \widetilde{E}'_{m_2}$ (only for $l \neq 1$), or
- $E'_1 \longrightarrow \cdots \longrightarrow E'_{m_1} \longrightarrow p'_{1,l} \longrightarrow p'_{1,s} \longrightarrow \widetilde{E}'_1 \longrightarrow \cdots \longrightarrow \widetilde{E}'_{m_2}$ (only for $l \neq 1$), or
- $E'_1 \longrightarrow \cdots \longrightarrow E'_{m_1} \longrightarrow p'_{1,i} \longrightarrow p'_{1,r} \longrightarrow \widetilde{E}'_1 \longrightarrow \cdots \longrightarrow \widetilde{E}'_{m_2}$ (only for $l = 1$), or
- $E'_1 \longrightarrow \cdots \longrightarrow E'_{m_1} \longrightarrow p'_{1,i} \longrightarrow p'_{1,s} \longrightarrow \widetilde{E}'_1 \longrightarrow \cdots \longrightarrow \widetilde{E}'_{m_2}$ (only for $l = 1$),

where all basic paths forming the paths $E'_1 \longrightarrow \cdots \longrightarrow E'_{m_1}$ and $\widetilde{E}'_1 \longrightarrow \cdots \longrightarrow \widetilde{E}'_{m_2}$ are exterior basic paths.

When $\widetilde{E}'_1 \subset \kappa(\langle f(v), f(p_1)\rangle)$ it follows that $p'_{1,i} \longrightarrow \widetilde{E}'_1$ for every $i \in \{2, 3, \dots, k\}$. Hence, by the minimality of the path, we are in the first case of the above list. In that case, if E'_{m_1} does not f'-cover another interior path it follows that $E_1 \longrightarrow \cdots \longrightarrow E_{m_1} \longrightarrow p_{1,i} \longrightarrow \widetilde{E}_1 \longrightarrow \cdots \longrightarrow \widetilde{E}_{m_2}$ is a path in the M-induced graph from E_1 to $\widetilde{E}_{m_2}$. If E'_{m_1} also f'-covers $p'_{1,z}$ then the M-induced graph contains the path $E_1 \longrightarrow \cdots \longrightarrow E_{m_1} \longrightarrow p_{l,z} \longrightarrow \widetilde{E}_1 \longrightarrow \cdots \longrightarrow \widetilde{E}_{m_2}$ from E_1 to $\widetilde{E}_{m_2}$.

When $\widetilde{E}'_1 \subset \kappa(\langle f(v), f(p_r)\rangle)$, $\widetilde{E}'_1 \subset \kappa(\langle f(v), f(p_s)\rangle)$ or $\widetilde{E}'_1 \subset \kappa(\langle f(v), f(p_i)\rangle)$, an analysis similar to the one above shows that, in all the cases, the M-induced graph contains a path from E_1 to $\widetilde{E}_{m_2}$. This ends the proof of the claim. □

Now we start the process of comparing paths in the $\mathcal{P}$-path and the M-induced graphs.

We will say that a basic path π of the M-induced graph is *admissible* if there exists a path in the M-induced graph beginning at some exterior basic path and ending at π. Let Δ be the set of all admissible basic paths. By the claim, $\mathcal{E} \subset \Delta$.

Claim 2. The set Δ is *transitive* in the following sense: any two elements of Δ can be joined by a path in the M-induced graph using only elements of Δ. Now we prove this claim. Notice that it holds for any two exterior basic paths $E, \widetilde{E}$ since, in the previous claim, we have proved that there is a path in the M-induced graph joining E and $\widetilde{E}$ (the interior basic paths contained in this path all belong to Δ by definition).

Let $p_{i,j} \in \Delta$ with $i, j \in \{1, 2, \dots, k\}$, $i \neq j$ be an interior basic path and let $E \in \mathcal{E}$. We will prove that there exists a path from E to $p_{i,j}$ and a path from $p_{i,j}$ to E, both containing only elements from Δ.

Since $\mathcal{E} \subset \Delta$, there exists a path γ from some exterior basic path $\widetilde{E}$ to $p_{i,j}$ and a path α from E to $\widetilde{E}$, both paths containing only elements from Δ. The concatenation $\alpha\gamma$ gives a path beginning at E and ending at $p_{i,j}$ containing only elements from Δ.

To prove that there is a path from $p_{i,j}$ to E containing only elements from Δ, we note that there is at least one arrow from $p_{i,j}$ to some exterior basic path $\widehat{E}$ in the M-induced graph. Indeed, since at most one element from $f(\{p_1, p_2, \dots, p_k\})$ does not belong to C_s, either $f(p_i)$ or $f(p_j)$ belong to C_s. Assume for definiteness that $f(p_j) \in C_s$. Then, $\langle f(v), f(p_j)\rangle \subset \langle C_s\rangle$ and all basic paths contained in $\langle f(v), f(p_j)\rangle$ are exterior basic paths of the $\mathcal{P}$-path graph. On the other hand, $f(\langle p_{i,j}\rangle) = \langle f(p_i), f(p_j)\rangle \supset \langle f(v), f(p_j)\rangle$. Hence, there is an arrow from $p_{i,j}$ to some exterior basic path $\widehat{E} \subset \langle f(v), f(p_j)\rangle$ in the $\mathcal{P}$-path graph. Since the vertices of the $\mathcal{P}$-path graph and the M-induced graph are identified and the $\mathcal{P}$-path graph is a subgraph of the M-induced graph, there is an arrow from $p_{i,j}$ to $\widehat{E}$ in the M-induced graph.

Concatenating the path $p_{i,j} \longrightarrow \widehat{E}$ with a path from $\widehat{E}$ to E, we obtain a path beginning at $p_{i,j}$ and ending at E, containing only elements from Δ.

To end the proof of the claim take interior paths $\pi, \widetilde{\pi} \in \Delta$ and let $E \in \mathcal{E}$. From the part already proven, there exist paths γ_1 from π to E and γ_2 from E to $\widetilde{\pi}$, containing only elements from Δ. The path $\gamma_1\gamma_2$ joins π with $\widetilde{\pi}$ and contains only elements from Δ. This ends the proof of Claim 2. □

Claim 3. The path $p_{r,s}$ belongs to Δ. In Subcase 1, $p_{r,s}$ was defined as an interior basic path covered by some exterior basic path. So, $p_{r,s} \in \Delta$ by definition.

Now we consider Subcase 2 when $l \neq 1$. We recall that since $M_P(\mathcal{P}')$ is irreducible, the $\mathcal{P}'$-path graph contains a path from some exterior basic path E' to $p'_{1,l}$, by Lemma 3.1(a,b). Then, the M-induced graph contains a path from E to $p_{i,l}$ for some $i \in \{1, 2, \dots, k\}$. By Lemma 7.3 it follows that $p_{i,l} \longrightarrow p_{r,s}$ and hence, by concatenating the two loops, we obtain a loop in the M-induced graph from E to $p_{r,s}$. Thus, again, $p_{r,s} \in \Delta$ by definition.

A similar argument works in Subcase 2 when $l = 1$, and the claim holds. □

Now let Y denote the set of basic paths of the M-induced graph that do not belong to Δ. Clearly Y is disjoint from $\mathcal{E}$ and so, it only contains interior basic paths. The columns of the matrix M corresponding to the elements of Y are identically zero. To see it notice that, by definition, there are no coverings from any element of Δ to any element of Y. Also, there are no coverings from any element of Y to any element of Y. Indeed, all elements of Y are interior basic paths and, by Lemma 7.3, the only interior basic path covered by another interior basic path is $p_{r,s}$ which belongs to Δ.

Let $\widetilde{M}_P(\mathcal{P})$ and $\widetilde{M}$ be the matrices obtained respectively from $M_P(\mathcal{P})$ and M by deleting the rows and the columns corresponding to the elements of Y. Clearly $\sigma(\widetilde{M}_P(\mathcal{P})) = \sigma(M_P(\mathcal{P}))$ and $\sigma(\widetilde{M}) = \sigma(M)$. On the other hand, the matrix $\widetilde{M}$ is irreducible because we have proved that the set Δ is transitive (i.e. any two elements of Δ can be joined by a path in the M-induced graph using only elements of Δ — see Lemma 3.1(a,b)). Observe also that $\widetilde{M} \geq \widetilde{M}_P(\mathcal{P})$ by the definition of M. Since $p_{r,s} \notin Y$ it follows that both matrices $\widetilde{M}$ and $\widetilde{M}_P(\mathcal{P})$, have a row corresponding to $p_{r,s}$. Moreover, since $1 \notin \{r, s\}$, from the definition of the matrix M it follows that the entries of the row of $\widetilde{M}$ corresponding to $p_{r,s}$ are strictly greater than the corresponding entries of $\widetilde{M}_P(\mathcal{P})$. So, by Lemma 3.1(d),

$$\sigma(M_P(\mathcal{P})) = \sigma(\widetilde{M}_P(\mathcal{P})) < \sigma(\widetilde{M}) = \sigma(M).$$

□

Lemma 7.5. *With the above notations $h(\mathcal{P}') = \log \sigma(M)$.*

Proof. Since $\mathcal{P}'$ is maximal we get $h(\mathcal{P}') = \log \sigma(M_P(\mathcal{P}'))$ from Remark 7.2. Thus, we have to prove that $\sigma(M_P(\mathcal{P}')) = \sigma(M)$. To do this we will use the matrix $\widetilde{M}$ from the proof of the previous lemma. Hence we have to prove that $\sigma(M_P(\mathcal{P}')) = \sigma(\widetilde{M})$ because $\sigma(\widetilde{M}) = \sigma(M)$.

Fix an external basic path $E' \in \mathcal{E}'$ and the corresponding external path $E \in \mathcal{E}$. The set Y defined in the proof of the previous lemma is disjoint from $\mathcal{E}$. Thus, the $\widetilde{M}$-induced graph has a vertex associated to E. Since $M_P(\mathcal{P}')$ and $\widetilde{M}$ are irreducible, by Lemma 3.1(c) it is enough to show that, for every $k \in \mathbb{N}$, the number of loops of length k in the $\mathcal{P}'$-path graph starting and ending at E' coincides with the number of loops of length k in the $\widetilde{M}$-induced graph starting and ending at E.

Let $\mathcal{A}'$ be the set of loops of length k starting and ending at E' in the $\mathcal{P}'$-path graph and let $\widetilde{\mathcal{A}}$ be the set of loops of length k starting and ending at E in the M-induced graph. In $\mathcal{A}'$ we introduce the equivalence relation $\sim'$ as follows. Given two loops from $\mathcal{A}'$ we say that they are $\sim'$-equivalent if they have the same elements of $\mathcal{E}'$ in the same position. Analogously, two loops from $\widetilde{\mathcal{A}}$ are said to be $\sim$-equivalent if they have the same elements of $\mathcal{E}$ in the same position.

We claim that, given two exterior basic paths $E'_1, E'_2 \in \mathcal{E}'$ and $m \geq 2$, the number of paths of the $\mathcal{P}'$-path graph of the form $E'_1 \longrightarrow \pi'_1 \longrightarrow \pi'_2 \longrightarrow \cdots \longrightarrow \pi'_{m-1} \longrightarrow E'_2$, where $\pi'_1, \pi'_2, \ldots, \pi'_{m-1}$ are interior basic paths coincides with the number of paths of the M-induced graph of the form $E_1 \longrightarrow \pi_1 \longrightarrow \pi_2 \longrightarrow \cdots \longrightarrow \pi_{m-1} \longrightarrow E_2$, where $\pi_1, \pi_2, \ldots, \pi_{m-1}$ are interior basic paths.

Clearly, if the claim holds, there is a bijection $\Psi\colon \mathcal{A}'/{\sim'} \longrightarrow \widetilde{\mathcal{A}}/{\sim}$ and for every $[A] \in \mathcal{A}'/{\sim'}$ it follows that $[A]$ and $\Psi([A])$ have the same number of elements. Thus the cardinality of $\mathcal{A}'$ and $\widetilde{\mathcal{A}}$ coincides and, hence, the lemma follows.

The rest of the proof is devoted to prove the claim.

First we prove the claim in Subcase 1. By Lemma 7.3, all paths of the $\mathcal{P}'$-path graph verifying the assumptions of the claim are of length 2 and of the form $E_1' \longrightarrow p_{1,i}' \longrightarrow E_2'$ with $i \in \{2, 3, \dots, k\}$. From the definition of the model (T', P', f') it follows that $E_2 \subset \langle f(v), f(p_1)\rangle \cup \langle f(v), f(p_i)\rangle$.

If $p_{1,i}'$ is the unique basic interior path f'-covered by E_1' then there is a unique i such that $E_1' \longrightarrow p_{1,i}' \longrightarrow E_2'$ is a path from E_1' to E_2' in the $\mathcal{P}'$-path graph, and the only path from E_1 to E_2 in the $\mathcal{P}$-path graph and hence in the M-induced graph is $E_1 \longrightarrow p_{1,i} \longrightarrow E_2$ (recall that the matrices $M_P(\mathcal{P})$ and M only differ in rows corresponding to paths $p_{i,j}$ with $1 < i < j$). By the definition of the sets Δ and Y from the proof of the previous lemma it follows that $E_1, p_{1,i}, E_2 \notin Y$. Hence $E_1 \longrightarrow p_{1,i} \longrightarrow E_2$ is the only path from E_1 to E_2 in the $\widetilde{M}$-induced graph. Thus, the claim follows in this case.

Assume now that E_1' f'-covers $p_{1,i}'$ and $p_{1,j}'$ with $1 < i < j$. In this case we clearly have the arrow $E_1 \longrightarrow p_{i,j}$ in the M-induced graph. When $E_2 \subset \langle f(v), f(p_i)\rangle$, as in the previous case, there is a unique i such that $E_1' \longrightarrow p_{1,i}' \longrightarrow E_2'$ is a path from E_1' to E_2' in the $\mathcal{P}'$-path graph. Moreover, by the definition of the matrix M, the entries of $M_P(\mathcal{P})$ and M corresponding to the row associated to the basic path $p_{i,j}$ and to the column associated to E_2 coincide. The fact that $E_2 \subset \langle f(v), f(p_i)\rangle$ implies that $p_{i,j}$ f-covers E_2 and, hence, there is a unique arrow from $p_{i,j}$ to E_2 in the M-induced graph. Consequently, as above, $E_1 \longrightarrow p_{i,j} \longrightarrow E_2$ is the only path from E_1 to E_2 in the $\widetilde{M}$-induced graph and the claim follows as in the previous case. Assume now that $E_2 \subset \langle f(v), f(p_1)\rangle$. In this case there are two paths from E_1' to E_2' in the $\mathcal{P}'$-path graph: $E_1' \longrightarrow p_{1,i}' \longrightarrow E_2'$ and $E_1' \longrightarrow p_{1,j}' \longrightarrow E_2'$. On the other hand, again by the definition of the matrix M, the entry of M corresponding to the row associated to the basic path $p_{i,j}$ and to the column associated to E_2 is 2, meaning that there are two labelled arrows from $p_{i,j}$ to E_2 in the M-induced graph. Consequently, there are two paths from E_1 to E_2 in the $\widetilde{M}$-induced graph: $E_1 \longrightarrow p_{i,j} \xrightarrow{1} E_2$ and $E_1 \longrightarrow p_{i,j} \xrightarrow{2} E_2$. This ends the proof of the claim in Subcase 1.

Now we consider Subcase 2 with $l \notin \{1, r, s\}$. In this situation, again by Lemma 7.3, all paths of the $\mathcal{P}'$-path graph verifying the assumptions of the claim are of length 2 or 3. When $m = 2$ and there is a path $E_1' \longrightarrow p_{1,i}' \longrightarrow E_2'$ with $i \in \{2, 3, \dots, k\}$ from E_1' to E_2' in the $\mathcal{P}'$-path graph the claim holds by the same arguments as in Subcase 1.

Assume now that there is a path from E_1' to E_2' in the $\mathcal{P}'$-path graph of length 3. By Lemma 7.3 this path must be $E_1' \longrightarrow p_{1,l}' \longrightarrow p_{1,i}' \longrightarrow E_2'$ with $i \in \{r, s\}$. If $p_{1,l}'$ is the unique basic interior path f'-covered by E_1' then, by the definition of M and Y, we have the following path in the $\widetilde{M}$-induced graph $E_1 \longrightarrow p_{1,l} \longrightarrow p_{r,s}$. When $E_2 \subset \langle f(v), f(p_i)\rangle$, $E_1' \longrightarrow p_{1,l}' \longrightarrow p_{1,i}' \longrightarrow E_2'$ is the only path of length 3 from E_1' to E_2' in the $\mathcal{P}'$-path graph. On the other hand, by using similar arguments to those, there is a unique arrow from $p_{r,s}$ to E_2 in the M-induced graph. Consequently, $E_1 \longrightarrow p_{1,l} \longrightarrow p_{r,s} \longrightarrow E_2$ is the only path from E_1 to E_2 in the $\widetilde{M}$-induced graph and the claim follows. Assume now that $E_2 \subset \langle f(v), f(p_1)\rangle$. In this case there are two paths from E_1' to E_2' in the $\mathcal{P}'$-path graph: $E_1' \longrightarrow p_{1,l}' \longrightarrow p_{1,r}' \longrightarrow E_2'$ and $E_1' \longrightarrow p_{1,l}' \longrightarrow p_{1,s}' \longrightarrow E_2'$. Again, by the definitions of the matrices M and Y, there are also two paths from E_1 to E_2 in the $\widetilde{M}$-induced graph: $E_1 \longrightarrow p_{1,l} \longrightarrow p_{r,s} \xrightarrow{1} E_2$ and $E_1 \longrightarrow p_{1,l} \longrightarrow p_{r,s} \xrightarrow{2} E_2$. Thus the claim follows in Subcase 2, with $l \notin \{1, r, s\}$; the loops from E_1' to E_2' in the $\mathcal{P}'$-path graph having length 3, and $p_{1,l}'$ being the unique basic interior path f'-covered by E_1'.

Now we consider the case when E_1' f'-covers $p_{1,l}'$ and $p_{1,j}'$ with $1 < l, j$ and $l \neq j$. In this case we clearly have the path $E_1 \longrightarrow p_{i,j} \longrightarrow p_{r,s}$ in the $\widetilde{M}$-induced graph and the claim follows by using the same arguments as in the previous case.

Now we assume that we are in Subcase 2 with $l \in \{r, s\}$ (since $1 \notin \{r, s\}$ we automatically have $l \neq 1$). We only study the case $l = r$. The other case follows in a similar way.

By Lemma 7.3, in this case we have paths of the $\mathcal{P}'$-path graph verifying the assumptions of the claim of arbitrary length $m \geq 2$. All these paths are of the form either

(i') $\left(E_1' \longrightarrow p_{1,r}'\right)\left(p_{1,r}' \longrightarrow p_{1,r}'\right)^{m-2}\left(p_{1,r}' \longrightarrow E_2'\right) =$
$E_1' \longrightarrow p_{1,r}' \longrightarrow p_{1,r}' \longrightarrow \cdots \longrightarrow p_{1,r}' \longrightarrow E_2'$, or

(ii') $\left(E_1' \longrightarrow p_{1,r}'\right)\left(p_{1,r}' \longrightarrow p_{1,r}'\right)^{m-3}\left(p_{1,r}' \longrightarrow p_{1,s}'\right)\left(p_{1,s}' \longrightarrow E_2'\right) =$
$E_1' \longrightarrow p_{1,r}' \longrightarrow p_{1,r}' \longrightarrow \cdots \longrightarrow p_{1,r}' \longrightarrow p_{1,s}' \longrightarrow E_2'$.

From Lemma 7.3 and the definition of Y it follows that the $\widetilde{M}$-induced graph has the path

(i) $(E_1 \longrightarrow p_{1,r} \longrightarrow p_{r,s})\,(p_{r,s} \longrightarrow p_{r,s})^{m-3}$ when E_1' only f'-covers $p_{1,r}'$, and

(ii) $(E_1 \longrightarrow p_{r,j} \longrightarrow p_{r,s})\,(p_{r,s} \longrightarrow p_{r,s})^{m-3}$ when E_1' f'-covers $p_{1,r}'$ and $p_{1,j}'$ with $1 < r, j$ and $r \neq j$.

Now, from the definitions of M and Y as before, if $E_2 \subset \langle f(v), f(p_r)\rangle \cup \langle f(v), f(p_s)\rangle$ it follows that either (i') or (ii') is the unique path of length m from E_1' to E_2' in the $\mathcal{P}'$-path graph, and concatenating the paths (i) or (ii) with $(p_{r,s} \longrightarrow E_2)$ we obtain a unique path of length m from E_1 to E_2 in the $\widetilde{M}$-induced graph and the claim holds. When $E_2 \subset \langle f(v), f(p_1)\rangle$ then both paths of length m, (i') and (ii'), occur in the $\mathcal{P}'$-path graph and concatenating the paths (i) or (ii) with the labelled arrows $\left(p_{r,s} \xrightarrow{1} E_2\right)$ and $\left(p_{r,s} \xrightarrow{2} E_2\right)$ we also obtain two paths from E_1 to E_2 in the $\widetilde{M}$-induced graph. So, the claim holds also in this case.

Finally we consider Subcase 2 with $l = 1$. Again from Lemma 7.3 we have paths of the $\mathcal{P}'$-path graph verifying the assumptions of the claim of arbitrary length $m \geq 2$. These paths are of the form $E_1' \longrightarrow p_{1,i}' \longrightarrow p_{1,l_2}' \longrightarrow p_{1,l_3}' \longrightarrow \cdots \longrightarrow p_{1,l_{m-1}}' \longrightarrow E_2'$, where $i \in \{2, 3, \ldots, k\}$, $l_j \in \{r, s\}$ for $j = 2, 3, \ldots, m-1$ and $E_2 \subset \langle f(v), f(p_r)\rangle \cup \langle f(v), f(p_s)\rangle \cup \langle f(v), f(p_1)\rangle$. Moreover, $l_{m-1} = r$ (respectively $l_{m-1} = s$) if $E_2 \subset \langle f(v), f(p_r)\rangle$ (respectively $E_2 \subset \langle f(v), f(p_s)\rangle$).

If $p_{1,i}'$ is the only path f'-covered by E_1', there are 2^{m-2} (respectively 2^{m-3}) paths of length m from E_1' to E_2' in the $\mathcal{P}'$-path graph when $E_2 \subset \langle f(v), f(p_1)\rangle$ (respectively $E_2 \subset \langle f(v), f(p_r)\rangle \cup \langle f(v), f(p_s)\rangle$). Moreover, by the definitions of M and Y (and Lemma 7.3), we have the following paths of length m from E_1 to E_2 in the $\widetilde{M}$-induced graph $E_1 \longrightarrow p_{1,i} \longrightarrow p_{r,s} \xrightarrow{s_3} p_{r,s} \xrightarrow{s_4} \ldots \xrightarrow{s_{m-1}} p_{r,s} \xrightarrow{s_m} E_2$, where $s_j \in \{1, 2\}$ for $j = 3, 4, \ldots, m-1$ and $s_m \in \{1, 2\}$ (respectively $s_m = 1$) when $E_2 \subset \langle f(v), f(p_1)\rangle$ (respectively $E_2 \subset \langle f(v), f(p_r)\rangle \cup \langle f(v), f(p_s)\rangle$). Consequently, there are 2^{m-2} (respectively 2^{m-3}) paths of length m from E_1 to E_2 in the $\widetilde{M}$-induced graph when $E_2 \subset \langle f(v), f(p_1)\rangle$ (respectively $E_2 \subset \langle f(v), f(p_r)\rangle \cup \langle f(v), f(p_s)\rangle$) and the claim holds.

The case when E_1' f'-covers $p_{1,i}'$ and $p_{1,j}'$ with $1 < i < j$ follows from a similar analysis. □

From Lemmas 7.4 and 7.5 it follows that inequality (8) is true and, hence, Theorem 7.1 is true in the case $f(v) \neq v$.

Case $f(v) = v$.

Since f is $P \cup \{v\}$-monotone it is locally injective at v. This means that f induces a permutation $\tau\colon \{1,2,\dots,k\} \longrightarrow \{1,2,\dots,k\}$ defined by: $f(p_i) \in C_{\tau(i)}$ for $i = 1,2,\dots,k$. We denote by B_i the set of all basic paths contained in the path $\langle p_{\tau(i)}, f(p_i)\rangle$. Observe that from the definition of τ, $p_{\tau(i)}, f(p_i) \in C_{\tau(i)}$. Hence, either $f(p_i) = p_{\tau(i)}$ and $B_i = \emptyset$ or all elements of B_i are exterior basic paths contained in $C_{\tau(i)}$.

Set

$$N := \{i \in \{1,2,\dots,k\} : B_i \neq \emptyset\} = \{i \in \{1,2,\dots,k\} : f(p_i) \neq p_{\tau(i)}\}.$$

Observe that $N \neq \emptyset$. Otherwise, $f(p_i) = p_{\tau(i)} \in \{p_1,p_2,\dots,p_k\}$ for every $p_i \in \{p_1,p_2,\dots,p_k\}$. So, since $|P| > k$, $\{p_1,p_2,\dots,p_k\}$ is a proper f- invariant subset of P; a contradiction.

We claim that either there exists $j \in N$ such that $\tau(j) \neq j$ (that is, $f(p_j) \neq p_{\tau(j)} \neq p_j$) or $\tau = \mathrm{Id}$, where Id denotes the identity function, and $N = \{1,2,\dots,k\}$ (that is, $p_i \neq f(p_i) \in C_i$ for $i = 1,2,\dots,k$). To prove the claim observe that either there is $j \in N$ such that $\tau(j) \neq j$ or $\tau\big|_N = \mathrm{Id}\big|_N$. We have to show that $\tau\big|_N = \mathrm{Id}\big|_N$ implies $N = \{1,2,\dots,k\}$. Assume that $\tau\big|_N = \mathrm{Id}\big|_N$. Since τ is a permutation and $\tau(N) = N$ it follows that $\tau(\{1,2,\dots,k\} \setminus N) \subset \{1,2,\dots,k\} \setminus N$. Set $\widetilde{N} := \{p_i : i \in \{1,2,\dots,k\} \setminus N\}$. Since $\widetilde{N} \subset \{p_1,p_2,\dots,p_k\} \subset P$ and $|P| > k$, $\widetilde{N}$ is a proper subset of P. Combining these results we have $f(p_i) = p_{\tau(i)} \in \widetilde{N}$ for every $p_i \in \widetilde{N}$. Consequently, $\widetilde{N}$ is f- invariant; a contradiction. This ends the proof of the claim.

Assume that $\tau = \mathrm{Id}$ and $N = \{1,2,\dots,k\}$. This implies that $p_2 \neq f(p_2) \in C_2$. Consequently, since C_2 cannot be f-invariant, there exists an exterior basic path in C_2 f-covering some interior basic path of the form $p_{2,j}$ with $2 \neq j$.

From the claim and the above observation, by relabelling the points p_i if necessary, we may assume that either $\tau(1) \neq 1 \in N$ or $\tau = \mathrm{Id}$ and there exist $w_1, w_2 \in \{2,3,\dots,k\}$, $w_1 \neq w_2$ such that some exterior basic path in C_{w_1} f-covers the interior basic path p_{w_1,w_2}.

We consider the pattern obtained from $\mathcal{P}$ by collapsing the edge $\langle v, p_1\rangle$ to a point. In this way we obtain a pattern $\mathcal{P}' = \big([T',P'],[\kappa \circ f \circ \kappa^{-1}\big|_{P'}]\big)$ (where κ is the standard projection from T to the tree T' obtained by collapsing the edge $\langle v, p_1\rangle$ of T and $P' = \kappa(P)$). As in the Case $f(v) \neq v$, $\mathcal{P}'$ is maximal, simplicial and $M_P(\mathcal{P}')$ is irreducible.

We will prove that $h(\mathcal{P}') > h(\mathcal{P})$. As in the case $f(v) \neq v$ this will be done with the help of two intermediate matrices (each of them "closer" to the matrix $M_P(\mathcal{P}')$) in a sequence of three lemmas. To this end we need to extend the notions of interior and exterior basic path, defined for the $\mathcal{P}$-path graph, to the $\mathcal{P}'$-path graph as we did in the Case $f(v) \neq v$.

Let f' be a map from T' to itself such that $\mathcal{P}' = [T',P',f']$ and (T',P',f') is a monotone model. Set $p'_i = \kappa(p_i)$ and $C'_i = \kappa(C_i)$ for $i = 1,2,\dots,k$. The basic path $\{p'_1, p'_j\}$ of (T',P') with $j \in \{2,3,\dots,k\}$ will be denoted by $p'_{1,j}$ (or $p'_{j,1}$) and called an *interior basic path of the $\mathcal{P}'$-path graph.* Every other basic path of the $\mathcal{P}'$-path graph will be called an *exterior basic path.* The set of exterior basic paths of the $\mathcal{P}'$-path graph will be denoted by $\mathcal{E}'$.

Now we define the first one of the intermediate matrices. To do it observe that, for any $i,j \in \{1,\dots,k\}$, $i \neq j$,

$$p_{i,j} \text{ } f\text{-covers } p_{u,v} \text{ if and only if } \{u,v\} = \{\tau(i),\tau(j)\}. \tag{9}$$

This means that the matrix $M_P(\mathcal{P})$ has a 1 in every entry that is in a row corresponding to an interior basic path $p_{i,j}$ with $i \neq j$ and in the column corresponding to $p_{\tau(i),\tau(j)}$. Every other entry in the same row and in the column corresponding to an interior basic path different from $p_{\tau(i),\tau(j)}$ has a 0. In particular, if we take i, j so that $1 \notin \{i, j\}$, the matrix $M_P(\mathcal{P})$ has a 0 in the columns corresponding to the paths $p_{\tau(1),\tau(i)}$ and $p_{\tau(1),\tau(j)}$ (in the row corresponding to $p_{i,j}$).

Let M be the matrix obtained from $M_P(\mathcal{P})$ by modifying the rows corresponding to the interior basic paths $p_{i,j}$ with $1 < i < j \leq k$ in the following way. We substitute the 1 appearing in the column corresponding to $p_{\tau(i),\tau(j)}$ by a 0 and we substitute the 0's in the columns corresponding to $p_{\tau(1),\tau(i)}$ and $p_{\tau(1),\tau(j)}$ by a 1.

The definition of the matrix M is motivated as follows. Let $p'_{1,i}$ (with $i \neq 1$) be an interior basic path of the $\mathcal{P}'$-path graph. By the definition of τ and (T', P', f'), $f'(p'_1) \in C'_{\tau(1)}$ and $f'(p'_i) \in C'_{\tau(i)}$. By definition $p'_{1,i}$ f'-covers $p'_{1,j}$ if and only if $p'_{1,j} \subset \langle f'(p'_1), f'(p'_j)\rangle$, which is equivalent to $j \in \{\tau(1), \tau(i)\}$. So the definition of M is natural taking into account that, as previously stated, we want that M is "closer" to $M_P(\mathcal{P}')$ than $M_P(\mathcal{P})$.

The relation between the entropy of $\mathcal{P}$ and the spectral radius of M is given by the following

Lemma 7.6. *With the above notations* $\log \sigma(M) \geq h(\mathcal{P})$.

The proof of Lemma 7.6 is based in the comparison of loops of the $\mathcal{P}$-path graph and the M-induced graph. To this end we need some more notation and two technical results. In what follows we identify the vertices of the M-induced graph with the vertices of the $\mathcal{P}$-path graph. In particular the notions of *interior* and *exterior* basic paths are extended to the M-induced graph.

We will denote by $\mathcal{B}^{\mathcal{E}}$ the set of paths $\alpha_0 \longrightarrow \alpha_1 \longrightarrow \cdots \longrightarrow \alpha_m$ in the $\mathcal{P}$-path graph such that $m \geq 2$, $\alpha_0, \alpha_m \in \mathcal{E}$ and $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$ are interior basic paths. In a similar way, $\mathcal{B}^{\mathcal{E}}_M$ will denote the set of paths $\beta_0 \longrightarrow \beta_1 \longrightarrow \cdots \longrightarrow \beta_m$ in the M-induced graph such that $m \geq 2$, $\beta_0, \beta_m \in \mathcal{E}$ and $\beta_1, \beta_2, \ldots, \beta_{m-1}$ are interior basic paths of the $\mathcal{P}$-path graph.

Lemma 7.7. *There exists an injective length preserving map* $\psi\colon \mathcal{B}^{\mathcal{E}} \longrightarrow \mathcal{B}^{\mathcal{E}}_M$ *such that the initial and final vertices of* $\psi(\alpha)$ *coincide with those of* α *for every* $\alpha \in \mathcal{B}^{\mathcal{E}}$.

Proof. Let $\alpha \in \mathcal{B}^{\mathcal{E}}$ be a loop of length $q \geq 2$. From (9) it follows that α is of the form

$$\alpha_0 \longrightarrow p_{a,b} \longrightarrow p_{\tau(a),\tau(b)} \longrightarrow \cdots \longrightarrow p_{\tau^{q-2}(a),\tau^{q-2}(b)} \longrightarrow \alpha_q.$$

If $q = 2$ then α is also a path in the M-induced graph and we set $\psi(\alpha) = \alpha$. Thus, in the rest of the proof we assume $q \geq 3$.

Note that $p_{\tau^{q-2}(a),\tau^{q-2}(b)}$ only f-covers exterior basic paths in $B_{\tau^{q-2}(a)} \cup B_{\tau^{q-2}(b)}$. So, either $\alpha_q \in B_{\tau^{q-2}(a)}$ or $\alpha_q \in B_{\tau^{q-2}(b)}$. Assume for definiteness that $\alpha_q \in B_{\tau^{q-2}(a)}$.

Set $t_0 = b$ and observe that $\alpha_0 \longrightarrow p_{a,t_0}$ is a path of length 1 in the M-induced graph with $t_0 \neq a = \tau^0(a)$.

We will inductively show that if $0 \leq i < q-2$ and

$$\alpha_0 \longrightarrow p_{a,t_0} \longrightarrow p_{\tau(a),t_1} \longrightarrow p_{\tau^2(a),t_2} \longrightarrow \cdots \longrightarrow p_{\tau^i(a),t_i}$$

is a path in the M-induced graph where $t_j \in \{1, 2, \ldots, k\} \setminus \{\tau^j(a)\}$ for $j = 0, 1, \ldots, i$, then there exists $t_{i+1} \in \{1, 2, \ldots, k\} \setminus \{\tau^{i+1}(a)\}$ such that

$$\alpha_0 \longrightarrow p_{a,t_0} \longrightarrow p_{\tau(a),t_1} \longrightarrow p_{\tau^2(a),t_2} \longrightarrow \cdots \longrightarrow p_{\tau^i(a),t_i} \longrightarrow p_{\tau^{i+1}(a),t_{i+1}}$$

is a path of length $i+2$ in the M-induced graph. If $1 \notin \{\tau^i(a), t_i\}$ then, by the definition of M,

$$\alpha_0 \longrightarrow p_{a,t_0} \longrightarrow p_{\tau(a),t_1} \longrightarrow p_{\tau^2(a),t_2} \longrightarrow \cdots \longrightarrow p_{\tau^i(a),t_i} \longrightarrow p_{\tau^{i+1}(a),\tau(1)}$$

is a path in the M-induced graph. Since $\tau^i(a) \neq 1$ and τ is a permutation it follows that $\tau^{i+1}(a) \neq \tau(1)$. Thus, the claim holds in this case by taking $t_{i+1} = \tau(1)$.

Assume that $1 \in \{\tau^i(a), t_i\}$. Then,

$$\alpha_0 \longrightarrow p_{a,t_0} \longrightarrow p_{\tau(a),t_1} \longrightarrow p_{\tau^2(a),t_2} \longrightarrow \cdots \longrightarrow p_{\tau^i(a),t_i} \longrightarrow p_{\tau^{i+1}(a),\tau(t_i)}$$

is a path in the M-induced graph. Since $\tau^i(a) \neq t_i$ and τ is a permutation it follows that $\tau^{i+1}(a) \neq \tau(t_i)$ and the claim follows by taking $t_{i+1} = \tau(t_i)$.

By the claim it follows that

$$\alpha_0 \longrightarrow p_{a,t_0} \longrightarrow p_{\tau(a),t_1} \longrightarrow p_{\tau^2(a),t_2} \longrightarrow \cdots \longrightarrow p_{\tau^{q-2}(a),t_{q-2}}$$

is a path in $\mathcal{B}_M^{\mathcal{E}}$. Thus, since $\alpha_q \in B_{\tau^{q-2}(a)}$,

$$\psi(\alpha) := \alpha_0 \longrightarrow p_{a,t_0} \longrightarrow p_{\tau(a),t_1} \longrightarrow p_{\tau^2(a),t_2} \longrightarrow \cdots \longrightarrow p_{\tau^{q-2}(a),t_{q-2}} \longrightarrow \alpha_q$$

is a path in $\mathcal{B}_M^{\mathcal{E}}$ of length q. This completes the definition of the map ψ.

Notice that α is uniquely determined by its length, the arrow $\alpha_0 \longrightarrow p_{i,j}$ and α_q. By the above construction this also determines uniquely $\psi(\alpha)$. So, ψ is injective. □

Given a loop $\alpha = \alpha_0 \longrightarrow \alpha_1 \longrightarrow \cdots \longrightarrow \alpha_m \longrightarrow \alpha_0$ in a combinatorial oriented generalized graph we define the *shift of* α, denoted by $S(\alpha)$, as the loop $\alpha_1 \longrightarrow \alpha_2 \longrightarrow \cdots \longrightarrow \alpha_m \longrightarrow \alpha_0 \longrightarrow \alpha_1$. For $\ell \geq 0$ we will also denote by S^ℓ the ℓ-fold shift. This means that $S^0(\alpha) = \alpha$ and if $j \equiv \ell \pmod{m}$ then $S^\ell(\alpha) = \alpha_j \longrightarrow \alpha_{j+1} \longrightarrow \cdots \longrightarrow \alpha_{m-1} \longrightarrow \alpha_0 \longrightarrow \alpha_1 \longrightarrow \cdots \longrightarrow \alpha_{j-1} \longrightarrow \alpha_j$.

We also denote by $\mathcal{EL}^m$ the set of all loops of length m in the $\mathcal{P}$-path graph that contain some exterior basic path and by $\mathcal{L}_M^m$ the set of all loops of length m in the M-induced graph.

Lemma 7.8. *There exists an injective map* $\phi\colon \mathcal{EL}^m \longrightarrow \mathcal{L}_M^m$.

Proof. We define the map ϕ with the help of the map ψ obtained in Lemma 7.7. To this end we introduce the following notation.

We will denote by $\mathcal{EP}$ the set of all paths in the $\mathcal{P}$-path graph that do not contain interior basic paths. Also, given a path $\alpha \in \mathcal{EL}^m$ we will denote by $i(\alpha)$ the number of consecutive interior basic paths at the beginning of α. That is, if $\alpha = \alpha_0 \longrightarrow \alpha_1 \longrightarrow \cdots \longrightarrow \alpha_{m-1} \longrightarrow \alpha_0$, then $i(\alpha)$ is defined so that $\alpha_{i(\alpha)} \in \mathcal{E}$ but $\alpha_0, \alpha_1, \ldots, \alpha_{i(\alpha)-1}$ are interior basic paths.

Now we define $\phi(\alpha)$ for $\alpha \in \mathcal{EL}^m$. We start with the case $i(\alpha) = 0$. If $\alpha \in \mathcal{EP}$ then, since the entries of the matrix M corresponding to rows and columns associated with exterior basic paths coincide with those of $M_P(\mathcal{P})$, it follows that α is also a path in the M-induced graph. So, we set $\phi(\alpha) = \alpha$.

Assume now that $\alpha \notin \mathcal{EP}$ (and $i(\alpha) = 0$). Then α can be written in concatenation form as $\alpha = \underline{\alpha}_1\underline{\beta}_1\underline{\alpha}_2\underline{\beta}_2 \cdots \underline{\alpha}_n\underline{\beta}_n\underline{\alpha}_{n+1}$ where $n \geq 1$, $\underline{\beta}_1, \underline{\beta}_2, \ldots, \underline{\beta}_n \in \mathcal{B}^{\mathcal{E}}$ and $\underline{\alpha}_1, \underline{\alpha}_2, \ldots, \underline{\alpha}_n, \underline{\alpha}_{n+1}$ are either empty paths or paths from $\mathcal{EP}$. We set

$$\phi(\alpha) := \underline{\alpha}_1\psi(\underline{\beta}_1)\underline{\alpha}_2\psi(\underline{\beta}_2) \cdots \underline{\alpha}_n\psi(\underline{\beta}_n)\underline{\alpha}_{n+1}.$$

Clearly $\phi(\alpha) \in \mathcal{L}_M^m$ since ψ is length preserving and, for every i, $\psi(\underline{\beta}_i)$ has the same initial and final vertices as $\underline{\beta}_i$.

When $i(\alpha) \neq 0$ then $\alpha \notin \mathcal{EP}$ and we have $i\left(S^{i(\alpha)}(\alpha)\right) = 0$. Hence $\phi\left(S^{i(\alpha)}(\alpha)\right)$ is already defined. We set

$$\phi(\alpha) := S^{m-i(\alpha)}\left(\phi\left(S^{i(\alpha)}(\alpha)\right)\right).$$

Clearly ϕ is a well defined map from $\mathcal{EL}^m$ to $\mathcal{L}_M^m$. To end the proof of the lemma we have to see that ϕ is injective. Observe that $\phi(\alpha) = S^{m-i(\alpha)}\left(\phi\left(S^{i(\alpha)}(\alpha)\right)\right)$ for every $\alpha \in \mathcal{EL}^m$ (independently on the fact that $i(\alpha)$ is positive or zero). Moreover, α starts with an exterior basic path if and only if $\phi(\alpha)$ also starts with an exterior basic path.

We claim that

$$\phi\left(S(\alpha)\right) = S\left(\phi(\alpha)\right).$$

To prove the claim we start with the case $i(\alpha) > 0$. Clearly, $i(S(\alpha)) = i(\alpha) - 1 \geq 0$ and, hence,

$$\begin{aligned} S\left(\phi(\alpha)\right) &= S\left(S^{m-i(\alpha)}\left(\phi\left(S^{i(\alpha)}(\alpha)\right)\right)\right) = S^{m-i(S(\alpha))}\left(\phi\left(S^{i(\alpha)-1}(S(\alpha))\right)\right) \\ &= \phi(S(\alpha)). \end{aligned}$$

Now we assume that $i(\alpha) = 0$. If $i(S(\alpha)) = \ell > 0$, then, in a similar way as before, $\alpha = \underline{\beta}_1\underline{\alpha}_2\underline{\beta}_2 \cdots \underline{\alpha}_n\underline{\beta}_n\underline{\alpha}_{n+1}$ where $n \geq 1$, $\underline{\beta}_1, \underline{\beta}_2, \dots, \underline{\beta}_n \in \mathcal{B}^{\mathcal{E}}$ and $\underline{\alpha}_2, \dots, \underline{\alpha}_n, \underline{\alpha}_{n+1}$ are either empty paths or paths from $\mathcal{EP}$. Clearly, $S^{\ell+1}(\alpha) = \underline{\alpha}_2\underline{\beta}_2 \cdots \underline{\alpha}_n\underline{\beta}_n\underline{\alpha}_{n+1}\underline{\beta}_1$ and $i\left(S^{\ell+1}(\alpha)\right) = 0$. Hence, from the definition of ϕ for the case $i(\alpha) = 0$,

$$\begin{aligned} \phi\left(S^{\ell+1}(\alpha)\right) &= \phi\left(\underline{\alpha}_2\underline{\beta}_2 \cdots \underline{\alpha}_n\underline{\beta}_n\underline{\alpha}_{n+1}\underline{\beta}_1\right) = \underline{\alpha}_2\psi(\underline{\beta}_2) \cdots \underline{\alpha}_n\psi(\underline{\beta}_n)\underline{\alpha}_{n+1}\psi(\underline{\beta}_1) \\ &= S^{\ell+1}\left(\psi(\underline{\beta}_1)\underline{\alpha}_2\psi(\underline{\beta}_2) \cdots \underline{\alpha}_n\psi(\underline{\beta}_n)\underline{\alpha}_{n+1}\right) = S^{\ell+1}\left(\phi(\alpha)\right) \end{aligned}$$

and thus,

$$\phi\left(S(\alpha)\right) = S^{m-\ell}\left(\phi\left(S^{\ell}(S(\alpha))\right)\right) = S^{m-\ell}\left(S^{\ell+1}\left(\phi(\alpha)\right)\right) = S\left(\phi(\alpha)\right).$$

Finally, if $i(\alpha) = i(S(\alpha)) = 0$, with the same notation as before we can write

$$\alpha = (\alpha_0 \longrightarrow \alpha_1)\underline{\alpha}_1\underline{\beta}_1\underline{\alpha}_2\underline{\beta}_2 \cdots \underline{\alpha}_n\underline{\beta}_n\underline{\alpha}_{n+1}.$$

Hence,

$$\begin{aligned} \phi\left(S(\alpha)\right) &= \phi\left(\underline{\alpha}_1\underline{\beta}_1\underline{\alpha}_2\underline{\beta}_2 \cdots \underline{\alpha}_n\underline{\beta}_n\underline{\alpha}_{n+1}(\alpha_0 \longrightarrow \alpha_1)\right) \\ &= \underline{\alpha}_1\psi(\underline{\beta}_1)\underline{\alpha}_2\psi(\underline{\beta}_2) \cdots \underline{\alpha}_n\psi(\underline{\beta}_n)\underline{\alpha}_{n+1}(\alpha_0 \longrightarrow \alpha_1) \text{ and} \\ \phi(\alpha) &= (\alpha_0 \longrightarrow \alpha_1)\underline{\alpha}_1\psi(\underline{\beta}_1)\underline{\alpha}_2\psi(\underline{\beta}_2) \cdots \underline{\alpha}_n\psi(\underline{\beta}_n)\underline{\alpha}_{n+1}. \end{aligned}$$

Consequently, $\phi\left(S(\alpha)\right) = S\left(\phi(\alpha)\right)$. This ends the proof of the claim.

To end the proof of the lemma we have to show that if $\alpha, \beta \in \mathcal{EL}^m$, $\alpha \neq \beta$ then $\phi(\alpha) \neq \phi(\beta)$. The statement clearly holds when one of α or β belongs to $\mathcal{EP}$ since $\phi\big|_{\mathcal{EP}}$ is the identity and $\phi(\gamma)$ contains interior basic paths of the $\mathcal{P}$-path graph whenever $\gamma \notin \mathcal{EP}$ (Lemma 7.7). So, we may assume that $\alpha, \beta \notin \mathcal{EP}$.

We assume for definiteness that $i(\alpha) \geq i(\beta)$. By way of contradiction we also assume that $\phi(\alpha) = \phi(\beta)$. Then, by the iterative use of the claim, $\phi\left(S^{i(\beta)}(\alpha)\right) = \phi\left(S^{i(\beta)}(\beta)\right)$. On the other hand, $S^{i(\beta)}(\beta) = 0$ and so, $S^{i(\beta)}(\beta)$ starts with an exterior basic path. Hence, $\phi\left(S^{i(\beta)}(\beta)\right) = \phi\left(S^{i(\beta)}(\alpha)\right)$ starts with an exterior basic path which implies that $S^{i(\beta)}(\alpha)$ also starts with an exterior basic path. Consequently, $i(S^{i(\beta)}(\alpha)) = i(S^{i(\beta)}(\beta)) = 0$. By the definition of ϕ and the injectivity of ψ (Lemma 7.7), $\phi\left(S^{i(\beta)}(\alpha)\right) = \phi\left(S^{i(\beta)}(\beta)\right)$ implies that $S^{i(\beta)}(\alpha) = S^{i(\beta)}(\beta)$; a contradiction. □

Now we are ready to prove Lemma 7.6.

Proof of Lemma 7.6. The proof is based on the comparison of the number of loops of length m in the M-induced graph and in the $\mathcal{P}$-path graph.

First we claim that the number of loops of a fixed length m in the $\mathcal{P}$-path graph that contain only interior basic paths is bounded above by $\frac{k(k-1)}{2}$ (recall that k is the cardinality of C_v). Let $\alpha = p_{i,j} \longrightarrow \alpha_1 \longrightarrow \alpha_2 \longrightarrow \cdots \longrightarrow \alpha_{m-1} \longrightarrow p_{i,j}$ be one of these loops. By (9), α is of the form $p_{i,j} \longrightarrow p_{\tau(i),\tau(j)} \longrightarrow p_{\tau^2(i),\tau^2(j)} \longrightarrow \cdots \longrightarrow p_{\tau^{m-1}(i),\tau^{m-1}(j)} \longrightarrow p_{i,j}$ and hence it is uniquely determined by $p_{i,j}$. Thus, the number of loops of fixed length m in the $\mathcal{P}$-path graph that contain only interior basic paths is bounded above by the number of interior basic paths which is $\frac{k(k-1)}{2}$ since $k = |C_v|$. This ends the proof of the claim.

From Lemma 3.1(a) it follows that, for every $m \in \mathbb{N}$, $\operatorname{tr}(M^m) = |\mathcal{L}_M^m|$ and $\operatorname{tr}(M_P(\mathcal{P})^m)$ is the number of loops of length m in the $\mathcal{P}$-path graph. The set of such loops is, by definition, $\mathcal{EL}^m$ together with the set of loops of length m in the $\mathcal{P}$-path graph that contain only interior basic paths. So, by the the claim and Lemma 7.8,

$$\begin{aligned}\operatorname{tr}(M_P(\mathcal{P})^m) &\leq |\mathcal{EL}^m| + \frac{k(k-1)}{2} = |\phi(\mathcal{EL}^m)| + \frac{k(k-1)}{2} \\ &\leq |\mathcal{L}_M^m| + \frac{k(k-1)}{2} = \operatorname{tr}(M^m) + \frac{k(k-1)}{2}.\end{aligned}$$

Consequently, by [5, Lemma 4.4.2],

$$\begin{aligned}\sigma(M) &= \limsup_{m\to\infty} \sqrt[m]{\operatorname{tr}(M^m)} = \limsup_{m\to\infty} \sqrt[m]{\operatorname{tr}(M^m) + \frac{k(k-1)}{2}} \\ &\geq \limsup_{m\to\infty} \sqrt[m]{\operatorname{tr}(M_P(\mathcal{P})^m)} = \sigma(M_P(\mathcal{P})),\end{aligned}$$

and the lemma follows from Remark 7.2. □

Now we consider the matrix $\widetilde{M}$ obtained from M in the following way. For any $1 < i < j \leq k$ we modify the entries corresponding to the row $p_{i,j}$ and to any column corresponding to an exterior basic path contained in B_1 by putting a 2 in each of these entries. As we did for the matrix M we identify the vertices of the $\widetilde{M}$-induced graph with the vertices of the M-induced graph (and, hence, with those of the $\mathcal{P}$-path graph). In particular the notions of *interior* and *exterior* basic paths are extended to the $\widetilde{M}$-induced graph.

Observe that, by definition, the coverings between the elements of $\mathcal{E}'$ in the $\mathcal{P}'$-path graph and the coverings between the elements of $\mathcal{E}$ in the M-induced and the $\widetilde{M}$-induced graphs are in one-to-one correspondence. Moreover, in view of the definition of τ, it follows that

(10) for every interior path $p_{i,j}$, there exists an exterior basic path $E \in B_1$ such that $p_{i,j} \longrightarrow E$ is an arrow in the $\widetilde{M}$-induced graph.

In the next two lemmas we will prove

$$\log \sigma(M) < \log \sigma(\widetilde{M}) = h(\mathcal{P}').$$

This, together with Lemma 7.6, will give Theorem 7.1 in the case $f(v) = v$.

Lemma 7.9. *With the above notations $\sigma(M) < \sigma(\widetilde{M})$.*

Proof. The proof follows the same approach as in Lemma 7.4.

First of all we claim that any pair of exterior basic paths of the $\mathcal{P}$-path graph (and, hence, of the $\widetilde{M}$-induced graph) can be joined by a path in the $\widetilde{M}$-induced graph. To prove the claim take $E, \widehat{E} \in \mathcal{E}$. Since $M_P(\mathcal{P}')$ is irreducible, it follows from Lemma 3.1(a,b) that there exists a path from E' to $\widehat{E}'$ in the $\mathcal{P}'$-path graph where $E', \widehat{E}'$ denote the elements of $\mathcal{E}'$ corresponding to E and $\widehat{E}$, respectively. If such a path does not contain interior basic paths, by the definition of $\widetilde{M}$, a corresponding path from E to $\widehat{E}$ also exists in the $\widetilde{M}$-induced graph. This ends the proof of the claim in this case.

Assume now that the above minimal path contains interior basic paths. Such a path can be written as a concatenation of paths $\gamma_1'\gamma_2'\cdots\gamma_m'$ such that each γ_i' begins and ends with an external basic path (γ_1' begins with E' and γ_m' ends with

$\widehat{E}'$), contains interior basic paths and all interior basic paths contained in each γ_i' are consecutive.

We will show that for each $\gamma_i' = E_1'^i \longrightarrow E_2'^i \longrightarrow \cdots \longrightarrow E_{n_i}'^i$ there exists a path γ_i from E_1^i to $E_{n_i}^i$ in the $\widetilde{M}$-induced graph, where E_1^i (respectively $E_{n_i}^i$) is the external basic path of the $\widetilde{M}$-induced graph that corresponds to $E_1'^i$ (respectively $E_{n_i}'^i$). In particular, γ_1 begins with E and γ_m ends with $\widehat{E}$. Thus, we can concatenate the paths γ_i to get a path $\gamma_1\gamma_2\cdots\gamma_m$ from E to $\widehat{E}$ and the claim follows.

Now we will show that for each path $\gamma_i' = E_1'^i \longrightarrow E_2'^i \longrightarrow \cdots \longrightarrow E_{n_i}'^i$ with the above properties there exists a path γ_i from E_1^i to $E_{n_i}^i$ in the $\widetilde{M}$-induced graph. By the definition of $\mathcal{P}'$, for every $i \in \{2, 3, \ldots, k\}$, $p_{1,i}'$ only $f'-$ covers the interior basic paths $p_{1,\tau(i)}'$ and $p_{1,\tau(1)}'$. Thus, the path γ_i' is of the form $E_1' \longrightarrow \cdots \longrightarrow E_{m_1}' \longrightarrow p_{1,\ell_0}' \longrightarrow p_{1,\ell_1}' \longrightarrow \cdots \longrightarrow p_{1,\ell_r}' \longrightarrow \widehat{E}_1' \longrightarrow \cdots \longrightarrow \widehat{E}_{m_2}'$ where $\widehat{E}_{m_2}' = E_{n_i}'^i$, all basic paths in $E_1' \longrightarrow \cdots \longrightarrow E_{m_1}'$ and $\widehat{E}_1' \longrightarrow \cdots \longrightarrow \widehat{E}_{m_2}'$ are exterior basic paths, $\ell_0 \in \{2, 3, \ldots, k\}$, $r \geq 0$ and $\ell_i \in \{\tau(\ell_{i-1}), \tau(1)\} \subset \{\tau^i(\ell_0), \tau(1), \tau^2(1), \ldots, \tau^i(1)\}$ for $i = 1, 2, \ldots, r$.

Since the coverings between the elements of $\mathcal{E}'$ in the $\mathcal{P}'$-path graph and the coverings between the elements of $\mathcal{E}$ in the M-induced and the $\widetilde{M}$-induced graphs are in one-to-one correspondence, $E_1 \longrightarrow \cdots \longrightarrow E_{m_1}$ and $\widehat{E}_1 \longrightarrow \cdots \longrightarrow \widehat{E}_{m_2}$ are paths in the $\widetilde{M}$-induced graph, where $E_1, \ldots, E_{m_1}, \widehat{E}_1, \ldots, \widehat{E}_{m_2}$ are the elements of $\mathcal{E}$ corresponding to $E_1', \ldots, E_{m_1}', \widehat{E}_1', \ldots, \widehat{E}_{m_2}'$, respectively. By the definition of $\mathcal{P}'$, $\widehat{E}_1 \in B_1 \cup B_{\ell_r}$.

Assume that there exists $j \notin \{1, \ell_0\}$ such that E_{m_1}' f'-covers $p_{1,j}'$. Then, clearly, E_{m_1} f-covers $p_{\ell_0,j}$ and, in the case $\widehat{E}_1 \in B_1$, it follows by the definition of $\widetilde{M}$ that $E_1 \longrightarrow \cdots \longrightarrow E_{m_1} \longrightarrow p_{\ell_0,j} \xrightarrow{1} \widehat{E}_1 \longrightarrow \cdots \longrightarrow \widehat{E}_{m_2}$ is a path in the $\widetilde{M}$-induced graph and we are done.

Assume now that $\widehat{E}_1 \in B_{\ell_r}$. Then, any interior basic path of the form $p_{\ell_r,a}$ f-covers $\widehat{E}_1$. So, by definition, the M-induced and $\widetilde{M}$-induced graphs contain the arrow $p_{\ell_r,a} \longrightarrow \widehat{E}_1$ and $E_1 \longrightarrow \cdots \longrightarrow E_{m_1} \longrightarrow p_{\ell_0,j} \longrightarrow p_{\tau(\ell_0),\tau(1)}$ is a path in the $\widetilde{M}$-induced graph. If $r = 0$ then $E_1 \longrightarrow \cdots \longrightarrow E_{m_1} \longrightarrow p_{\ell_0,j} \longrightarrow \widehat{E}_1 \longrightarrow \cdots \longrightarrow \widehat{E}_{m_2}$ is the path in the $\widetilde{M}$-induced graph that we are looking for. Assume now that $r \geq 1$. If $\ell_r = \tau^r(\ell_0)$, in a similar way to the inductive step in the proof of Lemma 7.7, there exist $t_1 = \tau(1), t_2, t_3, \ldots, t_r$ with $t_i \in \{1, 2, \ldots, k\} \setminus \{\tau^i(\ell_0)\}$ for $i = 1, 2, \ldots, r$ such that

$$\begin{aligned}
E_1 \longrightarrow \cdots \longrightarrow E_{m_1} \longrightarrow p_{\ell_0,j} \longrightarrow& \\
p_{\tau(\ell_0),t_1} \longrightarrow p_{\tau^2(\ell_0),t_2} \longrightarrow p_{\tau^3(\ell_0),t_3} \longrightarrow \cdots \longrightarrow p_{\tau^r(\ell_0),t_r} \longrightarrow& \\
\widehat{E}_1 \longrightarrow \cdots \longrightarrow \widehat{E}_{m_2}&
\end{aligned}$$

is a path in the $\widetilde{M}$-induced graph. If $\ell_r \neq \tau^r(\ell_0)$ then $\ell_r = \tau^m(1)$ with $1 \leq m \leq r$. As above, there exist $t_1 = \tau(\ell_0), t_2, t_3, \ldots, t_m$ such that

$$\begin{aligned}
E_1 \longrightarrow \cdots \longrightarrow E_{m_1} \longrightarrow p_{\ell_0,j} \longrightarrow& \\
p_{\tau(1),t_1} \longrightarrow p_{\tau^2(1),t_2} \longrightarrow p_{\tau^3(1),t_3} \longrightarrow \cdots \longrightarrow p_{\tau^m(1),t_m} \longrightarrow& \\
\widehat{E}_1 \longrightarrow \cdots \longrightarrow \widehat{E}_{m_2}&
\end{aligned}$$

is a path in the $\widetilde{M}$-induced graph and the claim follows when there exists $j \notin \{1, \ell_0\}$ such that E_{m_1}' f'-covers $p_{1,j}'$.

Now assume that, for every $j \notin \{1, \ell_0\}$, E_{m_1}' does not f'-cover $p_{1,j}'$. In this situation E_{m_1} f-covers p_{1,ℓ_0} and using arguments similar to those above, we also obtain a path from E_1 to $\widehat{E}_{m_2}$ in the $\widetilde{M}$-induced graph. This ends the proof of the claim.

We will say that a basic path π of the $\widetilde{M}$-induced graph is *admissible* if there exists a path in the $\widetilde{M}$-induced graph beginning at some exterior basic path and ending at π. Let Δ be the set of all admissible basic paths. By the claim, $\mathcal{E} \subset \Delta$.

We claim that the set Δ is *transitive*. We have to see that for every $\pi_1, \pi_2 \in \Delta$ there exists a path from π_1 to π_2 in the $\widetilde{M}$-induced graph using only elements of Δ. When $\pi \in \mathcal{E}$ this statement follows exactly as in Lemma 7.4.

Assume that $\pi \notin \mathcal{E}$. By (10) there exists an exterior basic path $E \in B_1$ such that $\pi \longrightarrow E$ is an arrow in the $\widetilde{M}$-induced graph. Since $\pi_2 \in \Delta$ there exists a basic path γ in the $\widetilde{M}$-induced graph from an exterior basic path $\widehat{E}$ to π_2 containing only elements of Δ (if $p_2 \in \mathcal{E}$ this path can be taken to be the empty path). By the previous claim there exists a basic path α from E to $\widehat{E}$ in the $\widetilde{M}$-induced graph containing only elements of Δ. Then the concatenated path $(\pi \longrightarrow E)\alpha\gamma$ is a path from π_1 to π_2 in the $\widetilde{M}$-induced graph containing only elements of Δ. This ends the proof that Δ is transitive.

Now let Y denote the set of basic paths of the $\widetilde{M}$-induced graph that do not belong to Δ. Clearly Y is disjoint from $\mathcal{E}$ and so, it only contains interior basic paths. The columns of the matrix $\widetilde{M}$ corresponding to the elements of Y are identically zero. To see this notice that, by definition, there are no arrows from any element of Δ to any element of Y. Also, there are no arrows from any element of Y to any element of Y. Indeed, all elements of Y are interior basic paths and by definition they have only arrows to interior basic paths of the form $p_{j,\tau(1)}$ that belong to Δ. Moreover, since $M \leq \widetilde{M}$ it follows that the columns of the matrix M corresponding to the elements of Y are also identically zero.

Denote by M' and $\widetilde{M}'$ the matrices obtained respectively from M and $\widetilde{M}$ by deleting the rows and the columns corresponding to the vertices in Y. As in Lemma 7.4, $\sigma(\widetilde{M}') = \sigma(\widetilde{M})$ and $\sigma(M') = \sigma(M)$. On the other hand, the matrix $\widetilde{M}'$ is irreducible because the set Δ is transitive (i.e. any two elements of Δ can be joined by a path in the M-induced graph using only elements of Δ — see Lemma 3.1(a,b)). Clearly, $\widetilde{M}' \geq M'$ because $\widetilde{M} \geq M$. We claim that $\widetilde{M}' \neq M'$. If the claim holds, by Lemma 3.1(d),

$$\sigma(\widetilde{M}) = \sigma(\widetilde{M}') > \sigma(M') = \sigma(M)$$

and the lemma is proved.

Now we prove that $\widetilde{M}' \neq M'$. In the case $\tau = \mathrm{Id}$ we know by construction that there exist $w_1, w_2 \in \{2, 3, \ldots, k\}$, $w_1 \neq w_2$ such that some exterior basic path f-covers the interior basic path p_{w_1,w_2}. Since this kind of arrows from the $\mathcal{P}$-path graph have been preserved in the M-induced and the $\widetilde{M}$-induced graphs, $p_{w_1,w_2} \in \Delta$. Thus, $\widetilde{M}' \neq M'$ because in the definition of $\widetilde{M}$ we have strictly increased the entries corresponding to the row of p_{w_1,w_2} and the columns corresponding to the elements in $B_1 \subset \mathcal{E} \subset \Delta$ (since $1 \in N$, $B_1 \neq \emptyset$). Hence, $\widetilde{M}' \neq M'$ in this case.

Assume now that $\tau \neq \mathrm{Id}$. It is enough to show that there exists $w \notin \{1, \tau(1)\}$ such that $p_{w,\tau(1)} \in \Delta$. Indeed: since $1 \notin \{w, \tau(1)\}$, in the definition of $\widetilde{M}$ we have strictly increased the entries corresponding to the row $p_{w,\tau(1)}$ in the columns corresponding to the elements in $B_1 \subset \Delta$ (and hence, $\widetilde{M}' \neq M'$).

The rest of the proof of the lemma will be devoted to show that, when $\tau \neq \mathrm{Id}$, there exists $w \notin \{1, \tau(1)\}$ such that $p_{w,\tau(1)} \in \Delta$.

Assume that there exists $E' \in \mathcal{E}'$ such that E' f'-covers $p'_{1,r}$ and $p'_{1,l}$ with $r \neq l$. Since τ is a permutation, one of r and l, say r for definiteness, verifies $\tau(r) \neq 1$. Also,

$\tau(r) \neq \tau(1)$ because $r \neq 1$. By the definition of M and $\widetilde{M}$, $E \longrightarrow p_{r,l} \longrightarrow p_{\tau(r),\tau(1)}$ is a path in the $\widetilde{M}$-induced graph and the claim follows from the definition of Δ by setting $w = \tau(r)$.

Now assume that every exterior basic path of $\mathcal{P}'$ f'-covers at most one interior basic path. There exists $E' \in \mathcal{E}'$ such that E' f'-covers a unique $p'_{1,r}$ with $r \neq 1$. In this case the $\widetilde{M}$-induced graph has the path $E \longrightarrow p_{1,r} \longrightarrow p_{\tau(r),\tau(1)}$. If $\tau(r) \neq 1$ then the claim follows from the definition of Δ by setting $w = \tau(r)$.

Suppose now that $\tau(r) = 1$. We know that the $\widetilde{M}$-induced graph has the path $E \longrightarrow p_{1,r} \longrightarrow p_{1,\tau(1)} \longrightarrow p_{\tau^2(1),\tau(1)}$. If $\tau^2(1) \neq 1$ then, again, the claim follows by setting $w = \tau^2(1)$.

We are left with the case $\tau^2(1) = 1$ and if an exterior basic path of $\mathcal{P}'$ f'-covers one interior basic path then this basic path is precisely $p'_{1,r}$ with $r = \tau(1)$. Clearly, in this situation, the only interior basic path f'-covered by $p'_{1,\tau(1)}$ is itself. Consequently, the set of vertices $\mathcal{E}' \cup \{p'_{1,\tau(1)}\}$ is invariant in the $\mathcal{P}'$-path graph (i.e. every path starting at $\mathcal{E}' \cup \{p'_{1,\tau(1)}\}$ has all vertices in $\mathcal{E}' \cup \{p'_{1,\tau(1)}\}$). This contradicts the irreducibility of $M_P(\mathcal{P}')$ (Lemma 3.1(a,b)). □

Lemma 7.10. *With the above notations* $\log \sigma(\widetilde{M}) = h(\mathcal{P}')$

In what follows we will denote by $\mathrm{Orb}_\tau(a)$ the set $\{a, \tau(a), \tau^2(a), \dots\}$. Clearly, since τ is a permutation over a set of cardinality k, $\mathrm{Orb}_\tau(a)$ has cardinality at most k for every a.

Proof of Lemma 7.10. From the proof of Lemma 7.5 we have to show that $\sigma(\widetilde{M}') = \sigma(M_P(\mathcal{P}'))$, where $\widetilde{M}'$ is the matrix from the proof of the previous lemma. Again by the proof of Lemma 7.5 it is enough to show that, given two exterior basic paths $E'_1, E'_2 \in \mathcal{E}'$ and $m \geq 2$, the number of paths of the $\mathcal{P}'$-path graph of the form $E'_1 \longrightarrow \pi'_1 \longrightarrow \pi'_2 \longrightarrow \cdots \longrightarrow \pi'_{m-1} \longrightarrow E'_2$, where $\pi'_1, \pi'_2, \dots, \pi'_{m-1}$ are interior basic paths, coincides with the number of paths of the $\widetilde{M}$-induced graph of the form $E_1 \longrightarrow \pi_1 \longrightarrow \pi_2 \longrightarrow \cdots \longrightarrow \pi_{m-1} \longrightarrow E_2$, where $\pi_1, \pi_2, \dots, \pi_{m-1}$ are interior basic paths.

We prove this in the case $\tau \neq \mathrm{Id}$. The case $\tau = \mathrm{Id}$ is simpler and we leave the details to the reader.

We start by proving the statement in the case $m = 2$. Let $E'_1 \longrightarrow p'_{1,i_1} \longrightarrow E'_2$ be a path of length 2 in the $\mathcal{P}'$-path graph from E'_1 to E'_2. The fact that the above path ends with $p'_{1,i_{m-1}} \longrightarrow E'_2$ implies that $E_2 \in B_1 \cup B_{i_{m-1}}$.

If p'_{1,i_1} is the unique interior basic path f'-covered by E'_1 then the above path is the unique path of length 2 in the $\mathcal{P}'$-path graph starting with E'_1 and ending with E'_2 and $E_1 \longrightarrow p_{1,i_1} \longrightarrow E_2$ is the unique path of length 2 in the $\widetilde{M}$-induced graph joining E_1 and E_2. Thus the lemma holds in this case.

Now assume that, additionally, E'_1 $f'-$covers p'_{1,b_1} with $b_1 \neq i_1$ ($1 \notin \{i_1, b_1\}$). If $E_2 \in B_{i_1}$ then, as before, $E'_1 \longrightarrow p'_{1,i_1} \longrightarrow E'_2$ is the unique path of length 2 in the $\mathcal{P}'$-path graph starting with E'_1 and ending with E'_2 and $E_1 \longrightarrow p_{i_1,b_1} \longrightarrow E_2$ is the unique path of length 2 in the $\widetilde{M}$-induced graph joining E_1 and E_2.

When $E_2 \notin B_{i_1}$ it follows that $E_2 \in B_1$ and there are exactly two paths of length 2 in the $\mathcal{P}'$-path graph starting with E'_1 and ending with E'_2 : $E'_1 \longrightarrow p'_{1,i_1} \longrightarrow E'_2$ and $E'_1 \longrightarrow p'_{1,b_1} \longrightarrow E'_2$. On the other and, from the definition of the matrix $\widetilde{M}$ it follows that there are exactly two paths of length 2 in the $\widetilde{M}$-induced graph starting with E_1 and ending with E_2 : $E_1 \longrightarrow p_{i_1,b_1} \xrightarrow{1} E_2$ and $E_1 \longrightarrow p_{i_1,b_1} \xrightarrow{2} E_2$. This ends the proof of the lemma in the case $m = 2$. So, in the rest of the proof we assume that $m \geq 3$.

By the definition of $\mathcal{P}'$, for every $i \in \{2,3,\ldots,k\}$, $p'_{1,i}$ only $f'-$ covers the interior basic paths $p'_{1,\tau(i)}$ and $p'_{1,\tau(1)}$. Thus, the path $E'_1 \longrightarrow \pi'_1 \longrightarrow \pi'_2 \longrightarrow \cdots \longrightarrow \pi'_{m-1} \longrightarrow E'_2$ is of the form

$$E'_1 \longrightarrow p'_{1,i_1} \longrightarrow p'_{1,i_2} \longrightarrow \cdots \longrightarrow p'_{1,i_{m-1}} \longrightarrow E'_2$$

with $i_1 \in \{2,3,\ldots,k\}$ and $i_2, i_3, \ldots, i_{m-1} \in \mathrm{Orb}_\tau(1) \cup \mathrm{Orb}_\tau(i_1)$ such that $i_j \notin \mathrm{Orb}_\tau(1)$ implies $i_1, i_2, \ldots, i_{j-1} \notin \mathrm{Orb}_\tau(1)$ for $j = 2,3,\ldots,m-1$. Moreover, in such case $i_j = \tau^{j-1}(i_1)$. Also, as in the case $m = 2$, $E_2 \in B_\ell$ with $\ell \in \{1, i_{m-1}\}$.

Concerning the $\widetilde{M}$-induced graph, from (9), the definition of M and the fact that in the definition of $\widetilde{M}$ we did not modify the entries corresponding only to interior paths, it follows that the path $E_1 \longrightarrow \pi_1 \longrightarrow \pi_2 \longrightarrow \cdots \longrightarrow \pi_{m-1} \longrightarrow E_2$ is of the form

$$E_1 \longrightarrow p_{a_1,b_1} \longrightarrow p_{\tau(1),t_2} \longrightarrow \cdots \longrightarrow p_{\tau(1),t_{m-1}} \longrightarrow E_2$$

with $a_1, b_1 \in \{1,2,3,\ldots,k\}$ and $t_2, t_3, \ldots, t_{m-1} \in \mathrm{Orb}_\tau(1) \cup \mathrm{Orb}_\tau(a_1) \cup \mathrm{Orb}_\tau(b_1)$ such that $t_j \notin \mathrm{Orb}_\tau(1)$ implies $t_2, t_3, \ldots, t_{j-1} \notin \mathrm{Orb}_\tau(1)$ for $j = 2,3,\ldots,m-1$. Moreover if $a_1 \notin \mathrm{Orb}_\tau(1)$ and $t_j \in \mathrm{Orb}_\tau(a_1)$ then $t_j = \tau^{j-1}(a_1)$ for $j = 2,3,\ldots,m-1$, and the same holds for b_1 instead of a_1.

Assume first that $\ell \notin \mathrm{Orb}_\tau(1)$. Then, $\ell = i_{m-1} = \tau^{m-2}(i_1)$, $i_1 \notin \mathrm{Orb}_\tau(1)$ and

$$E'_1 \longrightarrow p'_{1,i_1} \longrightarrow p'_{1,\tau(i_1)} \longrightarrow \cdots \longrightarrow p'_{1,\tau^{m-2}(i_1)} \longrightarrow E'_2$$

is a path of length m in the $\mathcal{P}'$-path graph starting with E'_1 and ending with E'_2.

If p'_{1,i_1} is the unique interior basic path f'-covered by E'_1 then the above path is the unique path of length m in the $\mathcal{P}'$-path graph starting with E'_1 and ending with E'_2 having all basic paths different from E'_1 and E'_2 interior and

$$E_1 \longrightarrow p_{1,i_1} \longrightarrow p_{\tau(1),\tau(i_1)} \longrightarrow \cdots \longrightarrow p_{\tau(1),\tau^{m-2}(i_1)} \longrightarrow E_2$$

is the unique path of length m in the $\widetilde{M}$-induced graph joining E_1 and E_2 and having all basic paths different from E_1 and E_2 interior. Thus the lemma holds in this case.

Now assume that E'_1 $f'-$covers p'_{1,b_1} with $b_1 \neq i_1$ ($1 \notin \{i_1, b_1\}$). Clearly,

$$E_1 \longrightarrow p_{i_1,b_1} \longrightarrow p_{\tau(1),\tau(i_1)} \longrightarrow \cdots \longrightarrow p_{\tau(1),\tau^{m-1}(i_1)} \longrightarrow E_2$$

is the unique path of length m in the $\widetilde{M}$-induced graph joining E_1 and E_2 and having all basic paths different from E_1 and E_2 interior. To prove the lemma in this case we have to show that

$$E'_1 \longrightarrow p'_{1,i_1} \longrightarrow p'_{1,\tau(i_1)} \longrightarrow \cdots \longrightarrow p'_{1,\tau^{m-2}(i_1)} \longrightarrow E'_2$$

is the unique path of length m in the $\mathcal{P}'$-path graph starting with E'_1 and ending with E'_2 and having all basic paths different from E'_1 and E'_2 interior. Observe that there is a unique path in the $\mathcal{P}'$-path graph of length $m-1$ starting with $E'_1 \longrightarrow p'_{1,b_1}$: $E'_1 \longrightarrow p'_{1,b_1} \longrightarrow p'_{1,b_2} \longrightarrow \cdots \longrightarrow p'_{1,b_{m-1}}$ with $b_{m-1} \in \mathrm{Orb}_\tau(1) \cup \mathrm{Orb}_\tau(b_1)$. If, additionally, $p'_{1,b_{m-1}}$ f'-covers some exterior basic path then this must belong to $B_1 \cup B_{b_{m-1}}$. Since $\ell \notin \mathrm{Orb}_\tau(1)$ it follows that $\ell \neq b_{m-1}$ when $b_{m-1} \in \mathrm{Orb}_\tau(1)$. On the other hand, when $b_{m-1} \notin \mathrm{Orb}_\tau(1)$, $b_1 \neq i_1$ implies $b_{m-1} = \tau^{m-2}(b_1) \neq \tau^{m-2}(i_1) = \ell$. Thus, in any case $\ell \notin \{b_{m-1}, 1\}$, and $E'_1 \longrightarrow p'_{1,b_1} \longrightarrow p'_{1,b_2} \longrightarrow \cdots \longrightarrow p'_{1,b_{m-1}} \longrightarrow E_2$ is not a path in the $\mathcal{P}'$-path graph because $E_2 \in B_\ell$. This ends the proof of the lemma in the case $\ell \notin \mathrm{Orb}_\tau(1)$.

Assume now that $\ell \in \mathrm{Orb}_\tau(1)$ and denote by p the $\tau-$period of 1. We also assume that $p > 2$ (that is, $\tau^2(1) \neq 1$). The proof in the case $p = 2$ works in a similar way with minor changes.

To deal with this case we introduce the following notation. For every $r \geq 2$ and $i \in \{1,2,3,\ldots,p-1\}$ we denote by β_i^r the number of paths of length r in the $\mathcal{P}'$-path graph of the form $E'_1 \longrightarrow \pi'_1 \longrightarrow \pi'_2 \longrightarrow \cdots \longrightarrow \pi'_{r-1} \longrightarrow p'_{1,\tau^i(1)}$ where $\pi'_1, \pi'_2, \dots, \pi'_{r-1}$ are interior basic paths. Analogously, for $r \geq 2$ and $i \in \{0, 2, 3, \dots, p-1\}$ we denote by α_i^r the number of paths of length r of the $\widetilde{M}$-induced graph of the form $E_1 \longrightarrow \pi_1 \longrightarrow \pi_2 \longrightarrow \cdots \longrightarrow \pi_{r-1} \longrightarrow p_{\tau(1),\tau^i(1)}$ where $\pi_1, \pi_2, \dots, \pi_{r-1}$ are interior basic paths. Also, given $E' \in \mathcal{E}'$ we set

$$\kappa(E') := \{i \in \{2, \dots, k\} : E' \ f' - \text{covers } p'_{1,i}\}.$$

Clearly $|\kappa(E')| \leq 2$.

We claim that, for every $r \geq 2$ and $i \in \{2, 3, \dots, p-1\}$, $\beta_i^r = \alpha_i^r$ and

$$\beta_1^r = |\kappa(E'_1) \setminus \operatorname{Orb}_\tau(1)| + \alpha_0^r + \sum_{i=2}^{p-1} \alpha_i^r.$$

First we will end the proof of the lemma in the case $\ell = \tau^j(1)$ with $j \in \{0, 1, 2, \dots, p-1\}$ by using the claim, and later we will prove the claim.

Denote by h'_m the number of paths of length m of the $\mathcal{P}'$-path graph of the form $E'_1 \longrightarrow \pi'_1 \longrightarrow \pi'_2 \longrightarrow \cdots \longrightarrow \pi'_{m-1} \longrightarrow E'_2$, where $\pi'_1, \pi'_2, \dots, \pi'_{m-1}$ are interior basic paths and by $\widetilde{h}_m$ the number of paths of of length m of the $\widetilde{M}$-induced graph of the form $E_1 \longrightarrow \pi_1 \longrightarrow \pi_2 \longrightarrow \cdots \longrightarrow \pi_{m-1} \longrightarrow E_2$, where $\pi_1, \pi_2, \dots, \pi_{m-1}$ are interior basic paths.

Assume $j = 0$ (i.e. $\ell = 1$). We claim that

$$h'_m = s + \sum_{i=1}^{p-1} \beta_i^{m-1}, \text{ and}$$
$$\widetilde{h}_m = 2\left(s + \sum_{i=2}^{p-1} \alpha_i^{m-1}\right) + \alpha_0^{m-1},$$

where $s = |\kappa(E'_1) \setminus \operatorname{Orb}_\tau(1)|$. We start by proving that $h'_m = s + \sum_{i=1}^{p-1} \beta_i^{m-1}$. Since $E_2 \in B_1$, every path counted in β_i^{m-1} (i.e. every path of length $m-1$ of the form $E'_1 \longrightarrow \pi'_1 \longrightarrow \pi'_2 \longrightarrow \cdots \longrightarrow \pi'_{m-2} \longrightarrow p'_{1,\tau^i(1)}$ where $\pi'_1, \pi'_2, \dots, \pi'_{m-1}$ are interior basic paths) gives a path that must be counted in h'_m. Let $t \in \kappa(E'_1) \setminus \operatorname{Orb}_\tau(1)$. Then, as in the case $\ell \notin \operatorname{Orb}_\tau(1)$,

$$E_1 \longrightarrow p_{t,t'} \longrightarrow p_{\tau(1),\tau(t)} \longrightarrow \cdots \longrightarrow p_{\tau(1),\tau^{m-2}(t)} \longrightarrow E_2$$

with $t' = 1$ when $s = 1$ and $t' \in \kappa(E'_1) \setminus \{t\}$ when $s = 2$ is a path of length m in the $\mathcal{P}'$-path graph from E'_1 to E'_2. Since $\tau^{m-2}(t) \notin \operatorname{Orb}_\tau(1)$ this path is different from the paths counted in β_i^{m-1}. This proves that $h'_m = s + \sum_{i=1}^{p-1} \beta_i^{m-1}$.

Now we prove that $\widetilde{h}_m = 2\left(s + \sum_{i=2}^{p-1} \alpha_i^{m-1}\right) + \alpha_0^{m-1}$. We are looking at the paths of length $m-1$ in the $\widetilde{M}$-induced graph of the form $E_1 \longrightarrow \pi_1 \longrightarrow \pi_2 \longrightarrow \cdots \longrightarrow \pi_{r-1} \longrightarrow p_{\tau(1),t}$ where $\pi_1, \pi_2, \dots, \pi_{r-1}$ are interior basic paths. From the definitions it follows that there are α_0^{m-1} of such paths with $t = 1$, α_i^{m-1} paths with $t = \tau^i(1)$ and $i \in \{2, 3, \dots, p-1\}$, and s paths with $t \notin \operatorname{Orb}_\tau(1)$. Since $E_2 \in B_1$, each of the α_0^{m-1} paths that have $t = 1$ gives a unique path of length m in the $\widetilde{M}$-induced graph from E_1 to E_2 having all basic paths different from E_1 and E_2 interior. For the rest of paths (which amount to $s + \sum_{i=2}^{p-1} \alpha_i^{m-1}$) it follows that $1 \notin \{\tau(1), t\}$. To see it recall that $1 \neq \tau(1)$ by assumption and $1 \neq t$ when $t \notin \operatorname{Orb}_\tau(1)$. Observe also that, since p is the τ-period of 1 and $p > 2$, $1 \neq \tau^i(1)$ for $i \in \{2, 3, \dots, p-1\}$. Hence, by the definition of the matrix $\widetilde{M}$, each of these paths gives two paths of length m in the $\widetilde{M}$-induced graph from E_1 to E_2 having all basic paths different from E_1 and E_2 interior: $E_1 \longrightarrow \pi_1 \longrightarrow \pi_2 \longrightarrow \cdots$

$\longrightarrow \pi_{r-1} \longrightarrow p_{\tau(1),t} \xrightarrow{1} E_2$ and $E_1 \longrightarrow \pi_1 \longrightarrow \pi_2 \longrightarrow \cdots \longrightarrow \pi_{r-1} \longrightarrow p_{\tau(1),t} \xrightarrow{2}$ E_2. This shows the above formula for $\widetilde{h}_m$.

By the two formulas above for h'_m and $\widetilde{h}_m$, and the claim we get

$$\begin{aligned} h'_m &= s + \beta_1^{m-1} + \sum_{i=2}^{p-1} \beta_i^{m-1} = s + \left(s + \alpha_0^{m-1} + \sum_{i=2}^{p-1} \alpha_i^{m-1} \right) + \sum_{i=2}^{p-1} \alpha_i^{m-1} \\ &= 2 \left(s + \sum_{i=2}^{p-1} \alpha_i^{m-1} \right) + \alpha_0^{m-1} = \widetilde{h}_m. \end{aligned}$$

When $j = 1$ (i.e. $\ell = \tau(1)$), in a similar way as in the case $j = 0$ we get $h'_m = \beta_1^{m-1}$ and $\widetilde{h}_m = |\kappa(E'_1) \setminus \operatorname{Orb}_\tau(1)| + \alpha_0^{m-1} + \sum_{i=2}^{p-1} \alpha_i^{m-1}$. By the claim $h'_m = \widetilde{h}_m$.

Finally, if $j \geq 2$, $h'_m = \beta_j^{m-1}$ and $\widetilde{h}_m = \alpha_j^{m-1}$ and the lemma follows again from the claim.

Now we prove the claim by induction on r. We will prove the claim assuming that E'_1 f'-covers a unique interior basic path p'_{1,i_1} (so, $|\kappa(E'_1) \setminus \operatorname{Orb}_\tau(1)| \leq 1$). The proof in the case when E'_1 f'-covers two interior basic paths follows in a similar way with the help of paths described above.

When $r = 2$ and $i_1 \neq \tau^{p-1}(1)$ there are only two paths of length r in the $\mathcal{P}'$-path graph starting with E'_1 and containing only interior basic paths except for E'_1: $E'_1 \longrightarrow p'_{1,i_1} \longrightarrow p'_{1,\tau(1)}$ and $E'_1 \longrightarrow p'_{1,i_1} \longrightarrow p'_{1,\tau(i_1)}$. In the $\widetilde{M}$-induced graph there is a unique path of length r starting with E_1 and containing only interior basic paths except for E_1: $E_1 \longrightarrow p_{1,i_1} \longrightarrow p_{\tau(1),\tau(i_1)}$.

If $i_1 \notin \operatorname{Orb}_\tau(1)$ then, from the definitions we get $|\kappa(E'_1) \setminus \operatorname{Orb}_\tau(1)| = 1$, $\beta_1^2 = 1$, $\alpha_0^2 = 0$ and $\alpha_i^2 = \beta_i^2$ for $i = 2, 3, \ldots, p-1$. So, the claim holds in this case.

If $i_1 = \tau^s(1)$ with $s \neq p-1$ then, $|\kappa(E'_1) \setminus \operatorname{Orb}_\tau(1)| = 0$, $\beta_1^2 = 1$, $\alpha_0^2 = 0$, $\alpha_{s+1}^2 = \beta_{s+1}^2 = 1$ and $\alpha_i^2 = \beta_i^2 = 0$ for $i = 2, 3, \ldots, p-1$, $i = s+1$. So, the claim also holds in this case.

Lastly, if $r = 2$ and $i_1 = \tau^{p-1}(1)$ there is a unique path of length 2 in the $\mathcal{P}'$-path graph (respectively in the $\widetilde{M}$-induced graph) starting with E'_1 (respectively E_1) and containing only interior basic paths except for the initial one: $E'_1 \longrightarrow p'_{1,\tau^{p-1}(1)} \longrightarrow p'_{1,\tau(1)}$ (respectively $E_1 \longrightarrow p_{1,\tau^{p-1}(1)} \longrightarrow p_{\tau(1),1}$). In this case we have $|\kappa(E'_1) \setminus \operatorname{Orb}_\tau(1)| = 0$, $\beta_1^2 = \alpha_0^2 = 1$ and $\alpha_i^2 = \beta_i^2 = 0$ for $i = 2, 3, \ldots, p-1$ and the claim also holds.

Now assume that the claim holds for $r \geq 2$ and prove it for $r+1$.

To do this note that from the definitions we get $\beta_i^{r+1} = \beta_{i-1}^r$ for $i \geq 2$, $\alpha_0^{r+1} = \alpha_{p-1}^r$ and $\alpha_i^{r+1} = \alpha_{i-1}^r$ for $i \geq 3$. Thus, by the induction hypothesis,

$$\beta_i^{r+1} = \beta_{i-1}^r = \alpha_{i-1}^r = \alpha_i^{r+1}$$

for $i \geq 3$.

On the other hand we have $\alpha_2^{r+1} = |\kappa(E'_1) \setminus \operatorname{Orb}_\tau(1)| + \alpha_0^r + \sum_{i=2}^{p-1} \alpha_i^r$. Indeed, in view of the definition of M and $\widetilde{M}$, each path of length r of the $\widetilde{M}$-induced graph of the form $E_1 \longrightarrow \pi_1 \longrightarrow \pi_2 \longrightarrow \cdots \longrightarrow \pi_{r-1} \longrightarrow p_{\tau(1),\tau^i(1)}$ with $i \in \{0, 2, \ldots, p-1\}$, gives the following path of length $r+1$: $E_1 \longrightarrow \pi_1 \longrightarrow \pi_2 \longrightarrow \cdots \longrightarrow \pi_{r-1} \longrightarrow p_{\tau(1),\tau^i(1)} \longrightarrow p_{\tau(1),\tau^2(1)}$. Clearly all these paths contribute to α_2^{r+1} and there are $\alpha_0^r + \sum_{i=2}^{p-1} \alpha_i^r$ of such paths. On the other hand, if $\kappa(E'_1) \setminus \operatorname{Orb}_\tau(1) = \{i_1\}$ then the path $E_1 \longrightarrow p_{1,i_1} \longrightarrow p_{\tau(1),\tau(i_1)} \longrightarrow \cdots \longrightarrow p_{\tau(1),\tau^{r-1}(i_1)}$ gives the path $E_1 \longrightarrow p_{1,i_1} \longrightarrow p_{\tau(1),\tau(i_1)} \longrightarrow \cdots \longrightarrow p_{\tau(1),\tau^{r-1}(i_1)} \longrightarrow p_{\tau(1),\tau^2(1)}$ of length $r+1$ in the $\widetilde{M}$-induced graph. Since this path also contributes to α_2^{r+1} and is different from the previous ones, we get the formula

$$\alpha_2^{r+1} = |\kappa(E_1') \setminus \mathrm{Orb}_\tau(1)| + \alpha_0^r + \sum_{i=2}^{p-1} \alpha_i^r.$$

So, by induction we have

$$\alpha_2^{r+1} = \beta_1^r = \beta_2^{r+1}.$$

In a similar way to the computation of α_2^{r+1} we get

$$\beta_1^{r+1} = |\kappa(E_1') \setminus \mathrm{Orb}_\tau(1)| + \sum_{i=1}^{p-1} \beta_i^r.$$

On the other hand, from above and the induction hypothesis, $\beta_{p-1}^r = \alpha_{p-1}^r = \alpha_0^{r+1}$ and $\beta_{i-1}^r = \alpha_i^{r+1}$ for $i = 2, \ldots, p-1$. Hence,

$$\beta_1^{r+1} = |\kappa(E_1') \setminus \mathrm{Orb}_\tau(1)| + \alpha_0^{r+1} + \sum_{i=2}^{p-1} \alpha_i^{r+1}.$$

This ends the proof of the lemma. □

Proof of Theorem 7.1. It follows from Lemmas 7.4 and 7.5 in the case $f(v) \neq v$ and from Lemmas 7.6, 7.9 and 7.10 in the case $f(v) = v$. □

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