

Singular Perturbations in the Quadratic Family with Multiple Poles

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Abstract

We consider the quadratic family of complex maps given by $q_c(z) = z^2 + c$ where c is the center of a hyperbolic component in the Mandelbrot set. Then, we introduce a singular perturbation on the corresponding bounded superattracting cycle by adding one pole to each point in the cycle. When $c = -1$ the Julia set of q_{-1} is the well known basilica and the perturbed map is given by $f_\lambda(z) = z^2 - 1 + \lambda/(z^{d_0}(z+1)^{d_1})$ where $d_0, d_1 \geq 1$ are integers, and λ is a complex parameter such that $|\lambda|$ is very small. We focus on the topological characteristics of the Julia and Fatou sets of f_λ that arise when the parameter λ becomes nonzero. We give sufficient conditions on the order of the poles so that for small λ the Julia sets consist of the union of homeomorphic copies of the unperturbed Julia set, countably many Cantor sets of concentric closed curves, and Cantor sets of point components that accumulate on them.

1 Introduction

In the last decade a number of papers have appeared that deal with rational maps obtained by perturbing a complex polynomial by adding a pole at some point in the Fatou set of the polynomial. This kind of perturbation of a polynomial is usually called a *singular perturbation* since the degree of the resulting rational map increases when the pole is added. As a consequence of this perturbation the structure of the Julia sets often change dramatically after the addition of the pole since new critical points appear close to the pole.

The most studied case is the singular perturbation of the polynomial z^n with $n \geq 2$ obtained by adding a pole at the origin. In this case the corresponding rational map is given by $z^n + \lambda/z^d$ where $d \geq 1$. See, for example, [4, 5, 6, 7]. For the unperturbed map (that is, when $\lambda = 0$) the Julia set is the unit circle. Points with modulus larger than one are attracted to the superattracting fixed point at infinity, while points with modulus smaller than one are attracted to the superattracting fixed point at the origin. When $\lambda \neq 0$ the superattracting fixed point at the origin is replaced by a pole. However, the rational map $z^n + \lambda/z^d$ inherits some properties of the polynomial. For example, infinity is still a superattracting fixed point. Since the origin is a pole of order d , there is an open neighborhood of 0 that is mapped onto a neighborhood of ∞ in a d -to-1 fashion. If the component of the basin of ∞ which contains ∞ is disjoint from this neighborhood around the origin we call this set the *trap door*, since any orbit that eventually enters the immediate basin of attraction of infinity must pass through the trap door at some iterate.

The first appearance of this type of singular perturbation was in 1988 ([12]) when McMullen used some members of this family as an example of a rational map whose Julia set is a Cantor set of simple closed curves. More precisely, McMullen showed that if the arithmetic condition $1/n + 1/d < 1$ is satisfied then, for λ sufficiently small, the critical values of $z^n + \lambda/z^d$ belong to the trap door and the Julia set of the map is a Cantor set of simple closed curves. When this arithmetic condition is not satisfied, it is not possible to control the behavior of the critical points. In particular in any neighborhood of $\lambda = 0$ there are infinitely many parameter values for which the associated Julia set is a Sierpiński curve. If parameter values are drawn from Sierpiński holes of different escape time, then the dynamics of the corresponding maps on their respective Sierpiński curve Julia sets are known to be dynamically different ([3]).

Another family of perturbed polynomials of recent interest is of the form $z^n + c + \lambda/z^d$ where c is chosen to be the center of a hyperbolic component of the corresponding Multibrot set and λ is chosen to be small ([1, 12]). In [9] the case when the function z^n is perturbed with 2 poles is studied. In [10] the quadratic function $z^2 + c$ is perturbed with a pole at the origin in the case when c belongs to a hyperbolic component of the Mandelbrot set but is not at the center.

In this article we consider a different case of singular perturbations to the quadratic polynomial $z^2 + c$ with multiple poles. We focus on the case when c lies at the center of a hyperbolic component of the Mandelbrot set and where the perturbation consists of the addition of multiple poles in place of its bounded superattracting cycle.

In general, if c is the center of a hyperbolic component of the Mandelbrot set such that the critical point 0 is in a cycle of period $N > 1$ given by $p_0 = 0, p_1 = f(p_0) = c, \dots, p_{N-1} = f(p_{N-2}), f(p_{N-1}) = p_0 = 0$, the function is given by

$$(1.1) \quad f_\lambda(z) = z^2 + c + \frac{\lambda}{\prod_{i=0}^{N-1} (z - p_i)^{d_i}}.$$

That is, we add a pole at each one of the points in the superattracting N -cycle. When $\lambda = 0$ we have the quadratic map. When λ becomes nonzero the degree of the map changes from 2 to $2 + d_0 + d_1 + \dots + d_{N-1}$ and new critical points are created. When λ is very small the map behaves like the quadratic family for z values outside small neighborhoods of the poles. For example, the point at infinity is still a superattracting fixed point and it has an immediate basin of attraction denoted B_λ . There are N disjoint open neighborhoods of the poles that are simply connected preimages of the immediate basin of attraction of infinity. These sets are the trap doors denoted by $T_i = T_i(\lambda)$ for $i = 0, \dots, N - 1$. If a point has an orbit that is attracted to infinity and the point is not in the immediate basin of attraction of infinity then its orbit must escape through one of the trap doors. We will show that there are a number of critical points surrounding each one of the poles that are mapped close to the next point in the cycle. When λ is small and for suitable choices of the order of the poles d_i , all these critical points are mapped inside the trap doors and escape to infinity. In these cases we know the fate of every critical point and this allows us to study the topological characteristics of the Julia sets of f_λ .

The Fatou set of f_λ , denoted by $F(f_\lambda)$, is defined to be the set of points at which the family of iterates of f_λ is a normal family in the sense of Montel. The complement of the Fatou set in the Riemann sphere is the *Julia set* of f_λ and is denoted by $J(f_\lambda)$. By definition, the Fatou set is open and the Julia set is closed. The Julia set is also the closure of the set of repelling periodic points of f_λ , and it is the set where f_λ has sensitive dependence on initial conditions. Both sets, $F(f_\lambda)$ and $J(f_\lambda)$, are completely invariant. In this paper we study the topological properties of the Julia and Fatou sets of f_λ and dynamics of f_λ restricted to these sets.

Figures 1-2 display the Julia set of f_λ for particular values of the parameters. Figure 1 shows the Julia set corresponding to the parameter $c = -1$ which is the center of the period two hyperbolic component of the Mandelbrot set. When $\lambda = 0$ the Julia set of $f_0(z) = z^2 - 1$ is the well known basilica which has a superattracting cycle of period two at $p_0 = 0$ and $p_1 = -1$. When $\lambda \neq 0$ the perturbed basilica is given by $f_\lambda(z) = z^2 - 1 + \lambda/(z^{d_0}(z + 1)^{d_1})$. Figure 2 shows the Julia set corresponding with the parameter $c = c_r \approx -0.12256 + 0.74486i$, i.e., c_r is defined to be the center of the hyperbolic component of period three in the portion of the Mandelbrot set that lies in the upper half plane. When $\lambda = 0$ the Julia set of $f_0(z) = z^2 + c_r$ is the well known Douady rabbit which has a superattracting cycle of period three at $p_0 = 0$, $p_1 = c_r$, and $p_2 = c_r^2 + c_r$. When $\lambda \neq 0$ the perturbed Douady rabbit is given by $f_\lambda(z) = z^2 + c_r + \lambda/(z^{d_0}(z - c_r)^{d_1}(z - c_r^2 - c_r)^{d_2})$.

The perturbation when λ becomes nonzero happens far away from the immediate basin of attraction of infinity and inside the bulbs of the Julia set for the unperturbed map. It is possible to find $\delta > 0$ such that for $|\lambda| < \delta$ the boundary of B_λ is homeomorphic to the Julia set of f_0 . This fact was shown in the case of a single pole perturbation in [1] using quasiconformal surgery. The same proof works well for f_λ but in this paper we present a different proof that uses holomorphic motions.

Theorem A (The Boundary of B_λ). *There exists $\delta > 0$ such that, if $|\lambda| < \delta$, the boundary of B_λ is homeomorphic to $\partial B_0 = J(f_0)$ and f_λ restricted to ∂B_λ is conjugate to f_0 on $J(f_0)$.*

By this result the structure of the Julia set of f_0 persists in f_λ as ∂B_λ when λ is small. However, the structure of $J(f_\lambda)$ inside the bounded components of ∂B_λ is far more complex. In Figure 1, note that the boundary of B_λ looks very similar to the basilica and in Figure 2 that the boundary of B_λ is difficult to distinguish from the Douady rabbit. Figure 3 shows a magnification of the Julia set (for the basilica parameter $c = -1$) of $f_{10^{-22}}$ near the two poles at $p_0 = 0$ and $p_1 = -1$. The two trap doors: T_0 containing the origin and T_1 containing -1, are visible in these plots. Also note doubly connected components of the Fatou set.

For sufficiently small values of λ and under a certain arithmetic condition concerning the order of the poles, we can describe the topological structure and the dynamics on the Julia and Fatou sets of f_λ .

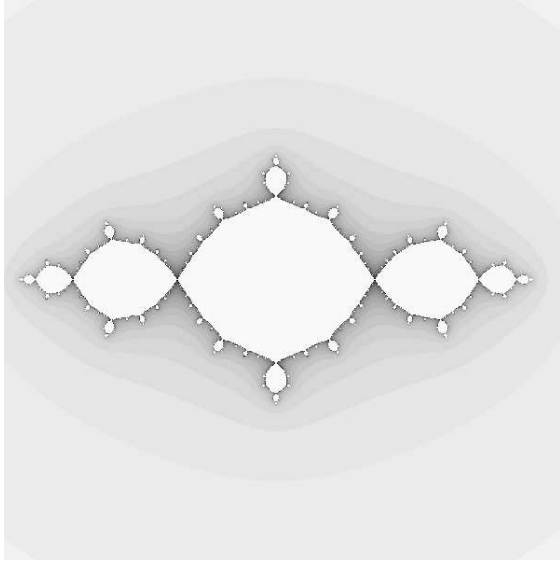
Recall that p_i is a pole of f_λ of order d_i for every $0 \leq i \leq N - 1$, with $p_0 = 0, p_1 = f_0(0), \dots, p_i = f_0^i(0)$.

Definition 1.1. *The natural numbers d_0, d_1, \dots, d_{N-1} satisfy the arithmetic condition \mathcal{I} when $2d_1 > d_0 + 2$ and $d_{i+1} > d_i + 1$ for every $1 \leq i \leq N - 1$.*

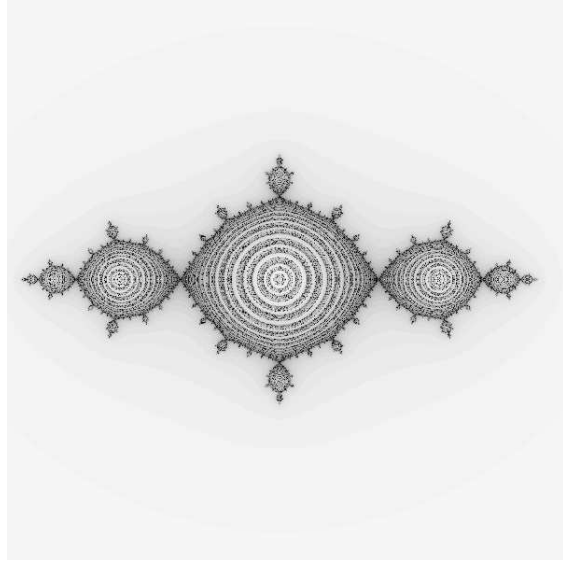
Theorem B (Structure of the Fatou set). *Suppose that $|\lambda|$ is sufficiently small and the natural numbers d_0, d_1, \dots, d_{N-1} satisfy the arithmetic condition \mathcal{I} . Then all the free critical orbits of f_λ escape to ∞ but the critical points themselves do not lie in B_λ . The Fatou set contains countably many simply connected components and annuli.*

Theorem C (Structure of the Julia set). *Suppose that $|\lambda|$ is sufficiently small and the natural numbers d_0, d_1, \dots, d_{N-1} satisfy the arithmetic condition \mathcal{I} . Then the Julia set contains countably many preimages of the boundary of B_λ , a countable collection of Cantor sets of closed curves, and an uncountable number of point components that accumulate on these curves.*

We will see from the proofs of Theorems B and C that we have more information about the Fatou and the Julia set. For instance, from the proof of Theorem B, we will know exactly which components of the Fatou set are simply connected (disks) and which components are doubly connected (annuli). In the same way, from the proof of Theorem C, we will understand the dynamics of f_λ restricted to its Julia set.



(a) Julia set for $\lambda = 0$. The basilica.



(b) Julia set for $\lambda = 10^{-22}$.

Figure 1: Dynamical planes of $f_\lambda(z) = z^2 - 1 + \lambda/(z^7(z+1)^5)$ for two different values of λ .

The paper is organized in the following manner. In §2 we approximate the location of the critical points and critical values of f_λ . In §3 we prove Theorem A using a holomorphic motion of $J(f_0)$. In §4 we study the topology of $F(f_\lambda)$ proving Theorem B and in §5 we investigate the topological structure and dynamics on $J(f_\lambda)$ proving Theorem C. In §6 we illustrate our results in detail for the example of the basilica where $c = -1$, $d_0 = 7$ and $d_1 = 5$.

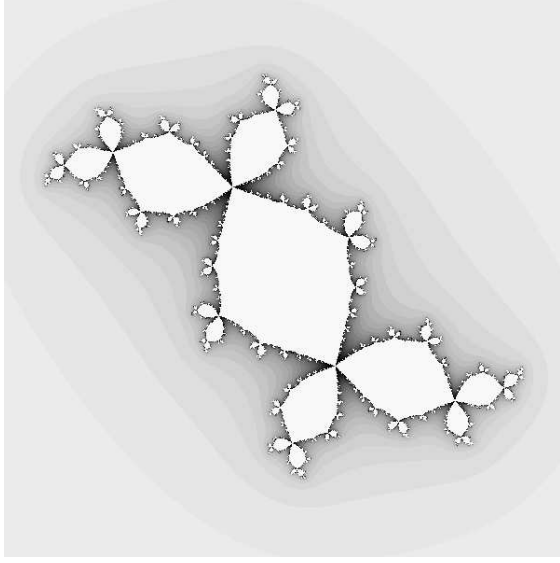
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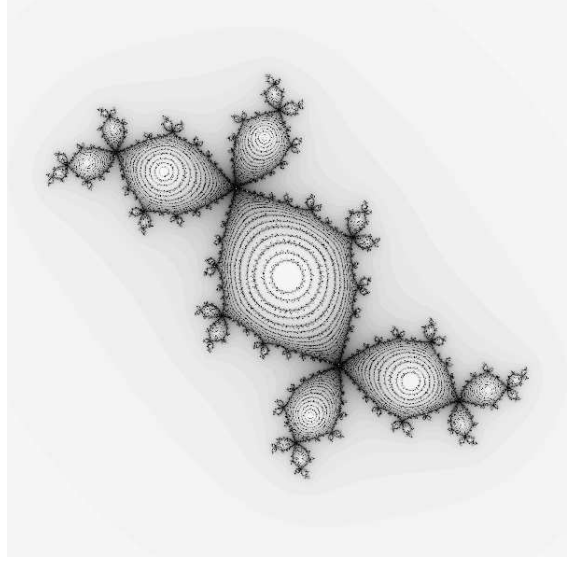
2 Preliminaries

In this section we compute the number, location, and approximate values of the critical points and critical values of f_λ (see Eq. 1.1).

Notice that when $\lambda \neq 0$, f_λ has degree $2 + d_0 + d_1 + \dots + d_{N-1}$ and so it has $2 + 2(d_0 + d_1 + \dots + d_{N-1})$ critical points counting multiplicities. Infinity is a critical point of order 1 and the poles located at p_i are critical points of order $d_i - 1$. That is a total of $d_0 + d_1 + \dots + d_{N-1} - N + 1$ critical points that are preimages of infinity. The fate of the



(a) Julia set for $\lambda = 0$. The Douady Rabbit. The parameter $c_r \approx -0.12256 + 0.74486i$.



(b) Julia set for $\lambda = 10^{-24}$.

Figure 2: Dynamical planes of $f_\lambda(z) = z^2 + c_r + \lambda/(z^{11}(z - c_r)^7(z - c_r^2 - c_r)^9)$ for two different values of λ .

other $d_0 + d_1 + \dots + d_{N-1} + N + 1$ “free” critical points determine the topology and the dynamics of the Julia set of f_λ .

Since the map f_λ is a singular perturbation of the quadratic map $z^2 + c$, these $1 + d_0 + d_1 + \dots + d_{N-1} + N$ critical points of f_λ are located close to the poles for sufficiently small values of the parameter λ . We denote by C_i the set of critical points of f_λ close to p_i . More precisely, we define

$$C_i = \{c_\lambda \mid c_\lambda \neq p_i \text{ is a critical point of } f_\lambda \text{ and } c_\lambda \rightarrow p_i \text{ as } \lambda \rightarrow 0\}$$

for every value $0 \leq i \leq N - 1$. The next lemma gives a precise description of the critical points and the critical values of f_λ when λ is small.

We write $f(\lambda) = g(\lambda) + o(h(\lambda))$ as $\lambda \rightarrow 0$ if and only if $\lim_{\lambda \rightarrow 0} \frac{f(\lambda) - g(\lambda)}{h(\lambda)} = 0$, and we write $f(\lambda) = g(\lambda) + O(h(\lambda))$ as $\lambda \rightarrow 0$ if and only if $\limsup_{\lambda \rightarrow 0} \left| \frac{f(\lambda) - g(\lambda)}{h(\lambda)} \right| < \infty$

Lemma 2.1. *Let λ be sufficiently small. The following statements hold:*

- (a) f_λ has $d_0 + 2$ critical points in C_0 approximately symmetrically distributed around $p_0 = 0$. Moreover, if $c_\lambda \in C_0$ there exists $k_0 \neq 0$ such that $c_\lambda = k_0 \lambda^{\frac{1}{d_0+2}} + o\left(\lambda^{\frac{1}{d_0+2}}\right)$ and $f_\lambda(c_\lambda) = p_1 + O\left(\lambda^{\frac{2}{d_0+2}}\right)$.
- (b) f_λ has $d_i + 1$ critical points in C_i approximately symmetrically distributed around p_i .

Moreover, if $c_\lambda \in C_i$ there exists $k_i \neq 0$ such that $c_\lambda = p_i + k_i \lambda^{\frac{1}{d_i+1}} + o\left(\lambda^{\frac{1}{d_i+1}}\right)$ and $f_\lambda(c_\lambda) = p_{i+1} + O\left(\lambda^{\frac{1}{d_i+1}}\right)$, for every $1 \leq i \leq N-1$.

Proof. A simple calculation shows that the critical points of f_λ satisfy

$$(2.1) \quad 2z^{d_0+2} \prod_{i=1}^{N-1} (z - p_i)^{d_i+1} = \lambda \left(\sum_{i=0}^{N-1} d_i \prod_{k=0; k \neq i}^{N-1} (z - p_k) \right).$$

When $\lambda = 0$ this equation has $d_0 + d_1 + \dots + d_{N-1} + N + 1$ solutions, the origin with multiplicity $d_0 + 2$ and p_i with multiplicity $d_i + 1$ for every $1 \leq i \leq N-1$. By continuity, for small enough values of $|\lambda|$, the $d_0 + d_1 + \dots + d_{N-1} + N + 1$ solutions to Eq. 2.1 become simple zeros of f'_λ that are approximately symmetrically distributed around the origin and the rest of the poles. As a consequence, when $|\lambda|$ is small, $d_0 + 2$ of the critical points of f_λ are grouped around $p_0 = 0$, near the pole at the origin, while $d_i + 1$ of the critical points are grouped around the pole p_i for every $1 \leq i \leq N-1$.

First we compute an approximation of the critical points near 0. We observe that solving Eq. (2.1) is equivalent to solving $T(z) = z$, where $T(z)$ is defined by

$$T(z) = \left(\lambda \frac{\sum_{i=0}^{N-1} d_i \prod_{k=0; k \neq i}^{N-1} (z - p_k)}{2 \prod_{i=1}^{N-1} (z - p_i)^{d_i+1}} \right)^{1/(d_0+2)} = \lambda^{\frac{1}{d_0+2}} R(z).$$

In the above expression we introduce the auxiliary function $R(z)$ and we observe that $R(z)$ does not depend on the variable λ . We also remark that there are $d_0 + 2$ possible different choices for the function T that are the $(d_0 + 2)$ branches of the map given by Eq. (2.1). Starting with the initial point 0 we find an approximate value, denoted by \tilde{c}_λ , of the critical point near the origin given by

$$\tilde{c}_\lambda = T(0) = k_0 \lambda^{\frac{1}{d_0+2}}, \quad \text{where } k_0 = \left(\frac{d_0}{2 \prod_{i=1}^{N-1} (-p_i)^{d_i}} \right)^{\frac{1}{d_0+2}}.$$

From the above expression, it is clear that the values of \tilde{c}_λ form the vertices of a regular polygon with $d_0 + 2$ sides centered at the origin. Since the critical point c_λ is a fixed point of the function T we can obtain an upper bound for the distance between the critical point c_λ and the approximate value \tilde{c}_λ . We have

$$|c_\lambda - \tilde{c}_\lambda| = |T(c_\lambda) - T(0)| \leq |T'(\xi)| |c_\lambda - 0| = |\lambda|^{\frac{1}{d_0+2}} |R'(\xi)| |c_\lambda|,$$

where ξ is a complex point in the segment joining the origin and c_λ . From the above expression we finally have that

$$\lim_{\lambda \rightarrow 0} \left| \frac{c_\lambda - \tilde{c}_\lambda}{\lambda^{\frac{1}{d_0+2}}} \right| \leq \lim_{\lambda \rightarrow 0} |R'(\xi)| \cdot |c_\lambda| = 0.$$

This proves that $c_\lambda = k_0 \lambda^{\frac{1}{d_0+2}} + o(\lambda^{\frac{1}{d_0+2}})$.

In the same way, we can compute an approximation of the critical points near the pole p_i , or in other words, in C_i . In this case solving Eq. (2.1) is equivalent to solving the equation $T(z) = z$ where

$$T(z) = p_i + \left(\lambda \frac{\sum_{i=0}^{N-1} d_i \prod_{k=0; k \neq i}^{N-1} (z - p_k)}{2 z^{d_0+2} \prod_{k=1, k \neq i}^{N-1} (z - p_k)^{d_k+1}} \right)^{1/(d_i+1)} = p_i + \lambda^{\frac{1}{d_i+1}} R(z).$$

In this case taking the initial value $T(p_i)$ we obtain an approximation of the critical points near p_i ,

$$\tilde{c}_\lambda = T(p_i) = p_i + k_i \lambda^{\frac{1}{d_i+1}}, \quad \text{where } k_i = \left(\frac{d_i}{2 p_i^{d_0+1} \prod_{k=1; k \neq i}^{N-1} (p_i - p_k)^{d_k}} \right)^{\frac{1}{d_i+1}}.$$

As before we can compute the distance between the critical point and its approximation

$$|c_\lambda - \tilde{c}_\lambda| = |T(c_\lambda) - T(p_i)| \leq |T'(\xi)| |c_\lambda - p_i| = |\lambda|^{\frac{1}{d_i+1}} |R'(\xi)| |c_\lambda - p_i|,$$

obtaining that $c_\lambda = \tilde{c}_\lambda + o(\lambda^{\frac{1}{d_i+1}})$.

Now, we turn our attention to critical values $v_\lambda = f_\lambda(c_\lambda)$. Let $c = c_\lambda$ be one of the $d_0 + d_1 + \dots + d_{N-1} + N + 1$ critical points of f_λ given by Eq. (2.1), we can rewrite this equation as

$$\frac{\lambda}{\prod_{i=0}^{N-1} (z - p_i)^{d_i}} = \frac{2z \prod_{i=0}^{N-1} (z - p_i)}{\sum_{i=0}^{N-1} d_i \prod_{k=0; k \neq i}^{N-1} (z - p_k)} = \frac{2z}{\sum_{i=0}^{N-1} \frac{d_i}{z - p_i}},$$

and then replace z by c_λ . Then the critical value $v_\lambda = f_\lambda(c_\lambda)$ corresponding to c_λ is given by

$$(2.2) \quad v_\lambda = f_\lambda(c) = c_\lambda^2 - c + \frac{\lambda}{\prod_{i=0}^{N-1} (c_\lambda - p_i)^{d_i}} = c_\lambda^2 + c + \frac{2c_\lambda}{\sum_{i=0}^{N-1} \frac{d_i}{c_\lambda - p_i}}.$$

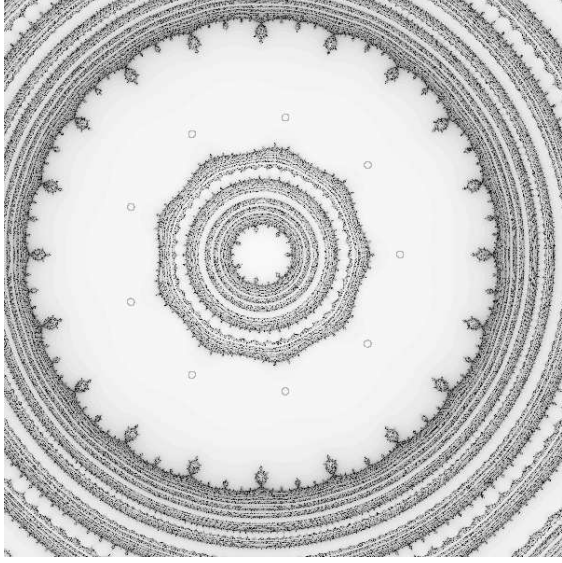
Then, we can compute an approximation for the critical values v_λ that we denote \tilde{v}_λ by computing $\tilde{v}_\lambda = f_\lambda(\tilde{c}_\lambda)$. As before we start with the critical points close to zero. For $c_\lambda \in C_0$, we have that $\tilde{c}_\lambda = k_0 \lambda^{\frac{1}{d_0+2}}$. Introducing this expression of \tilde{c}_λ in Eq. (2.2) we obtain

$$\tilde{v}_\lambda = p_1 + k_0^2 \lambda^{\frac{2}{d_0+2}} + \frac{2k_0 \lambda^{\frac{2}{d_0+2}}}{d_0/k_0 + \lambda^{\frac{1}{d_0+2}} \sum_{i=1}^{N-1} \frac{d_i}{k_0 \lambda^{\frac{1}{d_0+2}} - p_i}}.$$

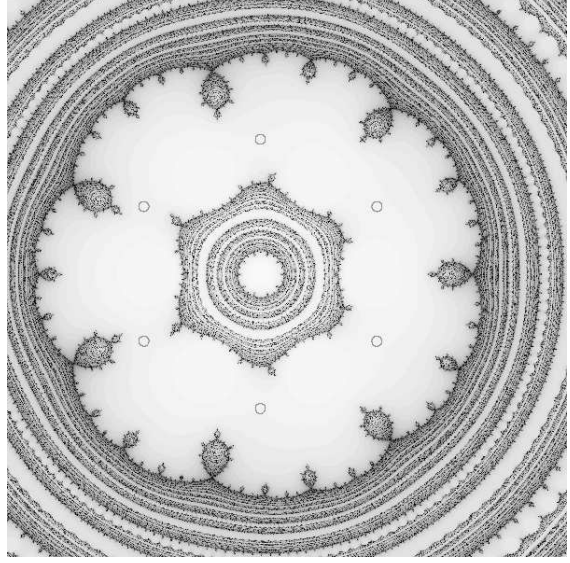
$$\text{Hence } \lim_{\lambda \rightarrow 0} \left| \frac{\tilde{v}_\lambda - p_1}{\lambda^{\frac{2}{d_0+2}}} \right| = |k_0|^2 (d_0 + 2)/d_0.$$

In the same way we obtain the corresponding result for $c_\lambda \in C_i$, using the approximation, $\tilde{c}_\lambda = k_i \lambda^{\frac{1}{d_i+1}}$, for $1 \leq i \leq N-1$.

□



(a) Magnification of the Julia set near $z = 0$. In the center we see T_0 and the 9 critical points in C_0 are located within the small circles.



(b) Magnification of the Julia set near $z = -1$. In the center we see T_1 and the 6 critical points in C_1 are located within the small circles.

Figure 3: Dynamical plane of $f_\lambda(z) = z^2 - 1 + 10^{-22}/(z^7(z+1)^5)$ near the two trap doors.

3 The immediate basin of attraction of infinity

In this section we prove that, for $|\lambda|$ sufficiently small, the boundary of B_λ is a homeomorphic copy of the Julia set of f_0 , i.e., the quadratic polynomial without the singular perturbation.

As is well known, there is a Böttcher coordinate φ_λ defined in a neighborhood of ∞ in B_λ that conjugates f_λ to $z \mapsto z^2$ in a neighborhood of ∞ . If none of the free critical points lie in B_λ , then it is well known that we may extend φ_λ so that it takes the entire immediate basin univalently onto $\mathbb{C} \setminus \overline{\mathbb{D}}^1$ and hence conjugates f_λ to $z \mapsto z^2$ on all B_λ . The following proposition shows that this occurs.

Proposition 3.1. *If $|\lambda|$ is sufficiently small, then the trap doors $T_i(\lambda)$ for $i = 0, \dots, N-1$ are disjoint from B_λ . Also, none of the free critical points lie in B_λ .*

Proof. Recall that the superattracting periodic orbit for f_0 is given by $p_0 = 0, p_1 = c, p_2, \dots, p_{N-1}$ where $f_0(p_{N-1}) = 0$. Let U_i denote the component of the interior of $K(f_0)$, the unperturbed filled Julia set, that contains p_i and γ_i the boundary of U_i , for $i = 0, \dots, N-1$. We have that f_λ^N preserves γ_i and is hyperbolic on this set since the critical orbit is periodic and hence bounded away from this set. There is an open annulus, A_i , containing γ_i such that $f_\lambda^N(A_i)$ is also an annulus and $A_i \subset f_\lambda^N(A_i)$. Similarly, provided $|\lambda|$ is sufficiently small, the larger annulus, $f_\lambda^N(A_i)$, is mapped completely over itself by f_λ^N . Hence B_λ cannot extend into the disk that is the bounded component of the complement of this annulus. Since the

¹We use $\overline{\mathbb{D}}$ to denote the closure of the unit disk in \mathbb{C} .

free critical points near p_i tend to p_i as λ tends to 0, this annulus may be chosen so that all the critical points near to p_i lie in this internal disk. Hence they do not lie in B_λ . \square

The idea of the proof of Theorem A is the following. For λ sufficiently small, we can define a holomorphic motion of B_0 parameterized by λ , obtaining B_λ as a result of this movement. Applying the Λ – *Lemma*, established by Mañé, Sad and Sullivan ([8]), we extend this holomorphic motion to the closure of B_0 . This will establish that the boundary of B_λ is a holomorphic motion of the boundary of B_0 which is equal to the Julia set of f_0 . First recall the definition of a holomorphic motion and the Λ – *Lemma* ([8]).

Definition.

Let $X \subset \hat{\mathbb{C}}$. We say that a map

$$\begin{aligned} H : X \times \mathbb{D} &\rightarrow \hat{\mathbb{C}} \\ (z, \lambda) &\rightarrow H(z, \lambda) \end{aligned}$$

is a holomorphic motion of X parameterized by \mathbb{D} if

- a) $H(z, 0) = z$ for all $z \in X$.
- b) the map $H(\cdot, \lambda) : X \rightarrow \hat{\mathbb{C}}$ is injective, for all $\lambda \in \mathbb{D}$.
- c) the map $H(z, \cdot) : \mathbb{D} \rightarrow \hat{\mathbb{C}}$ is holomorphic, for all $z \in X$.

Λ -Lemma. Let $H : X \times \mathbb{D} \rightarrow \hat{\mathbb{C}}$ be a holomorphic motion. Then, H is continuous, and it extends to a unique holomorphic motion $\bar{H} : \bar{X} \times \mathbb{D} \rightarrow \hat{\mathbb{C}}$, where \bar{X} is the closure of X .

Suppose $|\lambda|$ is chosen small enough so that Proposition 3.1 is satisfied and let $\delta > 0$ such that the Böttcher map φ_λ extends to the whole immediate basin of attraction B_λ for all $|\lambda| < \delta$. Let \mathbb{D}_δ denote the set of complex parameters λ with modulus smaller than δ . We can define now a holomorphic motion of B_0 . More precisely, consider the following map:

$$\begin{aligned} H : B_0 \times \mathbb{D}_\delta &\rightarrow \hat{\mathbb{C}} \\ (z, \lambda) &\rightarrow \varphi_\lambda^{-1} \circ \varphi_0(z). \end{aligned}$$

Next we verify that H is a holomorphic motion. By construction, we have that $H(z, 0) = \varphi_0^{-1} \circ \varphi_0(z) = z$. If we fix the parameter λ we can see that the map $H(\cdot, \lambda)$ is injective. This is immediate since the Böttcher mapping φ_λ is conformal. Finally, if we fix a point $z \in B_0$ we can see that $H(z, \cdot) : \mathbb{D} \rightarrow \mathbb{C}$ is a holomorphic map. In this case this map is a composition of holomorphic maps, since the Böttcher map depends analytically on parameters.

Geometrically, if we fix $\lambda \in \mathbb{D}$, the map $z \rightarrow H(z, \lambda)$ sends points in B_0 to points in B_λ according to the Böttcher coordinates (see Figure 4).

Finally, applying the Λ -Lemma to H , we obtain a new holomorphic motion $\bar{H} : \bar{B}_0 \times \mathbb{D} \rightarrow \hat{\mathbb{C}}$. Hence, it follows that the boundary of B_λ is the continuous image under \bar{H} of the Julia set of f_0 . Finally, interchanging the role of 0 and λ we obtain that \bar{H} is a homeomorphism. This concludes the proof of Theorem A.

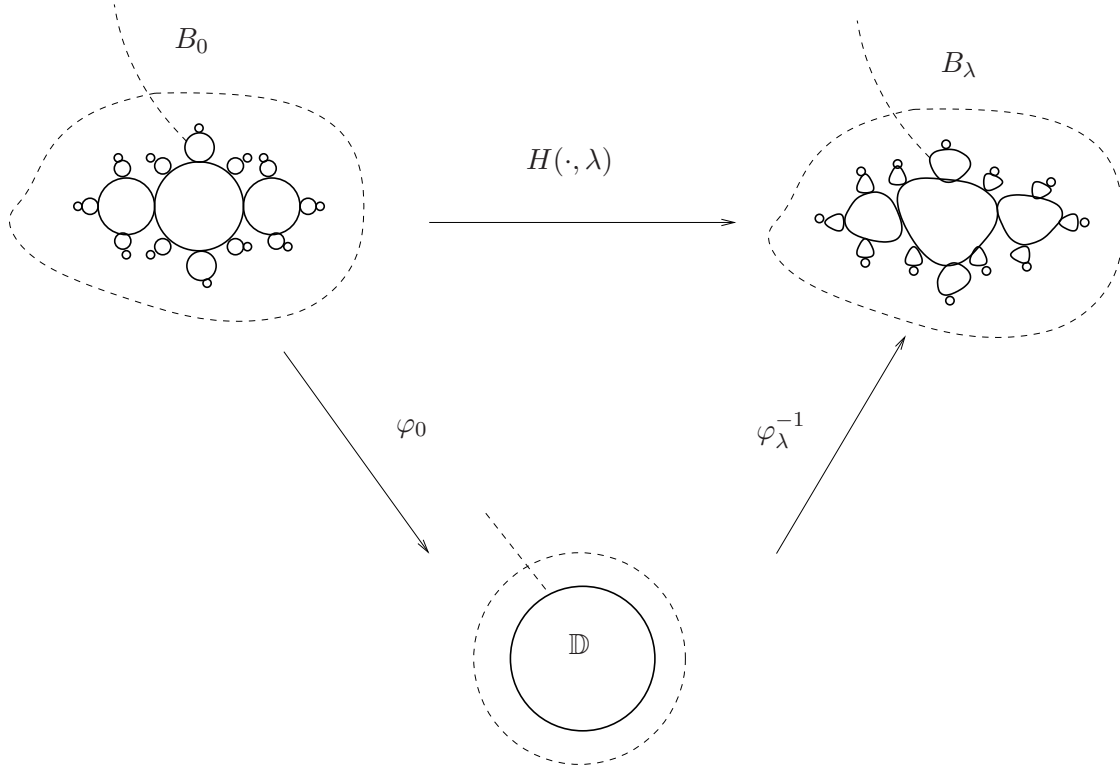


Figure 4: Sketch of the Holomorphic motion H . Geometrically, H sends equipotentials and rays from B_0 to B_λ according to Böttcher coordinates.

4 The Fatou set

In this section we will prove Theorem B that establishes the main properties of the Fatou set of f_λ for small enough values of the parameter λ where arithmetic condition \mathcal{I} is satisfied.

Lemma 4.1. *For λ sufficiently small and d_0, d_1, \dots, d_{N-1} satisfying arithmetic condition \mathcal{I} , the critical points in C_i are mapped inside the trap doors T_{i+1} , for $i = 0, \dots, N-1$, and escape to infinity.*

Proof. Let $c = c_\lambda$ be one of the $d_0 + d_1 + \dots + d_{N-1} + N + 1$ critical points of f_λ . Then the critical value v_λ corresponding to c_λ from equation 2.2 is given by

$$v_\lambda = c_\lambda^2 + c + \frac{2c_\lambda}{\sum_{i=0}^{N-1} \frac{d_i}{c_\lambda - p_i}}.$$

From the above expression it follows that, as $\lambda \rightarrow 0$, if $c_\lambda \rightarrow p_i$, then the corresponding critical value $v_\lambda \rightarrow p_{i+1}$, since $p_{i+1} = p_i^2 + c$. Hence, critical values corresponding to critical points in C_0 , are close to p_1 . In the same way, critical values corresponding to critical points in C_1 are close to p_2 and so on.

The fact that the critical values tend to the points p_0, p_1, \dots, p_{N-1} does not mean that the corresponding critical values belong to the trap doors T_0, T_1, \dots, T_{N-1} , respectively. However, when the order of the poles d_0, d_1, \dots, d_{N-1} satisfy the arithmetic condition \mathcal{I} (see definition 1.1) we will see that if λ is sufficiently small, all the free critical points of f_λ map within one of the trap doors after exactly one iterate of f_λ .

First, observe that all free critical points belong to the basin of attraction of infinity. Let $c_\lambda \in C_0$ and let v_0 denote the corresponding critical value. As $\lambda \rightarrow 0$, \tilde{v}_0 approaches p_1 at a rate of $\lambda^{2/(d_0+2)}$. Hence, we can write $\tilde{v}_0 = p_1 + k'_0 \lambda^{2/(d_0+2)}$. Introducing this value in f_λ we can see that the image of this point is close to ∞ when $2d_1 > d_0 + 2$ in the following expression

$$f_\lambda(\tilde{v}_0) = (\tilde{v}_0)^2 + c + \frac{\lambda}{\prod_{i=0}^{N-1} (\tilde{v}_0 - p_i)^{d_i}} =$$

$$(\tilde{v}_0)^2 + c + \frac{\lambda}{\lambda^{2d_1/(d_0+2)}} \cdot \frac{1}{k'_0 \prod_{i=0; i \neq 1}^{N-1} (\tilde{v}_0 - p_i)^{d_i}} \rightarrow \infty \text{ as } \lambda \rightarrow 0.$$

As λ tends to 0, note that if $f_\lambda(\tilde{v}_0)$ is close to ∞ , then \tilde{v}_0 is inside of the corresponding trap door T_1 . In the same way, for sufficiently small values of λ , if $d_{i+1} > d_i + 1$ then the image of the critical value corresponding to a critical point in C_i is close to ∞ . Hence the critical value, corresponding to a critical point in C_i , belongs to T_{i+1} . \square

To establish the topological characteristics of the Julia and Fatou sets of f_λ we use the Riemann-Hurwitz formula (see, for example, [11]).

Theorem 4.2. (*Riemann-Hurwitz formula*) *Let U and V be connected regions in the complex plane such that the connectivity (number of boundary components) of U is u and the connectivity of V is v and such that a map $g : U \rightarrow V$ is proper of degree k and contains n critical points in U . Then,*

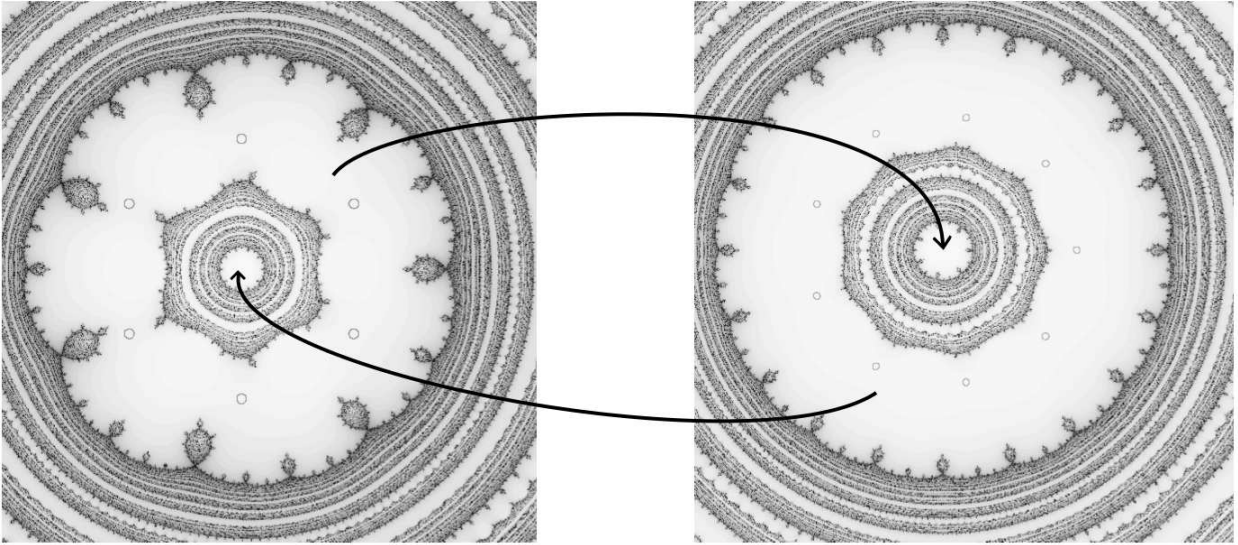
$$u - 2 = k(v - 2) + n.$$

Since all the critical points of f_λ are attracted to infinity, the Fatou set is the basin of attraction of infinity. Thus we can study the Fatou set by considering the connectivity of the connected components in the Fatou set. To do so, first we study the preimages of B_λ , second the preimages of the trap doors T_i and finally the rest of the components of the Fatou set.

In our case, B_λ is a simply connected domain and there are N trap doors, that is N disjoint preimages of B_λ that are also simply connected. Each trap door T_i surrounds the pole at p_i , for every $0 \leq i \leq N-1$. The pole p_i is a critical point of order $d_i - 1$ and infinity is a critical point of order 1. Then all the preimages of points in B_λ lie in $B_\lambda \cup T_0 \cup T_1 \cup \dots \cup T_{N-1}$. The set B_λ is mapped 2-to-1 onto itself and the sets T_i are mapped at least d_i -to-1 onto B_λ because $p_i \in T_i$, for every $0 \leq i \leq N-1$. Since their boundaries are mapped to ∂B_λ we have that the trap doors are bounded by inverted copies of the boundary of B_0 and not by

quasicircles (see Figure 5). Since the degree of the map f_λ is $2 + d_0 + d_1 + \dots + d_{N-1}$ we have that each trap door T_i is mapped onto B_λ with exactly degree d_i under f_λ .

It is easy to see that B_λ is disjoint from T_i and $T_i \cap T_j = \emptyset$ for $i \neq j$ because they are in different components of the homeomorphic copy of the filled Julia set of f_0 . Since the boundaries of the T_i are in $J(f_\lambda)$ we have that the Julia set is disconnected and then, by a classical result from complex dynamics (see, for example, [11]), it must consist of uncountably many distinct connected components.



(a) Magnification of the Julia set. On the left hand side, the trap door near $z = 0$ is the central white region. On the right hand side, the trap door near $z = -1$ is the central white region. The small circles show the location of the critical points and the arrows show the annulus that maps to each trap door under one application the map f_λ .

Figure 5: Dynamical plane for $f_\lambda(z) = z^2 - 1 + 10^{-22}/(z^7(z+1)^5)$ near the two trap doors.

Fix d_0, d_1, \dots, d_{N-1} which satisfy arithmetic condition \mathcal{I} (see definition 1.1) and let λ be chosen sufficiently small. Now we can study the preimages of the trap doors T_i for every $0 \leq i \leq N-1$. There are two different cases: the preimage of T_1 and the preimages of the rest of the trap doors. We consider each case separately.

Let $i \neq 1$ and consider the trap door T_i . From Theorem A, we know that ∂B_λ is homeomorphic to $J(f_0)$ and that the dynamics of f_λ are conjugate to the dynamics of f_0 restricted to the respective sets. Consider the components of the filled unperturbed Julia set that do not contain a point of in the superattracting cycle. There is no change to the dynamics of these components because λ is small enough so that the singular perturbation occurs outside of those components. Thus, for $|\lambda|$ small enough, there is a preimage of T_i that is simply connected and contains the preimage of p_{i-1} that is located inside a bulb that corresponds to a bulb outside of the superattracting N -cycle in the unperturbed map. We also have d_k preimages of T_i that are simply connected and are symmetrically distributed around each pole p_k where $k \neq i-1$. Hence we have $1 + d_0 + \dots + d_{i-1} + d_{i+1} + \dots + d_{N-1}$

preimages of T_i . The degree of f_λ is $2 + d_0 + \dots + d_{N-1}$ and it follows that the other $d_i + 1$ preimages of T_i are in an annulus that surrounds p_{i-1} and contains the $d_i + 1$ critical points in C_{i-1} .

To show that these preimages of T_i are in an annulus we apply the Riemann-Hurwitz formula. If we assume that the preimages are not connected we get a contradiction on the number of preimages of points of the trap doors. Suppose that the $d_i + 1$ critical points that surround p_{i-1} (the critical points in C_{i-1}) and are mapped inside T_i are not in one connected component. Suppose that there are two groups of critical points in different simply connected components of $f_\lambda^{-1}(T_i)$, one with u critical points and one with v critical points such that $u + v = d_i + 1$. Notice that there must be $d_i + 1$ preimages of points in T_i in these components (since one preimage of T_i is at $-p_{i-1}$ and the other preimages are around the poles). Then the component that contains u critical points contains at least $u + 1$ preimages of points in T_i and the other component contains at least $v + 1$ preimages of points in T_i . This gives a contradiction.

We now consider the preimages of T_1 . In this case we have d_k preimages of T_1 that are simply connected and are symmetrically distributed around each pole p_k where $k \neq 0$. Thus, there are $d_1 + \dots + d_{N-1}$ preimages of T_1 . Using the same argument as before, the other preimage of T_1 is now an annulus. In this case the annulus contains $d_0 + 2$ critical points (the critical points in S_0), and using the Riemann-Hurwitz formula, the map f_λ maps this annulus onto the trap door T_1 with degree $d_0 + 2$.

Finally, a simple argument describes the connectivity of all further preimages. Since all the critical points are either already accounted for or are contained in one of B_λ (the points of infinity), T_i (the poles), or in the annuli around p_i (the critical points in C_{i-1}) the Riemann-Hurwitz formula indicates that the preimage of each annulus is another annulus and the preimages of B_λ are simply connected domains.

5 The Julia set

In this section, we use the previous propositions to construct countably many Cantor sets of simple closed curves. We show that there are N such sets inside of the components which correspond to the connected components of the unperturbed Julia set which contain the superattracting cycle and infinitely many other such sets in the preimages of those sets. However, this is an incomplete description since it is known that repelling periodic points are dense in the Julia set. The boundary of B_λ contains repelling periodic points exactly as the basilica. We show that the remainder of the Julia set consists of Cantor sets of point components which contain the other repelling periodic points.

5.1 Cantor sets of simple closed curves

Recall that when $\lambda = 0$, f_λ has a superattracting cycle of period N given by p_0, p_1, \dots, p_{N-1} . Denote the immediate basin of attraction of p_i under the appropriate iterate of f_λ when

$\lambda = 0$ by U_i and the boundary of each U_i by γ_i . The set U_0 maps 2-to-1 over the set U_1 and U_j maps 1-to-1 over U_{j+1} . The boundary curves γ_i map over each other in an analogous way.

When $\lambda \neq 0$, the sets U_i and their boundaries persist due to the holomorphic motion argument presented earlier (see § 3). So, for f_λ there are analogous regions $U_0(\lambda), \dots, U_{N-1}(\lambda)$ and boundary curves $\gamma_i(\lambda)$ for $i = 0, \dots, N-1$. The dynamics on the boundary curves is the same as the unperturbed case, however the dynamics within the $U_i(\lambda)$ regions is very different due to the presence of the poles located at each point of the superattracting cycle of f_0 . The boundary of the trap door at p_i maps as a d_i -to-1 covering of the boundary of B_λ . Therefore, we call the boundary of the trap door within the set U_i a d_i -fold inverted copy of the Julia set of f_0 .

Inside $U_{i+1}(\lambda)$ there is a preimage of the complement of $U_i(\lambda)$. We denote this set $S_{i+1}(\lambda)$. Denote the boundary of $S_{i+1}(\lambda)$ by $\xi_{i+1}(\lambda)$. Note that the set $S_{i+1}(\lambda)$ contains the trap door in $U_{i+1}(\lambda)$ and also much more. In particular, it contains all of the “decorations” that point toward the pole at p_{i+1} . However, by the previous section, $S_{i+1}(\lambda)$ cannot contain any critical points except the pole p_{i+1} . Indeed, by the Riemann-Hurwitz formula, the preimage of the set $S_{i+1}(\lambda)$ is an annulus denoted by \mathcal{A}_i in the set $U_i(\lambda)$ which surrounds the set $S_i(\lambda)$. This annulus contains the free critical points that surround the pole p_i .

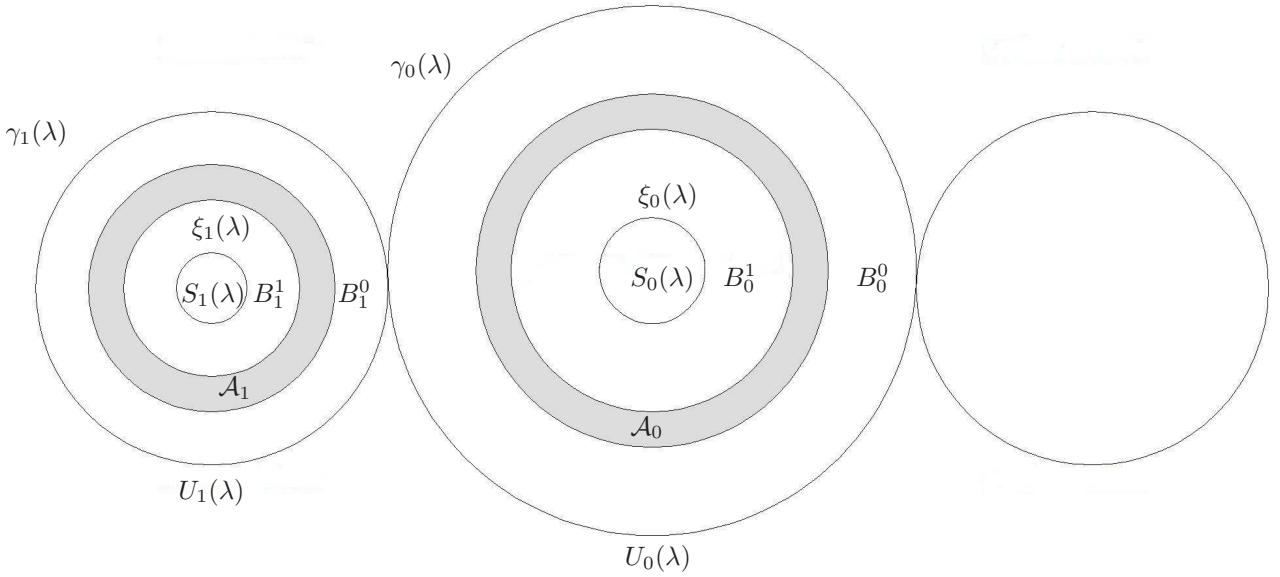


Figure 6: The regions $U_0(\lambda)$ and $U_1(\lambda)$ in the case where the unperturbed Julia set is the basilica.

Now, consider the case where the unperturbed map is the basilica, i.e., $f_\lambda(z) = z^2 - 1 + \lambda/(z^{d_0}(z+1)^{d_1})$ where d_0 and d_1 satisfy the arithmetic condition \mathcal{I} . (The cases for all other cycle lengths can be proven in the same way as the following case. We restrict to this case so that the notation is more manageable). The sets and boundary curves described

above are displayed in Figure 6. By the approximations presented earlier, we know that all of the free critical points map inside of the trap doors (see section § 4). Using the argument involving the Riemann-Hurwitz formula, the preimage of set $S_0(\lambda)$ is the annulus \mathcal{A}_1 and the preimage of the set $S_1(\lambda)$ is the annulus \mathcal{A}_0 . Consider the annulus bounded by the curves $\gamma_0(\lambda)$ and $\xi_0(\lambda)$. Call this annulus \mathcal{Q}_0 . Similarly, the annulus bounded by $\gamma_1(\lambda)$ and $\xi_1(\lambda)$ is denoted \mathcal{Q}_1 . Both boundary curves of \mathcal{Q}_0 map over the curve $\gamma_1(\lambda)$: the curve $\gamma_0(\lambda)$ maps over it twice and f_λ maps $\xi_0(\lambda)$ d_0 -to-1 over $\gamma_1(\lambda)$. Thus, \mathcal{Q}_0 is mapped $d_0 + 2$ times over \mathcal{Q}_1 . Similarly, \mathcal{Q}_1 is mapped $d_1 + 1$ times over \mathcal{Q}_0 .

Each \mathcal{Q}_i can be divided into three subannuli: B_i^0, B_i^1 , and \mathcal{A}_i . Consider the two annuli B_0^0 and B_0^1 inside of \mathcal{Q}_0 . Since the boundary curves $\gamma_0(\lambda)$ and $\xi_0(\lambda)$ are both mapped by f_λ over $\gamma_1(\lambda)$ it follows that both B_0^0 and B_0^1 map over the annulus \mathcal{Q}_1 : B_0^0 with degree 2 and B_0^1 with degree d_0 . Thus, there is a preimage of the annulus \mathcal{A}_1 inside of both B_0^0 and B_0^1 . Similarly, there will be a preimage of the annulus \mathcal{A}_0 in both the annuli B_1^0 and B_1^1 . We continue this process inductively infinitely many times. This construction yields a pair of Cantor sets of simple closed curves in $U_0(\lambda)$ and $U_1(\lambda)$ as in [12]. We call these Cantor sets of simple closed curves $\Lambda_0(\lambda)$ and $\Lambda_1(\lambda)$. In each $\Lambda_i(\lambda)$, there are countably many “boundary” curves which eventually map to γ_i and also an uncountable collection of buried curves that map to other buried curves in $\Lambda_0(\lambda) \cup \Lambda_1(\lambda)$ but never to either γ_i curve.

For each component corresponding to a connected component of the unperturbed Julia set, there will be a preimage of either $\Lambda_0(\lambda)$ or $\Lambda_1(\lambda)$. Note that any point in this portion of the Julia set will eventually map to either $\Lambda_0(\lambda)$ or $\Lambda_1(\lambda)$ and then remain in the union of $U_0(\lambda)$ and $U_1(\lambda)$ for all further iterates. This series of preimages produces countably many Cantor sets of simple closed curves in the Julia set of f_λ . We have shown:

Proposition 5.1. *Let λ be sufficiently small, $c \neq 0$ be chosen to be the center of a hyperbolic component of the Mandelbrot set, and suppose d_0, d_1, \dots, d_{N-1} satisfy the arithmetic condition \mathcal{I} . Then the Julia set of f_λ contains countably many Cantor sets of simple closed curves.*

There is actually much more structure in the Julia set of the family f_λ . In the above argument, the sets Λ_i were generated using preimages of the curves γ_i which is only a small part of the boundary of B_λ . If we take preimages of the full boundary of B_λ , we can see that the curves ξ_i are actually d_i -fold inverted copies of $J(f_0)$ rather than simple closed curves. We consider the preimages of the other components to be “decorations” on the boundary curves of Λ_i . The d_i decorations attached to the ξ_i all point toward the associated pole at p_i .

We continue with the example of the perturbed basilica. In $S_0(\lambda)$ we can observe d_0 large decorations that correspond to preimages of U_0 . The preimage of $S_0(\lambda)$ in U_0 is the annulus \mathcal{A}_1 . The inner boundary of \mathcal{A}_1 maps onto ξ_0 with degree d_1 . Thus there are $d_0 d_1$ large decorations that point inside of the annulus \mathcal{A}_1 . The outer boundary of \mathcal{A}_1 has degree 1 so it maps once over ξ_0 . It has d_0 large decorations that point toward the interior of the annulus \mathcal{A}_1 . On the left-hand side of Figure 5, we can see 7 large decorations on the outer boundary of \mathcal{A}_1 . There is a slight complication in the number of large decorations inside the

annulus \mathcal{A}_0 . The outer boundary maps to ξ_1 with degree 2 while the inner boundary maps to ξ_1 with degree d_0 . Thus there are 10 preimages of $U_1(\lambda)$ pointing toward the interior of the annulus. On the right-hand side of Figure 5, there are actually 20 “large decorations.” The additional 10 decorations correspond to preimages of the component containing the point $z = 1$. Notice in the unperturbed basilica that $f_0(-z) = f_0(z)$. Thus, the component containing the point $z = 1$ and the component containing the point $z = -1$ are the same size.

There are decorations on all curves which eventually map to the boundary of B_λ . All of the buried curves in Λ_i and their preimages never map to the boundary of B_λ . Therefore, these are simple closed curves, i.e., curves without decorations. This can be proven by performing quasiconformal surgery on the component of the unperturbed filled Julia set containing 0, just as in [1]. What remains is conjugate to a map of the form, $z^3 + C/z^3$ and is a Cantor set of quasicircles since the critical values lie within the trap door. Adding back the decorations will result in attachments on only those curves which eventually map to the boundary of the immediate basin of attraction of infinity. Therefore buried curves must be quasicircles. We have just shown:

Proposition 5.2. *In any preimage of a Cantor set of simple closed curves, $\Lambda_i(\lambda)$*

1. *Each unburied curve comes with attachments which are preimages of ∂B_λ .*
2. *Each buried curve is a simple closed curve without decorations.*

5.2 Cantor sets of point components

In the previous sections, the only portions of the Julia set that could contain periodic points were located in ∂B_λ or in one of the Cantor sets of closed curves, $\Lambda_0(\lambda), \Lambda_1(\lambda), \dots, \Lambda_{N-1}(\lambda)$. The countable many preimages of ∂B_λ and the preimages of the $\Lambda_i(\lambda)$ cannot contain periodic points since points within these sets eventually map to ∂B_λ or to the union of the $\Lambda_i(\lambda)$ and then remain there for all subsequent iterates. Thus, our description of the Julia set is incomplete because it is known that repelling periodic points are dense in the Julia set. In this section, we prove that the remainder of the Julia set consists of an uncountable collection of point components.

Recall that the point p_i of the superattracting cycle is contained in the component of $K(f_0)$ denoted U_i and that these sets persist (but not the super-attracting cycle) when $\lambda \neq 0$ due to the holomorphic motions argument (see § 3). The filled Julia set of f_0 consists of infinitely many other such open disks. For $j > N - 1$, let U_j denote a unique open disk. The way that the U_j are indexed is not important to the argument that follows. Let $U_j(\lambda)$ denote the corresponding open disk for f_λ . We shall consider just those points of the Julia set that are not in ∂B_λ or any of its preimages. We assign an itinerary to each point in this set.

We assign an itinerary to each point z in the usual way, i.e., $S(z) = (s_0 s_1 \dots)$ where $s_j = \ell$ if and only if $f_\lambda^\ell(z) \in U_\ell(\lambda)$. Since, by assumption, the orbit of z never lands on ∂B_λ there is no ambiguity in this definition.

Let \mathcal{G}_λ be the set of points in the Julia set of f_λ that do not lie either in ∂B_λ or in any of its preimages or in one of the Cantor sets $\Lambda_i(\lambda)$ or any of its preimages. We say that the itinerary $s = (s_0 s_1 \dots)$ is *allowable* if it corresponds to a point $z \in \mathcal{G}_\lambda$. Which sequences are allowable clearly depends on the choice of how the components of $K(f_0)$ are indexed.

Suppose $s = (s_0 s_1 \dots)$ is an allowable itinerary. Suppose some entry of the sequence, s_j , is an element D_i of the set $\mathcal{D} = \{0, 1, \dots, (N-1)\}$. Then either s_{j+1} is D_{i+1} or it is not. If it is not, then we call s_j a *departure index*, because this entry is where the itinerary departs from the itinerary of any 2-block in the superattracting N -cycle of f_0 , i.e., any 2-block in the itinerary $0\ 1\ \dots\ (N-1)$. Note that since we are working modulo N , if the itinerary is following the itinerary of the unperturbed map, the entry we would expect to follow the entry $(N-1)$ is 0. Furthermore, suppose the entry s_j is any entry which is not an element of \mathcal{D} . The orbit of such a point would follow the dynamics of the unperturbed map until an entry in its itinerary is from the set \mathcal{D} . This is due to the fact that the poles are located along the original superattracting cycle. The behavior far from the poles is not altered by the perturbation. Once the itinerary of this orbit contains an element from the set \mathcal{D} , it could either follow the cycle for any finite number of entries or could immediately depart from the cycle. For this reason, there will be infinitely many entries in an allowable itinerary from the set \mathcal{D} and, by the same logic, infinitely many of these entries must be departure indices since an itinerary for a point from the set \mathcal{G}_λ cannot eventually end in the repeating sequence $0\ 1\ \dots\ (N-1)$.

Proposition 5.3. *Suppose $s = (s_0 s_1 \dots)$ is an allowable itinerary that does not end in the repeating sequence $012\dots(N-1)$. Then the set of points with itinerary s is a Cantor set in $J(f_\lambda)$. Moreover, every point in this Cantor set is a point component of $J(f_\lambda)$.*

Before the proof of this statement, we will give two examples. As before, we use the example of the perturbed basilica. In both of these examples $\mathcal{D} = \{0, 1\}$. In other words, an itinerary has the ability to depart from the itinerary of the superattracting cycle whenever the corresponding orbit enters the component containing either $z = 0$ or $z = -1$.

Example 1: The set with itinerary $(\overline{02})$.

Consider the set of points in the Julia set with itinerary $(\overline{02})$. In this example, every entry that is a 0 is a departure index since 0 should be followed by a 1 if it is following the superattracting 2-cycle in the unperturbed basilica. Every entry that is a 2 is not a departure index. This set is allowable because the set U_0 maps over all of $\hat{\mathbb{C}}$ and the set U_2 maps in a 1-to-1 fashion over U_0 . First consider the set with partial itinerary 02, which we will denote W_{02} . Since ξ_0 maps d_0 -to-1 over γ_1 , there are d_0 small copies of U_0 in the interior of the set S_0 . Attached to each one of those copies of U_0 is one copy of the set U_2 . Thus, the set with partial itinerary 02 consists of d_0 small subdisks in the set U_0 . The set with partial itinerary 202 consists of d_0 subdisks inside of U_2 since U_2 maps in a 1-to-1 fashion over U_0 . Next we turn to the set with partial itinerary 0202, i.e., the set in U_0 that maps over the d_0 subdisks in U_2 with partial itinerary 202. Note that each of the subdisks in W_{02} maps over U_2 . Thus, there are d_0 subdisks nested within each of the subdisks in W_{02} that

map to W_{202} . It follows that W_{0202} consists of $(d_0)^2$ subdisks. Continuing in this fashion, every other iterate will yield a multiple of d_0 smaller, nested subdisks. By standard results in complex dynamics, the intersection of all of these sets yields a Cantor set of points with itinerary $(\overline{02})$. Any point in the collection is surrounded by arbitrarily small annuli that lie in the Fatou set (preimages of the annulus \mathcal{A}_0) thus each point in this Cantor set is actually a point component of the Julia set. See Figure 5.2 for a sketch of the first few nested disks.

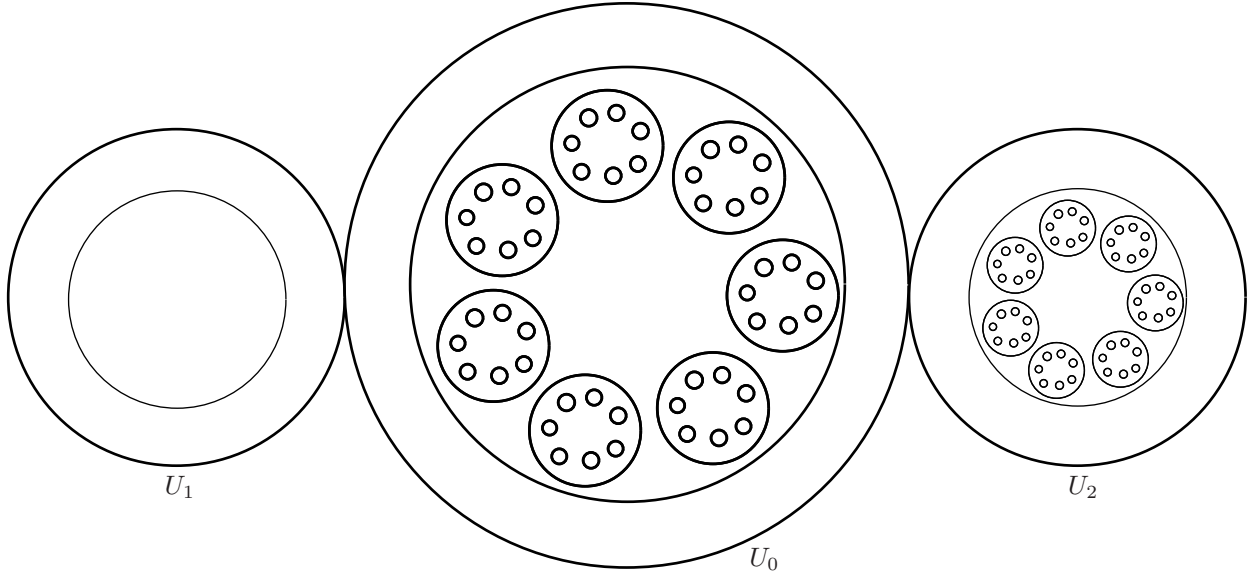


Figure 7: Sketch of some of the nested disks which generate a Cantor set of point components in Example 1.

Example 2: The set with itinerary $(\overline{012})$.

Next, we consider the set of points whose itinerary is $(\overline{012})$. In this example, every entry that is a 1 is a departure index since 1 should be followed by a 0 (rather than a 2) if it is following the superattracting 2-cycle in the unperturbed basilica. The remaining 0 and 2 entries are not departure indices. This allowable sequence is different from the first example because the partial itinerary 01 follows the attracting cycle before the sequence departs from the cycle. Allowable itineraries may follow the cycle for any finite number of iterates provided there is a departure index afterwards. This makes the construction slightly more complex.

First consider the set W_{12} , i.e., the set with partial itinerary 12. Since the curve ξ_1 in U_1 maps to γ_0 and the set U_2 is attached to γ_0 , there will be d_1 disks in the interior of S_1 that map to U_2 . Next, we consider the set W_{012} . Since the annulus \mathcal{A}_0 maps in a $(2 + d_0)$ -to-1 fashion over the set S_1 , the set W_{012} consists of $2d_1 + d_0d_1$ disks which are some of the decorations that are in the interior of the annulus \mathcal{A}_0 : $2d_1$ on the outer boundary of the annulus and d_0d_1 on the inner boundary of the annulus. Since U_2 maps 1-to-1 over U_0 , W_{2012}

consists of the same number of disks within the set U_2 . Continuing in this fashion, we find another Cantor set of points with this itinerary since all of the disks are nested within each other. Just as in the first example, these are point components of the Julia set since each point is surrounded by arbitrarily small annuli in the Fatou set. See Figure 5.2 for a sketch of the first few nested disks.

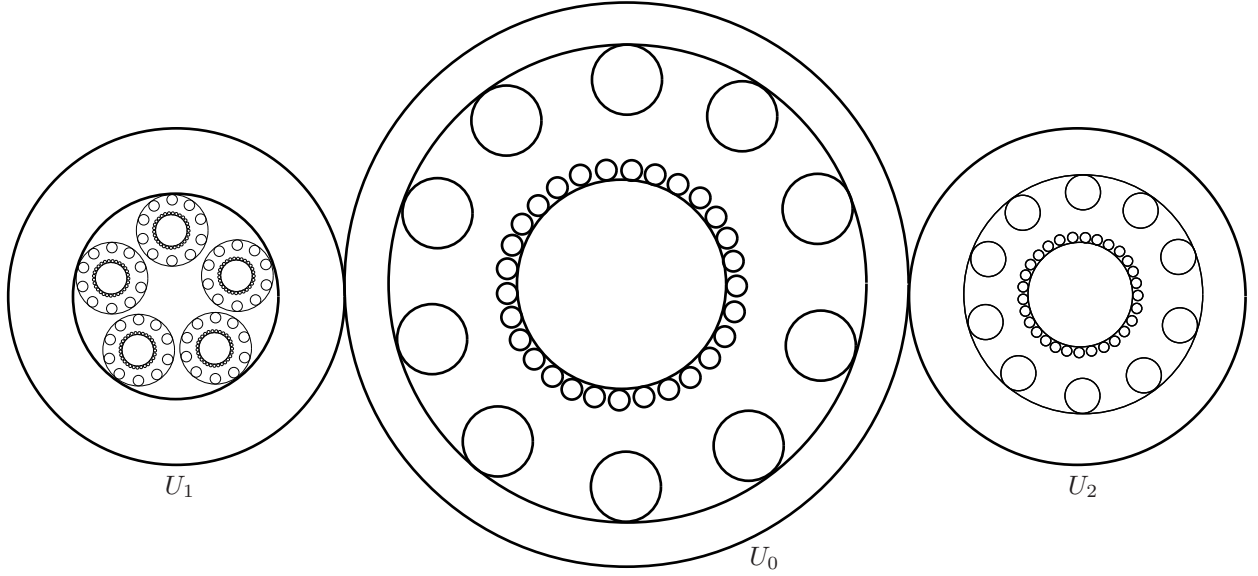


Figure 8: Sketch of some of the nested disks which generate a Cantor set of point components in Example 2.

Now we turn to the proof of the proposition which follows in the same way as the above examples.

Proof. Suppose that $s = (s_0 s_1 s_2 \dots)$ is an allowable itinerary. Since $s \in \mathcal{G}$, s must contain infinitely many entries D_i from the set \mathcal{D} . Since \mathcal{D} is a set with finitely many members, at least one of the D_i occurs infinitely often in the sequence s . Without loss of generality, assume that the entry 0 occurs infinitely many times in s and also that the first entry of s is 0, i.e., $s = (0 s_1 s_2 \dots)$. We can rewrite the sequence s as $(0 \tau_1 0 \tau_2 0 \dots)$ where τ_i denotes all of the nonzero entries between consecutive 0's.

There are only four distinct types of 2-blocks of consecutive entries that can occur in the sequence s and, more generally, in any allowable itinerary. The first is a block of the form 01. When this 2-block occurs, there are $d_0 + 2$ disks within U_0 that map to U_1 . When the 2-block $D_i D_{i+1}$ occurs, there are $d_i + 1$ disks within U_{D_i} that map to $U_{D_{i+1}}$. In this case when $D_i = N - 1$ then $D_{i+1} = 0$. Next, when the 2-block takes the form $D_i s_*$, where $s_* \neq D_{i+1}$, there are d_i disks within U_{D_i} that map within U_{s_*} . Finally, when the 2-block is of the form $s_i s_{i+1}$, where $s_i \notin \mathcal{D}$, there is only 1 disk in U_{s_i} that maps to $U_{s_{i+1}}$.

As before, we can consider those points with partial itinerary $0 \tau_1 0$. We claim that the set of points in U_0 following this partial itinerary corresponds to disks nested within each other. In order to compute the number of such disks it is enough to consider the product of the number of disks resulting from each 2-block in the partial itinerary. In the next step, we consider the points with partial itinerary $0 \tau_1 0 \tau_2 0$. We can determine the number of disks with this partial itinerary by creating a product similar to the one in the first step. These disks are nested within those of partial itinerary $0 \tau_1 0$. Continuing in this fashion, standard arguments from complex dynamics yield a Cantor set of points with the itinerary s . If the itinerary s begins with an entry different from zero, all we need do is pull the Cantor set of points back into the corresponding U_i . Like in both examples presented earlier, there are arbitrarily small Fatou components surrounding each of these points, preimages of the annuli \mathcal{A}_i , hence they are point components of the Julia set. \square

6 The perturbed basilica

In the final section, we turn to the example of the perturbed basilica once more to explore some of the details regarding the arithmetic condition \mathcal{I} . We consider the function

$$f_\lambda(z) = z^2 - 1 + \frac{\lambda}{z^{d_0}(z+1)^{d_1}}.$$

All of the theorems of this paper can be proven when the arithmetic condition \mathcal{I} is satisfied and for sufficiently small values of $|\lambda|$. In this case, the arithmetic condition is

$$(6.1) \quad \begin{cases} 2d_1 > d_0 + 2 \\ d_0 > d_1 + 1. \end{cases}$$

Figure 9 shows a graphical representation of the values of d_0 and d_1 that satisfy \mathcal{I} . In the figure, it is clear that the smallest values that satisfy \mathcal{I} are $d_0 = 7, d_1 = 5$. Notice that since the region is unbounded, there is an infinite collection of pairs of numbers that will also satisfy \mathcal{I} . Hereafter, we restrict to the specific case where $d_0 = 7, d_1 = 5$.

Notice that the degree f_λ is 14 and so there are 26 critical points counted with multiplicity. Infinity is a critical point of order 1, 0 is a critical point of order 6 and -1 of order 4. The other critical points of f_λ that satisfy the equation

$$2c^9(c+1)^6 = \lambda(7(c+1) + 5c)$$

and we see that when $\lambda \rightarrow 0$ there are 9 critical points that approach 0, and 6 critical points that approach -1 . The critical values corresponding to these 15 “free” critical points are determined by

$$v = f_\lambda(c) = c^2 - 1 + \frac{2c^2(c+1)}{7(c+1) + 5c}.$$

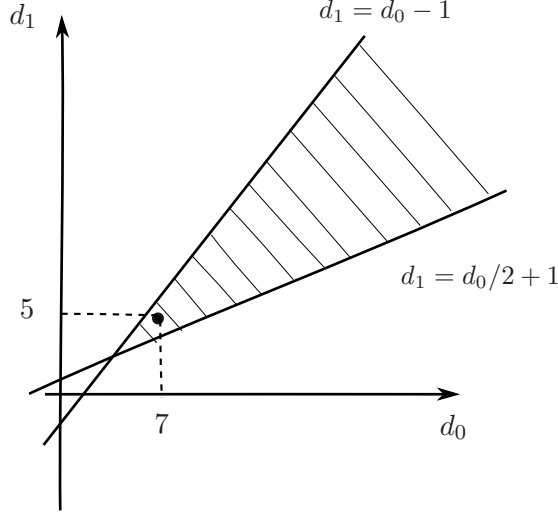


Figure 9: Sketch of the region where arithmetic condition \mathcal{I} is satisfied.

Note that if $c \rightarrow 0$ then $v \rightarrow -1$ and if $c \rightarrow -1$ then $v \rightarrow 0$.

Let C_0 denote the set of 9 critical points around 0 and C_{-1} denote the 6 critical points around -1 . Let c_0 be one of the critical points in C_0 and c_{-1} be one of the critical points in C_{-1} . Then, we can write

$$c_0 \approx \left(\frac{7}{2}\lambda\right)^{1/9} \quad c_{-1} \approx -1 + \left(\frac{5}{2}\lambda\right)^{1/6}.$$

We can find the critical values corresponding to c_0 and c_{-1} , respectively, by computing $f_\lambda(c_0)$ and $f_\lambda(c_{-1})$. We get

$$v_0 = f_\lambda(c_0) \approx -1 + \frac{9}{7} \left(\frac{7}{2}\lambda\right)^{2/9} = \tilde{v}_0$$

and

$$v_{-1} = f_\lambda(c_{-1}) \approx -\frac{12}{5} \left(\frac{5}{2}\lambda\right)^{1/6} = \tilde{v}_{-1}.$$

It is easy to check that when $\lambda \rightarrow 0$ we have $v_0 \rightarrow -1$ at a rate proportional to $\lambda^{2/9}$ and $v_{-1} \rightarrow 0$ at a rate proportional to $\lambda^{1/6}$. We can compute now the images of the critical values to see if they are close to infinity when λ is small. If this happens then we deduce that for λ small the critical values v_0 and v_{-1} lie in the trap doors around -1 and 0 , respectively.

$$\omega_0 = f_\lambda(\tilde{v}_0) = \tilde{v}_0^2 - 1 + \frac{\lambda}{\tilde{v}_0^7 \left(\frac{9}{7} \left(\frac{7}{2}\lambda\right)^{2/9}\right)^5} \rightarrow \infty$$

and

$$\omega_{-1} = f_\lambda(\tilde{v}_{-1}) = \tilde{v}_{-1}^2 - 1 + \frac{\lambda}{\left(-\frac{12}{5} \left(\frac{5}{2}\lambda\right)^{1/6}\right)^7 \tilde{v}_{-1}^5} \rightarrow \infty.$$

From these two equations we see that ω_0 and ω_{-1} tend to infinity when λ tends to zero so we conclude that v_0 and v_{-1} lie in the trap doors as we wanted to show.

This implies that we know the fate of the orbit of each critical point of f_λ . We have 9 critical points that surround the origin and are mapped inside T_1 and 6 critical points that surround -1 and are mapped inside T_0 (see Figure 5). These two trap doors are disjoint preimages of the immediate basin of infinity B_λ . Notice that B_λ is mapped to itself in a 2-to-1 fashion, the set T_0 is mapped 7-to-1 onto B_λ and T_1 is mapped 5-to-1 onto B_λ .

Finally, in the special case of the basilica we can give a general idea for the size of $|\lambda|$ required for our results to hold. To generate this estimate we need to know the approximate sizes of the immediate basin of attraction of infinity, B_λ , and the two trap doors: T_0 and T_1 .

First we approximate the size of B_λ . From f_λ it is easy to see that if $|\lambda| < 1/10$ then B_λ contains the open set $D_\infty = \{z \in \mathbb{C}; |z| > 4\}$. Second we can find a small disk, $D_0 = \{z \in \mathbb{C}; |z| < \delta\}$, that is completely contained in the trap door T_0 . This disk consists of the points that will map to D_∞ after one application of f_λ . When z is close to 0, we have that

$$|f_\lambda(z)| \geq \frac{|\lambda|}{|z|^7} - 1 > 4,$$

from the above expression taking $\delta = \left(\frac{|\lambda|}{5}\right)^{1/7}$ we see that the trap door T_0 contains the open disk $D_0 = \{z \in \mathbb{C}; |z| < \left(\frac{|\lambda|}{5}\right)^{1/7}\}$. In the same way we can find a second disk, D_1 centered at -1 , and completely contained in the trap door T_1 , given by $D_1 = \{z \in \mathbb{C}; |z+1| < \left(\frac{|\lambda|}{4}\right)^{1/5}\}$.

Now we can find the approximate size of λ . The idea is to require that the critical values, corresponding to critical points close to zero, are mapped inside the trap door T_1 . The critical points close to 0 are mapped to

$$v_0 \approx -1 + \frac{9}{7} \left(\frac{7}{2}\lambda\right)^{2/9}.$$

From the above estimate of T_1 requiring that $v_0 \in D_1 \subset T_1$ we have

$$\frac{9}{7} \left(\frac{7}{2}|\lambda|\right)^{2/9} < \left(\frac{|\lambda|}{4}\right)^{1/5}.$$

We also require that the critical values, corresponding to critical points close to -1 , are mapped inside the trap door T_0 . The critical points close to -1 are mapped to

$$v_{-1} \approx -\frac{12}{5} \left(\frac{5}{2}\lambda\right)^{1/6},$$

these values are in $D_0 \subset T_0$ when

$$\frac{12}{5} \left(\frac{5}{2} |\lambda| \right)^{1/6} < \left(\frac{|\lambda|}{5} \right)^{1/7}.$$

Thus, we can apply our results as long as $|\lambda|$ satisfies both inequalities, in this concrete case, when $|\lambda| \approx 10^{-22}$.

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