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GENERALIZED FRIEDMANN-ROBERTSON-WALKER HAMILTONIAN SYSTEMS: PERIODIC ORBITS AND NON–INTEGRABILITY

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ABSTRACT. The averaging theory of first order is applied to study a generalization of the Friedmann-Robertson-Walker Hamiltonian systems with three parameters. Two main results are proved. First, we provide sufficient conditions on the three parameters of the generalized system to guarantee the existence of continuous families of periodic orbits parameterized by the energy, and these families are given up to first order in a small parameter. Second, using such periodic orbits we provide information about the non-integrability of these Hamiltonian systems.

1. Introduction

The dynamics of the universe is an area of the astrophysics where the application of modern results coming from dynamical systems has been revealed very fruitful, specially in galactic dynamics see for instance the articles [2, 6, 9, 10, 15] and the references quoted there.

Calzeta and Hasi in [3] present analytical and numerical evidence of the existence of chaotic motion for the simplified Friedmann-Robertson-Walker Hamiltonian

$$H = \frac{1}{2}(p_Y^2 - p_X^2) + \frac{1}{2}(Y^2 - X^2) + \frac{b}{2}\,X^2Y^2, \tag{1}$$

which modelates a universe, filled with a conformally coupled but massive real scalar field. Although this model is too simplified to be considered realistic, its simplicity itself makes it an interesting testing ground for the implications of chaos in cosmology, either classical, semiclassical or quantum, see for more details [3]. Similar models have been used by Hawking [4] and Page [8] to discuss the relationship between the cosmological and thermodynamic arrow of time, in the framework of quantum cosmology.

In problems of galactic dynamics it is usual to consider potentials of the form $V(x^2, y^2)$, i.e. potentials exhibiting a reflection symmetry with respect to both axes, see [12] and the previous articles mentioned on galactic dynamics. For this reason here we generalize the Calzeta–Hasi's model as follows

$$H = \frac{1}{2}(p_Y^2 - p_X^2) + \frac{1}{2}(Y^2 - X^2) + \frac{a}{4}\,X^4 + \frac{b}{2}\,X^2Y^2 + \frac{c}{4}\,Y^4. \tag{2}$$

To determine salient features of the orbital structure of non–integrable Hamiltonian systems is a fundamental topic in the study of its dynamics. Knowledge

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of the existence and stability of periodic orbits, for example, is very important for clarifying the general understanding of the phase–space structure.

We have two main objectives. First, we shall provide sufficient conditions on the three parameters a, b and c of the Hamiltonian system with Hamiltonian (2) to guarantee the existence of continuous families of periodic orbits parameterized by the energy, and these families are given explicitly up to first order in a small parameter. Second, using such periodic orbits we provide information about the non–integrability of these Hamiltonian systems. Moreover the tools for proving these results can be applied to arbitrary Hamiltonian systems.

The first objective of this paper is to study analytically the periodic orbits of the two degree of freedom Hamiltonian system defined by Hamiltonian (2). We shall use the averaging theory for computing an explicit analytic approximation of four families of periodic orbits parameterized by the energy level H=h. The Hamiltonian system associated to Hamiltonian (2) is

$$
\begin{aligned}
\dot{X} &= -p_X, \\
\dot{Y} &= p_Y, \\
\dot{p}_X &= X - (a\,X^3 + b\,XY^2), \\
\dot{p}_Y &= -Y - (b\,X^2Y + c\,Y^3).
\end{aligned} \tag{3}
$$

Doing the rescaling of the variables

$$X = \sqrt{\varepsilon}x,\ Y = \sqrt{\varepsilon}y,\ p_X = \sqrt{\varepsilon}p_x,\ p_Y = \sqrt{\varepsilon}p_y,$$

the Hamiltonian system (3) becomes the new Hamiltonian system

$$
\begin{aligned}
\dot{x} &= -p_x, \\
\dot{y} &= p_y, \\
\dot{p}_x &= x - \varepsilon(a\,x^3 + b\,xy^2), \\
\dot{p}_y &= -y - \varepsilon(b\,x^2y + c\,y^3),
\end{aligned} \tag{4}
$$

with Hamiltonian

$$H = \frac{1}{2}(p_y^2 - p_x^2) + \frac{1}{2}(y^2 - x^2) + \varepsilon\left(\frac{a}{4}\,x^4 + \frac{b}{2}\,x^2y^2 + \frac{c}{4}\,y^4\right). \tag{5}$$

Periodic orbits are the most simple non-trivial solutions of a differential system. Their study is of special interest because the motion in their neighborhood can be determined by their kind of stability. Moreover, the existence of isolated periodic orbits in the energy levels of a Hamiltonian system with multipliers distinct from 1 force, under convenient assumptions the nonexistence of second first integrals of class C^1 independent of the Hamiltonian. In short, the study of the periodic orbits for a Hamiltonian system is relevant for these reasons. All the notions mentioned in this paragraph will be defined later on. We shall use the averaging theory of first order as it is stated in section 2 for studying the periodic orbits of the Hamiltonian system (4) in every energy level H=h. Our main result on the periodic orbits is the next one.

Theorem 1. *At every energy level $H = h$ with $h \neq 0$ the generalized Friedmann-Robertson-Walker Hamiltonian system* (4) *has at least one, two or three periodic solutions if one, two or three of the following conditions hold:*

(1) *$h(b+c)(a+2b+c) < 0$, $h(a+b)(a+2b+c) > 0$ and $b \neq 0$, the corresponding periodic solution is unstable if $b(a+2b+c) > 0$, and linear stable if $b(a+2b+c) < 0$;*
(2) *$h(b+3c)(3a+2b+3c) < 0$, $h(3a+b)(3a+2b+3c) > 0$ and $b \neq 0$, the corresponding periodic solution is unstable if $b(3a+2b+3c) < 0$, and linear stable if $b(3a+2b+3c) > 0$;*
(3) *$h < 0$, $b \neq 0$ and $(a+b)(3a+b) \neq 0$, the corresponding periodic solution is unstable if $(a+b)(3a+b) < 0$, and linear stable if $(a+b)(3a+b) > 0$; and*
(4) *$h > 0$, $b \neq 0$ and $(b+c)(b+3c) \neq 0$, the corresponding periodic solution is unstable if $(b+c)(b+3c) < 0$, and linear stable if $(b+c)(b+3c) > 0$.*

For the Hamiltonian (1) studied by Calzeta and Hasi [3] we have the following result, which follows directly from Theorem 1.

Corollary 2. *At every energy level $H = h$ with $bh \neq 0$ the Friedmann-Robertson-Walker Hamiltonian system* (4) *with $a = c = 0$ has at least one periodic solution.*

We can be more precise than in the statement of Theorem 1. Thus we consider the following seven hyperplanes

$$h = 0, a+b = 0, 3a+b = 0, b+c = 0, b+3c = 0, a+2b+c = 0, 3a+2b+c = 0,$$

in the 4-dimensional space of parameters $(h, a, b, c) \in \mathbb{R}^4$. These seven hyperplanes separate the parameter space $\mathbb{R}^4$ into $2^7 = 128$ open regions. Every one of these open regions will be denoted by the seven signs of the seven hyperplanes. Thus when we write $+++++++, 4$ this means that in the region

$$h > 0, a+b > 0, 3a+b > 0, b+c > 0, b+3c > 0, a+2b+c > 0, 3a+2b+3c > 0,$$

there is only one periodic orbit provided by the fourth condition of Theorem 1. On the other hand, ++++-++, 2, 4 this means that in the region

$$h > 0, a+b > 0, 3a+b > 0, b+c > 0, b+3c < 0, a+2b+c > 0, 3a+2b+3c > 0,$$

there are two periodic orbits provided by the second and fourth conditions of Theorem 1.

Note that either condition (3) or (4) of Theorem 1 always occurs in every one of the 128 open regions.

Now we summarize the number of periodic orbits in every one of the 128 open regions:

+ + + + + + +	4	− − − − − − −	3	− + + + + + +	3	+ − + + + + +	4
+ + − + + + +	4	+ + + − + + +	1, 4	+ + + + − + +	2, 4	+ + + + + − +	4
+ + + + + + −	4	− − + + + + +	1, 3	+ − − + + + +	4	+ + − − + + +	1, 4
+ + + − − + +	1, 2, 4	+ + + + − − +	2, 4	+ + + + + − −	4	− + − + + + +	2, 3
+ − + − + + +	4	+ + − + − + +	4	+ + + − + − +	4	+ + + + − + −	4
− + + − + + +	3	+ − + + − + +	2, 4	+ + − + + − +	4	+ + + − + + −	1, 4
− + + + − + +	1, 3	+ − + + + − +	1, 4	+ + − + + + −	2, 4	− + + + + − +	3
+ − + + + + −	4	− + + + + + −	3	+ + − − − − −	4	− + + − − − −	1, 2, 3
− − + + − − −	2, 3	− − − + + − −	3	− − − − + + −	3	− − − − − + +	3
+ − + − − − −	4	− + − + − − −	3	− − + − + − −	3	− − − + − + −	1, 3
− − − − + − +	2, 3	+ − − + − − −	1, 4	− + − − + − −	1, 3	− − + − − + −	2, 3
− − − + − − +	3	+ − − − + − −	2, 4	− + − − − + −	3	− − + − − − +	3
+ − − − − + −	4	− + − − − − +	1, 3	+ − − − − − +	4	− − − + + + +	1, 2, 3
− − + − + + +	3	− − + + − + +	1, 3	− − + + + − +	2, 3	− − + + + + −	1, 3
− + − − + + +	2, 3	− + − + − + +	3	− + − + + − +	2, 3	− + − + + + −	3
− + + − − + +	3	− + + − + − +	1, 3	− + + − + + −	3	− + + + − − +	3
− + + + − + −	2, 3	− + + + + − −	3	+ − − − + + +	4	+ − − + − + +	4
+ − − + + − +	1, 4	+ − − + + + −	2, 4	+ − + − − + +	2, 4	+ − + − + − +	4
+ − + − + + −	4	+ − + + − − +	1, 2, 4	+ − + + − + −	4	+ − + + + − −	1, 4
+ + − − − + +	1, 4	+ + − − + − +	4	+ + − − + + −	1, 2, 4	+ + − + − − +	4
+ + − + − + −	4	+ + − + + − −	2, 4	+ + + − − − +	2, 4	+ + + − − + −	1, 4
+ + + − + − −	4	+ + + + − − −	4	+ + + − − − −	4	+ + − + − − −	4
+ + − − + − −	2, 4	+ + − − − + −	4	+ + − − − − +	4	+ − + + − − −	1, 4
+ − + − + − −	4	+ − + − − + −	4	+ − + − − − +	2, 4	+ − − + + − −	1, 4
+ − − + − + −	4	+ − − + − − +	1, 4	+ − − − + + −	2, 4	+ − − − + − +	4
+ − − − − + +	4	− + + + − − −	2, 3	− + + − + − −	1, 3	− + + − − + −	2, 3
− + + − − − +	1, 3	− + − + + − −	3	− + − + − + −	3	− + − + − − +	3
− + − − + + −	3	− + − − + − +	1, 2, 3	− + − − − + +	3	− − + + + − −	3
− − + + − + −	1, 2, 3	− − + + − − +	3	− − + − + + −	3	− − + − + − +	3
− − + − − + +	3	− − − + + + −	1, 3	− − − + + − +	2, 3	− − − + − + +	1, 3
− − − − + + +	2, 3	+ − − − − − −	4	− + − − − − −	1, 3	− − + − − − −	2, 3
− − − + − − −	3	− − − − + − −	3	− − − − − + −	3	− − − − − − +	3

Theorem 1 is proved in section 3. Then, as a consequence of the existence of these periodic orbits, we can show our second result about the non–integrability of the Friedmann-Robertson-Walker systems (4) in the sense of Liouville–Arnold for any second first integral of class $\mathcal{C}^1$.

It is well known that integrable and non–integrable Hamiltonian systems can have infinitely many periodic orbits. However in general it is not easy to prove the existence of families of periodic orbits in an analytical way, specially if the Hamiltonian system is non–integrable. We can use the families of periodic orbits find in Theorem 1 to prove our second main result about the $\mathcal{C}^1$ non–integrability in the sense of Liouville–Arnold of Hamiltonian system (4). See section 4 for a precise definition of a Liouville–Arnold integrable Hamiltonian system.

Theorem 3. *The Hamiltonian system* (4) *with Hamiltonian H cannot have a $\mathcal{C}^1$ second first integral G such that the gradients of H and G are linearly independent at each point of the periodic orbits found in Theorem 1.*

Theorem 3 is proved in section 4.

Our study on the non Liouville–Arnold integrability uses isolated periodic orbits in the Hamiltonian levels, but other studies as the Moser–Holmes proof (see [7, 5]) use transverse homoclinic orbits. We remark that both our method and Moser-Holmes method work for Hamiltonian systems which are close to integrable systems.

2. The averaging theory

Now we shall present the basic results from averaging theory that we need for proving the results of this paper.

The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for the proof see Theorems 11.5 and 11.6 of Verhulst [14].

Consider the differential equation

$$\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{6}$$

with $\mathbf{x} \in D$, where D is an open subset of $\mathbb{R}^n$, $t \geq 0$. Moreover we assume that both $F_1(t, \mathbf{x})$ and $F_2(t, \mathbf{x}, \varepsilon)$ are $T-$periodic in t. We also consider in D the averaged differential equation

$$\dot{\mathbf{y}} = \varepsilon f_1(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{7}$$

where

$$f_1(\mathbf{y}) = \frac{1}{T} \int_0^T F_1(t, \mathbf{y}) dt.$$

Under certain conditions, equilibrium solutions of the averaged equation turn out to correspond with $T-$periodic solutions of equation (6).

Theorem 4. *Consider the two initial value problems (6) and (7). Suppose:*

(i) *F_1, its Jacobian $\partial F_1/\partial x$, its Hessian $\partial^2 F_1/\partial x^2$, F_2 and its Jacobian $\partial F_2/\partial x$ are defined, continuous and bounded by a constant independent of ε in $[0, \infty) \times D$ and $\varepsilon \in (0, \varepsilon_0]$.*

(ii) *F_1 and F_2 are $T-$periodic in t (T independent of ε).*

Then the following statements hold.

(a) *If p is an equilibrium point of the averaged equation* (7) *and*

$$\det \left(\frac{\partial f_1}{\partial \mathbf{y}} \right) \bigg|_{\mathbf{y}=p} \neq 0,$$

then there exists a $T-$periodic solution $\varphi(t, \varepsilon)$ of equation (6) *such that $\varphi(0, \varepsilon) \to p$ as $\varepsilon \to 0$.*

(b) *The stability or instability of the limit cycle $\varphi(t, \varepsilon)$ is given by the stability or instability of the equilibrium point p of the averaged system* (7). *In fact the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\varphi(t, \varepsilon)$.*

We point out the main facts in order to prove Theorem 4(b), for more details see Section 6.3 and 11.8 in [14].

3. Proof of Theorem 1

Periodic orbits of a Hamiltonian system of more than one degree of freedom are generically on cylinders filled with periodic orbits in the phase space (for more details see [1]), then we will not be able to apply directly the Averaging Theorem of section 2 to a Hamiltonian system because the Jacobian of the corresponding function f_1 at the fixed point a will be always zero. This problem will be solved by fixing an energy level, where the periodic orbits generically are isolated.

In the variables (x, y, p_x, p_y) we consider the Hamiltonian system (3).

Let $\mathbb{R}^+ = [0,\infty)$ and $\mathbb{S}^1$ the circle. We do the change of variables $(x, y, p_x, p_y) \to (r, \theta, s, \alpha) \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{S}^1$ defined by

$$x = r\cos\theta, \quad p_x = r\sin\theta, \quad y = s\cos(\alpha-\theta), \quad p_y = s\sin(\alpha-\theta).$$

Note that it is not canonical, so we loss the Hamiltonian structure of the differential equations. The differential system in the new variables become

$$\begin{aligned}
\dot r &= -\varepsilon\, r\sin\theta\,\cos\theta\left(ar^2\cos^2\theta + bs^2\cos^2(\alpha-\theta)\right),\\
\dot\theta &= 1-\varepsilon\cos^2\theta\left(ar^2\cos^2\theta + bs^2\cos^2(\alpha-\theta)\right),\\
\dot s &= -\varepsilon\frac{1}{4}s\,\left(br^2+cs^2+cs^2\cos(2(\alpha-\theta))+br^2\cos(2\theta)\right)\sin(2(\alpha-\theta)),\\
\dot\alpha &= \varepsilon\left(-cs^2\cos^4(\alpha-\theta) - b(r^2+s^2)\cos^2\theta\cos^2(\alpha-\theta) - ar^2\cos^4\theta\right),
\end{aligned}\tag{8}$$

having the first integral

$$H = \frac{1}{2}\left(s^2-r^2\right)+\frac{\varepsilon}{4}\left(cs^4\cos^4(\alpha-\theta)+2br^2s^2\cos^2(\alpha-\theta)\cos^2\theta + ar^4\cos^4\theta\right).\tag{9}$$

In order that the right hand side of the differential system (8) be periodic with respect to the independent variable, we change the old independent variable t by the new independent variable θ, for obtaining the periodicity necessary for applying the averaging theory. Dividing system (8) by $\dot\theta$ omitting the $\dot\theta$ equation, system (8) goes over to

$$\begin{aligned}
r' &= -\varepsilon\, r\sin\theta\,\cos\theta\left(ar^2\cos^2\theta + bs^2\cos^2(\alpha-\theta)\right)+O(\varepsilon^2),\\
s' &= -\frac{\varepsilon}{4}s\,\left(br^2+cs^2+cs^2\cos(2(\alpha-\theta))+br^2\cos(2\theta)\right)\sin(2(\alpha-\theta))+O(\varepsilon^2),\\
\alpha' &= \varepsilon\left(-cs^2\cos^4(\alpha-\theta) - b(r^2+s^2)\cos^2\theta\cos^2(\alpha-\theta) - ar^2\cos^4\theta\right)+O(\varepsilon^2),
\end{aligned}\tag{10}$$

where the prime denotes the derivative with respect to the new independent variable θ. System (10) is 2π–periodic in the variable θ. However as the differential system (10) comes from a Hamiltonian system, as we mentioned before, its periodic orbits are not isolated in the set of all periodic orbits of system (10). Consequently, in order to use the averaging theory for studying its periodic orbits, we restrict the differential system (10) to every fixed energy level $H(r,\theta,s,\alpha)=h$. Then in such energy levels, we can put s in function of h, θ, r and α and substitute s in (10), and we will be able to apply Theorem 4. For s we get

$$s = \sqrt{2h+r^2}+O(\varepsilon).$$

As we will apply averaging of first order, we do not need more information on s. Substituting s in equation (10), this becomes

$$\begin{aligned}
r' &= -\varepsilon\, r\sin\theta\,\cos\theta\left(ar^2\cos^2\theta + b(2h+r^2)\cos^2(\alpha-\theta)\right)+O(\varepsilon^2),\\
\alpha' &= \varepsilon\left(-c(2h+r^2)\cos^4(\alpha-\theta) - 2b(h+r^2)\cos^2(\alpha-\theta)\cos^2\theta - ar^2\cos^4\theta\right)+O(\varepsilon^2).
\end{aligned}$$

If we write the previous system as a Taylor series of first order in ε we get

$$\begin{aligned}
r' &= \varepsilon F_{11}(\theta,r,\alpha)+O(\varepsilon^2),\\
\alpha' &= \varepsilon F_{12}(\theta,r,\alpha)+O(\varepsilon^2).
\end{aligned}\tag{11}$$

We see that system (11) has the canonical form (6) for applying the averaging theory and satisfies the assumptions of Theorem 4 for $|\varepsilon| > 0$ sufficiently small, with $T = 2\pi$ and $F_1 = (F_{11}, F_{12})$ which are analytical functions.

Averaging the function F_1 with respect to the variable θ we obtain

$$f_1(r,\alpha) = (f_{11}(r,\alpha), f_{12}(r,\alpha)) = \frac{1}{2\pi}\int_0^{2\pi} \left(F_{11}(\theta,r,\alpha), F_{12}(\theta,r,\alpha)\right) d\theta\,, \tag{12}$$

where

$$\begin{aligned} f_{11}(r,\alpha) &= -\frac{1}{8}\, b\, r(2h+r^2)\sin(2\alpha),\\ f_{12}(r,\alpha) &= \frac{1}{8}\left(-6ch - 3(a+c)r^2 - 4b(h+r^2) - 2b(h+r^2)\cos(2\alpha)\right). \end{aligned} \tag{13}$$

We have to find the zeros (r^*, α^*) of the function $f_1(r,\alpha)$, and to check that the Jacobian determinant at these points is not zero, i.e.

$$\det\left(\left.\frac{\partial(f_{11}, f_{12})}{\partial(r,\alpha)}\right|_{(r,\alpha)=(r^*,\alpha^*)} \right) \neq 0. \tag{14}$$

From $f_{11}(r,\alpha) = 0$ we obtain that either

$$\alpha = 0, \pm\pi/2, \pi, \text{ or } r = \sqrt{-2h} \text{ when } h < 0. \tag{15}$$

We look for the solutions of $f_{12}(r,\alpha) = 0$ at these solutions of (15). We obtain seven possible solutions (α^*, r^*, s^*) with $r^* \geq 0$ and $s^* \geq 0$, namely

$$\begin{gathered} \left(0, \sqrt{\frac{-2h(b+c)}{a+2b+c}}, \sqrt{\frac{2h(a+b)}{a+2b+c}}\right), \left(\pm\frac{\pi}{2}, \sqrt{\frac{-2(b+3c)h}{3a+2b+3c}}, \sqrt{\frac{2(3a+b)h}{3a+2b+3c}}\right),\\ \left(\pm\frac{1}{2}\arccos\left(\frac{-3a-2b}{b}\right), \sqrt{-2h}, 0\right), \left(\pm\frac{1}{2}\arccos\left(\frac{-3a-2b}{b}\right), 0, \sqrt{2h}\right). \end{gathered} \tag{16}$$

But the solutions with $\pm$ provide two different initial conditions of the same periodic orbit. So we only have four different periodic orbits.

Finally we calculate the determinant (14) of the Jacobian matrix

$$\begin{pmatrix} -\frac{1}{8}b\left(2h+3r^2\right)\sin(2\alpha) & -\frac{1}{4}br\left(2h+r^2\right)\cos(2\alpha)\\ -\dfrac{1}{4}r\left(3a+4b+3c+2b\cos(2\alpha)\right) & \dfrac{1}{2}b\left(h+r^2\right)\sin 2\alpha \end{pmatrix} \tag{17}$$

at the four solutions (r^*, α^*, s^*) given in (16). The determinants are respectively given by

$$\begin{aligned} &\frac{3b(a+b)(b+c)h^2}{4(a+2b+c)}, \quad -\frac{b(3a+b)(b+3c)h^2}{4(3a+2b+3c)},\\ &\frac{3}{4}(a+b)(3a+b)h^2, \quad \frac{3}{8}(b+c)(b+3c)h^2. \end{aligned} \tag{18}$$

To have the solutions (16) defined and the above determinants different from zero, we must have one of the following four conditions

(1) $h(b+c)(a+2b+c) < 0$, $h(a+b)(a+2b+c) > 0$ and $b \neq 0$;
(2) $h(b+3c)(3a+2b+3c) < 0$, $h(3a+b)(3a+2b+3c) > 0$ and $b \neq 0$;
(3) $h < 0$, $b \neq 0$ and $(a+b)(3a+b) \neq 0$; and
(4) $h > 0$, $b \neq 0$ and $(b+c)(b+3c) \neq 0$.

We conclude that under each of the four cases, the solutions (r^*, α^*, s^*) of (16) provide a periodic solution of system (11), and consequently of system (4).

According to Theorem 4(b), for completing the proof of Theorem 1 we need to study the kind of stability of the found periodic orbits. For this we only need to study the eigenvalues of the Jacobian matrix (17) at the different solutions (r^*, α^*, s^*) of (16), which are respectively

(1) $\pm\sqrt{3}h\sqrt{-B(A+B)(B+C)/(A+2B+C)}/2$;
(2) $\pm h\sqrt{B(3A+B)(B+3C)/(3A+2B+3C)}/2$;
(3) $\pm\sqrt{3}h\sqrt{-(A+B)(3A+B)}/2$; and
(4) $\pm\sqrt{3}h\sqrt{-(B+C)(B+3C)}/4$.

according with the previous four conditions. Then, from statement (b) of Theorem 4, it follows the stability of the periodic orbits described in Theorem 1.

4. Proof of Theorem 3

We recall that a Hamiltonian system with Hamiltonian H of two degrees of freedom is *integrable in the sense of Liouville–Arnold* if it has a first integral C independent with H (i.e. the gradient vectors of H and C are *independent* in all the points of the phase space except perhaps in a set of zero Lebesgue measure), and in *involution* with H (i.e. the parenthesis of Poisson of H and C is zero).

We consider the autonomous differential system

$$\dot{x} = f(x), \tag{19}$$

where $f\colon U \to \mathbb{R}^n$ is $\mathcal{C}^2$, U is an open subset of $\mathbb{R}^n$ and the dot denotes the derivative respect to the time t. We write its general solution as $\phi(t, x_0)$ with $\phi(0, x_0) = x_0 \in U$ and t belonging to its maximal interval of definition.

We say that $\phi(t, x_0)$ is T*-periodic* with $T > 0$ if and only if $\phi(T, x_0) = x_0$ and $\phi(t, x_0) \neq x_0$ for $t \in (0, T)$. The variational equation associated to the T-periodic solution $\phi(t, x_0)$ is

$$\dot{M} = \left(\frac{\partial f(x)}{\partial x}\Big|_{x=\phi(t,x_0)}\right)M, \tag{20}$$

where M is an $n \times n$ matrix. The *monodromy matrix* associated to the T-periodic solution $\phi(t, x_0)$ is the solution $M(T, x_0)$ of (20) satisfying that $M(0, x_0)$ is the identity matrix. The eigenvalues λ of the monodromy matrix associated to the periodic solution $\phi(t, x_0)$ are called the *multipliers* of the periodic orbit.

For an autonomous differential system, one of the multipliers is always 1, and its corresponding eigenvector is tangent to the periodic orbit.

A periodic solution of an autonomous Hamiltonian system always has two multipliers equal to one. One multiplier is 1 because the Hamiltonian system is autonomous, and another one is 1 due to the existence of the first integral given by the Hamiltonian.

Theorem 5. *If a Hamiltonian system with two degrees of freedom and Hamiltonian H is Liouville–Arnold integrable, and C is a second first integral such that the gradients of H and C are linearly independent at each point of a periodic orbit of the system, then all the multipliers of this periodic orbit are equal to* 1.

Theorem 5 is due to Poincaré see section 36 of [11]. It gives us a tool to study the non Liouville–Arnold integrability, independently of the class of differentiability of the second first integral. The main problem for applying this theorem is to find periodic orbits having multipliers different from 1.

Proof of Theorem 3. We consider the four families of periodic orbits given in Theorem 1. Since the Jacobians (18) corresponding to the four periodic orbits depend on the 3 parameters a, b, c and on the energy level h, they are in general different from 1, and these Jacobians are the product of two multipliers of these periodic orbits. Therefore, it follows that such two multipliers are in general different from 1. Hence, Theorem 3 follows from Theorem 5. □

5. Conclusions

We have used two important tools of the area of dynamical systems. First the averaging theory for studying analytically the existence of periodic orbits and their stability adapted to Hamiltonian systems. The main results on the periodic orbits of the Hamiltonian system (4) are summarized in Theorem 1. The second tool based in Theorem 5 of Poincaré allows to study the $\mathcal{C}^1$ non–integrability in the sense of Liouville–Arnold of the Hamiltonian systems. Theorem 3 summarizes this result for our Hamiltonian system (4) .

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References

[1] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, Benjamin, Reading, Masachusets, 1978.

[2] C. Belmonte, D. Boccaletti and G. Pucacco, On the orbit structure of the logarithm potential, The Astrophysical J. **669** (2007), 202–217.

[3] E. Calzeta and C.E. Hasi, *Chaotic Friedmann-Robertson-Walker cosmology*, Class. Quantum Grav. **10** (1993), 1825–1841.

[4] S.W. Hawking, *Arrow of time in cosmology*, Phys. Rev. D **32** (1985), 2489–2495.

[5] P. Holmes, *Proof of non-integrability for the Hénon-Heiles Hamiltonian near an exceptional integrable case*, Physica D: Nonlinear Phenomena **5** (1982), 335–347.

[6] D. Merritt and M. Valluri, *Chaos and mixing in triaxial stellar systems*, The Astrophysical J. **471** (1996), 82–105.

[7] J. Moser, *Stable and random motions in dynamical systems. With special emphasis on celestial mechanics*, Annals of Mathematics Studies, No. **77**, Princeton University Press, Princeton, N. J, 1973.

[8] D. Page, *Will entropy decrease if the universe recollapses?* Phys. Rev. D **32** (1991), 2496–2499.

[9] Y. Papaphilippou and J. Laskar, *Frequency map analysis and global dynamics in a galactic potential with two degrees of freedom*, Astronomy and Astrophysics **307** (1996), 427–449.

[10] Y. Papaphilippou and J. Laskar, *Global dynamics of triaxial galactic models though frequency analysis*, Astronomy and Astrophysics **329** (1998), 451–481.

[11] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Vol. I, Gauthier-Villars, Paris 1899.

[12] G. Pucacco, D. Boccaletti and C. Belmonte, *Quantitative predictions with detuned normal forms*, Celest. Mech. Dyn. Astr. **102** (2008), 163–176.

[13] M. Schwarzschild, *A numerical model for a triaxial stellar system in dynamical equilibrium*, Astrophysical J. **232** (1979), 236–247.

[14] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, Universitext Springer Verlag, 1996.

[15] H.S. Zhao, C.M. Carollo and T. De Zeeuw, *Can galactic nuclei be non-axisymmetric? The parameter space of power-law discs*, Monthly Notices of the Royal Astronomical Society **304** (1999), 457–464.

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