This is a preprint of: "Partially periodic point free self-maps on surfaces, graphs, wedge sums and products of spheres", Jaume Llibre, Víctor F. Sirvent, *J. Difference Equ. Appl.*, vol. 19(10), 1654–1662, 2013.
DOI: [10.1080/10236198.2013.771634]

PARTIALLY PERIODIC POINT FREE SELF–MAPS ON GRAPHS, SURFACES AND OTHER SPACES

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ABSTRACT. Let (X, f) be a topological dynamical system. We say that it is partially periodic point free up to period n, if f does not have periodic points of periods smaller than $n+1$. When X is a compact connected surface, a connected compact graph, or $\mathbb{S}^{2m} \vee \mathbb{S}^m \vee \cdots \vee \mathbb{S}^m$, we give conditions on X, so that there exist partially periodic point free maps up to period n.

We also introduce the notion of a Lefschetz partially periodic point free map up period n. This is a weaker concept than partially periodic point free up period n. We characterize the Lefschetz partially periodic point free self–maps for the manifolds $\mathbb{S}^n \times \overset{k}{\cdots} \times \mathbb{S}^n$, $\mathbb{S}^n \times \mathbb{S}^m$ with $n \neq m$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{O}P^n$.

1. INTRODUCTION

Let X be a topological space and let $f : X \to X$ be a continuous map. A (*discrete*) *topological dynamical system* is formed by the pair (X, f).

We say that $x \in X$ is a *periodic point of period* k if $f^k(x) = x$ and $f^j(x) \neq x$ for $j = 1, \ldots, k-1$. We denote by $\text{Per}(f)$ the set of all periods of f.

The set $\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$ is called the *orbit* of the point $x \in X$. To study *the dynamics of a map* f is to study all the different kind of orbits of f. If x is a periodic point of f of period k, then its *orbit* is $\{x, f(x), f^2(x), \ldots, f^{k-1}(x)\}$, and it is called a *periodic orbit*.

Usually the periodic orbits play an important role in the dynamics of a discrete dynamical system, for studying them we can use topological information. One of the best known results in this direction are the results contained in the well known paper entitled *Period three implies chaos* for continuous self–maps on the interval, see [18].

If $\text{Per}(f) = \emptyset$ then we say that the map f is *periodic point free*. There are several papers studying different classes of periodic point free self–maps on the annulus see [11, 16], or on the 2–dimensional torus see [3, 12, 17].

If $\text{Per}(f) \cap \{1, 2, \ldots, n\} = \emptyset$ then we say that the map f is *partially periodic point free up to period* n. If $n = 1$, we say that f is *fixed point free*. There are

2010 *Mathematics Subject Classification.* 37C25, 37E15, 37E25.
Key words and phrases. Periodic point, Lefschetz number, connected compact graph, connected compact surface, orientable surface, non–orientable surface.

some papers studying different classes of partially periodic point free self–maps see [6, 21, 24].

Let n be the topological dimension of a compact polyhedron X. We denote by $H_k(X, \mathbb{Q})$, for $0 \le k \le n$, the homology linear spaces of X with coefficients over the rational numbers. These spaces are finite dimensional vector spaces over $\mathbb{Q}$. Given a continuous map $f : X \to X$, it induces linear maps $f_{*k} : H_k(X, \mathbb{Q}) \to H_k(X, \mathbb{Q})$, for $0 \le k \le n$. all the entries of the matrices f_{*k} are integer numbers.

The *Lefschetz number* $L(f)$ is defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \text{trace}(f_{*k}).$$

One of the main results connecting the algebraic topology with the fixed point theory is the *Lefschetz Fixed Point Theorem* which establishes the existence of a fixed point if $L(f) \neq 0$, see for instance [4]. If we consider the Lefschetz number of f^m, i.e. $L(f^m)$, then in general it is not true that $L(f^m) \neq 0$ implies that f has a periodic point of period m; it only implies the existence of a periodic point with period a divisor of m.

We say that a continuous map $f : X \to X$ is *Lefschetz periodic point free* if $L(f^m) = 0$ for all $m \ge 1$. We say that the map f is *Lefschetz partially periodic point free up to period* n if $L(f^m) = 0$ for all $1 \le m \le n$. If $n = 1$, we say that f is *Lefschetz fixed point free.* These are weaker notions of periodic point free and partially periodic point free up to period n, since Lefschetz periodic point free is a necessary condition to be a periodic point free, but not sufficient as it is shown by continuous self-maps on degree 1 on the circle, (*cf.* [1]). The Lefschetz periodic free maps on some connected compact manifolds have been studied in [10, 19].

One of the goals of this paper is to study the self–continuous maps on graphs and closed surfaces and other manifolds which are (Lefschetz) partially periodic point free up to period n.

A *graph* is a union of *vertices* and *edges*, which are homeomorphic to the closed interval, and have mutually disjoint interiors. The endpoints of the edges are vertices (not necessarily different) and the interiors of the edges are disjoint from the vertices.

Here a *closed surface* means a connected compact surface without boundary, orientable or not. More precisely, *an orientable connected compact surface without boundary of genus* $g \ge 0$, $\mathbb{M}_g$, is homeomorphic to the sphere if $g = 0$, to the torus if $g = 1$, or to the connected sum of g copies of the torus if $g \ge 2$. *An orientable connected compact surface with boundary of genus* $g \ge 0$, $\mathbb{M}_{g,b}$, is homeomorphic to $\mathbb{M}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows $\mathbb{M}_{g,0} = \mathbb{M}_g$.

A non–orientable connected compact surface without boundary of genus $g \ge 1$, $\mathbb{N}_g$, is homeomorphic to the real projective plane if $g = 1$, or to the connected sum of g copies of the real projective plane if $g > 1$. *A non–orientable connected compact surface with boundary of genus* $g \geq 1$, $\mathbb{N}_{g,b}$, is homeomorphic to $\mathbb{N}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows $\mathbb{N}_{g,0} = \mathbb{N}_g$.

Theorem 1. *Let $f : X \to X$ be a continuous map partially periodic point free up to period n. Assume that the induced map in the first homology space $f_{*1} : H_1(X, \mathbb{Q}) \to H_1(X, \mathbb{Q})$ is invertible. Then the following statements hold.*

- (a) *Let X be a connected compact graph such that $\dim H_1(X, \mathbb{Q}) = r$. If $n \geq 2$ then $n < r$.*
- (b) *Let $X = \mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components. Let d be the degree of f if $b = 0$.*
  - (b.1) *If $b = 0$, $d \neq 0$ and $n \geq 3$, then $n < 2g$ (consequently $g > 1$).*
  - (b.2) *If either $b = d = 0$ or $b > 0$, and $n \geq 2$ then $n < 2g + b - 1$.*
- (c) *Let $X = \mathbb{N}_{g,b}$ be a non–orientable connected compact surface of genus $g \geq 1$ with $b \geq 0$ boundary components. If $n \geq 2$ then $n < g + b - 1$.*

All the results stated in this introduction are proved in section 2.

One of the first results that show the relation between the topology of a compact topological space M and the existence of periodic points of a homeomorphism $f : M \to M$ is due to Fuller [8]. In particular, from Fuller's result it follows that if $g \geq 1$ and $f : \mathbb{M}_g \to \mathbb{M}_g$ is a homeomorphism then $\text{Per}(f) \cap \{1, 2, \ldots, 2g\} \neq \emptyset$, see for more details [7].

For homeomorphisms there are results improving Theorem 1 without boundary components by Wang [24], and with boundary components by Chas [5]. In fact, the proof of Theorem 1 uses ideas of [24].

The technique of using Lefschetz numbers to obtain information about the periods of a map is also used in many other papers, see for instance the book of Jezierski and Marzantowicz [15], the article of Gierzkiewicz and Wójcik [**?**] and the references quoted in both.

Given topological spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the *wedge sum* $X \vee Y$ is the quotient of the disjoint union X and Y obtained by identifying x_0 and y_0 to a single point (see for more details pp. 10 of [14]). The wedge sum is also known as "one point union". For example, $\mathbb{S}^1 \vee \mathbb{S}^1$ is homeomorphic to the figure "8," two circles touching at a point. We can think the wedge sums of spheres as generalization of graphs in higher dimensions.

Theorem 2. *Let $X = \mathbb{S}^{2m} \vee \mathbb{S}^m \vee \overset{s-times}{\cdots} \vee \mathbb{S}^m$, and $f : X \to X$ be a continuous map partially periodic point free up to period n. Assume that the induced map in the m-homology space $f_{*m} : H_m(X, \mathbb{Q}) \to H_m(X, \mathbb{Q})$ is invertible.*

- (a) *If m is odd then $n < s$.*
- (b) *If m is even, s odd and the degree of f is -1 then $n < s$.*

Theorem 3. *Let* $X = \mathbb{S}^n \times \overset{k-times}{\cdots} \times \mathbb{S}^n$ *and* $f : X \to X$ *be a continuous map.*

(a) *If* n *is odd,* f *is Lefschetz periodic point free if and only if* 1 *is an eigenvalue of* f_{*n}.

(b) *If* n *is even,* f *is Lefschetz partially periodic point free up to period* m *if* -1 *is an eigenvalue of* f_{*n}^l, *for* $1 \leq l \leq m$. *Moreover* f *is not Lefschetz periodic point free. Additionally, if* $m \geq 2$ *then* $k \geq 3$.

If $X = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ the previous result is well know, see [13].

In [10] the case $k = 2$ is considered. Necessary and sufficient conditions are given for a self-continuous map to be Lefschetz periodic free point. However those conditions are stated in a different manner, they are special cases of the present theorem.

In the following propositions we shall consider continuous self maps when the space X is $\mathbb{S}^n \times \mathbb{S}^m$, with $n \neq m$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{O}P^n$; the n-dimensional projective plane over the complex numbers, the quaternions and the octonions. We give conditions when they are Lefschetz (partially) periodic point free. The results presented here are generalizations of results contained in [10].

Proposition 4. *Let* $X = \mathbb{S}^n \times \mathbb{S}^m$ *and* $f : X \to X$ *be a continuous map with induced maps on homology* $f_{*n} = a$, $f_{*m} = b$ *and* $f_{*n+m} = d$.

(a) *If* n *and* m *are even and* $1 + a + b + d \neq 0$, f *has fixed points. If* $1 + a + b + d = 0$, *then the map* f *is Lefschetz fixed point free and it is not Lefschetz partially periodic point free up to period* 2.

(b) *If* n *and* m *are not booth even and if* f *is Lefschetz partially periodic point free up to period* 2, *then* f *is Lefschetz periodic point free.*

Proposition 5. *Let* $X = \mathbb{C}P^n$ *be the* n*-dimensional complex projective space and* $f : X \to X$ *be a continuous map with induced map in homology* $f_{*2} = a$. *Then the map* f *is Lefschetz fixed point free if* $a = -1$, *otherwise the map* f^m *has fixed points for all* $m \geq 1$.

Proposition 6. *Let* $X = \mathbb{H}P^n$ $(X = \mathbb{O}P^n$, *respec.) be the* n*-dimensional quaternion (octonionic, respec.) projective space and* $f : X \to X$ *be a continuous map with induced map in homology* $f_{*4} = a$, $(f_{*8} = a$, *respec.). Then the map* f *is Lefschetz fixed point free if* $a = -1$, *otherwise the map* f^m *has fixed points for all* $m \geq 1$.

For more information on the complex projective spaces $\mathbb{C}P^n$ or the quaternion projective spaces $\mathbb{H}P^n$ see for instance [23], and for the octonionic projective spaces $\mathbb{O}P^n$, see [2].

2. Proof of Theorems and Propositions

We separate the proof for the different statements of Theorem 1.

Proof of statement (*a*) *of Theorem 1.* Let X be a connected compact graph. Since the continuous map $f : X \to X$ does not have periodic points up to period n, by the Lefschetz fixed point theorem we have

$$L(f) = L(f^2) = \cdots = L(f^n) = 0. \tag{1}$$

Due to the fact that X is connected we know that $f_{*0} = (1)$, i.e. f_{*0} is the identity of $H_0(X, \mathbb{Q}) = \mathbb{Q}$, for more details see [20, 23]. From the definition of the Lefschetz number and if $\alpha_j = \text{trace}(f_{*1}^j)$ we have

$$L(f^j) = \text{trace}(f_{*0}^j) - \text{trace}(f_{*1}^j) = 1 - \alpha_j = 0.$$

Therefore, $\alpha_j = 1$ for $1 \le j \le n$. Let $r = \dim H_1(X, \mathbb{Q})$ and $\lambda_1, \cdots, \lambda_r$ be the eigenvalues of f_{*1}, so

$$\alpha_j = \sum_{i=1}^{r} \lambda_i^j.$$

The characteristic polynomial of f_{*1} is

$$p(x) = \prod_{j=1}^{r} (x - \lambda_j) = x^r - a_1 x^{r-1} + \cdots + (-1)^r a_r. \tag{2}$$

Due to Newton's formulae for symmetric polynomials (see for instance [22]), we have

$$\begin{array}{rcl} \alpha_1 - a_1 & = & 0, \\ \alpha_2 - a_1\alpha_1 + 2a_2 & = & 0, \\ \alpha_3 - a_1\alpha_2 + a_2\alpha_1 - 3a_3 & = & 0, \\ & \vdots & \\ \alpha_r + \displaystyle\sum_{i=1}^{r-1} (-1)^i a_i \alpha_{r-i} + (-1)^r r a_r & = & 0. \end{array} \tag{3}$$

So we get $a_1 = \alpha_1 = 1$, $a_2 = 0$, and by induction $a_j = 0$ for $2 \le j \le n$. Since $a_r = \det(f_{*1}) \neq 0$, if $n \ge 2$ then $n < r$, and statement (a) is proved. □

Proof of statement (b) *of Theorem 1.* Let $X = \mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \ge 0$ with $b \ge 0$ boundary components. Since the continuous map $f : X \to X$ does not have periodic points up to period n, by the Lefschetz fixed point theorem we have (1).

Suppose that the degree of f is d if $b = 0$. We recall the homological spaces of $\mathbb{M}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$, $n_1 = 2g + b - 1$ if $b > 0$, $n_2 = 1$ if $b = 0$, and $n_2 = 0$ if $b > 0$; and the induced linear maps $f_{*0} = (1)$, $f_{*2} = (d)$ if $b = 0$, and $f_{*2} = 0$ if $b > 0$ (see for additional details [20, 23]). In the next computations we must take $d = 0$ if $b > 0$.

From the definition of the Lefschetz number and if $\alpha_j = \text{trace}(f_{*1}^j)$ we have

$$L(f^j) = \text{trace}(f_{*0}^j) - \text{trace}(f_{*1}^j) + \text{trace}(f_{*2}^j) = 1 - \alpha_j + d^j = 0.$$

Therefore, $\alpha_j = 1+d^j$ for $1 \leq j \leq n$. Now the characteristic polynomial of f_{*1} is (2) with $r = n_1$. Using the Newton's formulae (3) we get $a_1 = 1+d$, $a_2 = d$, $a_3 = 0$, and by induction $a_j = 0$ for $3 \leq j \leq n$. Note that if $b > 0$ then $a_j = 0$ for $2 \leq j \leq n$. Since $a_{n_1} = \det(f_{*1}) \neq 0$, if $n \geq 3$ then $n < n_1$, and statement (b) is proved. $\square$

Proof of statement (c) *of Theorem 1.* Let $X = \mathbb{N}_{g,b}$ be a non–orientable connected compact surface of genus $g \geq 1$ with $b \geq 0$ boundary components. Since the continuous map $f : X \to X$ does not have periodic points up to period n, by the Lefschetz fixed point theorem we have (1).

We recall the homological groups of $\mathbb{N}_g$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{N}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = g+b-1$ and $n_2 = 0$; and the induced linear map $f_{*0} = (1)$ (see again for additional details [20, 23]).

From the definition of the Lefschetz number and if $\alpha_j = \text{trace}(f_{*1}^j)$ we have

$$L(f^j) = \text{trace}(f_{*0}^j) - \text{trace}(f_{*1}^j) = 1 - \alpha_j = 0.$$

Therefore, $\alpha_j = 1$ for $1 \leq j \leq n$. Now the characteristic polynomial of f_{*1} is (2) with $r = n_1$. Using the Newton's formulae (3) we get $a_1 = 1$, $a_2 = 0$, and by induction $a_j = 0$ for $2 \leq j \leq n$. Since $a_{n_1} = \det(f_{*1}) \neq 0$, if $n \geq 2$ then $n < n_1$, and statement (c) is proved. $\square$

Proof of Theorem 2. Let $X = \mathbb{S}^{2m} \vee \mathbb{S}^m \vee \overset{s-\text{times}}{\cdots} \vee \mathbb{S}^m$. Using the properties of the wedge sum (see pp. 160 of [14]), the homology spaces of X with coefficients in $\mathbb{Q}$ are:

$$H_k(X, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = n_{2m} = 1$, $n_m = s$ and 0 otherwise. So the non-trivial induced linear maps are f_{*0}, f_{*m} and f_{*2m}, where $f_{*0} = 1$ and $f_{*2m} = (d)$, the degree of f.

We adapt here the argument used in the proof of Theorem 1. Using the fact that m is odd, the Lefschetz numbers of the iterates of f are:

$$L(f^j) = \text{trace}(f_{*0}^j) - \text{trace}(f_{*m}^j) + \text{trace}(f_{*2m}^j) = 1 - \alpha_j + d^j,$$

where $\alpha_j = \text{trace}(f_{*m}^j)$. If f is partially periodic point free up to period n. Then $L(f^j) = 0$ for $1 \leq j \leq n$. Now the characteristic polynomial of f_{*m} is (2) with $r = n_m = s$. Using the Newton's formulae (3) we get $a_1 = 1+d$, $a_2 = d$ and $a_j = 0$ for $3 \leq j \leq n$. Since $a_s = \det(f_{*m}) \neq 0$ then $n < n_m = s$. This completes the proof of statement (a).

If m is even, then $L(f^j) = 1 + \alpha_j + d^j$. Assuming $L(f^j) = 0$ for $0 \leq j \leq n$, using the Newton's formulae (3) and induction we get

$$a_j = (-1)^j (1 + d + \cdots + d^j),$$

for $1 \le j \le n$. If $d = -1$ and j odd, then $a_j = 0$. So, if s is odd and $s \le n$, then $a_s = 0$. Which contradicts the hypothesis of $\det(f_{*m}) \neq 0$. Therefore $n < s$. This completes the proof of statement (b). □

Proof of Theorem 3. Let $X = \mathbb{S}^n \times \overset{k-\text{times}}{\cdots} \times \mathbb{S}^n$. According to the Künneth theorem, the homological groups of X over $\mathbb{Q}$ are

$$H_{jn}(X, \mathbb{Q}) = \mathbb{Q} \oplus \overset{c(k,j)}{\cdots} \oplus \mathbb{Q}$$

where $c(k, j) = \binom{k}{j} = k!/(j!(k-j)!)$, for $0 \le j \le k$; and $H_l(X, \mathbb{Q})$ is trivial if l is not of the form jn.

Taking the inclusion of $\mathbb{Q}$ into $\mathbb{C}$, we get

$$H_{jn}(X, \mathbb{C}) = \mathbb{C} \oplus \overset{c(k,j)}{\cdots} \oplus \mathbb{C}.$$

Using the fact that the cohomology $H^{jn}(X, \mathbb{C})$ is the dual of $H_{jn}(X, \mathbb{C})$, we get $\{H^{jn}(X, \mathbb{C})\}_{j=0}^k$ is an exterior algebra over $\mathbb{C}$ with k generators of dimension 1, for more details see [23][pp. 192]. Since $H^{jn}(X, \mathbb{C})$ is torsion free for $1 \le j \le k$, we compute the Lefschetz numbers over $\{H_{jn}(X, \mathbb{C})\}_{j=0}^k$.

Let $A = (a_{i,j})_{1 \le i,j \le k}$ be a complex matrix that represents of f^{*n}, the induced map on the cohomology group $H^n(X, \mathbb{C})$. Since f_{*n} is the dual of f^{*n}, the trace of A, or f_{*n}, is $\sum_{i=1}^k a_{ii}$. By the exterior algebra structure the trace of A^2, or f_{*2n}, is $\sum_{i<j} a_{ii}a_{jj}$ and the trace A^j is

$$\sum_{i_1 < \cdots < i_j} a_{i_1 i_1} \cdots a_{i_j i_j}.$$

So, using the fact that n is odd, the Lefschetz number of f is

$$\begin{aligned} L(f) = {} & 1 + (-1)^n \textstyle\sum_{i=1}^k a_{ii} + (-1)^{2n} \sum_{i<j} a_{ii}a_{jj} + \cdots + (-1)^{nk} a_{11}a_{22} \cdots a_{kk} \\ = {} & (1 - a_{11})(1 - a_{22}) \cdots (1 - a_{kk}) \\ = {} & \det(Id - A). \end{aligned}$$

Similarly the trace of f^l_{*jn} is

$$\sum_{i_1 < \cdots < i_j} a^l_{i_1 i_1} \cdots a^l_{i_j i_j}.$$

So $L(f^l) = \det(Id - A^l)$. Therefore $L(f) = \cdots = L(f^m) = 0$ if and only if 1 is an eigenvalue of A^l for $1 \le l \le m$. Hence f is Lefschetz periodic point free if and only if 1 is an eigenvalue of f_{*n}. This completes the proof so statement (a).

If n is even then a similar computation shows that $L(f^l) = \det(Id + A^l)$. Therefore $L(f) = \cdots = L(f^m) = 0$ if and only if -1 is an eigenvalue of A^l for $1 \le l \le m$.

The matrix A is $k\times k$, if -1 is an eigenvalue of A^l for $0\le l\le k$; then $(-1)^{1/l}$ is an eigenvalue of A. We consider A^s, with $s=2k!$, then its eigenvalues are of the form

$$(-1)^{2k!/l}=(-1)^{2k(k-1)\cdots(l+1)(l-1)\cdots 2}=1.$$

Hence 1 is the only eigenvalue of A^s. So $L(f^s)\neq 0$, therefore f is not Lefschetz free periodic point.

If f is Lefschetz periodic point free up to order m, with $m\ge 2$ then -1 and $\sqrt{-1}$ are roots of the characteristic polynomial of A, so its degree should be grater than or equal to 3. Since the degree of the characteristic polynomial of the matrix A is k, then $k\ge 3$, if $m\ge 2$. □

Proof of Proposition 4. The homological groups of $\mathbb{S}^n\times\mathbb{S}^m$ over $\mathbb{Q}$ are $H_l(\mathbb{S}^n\times\mathbb{S}^m,\mathbb{Q})=\mathbb{Q}$, if $l=0,n,m$ or $n+m$ and trivial for the other values of l. If the induced maps on homology f are $f_{*n}=a$, $f_{*m}=b$ and $f_{*n+m}=d$; the maps f_{*l} are trivial if l is different from $0,n,m$ and $n+m$, the Lefschetz numbers for the iterates of f are

$$L(f^l)=1+(-1)^n a^l+(-1)^m b^l+(-1)^{n+m}d^l,$$

for all $l\ge 1$.

If n and m are even we get that $L(f^2)=1+a^2+b^2+d^2$, so $L(f^2)\neq 0$. So f is not Lefschetz periodic point free up to period 2 and it is Lefschetz fixed point free if $1+a+b+d=0$. This completes the proof of statement (a).

If n is even and m odd then the solution of the linear system $L(f)=L(f^2)=0$ is $a=b$ and $d=1$, or $a=d$ and $b=1$, so this implies that $L(f^l)=0$ for $l>2$. Similarly for the other cases when n and m are not simultaneously even. Therefore f is Lefschetz periodic point free. This completes the proof of statement (b). □

Proof of Proposition 5. The homological groups of $\mathbb{C}P^n$ over $\mathbb{Q}$ are $H_{2l}(\mathbb{C}P^n,\mathbb{Q})=\mathbb{Q}$, if $0\le l\le n$ and it is trivial otherwise. So the induced maps on homology are $f_{*2l}=a^l$ if $0\le l\le n$, with $a\in\mathbb{Z}$, and $f_{*l}=0$ if l is odd. Therefore $f^m_{*2l}=(a^l)^m$ for $0\le l\le n$. Hence the Lefschetz numbers of the iterates of f are:

$$\begin{array}{rcl} L(f) &=& 1+a+\cdots+a^n,\\ L(f^2) &=& 1+a^2+\cdots+a^{2n},\\ &\vdots& \\ L(f^m) &=& 1+a^m+\cdots+a^{mn}. \end{array}$$

If $a\neq -1$ then $L(f^m)=(1-a^{m(n+1)})/(1-a^m)\neq 0$ for all m, so f^m has fixed points. Similarly if $a=-1$ and n even. If $a=-1$ and n odd then $L(f)=0$ and $L(f^2)\neq 0$ so, f is Lefschetz fixed point free. □

Proof of Proposition 6. The homology groups of $\mathbb{H}P^n$ over $\mathbb{Q}$ are $H_{4l}(\mathbb{H}P^n, \mathbb{Q}) = \mathbb{Q}$, if $0 \leq l \leq n$ and it is trivial otherwise. So the induced maps on homology are $f_{*4l} = a^l$ if $0 \leq l \leq n$, with $a \in \mathbb{Z}$, and $f_{*l} = 0$ if l is not a multiple of 4.

On the other hand the homology groups of $\mathbb{O}P^n$ over $\mathbb{Q}$ are $H_{8l}(\mathbb{O}P^n, \mathbb{Q}) = \mathbb{Q}$, if $0 \leq l \leq n$ and otherwise it is trivial. So the induced maps on homology are $f_{*8l} = a^l$ if $0 \leq l \leq n$, with $a \in \mathbb{Z}$, and $f_{*l} = 0$ otherwise.

Now the computation follows in a similar manner as in the case of $\mathbb{C}P^n$. □

Acknowledgements

The authors would like to thank Matija Cencelj for pointed out the space considered in Theorem 2. The second author would like to thank the Department of Mathematics at the Universitat Autònoma de Barcelona for its hospitality. The research of this article was partially done during a visit of the second author to this institution.

The first author is partially supported by a MICINN/FEDER grant number MTM2008–03437, by an AGAUR grant number 2009SGR–410 and by ICREA Academia.

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