

LIMIT CYCLES FOR A GENERALIZATION OF POLYNOMIAL LIÉNARD DIFFERENTIAL SYSTEMS

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ABSTRACT. We study the number of limit cycles of the polynomial differential systems of the form

$$\dot{x} = y - f_1(x)y, \quad \dot{y} = -x - g_2(x) - f_2(x)y,$$

where $f_1(x) = \varepsilon f_{11}(x) + \varepsilon^2 f_{12}(x) + \varepsilon^3 f_{13}(x)$, $g_2(x) = \varepsilon g_{21}(x) + \varepsilon^2 g_{22}(x) + \varepsilon^3 g_{23}(x)$ and $f_2(x) = \varepsilon f_{21}(x) + \varepsilon^2 f_{22}(x) + \varepsilon^3 f_{23}(x)$ where f_{1i} , f_{2i} and g_{2i} have degree l , n and m respectively for each $i = 1, 2, 3$, and ε is a small parameter. Note that when $f_1(x) = 0$ we obtain the generalized polynomial Liénard differential systems. We provide an accurate upper bound of the maximum number of limit cycles that this differential system can have bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of third order.

1. INTRODUCTION

The second part of the 16th Hilbert’s problem wants to find an upper bound on the maximum number of limit cycles that the class of all polynomial vector fields with a fixed degree can have. In this paper we will try to give a partial answer to this problem for the class of polynomial differential systems

$$(1) \quad \dot{x} = y - f_1(x)y, \quad \dot{y} = -x - g_2(x) - f_2(x)y,$$

where $f_1(x) = \varepsilon f_{11}(x) + \varepsilon^2 f_{12}(x) + \varepsilon^3 f_{13}(x)$, $g_2(x) = \varepsilon g_{21}(x) + \varepsilon^2 g_{22}(x) + \varepsilon^3 g_{23}(x)$ and $f_2(x) = \varepsilon f_{21}(x) + \varepsilon^2 f_{22}(x) + \varepsilon^3 f_{23}(x)$ where f_{1i} , f_{2i} and g_{2i} have degree l , n and m respectively for each $i = 1, 2, 3$, and ε is a small parameter. When $f_1(x) = 0$ these systems coincide with the generalized polynomial Liénard differential systems

$$(2) \quad \dot{x} = y, \quad \dot{y} = -g(x) - f(x)y,$$

where $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m , respectively. The classical polynomial Liénard differential systems are

$$(3) \quad \dot{x} = y, \quad \dot{y} = -x - f(x)y,$$

where $f(x)$ is a polynomial in the variable x of degree n . For these systems in 1977 Lins, de Melo and Pugh [15] stated the conjecture that if $f(x)$ has degree $n \geq 1$ then system (3) has at most $[n/2]$ limit cycles. They prove this conjecture for $n = 1, 2$. The conjecture for $n = 3$ has been proved recently by Chengzi Li and Llibre in [16]. For $n \geq 5$ the conjecture is not true, see De Maesschalck and Dumortier [7] and Dumortier, Panazzolo and Roussarie [8]. So it remains to know if the conjecture is true or not for $n = 4$.

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Many of the results on the limit cycles of polynomial differential systems have been obtained by considering limit cycles which bifurcate from a single degenerate singular point (i.e., from a Hopf bifurcation), that are called *small amplitude limit cycles*, see for instance [20]. There are partial results concerning the maximum number of small amplitude limit cycles for Liénard polynomial differential systems. Of course, the number of small amplitude limit cycles gives a lower bound for the maximum number of limit cycles that a polynomial differential system can have.

There are many results concerning the existence of small amplitude limit cycles for the following generalized Liénard polynomial differential system (2). We denote by $H(m, n)$ the number of limit cycles that systems (2) can have. This number is usually called the *Hilbert number* for systems (2).

- (i) In 1928 Liénard [14] proved that if $m = 1$ and $F(x) = \int_0^x f(s)ds$ is a continuous odd function, which has a unique root at $x = a$ and is monotone increasing for $x \geq a$, then equation (2) has a unique limit cycle.
- (ii) In 1973 Rychkov [26] proved that if $m = 1$ and $F(x)$ is an odd polynomial of degree five, then equation (2) has at most two limit cycles.
- (iii) In 1977 Lins, de Melo and Pugh [15] proved that $H(1, 1) = 0$ and $H(1, 2) = 1$.
- (iv) In 1998 Coppel [6] proved that $H(2, 1) = 1$.
- (v) Dumortier, Li and Rousseau in [11] and [9] proved that $H(3, 1) = 1$.
- (vi) In 1997 Dumortier and Chengzhi [10] proved that $H(2, 2) = 1$.
- (vii) In 2011 Chengzhi Li and Llibre [16] proved that $H(1, 3) = 1$.

Up to now and as far as we know only for these five cases ((iii)-(vii)) the Hilbert number for systems (2) has been determined.

The maximum number of small amplitude limit cycles for systems (2) is denoted by $\hat{H}(m, n)$. Blows, Lloyd and Lynch, [3], [21] and [22] have used inductive arguments in order to prove the following results.

- (I) If g is odd then $\hat{H}(m, n) = [n/2]$.
- (II) If f is even then $\hat{H}(m, n) = n$, whatever g is.
- (III) If f is odd then $\hat{H}(m, 2n + 1) = [(m - 2)/2] + n$.
- (IV) If $g(x) = x + g_e(x)$, where g_e is even then $\hat{H}(2m, 2) = m$.

Christopher and Lynch [5], [23], [24], [25] have developed a new algebraic method for determining the Liapunov quantities of systems (2) and proved some other bounds for $\hat{H}(m, n)$ for different m and n .

- (V) $\hat{H}(m, 2) = [(2m + 1)/3]$.
- (VI) $\hat{H}(2, n) = [(2n + 1)/3]$.
- (VII) $\hat{H}(m, 3) = 2[(3m + 2)/8]$ for all $1 < m \leq 50$.
- (VIII) $\hat{H}(3, n) = 2[(3n + 2)/8]$ for all $1 < m \leq 50$.
- (IX) $\hat{H}(4, k) = \hat{H}(k, 4)$, $k = 6, 7, 8, 9$ and $\hat{H}(5, 6) = \hat{H}(6, 5)$.

In 1998 Gasull and Torregrosa [12] obtained upper bounds for $\hat{H}(7, 6)$, $\hat{H}(6, 7)$, $\hat{H}(7, 7)$ and $\hat{H}(4, 20)$.

In 2006 Yu and Han in [28] give some accurate values of $\hat{H}(m, n) = \hat{H}(n, m)$, for $n = 4$, $m = 10, 11, 12, 13$; $n = 5$, $m = 6, 7, 8, 9$; $n = 6$, $m = 5, 6$, see also [18] for a table with all the specific values.

In 2010 Llibre, Mereu and Teixeira [18] compute the maximum number of limit cycles $\tilde{H}_k(m, n)$ of systems (2) which bifurcate from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$, using the averaging theory of order k , for $k = 1, 2, 3$.

In [19] the authors studied using the averaging theory of first and second order the more general system

$$(4) \quad \begin{aligned} \dot{x} &= y - \varepsilon(g_{11}(x) + f_{11}(x)y) - \varepsilon^2(g_{12}(x) + f_{12}(x)y), \\ \dot{y} &= -x - \varepsilon(g_{21}(x) + f_{21}(x)y) - \varepsilon^2(g_{22}(x) + f_{22}(x)y), \end{aligned}$$

where $g_{1i}, f_{1i}, g_{2i}, f_{2i}$ have degree l, k, m and n respectively for each $i = 1, 2$, and ε is a small parameter. Using the averaging method of first and second order they proved the following result.

Theorem 1. *For $|\varepsilon|$ sufficiently small the maximum number of limit cycles of the generalized Liénard polynomial differential systems (4) bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$ using the averaging theory of second order is:*

$$(5) \quad \lambda = \max \left\{ \mu + [(m-1)/2], \mu + [l/2], [(n-1)/2] + [m/2], [k/2] + [m/2] - 1, \right. \\ \left. [(n-1)/2] + [(l-1)/2] + 1, [k/2] + [(l-1)/2] \right\},$$

with $\mu = \min\{[n/2], [(k-1)/2]\}$.

In Alavez-Ramirez et al. in [2] they studied the polynomial differential system

$$(6) \quad \begin{aligned} \dot{x} &= y - \varepsilon g_{11}(x) - \varepsilon^2 g_{12}(x), \\ \dot{y} &= -x - \varepsilon(g_{21}(x) + f_{21}(x)y) - \varepsilon^2(g_{22}(x) + f_{22}(x)y), \end{aligned}$$

where g_{1i}, g_{2i}, f_{2i} have degree l, m and n respectively for each $i = 1, 2$, and ε is a small parameter. They proved the following result.

Theorem 2. *For $|\varepsilon|$ sufficiently small the maximum number of limit cycles of the generalized Liénard polynomial differential systems (6) bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$ using the averaging theory of third order is*

$$\frac{1}{2} (\max\{O(m+n), E(l+m) - 1\} - 1),$$

where $O(k)$ is the largest odd integer $\leq k$, and $E(k)$ is the largest even integer $\leq k$.

In the present paper we study system (1). We define Λ by

$$\begin{aligned} \Lambda = \max\{ & 2(\lfloor (n-1)/2 \rfloor + 2\lfloor m/2 \rfloor) + 1, \\ & 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor + \lfloor (m-1)/2 \rfloor) + 2, \\ & 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor + \lfloor l/2 \rfloor) + 1, \\ & 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor + \lfloor (l-1)/2 \rfloor) + 2, \\ & 2(\lfloor (n-1)/2 \rfloor + \lfloor (m-1)/2 \rfloor + \lfloor l/2 \rfloor) + 2, \\ & 2(\lfloor (n-1)/2 \rfloor + 2\lfloor (l-1)/2 \rfloor) + 4, \\ & 2(\lfloor (n-1)/2 \rfloor + \lfloor (m-1)/2 \rfloor + \lfloor (l-1)/2 \rfloor) + 3, \\ & 2(\lfloor (n-1)/2 \rfloor + \lfloor l/2 \rfloor + \lfloor (l-1)/2 \rfloor) + 3, \\ & 2(\lfloor (n-1)/2 \rfloor + \lfloor n/2 \rfloor) + 2, \\ & 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor) + m, \\ & 2(\lfloor (n-1)/2 \rfloor + \lfloor (l-1)/2 \rfloor) + m + 2, \\ & 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor) + l + 1, \\ & 2(\lfloor (n-1)/2 \rfloor + \lfloor (l-1)/2 \rfloor) + l + 3\}. \end{aligned}$$

Using the averaging method of third order we will show the following result that is the main result of the paper.

Theorem 3. *For $|\varepsilon|$ sufficiently small the maximum number of limit cycles of the generalized Liénard polynomial differential systems (1) bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of third order is at most Λ .*

The proof of Theorem 3 is given in section 3.

The results that we shall use from the averaging theory of third order for computing limit cycles are presented in section 2.

2. THE AVERAGING THEORY OF FIRST AND SECOND ORDER

The averaging theory for studying specifically limit cycles up to third order in ε was developed in [4]. It is summarized as follows.

Consider the differential system

$$(7) \quad \dot{x} = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 F_3(t, x) + \varepsilon^4 R(t, x, \varepsilon),$$

where $F_1, F_2, F_3: \mathbb{R} \times D \rightarrow \mathbb{R}$, $R: \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \rightarrow \mathbb{R}$ are continuous functions, T -periodic in the first variable, and D is an open subset of \mathbb{R} . Assume that the following conditions hold.

- (i) $F_1(t, \cdot) \in C^2(D)$, $F_2(t, \cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, F_3, R are locally Lipschitz with respect to x , and R is twice differentiable with respect to ε .

We define $F_{k0}: D \rightarrow \mathbb{R}$ for $k = 1, 2$ as

$$\begin{aligned} F_{10}(z) &= \frac{1}{T} \int_0^T F_1(s, z) ds, \\ F_{20}(z) &= \frac{1}{T} \int_0^T [D_z F_1(s, z) y_1(s, z) + F_2(s, z)] ds, \\ F_{30}(z) &= \frac{1}{T} \int_0^T \left(\frac{1}{2} \frac{\partial^2 F_1}{\partial z^2}(s, z) y_1(s, z)^2 + \frac{1}{2} \frac{\partial F_1}{\partial z}(s, z) y_2(s, z) \right. \\ &\quad \left. + \frac{\partial F_2}{\partial z}(s, z) y_1(s, z) + F_3(s, z) \right) ds, \end{aligned}$$

where

$$\begin{aligned} y_1(s, z) &= \int_0^s F_1(t, z) dt, \\ y_2(s, z) &= 2 \int_0^s \left(\frac{\partial F_1}{\partial z}(t, z) \int_0^t F_1(r, z) dr + F_2(t, z) \right) dt. \end{aligned}$$

- (ii) For $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, there exists $a_\varepsilon \in V$ such that $F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) + \varepsilon^2 F_{30}(a_\varepsilon) = 0$ and $d_B(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}, V, a_\varepsilon) \neq 0$.

Then for $|\varepsilon| > 0$ sufficiently small there exists a T -periodic solution $\phi(\cdot, \varepsilon)$ of the system such that $\phi(0, a_\varepsilon) \rightarrow a_\varepsilon$ when $\varepsilon \rightarrow 0$.

The expression $d_B(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}: V \rightarrow \mathbb{R}$ at the fixed point a_ε is not zero. A sufficient condition in order that this inequality is true is that the Jacobian of the function $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ at a_ε is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are mainly the zeros of F_{10} for ε sufficiently small. In this case the previous result provides the *averaging theory of first order*.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case the previous result provides the *averaging theory of second order*.

If F_{10} and F_{20} are identically zero and F_{30} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are mainly the zeros of F_{30} for ε sufficiently small. In this case the previous result provides the *averaging theory of third order*.

3. PROOF OF THEOREM 3

We shall need the third order averaging theory to prove Theorem 3. We write system (1) in polar coordinates (r, θ) where

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r > 0.$$

In this way system (1) will become written in the standard form for applying the averaging theory. If we write

$$(8) \quad \begin{aligned} f_{21}(x) &= \sum_{i=0}^n a_{i,2}x^i, & f_{11}(x) &= \sum_{i=0}^l a_{i,1}x^i, & g_{21}(x) &= \sum_{i=0}^m b_{i,2}x^i, \\ f_{22}(x) &= \sum_{i=0}^n c_{i,2}x^i, & f_{12}(x) &= \sum_{i=0}^l c_{i,1}x^i, & g_{22}(x) &= \sum_{i=0}^m d_{i,2}x^i, \\ f_{23}(x) &= \sum_{i=0}^n p_{i,2}x^i, & f_{13}(x) &= \sum_{i=0}^l p_{i,1}x^i, & g_{23}(x) &= \sum_{i=0}^m q_{i,2}x^i, \end{aligned}$$

then system (1) becomes

$$(9) \quad \begin{aligned} \dot{r} &= -\varepsilon(A + \varepsilon B + \varepsilon^2 C), \\ \dot{\theta} &= -1 - \frac{\varepsilon}{r}(A_1 + \varepsilon B_1 + \varepsilon^2 C_1) \end{aligned}$$

where

$$\begin{aligned} A &= \sum_{i=0}^n a_{i,2}r^{i+1} \cos^i \theta \sin^2 \theta + \sum_{i=0}^m b_{i,2}r^i \cos^i \theta \sin \theta + \sum_{i=0}^l a_{i,1}r^{i+1} \cos^{i+1} \theta \sin \theta, \\ B &= \sum_{i=0}^n c_{i,2}r^{i+1} \cos^i \theta \sin^2 \theta + \sum_{i=0}^m d_{i,2}r^i \cos^i \theta \sin \theta + \sum_{i=0}^l c_{i,1}r^{i+1} \cos^{i+1} \theta \sin \theta, \\ C &= \sum_{i=0}^n p_{i,2}r^{i+1} \cos^i \theta \sin^2 \theta + \sum_{i=0}^m q_{i,2}r^i \cos^i \theta \sin \theta + \sum_{i=0}^l p_{i,1}r^{i+1} \cos^{i+1} \theta \sin \theta, \end{aligned}$$

and

$$\begin{aligned} A_1 &= \sum_{i=0}^n a_{i,2}r^{i+1} \cos^{i+1} \theta \sin \theta + \sum_{i=0}^m b_{i,2}r^i \cos^{i+1} \theta - \sum_{i=0}^l a_{i,1}r^{i+1} \cos^i \theta \sin^2 \theta, \\ B_1 &= \sum_{i=0}^n c_{i,2}r^{i+1} \cos^{i+1} \theta \sin \theta + \sum_{i=0}^m d_{i,2}r^i \cos^{i+1} \theta - \sum_{i=0}^l c_{i,1}r^{i+1} \cos^i \theta \sin^2 \theta, \\ C_1 &= \sum_{i=0}^n p_{i,2}r^{i+1} \cos^{i+1} \theta \sin \theta + \sum_{i=0}^m q_{i,2}r^i \cos^{i+1} \theta - \sum_{i=0}^l p_{i,1}r^{i+1} \cos^i \theta \sin^2 \theta. \end{aligned}$$

Now taking θ as the new independent variable, system (9) becomes

$$\frac{dr}{d\theta} = \varepsilon F_1(r, \theta) + \varepsilon^2 F_2(r, \theta) + \varepsilon^3 F_3(r, \theta) + O(\varepsilon^4),$$

where

$$F_1(r, \theta) = A, \quad F_2(r, \theta) = B - \frac{1}{r}AA_1, \quad F_3(r, \theta) = C - \frac{1}{r}(BA_1 + AB_1) + \frac{1}{r^2}A_1^2A.$$

It was proved in [19] that F_{10} is identically zero if and only if

$$a_{2i,2} = 0 \quad \text{for } i = 0, \dots, [n/2].$$

Moreover, F_{20} is identically zero if and only if

$$\begin{aligned} c_{2k,2} = & \sum_{i+j=t, i \text{ even}, j \text{ odd}} S_{1,i,j} a_{i,2} b_{j,2} + \sum_{i+j=t, i \text{ odd}, j \text{ even}} S_{2,i,j} a_{i,2} b_{j,2} \\ & + \sum_{i+j=t, i \text{ even}, j \text{ even}} S_{3,i,j} a_{i,2} a_{j,1} + \sum_{i+j=t-1, i \text{ odd}, j \text{ odd}} S_{4,i,j} a_{i,2} a_{j,1}, \end{aligned}$$

where t varies from 0 to λ given in the statement of Theorem 1, and $S_{s,i,j} \geq 0$ for $s = 1, \dots, 4$.

In order to apply the third order averaging method we need to compute the corresponding function $F_{30}(r)$ that we rewrite as

$$F_{30}(r) = F_{30}^1(r) + F_{30}^2(r) + F_{30}^3(r) + F_{30}^4(r),$$

with

$$\begin{aligned} F_{30}^1(r) &= \frac{1}{4\pi} \int_0^{2\pi} \frac{\partial^2 F_1}{\partial r^2}(r, \theta) y_1(r, \theta)^2 d\theta, \\ F_{30}^2(r) &= \frac{1}{4\pi} \int_0^{2\pi} \frac{\partial F_1}{\partial r}(r, \theta) y_2(r, \theta) d\theta, \\ F_{30}^3(r) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial F_2}{\partial r}(r, \theta) y_1(r, \theta) d\theta, \\ F_{30}^4(r) &= \frac{1}{2\pi} \int_0^{2\pi} F_3(r, \theta) d\theta, \end{aligned} \tag{10}$$

where it was proved in [19] that using the integrals of the appendix we obtain that $y_1 = y_1(\theta, r)$ is equal to

$$\begin{aligned} y_1 = & \sum_{i=0}^{[(n-1)/2]} a_{2i+1,2} r^{2i+2} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \sin((2s+1)\theta) + \sum_{i=0}^m \frac{b_{i,2}}{i+1} r^i (1 - \cos^{i+1} \theta) \\ & + \sum_{i=0}^l \frac{a_{i,1}}{i+2} r^{i+1} (1 - \cos^{i+2} \theta), \end{aligned} \tag{11}$$

where $\tilde{\gamma}_{i,l}$ are constant. Again from [19] we have that

$$\begin{aligned} \frac{\partial}{\partial r} F_1(\theta, r) = & \sum_{i=0}^{[(n-1)/2]} (2i+2) a_{2i+1,2} r^{2i+1} \cos^{2i+1} \theta (1 - \cos^2 \theta) \\ & + \sum_{i=0}^m i b_{i,2} r^{i-1} \cos^i \theta \sin \theta + \sum_{i=0}^l (i+1) a_{i,1} r^i \cos^{i+1} \theta \sin \theta. \end{aligned}$$

We also note that $F_2(r, \theta)$ is equal to

$$\begin{aligned} (12) \quad & \sum_{i=0}^n c_{i,2} r^{i+1} \cos^i \theta \sin \theta + \sum_{i=0}^m d_{i,2} r^i \cos^i \theta \sin \theta + \sum_{i=0}^l c_{i,1} r^{i+1} \cos^{i+1} \theta \sin \theta \\ & - \sum_{i=0}^n \sum_{j=0}^n a_{i,2} a_{j,2} r^{i+j+1} \cos^{i+j+1} (1 - \cos^2 \theta) \sin \theta \\ & - 2 \sum_{i=0}^n \sum_{j=0}^m a_{i,2} b_{j,2} r^{i+j} \cos^{i+j+1} \theta (1 - \cos^2 \theta) \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=0}^n \sum_{j=0}^l a_{i,2} a_{j,1} r^{i+j+1} \cos^{i+j} \theta (1 - 3 \cos^2 \theta + 2 \cos^4 \theta) \\
& - \sum_{i=0}^m \sum_{j=0}^m b_{i,2} b_{j,2} r^{i+j-1} \cos^{i+j+1} \theta \sin \theta \\
& + \sum_{i=0}^m \sum_{j=0}^l b_{i,2} a_{j,1} r^{i+j} \cos^{i+j} \theta (1 - 2 \cos^2 \theta) \sin \theta \\
& + \sum_{i=0}^l \sum_{j=0}^l a_{i,1} a_{j,1} r^{i+j+1} \cos^{i+j+1} \theta (1 - \cos^2 \theta) \sin \theta.
\end{aligned}$$

The proof of Theorem 3 will be a direct consequence of the next auxiliary lemmas.

For an explicit expression of the polynomial $F_{30}^1(r)$ we refer the reader to the proof of Lemma 4.

Lemma 4. *The integral $F_{30}^1(r)$ is a polynomial in the variable r of degree less than or equal to λ_1 given by*

$$\begin{aligned}
\lambda_1 = \max\{ & 2(\lfloor (n-1)/2 \rfloor + 2\lfloor m/2 \rfloor) + 1, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor + \lfloor (m-1)/2 \rfloor) + 2, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor + \lfloor l/2 \rfloor) + 1, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor + \lfloor (l-1)/2 \rfloor) + 2, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor (m-1)/2 \rfloor + \lfloor l/2 \rfloor) + 2, \\
& 2(\lfloor (n-1)/2 \rfloor + 2\lfloor (l-1)/2 \rfloor) + 4, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor (m-1)/2 \rfloor + \lfloor (l-1)/2 \rfloor) + 3, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor l/2 \rfloor + \lfloor (l-1)/2 \rfloor) + 3\}.
\end{aligned}$$

Proof. We first note that

$$\begin{aligned}
\frac{\partial^2}{\partial r^2} F_1(\theta, r) &= \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} (2i+2)(2i+1) a_{2i+1,2} r^{2i} \cos^{2i+1} \theta (1 - \cos^2 \theta) \\
&+ \sum_{i=0}^m i(i-1) b_{i,2} r^{i-2} \cos^i \theta \sin \theta + \sum_{i=0}^l (i+1) i a_{i,1} r^{i-1} \cos^{i+1} \theta \sin \theta,
\end{aligned}$$

and $y_1(r, \theta)^2$ is equal to

$$\begin{aligned}
& \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} a_{2i+1,2} a_{2j+1,2} r^{2i+2j+4} \sum_{s=0}^{i+1} \sum_{r=0}^{j+1} \tilde{\gamma}_{i,s} \sin((2s+1)\theta) \tilde{\gamma}_{j,r} \sin((2r+1)\theta) \\
& + 2 \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^m \frac{a_{2i+1,2} b_{j,2}}{j+1} r^{2i+j+2} (1 - \cos^{j+1} \theta) \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \sin((2s+1)\theta) \\
& + 2 \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^l \frac{a_{2i+1,2} a_{j,1}}{j+2} r^{2i+j+3} (1 - \cos^{j+2} \theta) \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \sin((2s+1)\theta) \\
& + \sum_{i=0}^m \sum_{j=0}^m \frac{b_{i,2} b_{j,2}}{(i+1)(j+1)} r^{i+j} (1 - \cos^{i+1} \theta) (1 - \cos^{j+1} \theta)
\end{aligned}$$

$$\begin{aligned}
& + 2 \sum_{i=0}^m \sum_{j=0}^l \frac{b_{i,2} a_{j,1}}{(i+1)(j+2)} r^{i+j+1} (1 - \cos^{i+1} \theta) (1 - \cos^{j+2} \theta) \\
& + \sum_{i=0}^l \sum_{j=0}^l \frac{a_{i,1} a_{j,1}}{(i+2)(j+2)} r^{i+j+2} (1 - \cos^{i+2} \theta) (1 - \cos^{j+2} \theta).
\end{aligned}$$

Hence, using the integrals which are zero in the formulae in the appendix and the explicit formula of $F_{30}^1(r)$ given in (10) we obtain that $4\pi F_{30}^1(r)$ is equal to

$$\begin{aligned}
& \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^m \sum_{j=0}^m (2t+2)(2t+1) a_{2t+1,2} \frac{b_{i,2} b_{j,2}}{(i+1)(j+1)} r^{2t+i+j} \\
& \int_0^{2\pi} \cos^{2t+1} \theta (1 - \cos^2 \theta) (1 - \cos^{i+1} \theta) (1 - \cos^{j+1} \theta) d\theta \\
& + 2 \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^m \sum_{j=0}^l (2t+2)(2t+1) a_{2t+1,2} \frac{b_{i,2} a_{j,1}}{(i+1)(j+2)} r^{2t+i+j+1} \\
& \int_0^{2\pi} \cos^{2t+1} \theta (1 - \cos^2 \theta) (1 - \cos^{i+1} \theta) (1 - \cos^{j+2} \theta) d\theta \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^l \sum_{j=0}^l (2t+2)(2t+1) a_{2t+1,2} \frac{a_{i,1} a_{j,1}}{(i+2)(j+2)} r^{2t+i+j+2} \\
& \int_0^{2\pi} \cos^{2t+1} \theta (1 - \cos^2 \theta) (1 - \cos^{i+2} \theta) (1 - \cos^{j+2} \theta) d\theta \\
& + 2 \sum_{t=0}^m \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^m t(t-1) b_{t,2} \frac{a_{2i+1,2} b_{j,2}}{j+1} r^{t+2i+j} \\
& \int_0^{2\pi} \cos^t \theta \sin \theta (1 - \cos^{j+1} \theta) \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \sin((2s+1)\theta) d\theta \\
& + 2 \sum_{t=0}^m \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^l t(t-1) b_{t,2} \frac{a_{2i+1,2} a_{j,1}}{j+2} r^{t+2i+j+1} \\
& \int_0^{2\pi} \cos^t \theta \sin \theta (1 - \cos^{j+2} \theta) \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \sin((2s+1)\theta) d\theta \\
& + 2 \sum_{t=0}^l \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^m t(t+1) a_{t,1} \frac{a_{2i+1,2} b_{j,2}}{j+1} r^{t+2i+j+1} \\
& \int_0^{2\pi} \cos^{t+1} \theta \sin \theta (1 - \cos^{j+1} \theta) \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \sin((2s+1)\theta) d\theta \\
& + 2 \sum_{t=0}^l \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^l t(t+1) a_{t,1} \frac{a_{2i+1,2} a_{j,1}}{j+2} r^{t+2i+j+2}
\end{aligned}$$

$$\int_0^{2\pi} \cos^{t+1} \theta \sin \theta (1 - \cos^{j+2} \theta) \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \sin((2s+1)\theta) d\theta.$$

Then, now using the integrals which are not zero in the formulae in the appendix we conclude that $F_{30}^1(r)$ is equal to

$$\begin{aligned} & \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[m/2]} \sum_{j=0}^{[m/2]} \rho_1 a_{2t+1,2} b_{2i,2} b_{2j,2} r^{2t+2i+2j} \\ & + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[m/2]} \sum_{j=0}^{[(m-1)/2]} \rho_2 a_{2t+1,2} b_{2i,2} b_{2j+1,2} r^{2t+2i+2j+1} \\ & + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[m/2]} \sum_{j=0}^{[l/2]} \rho_3 a_{2t+1,2} b_{2i,2} a_{2j,1} r^{2t+2i+2j+1} \\ & + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[m/2]} \sum_{j=0}^{[(l-1)/2]} \rho_4 a_{2t+1,2} b_{2i,2} a_{2j+1,1} r^{2t+2i+2j+2} \\ & + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[(m-1)/2]} \sum_{j=0}^{[l/2]} \rho_5 a_{2t+1,2} b_{2i+1,2} a_{2j,1} r^{2t+2i+2j+2} \\ & + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[(m-1)/2]} \sum_{j=0}^{[(l-1)/2]} \rho_6 a_{2t+1,2} b_{2i+1,2} a_{2j+1,1} r^{2t+2i+2j+3} \\ & + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[l/2]} \sum_{j=0}^{[(l-1)/2]} \rho_7 a_{2t+1,2} a_{2i,1} a_{2j+1,1} r^{2t+2i+2j+3} \\ & + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[(l-1)/2]} \sum_{j=0}^{[(l-1)/2]} \rho_8 a_{2t+1,2} a_{2i+1,1} a_{2j+1,1} r^{2t+2i+2j+4} \\ & + \sum_{t=0}^{[m/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[m/2]} \rho_9 b_{2t,2} a_{2i+1,2} b_{2j,2} r^{2t+2i+2j+1} \\ & + \sum_{t=0}^{[m/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(m-1)/2]} \rho_{10} b_{2t,2} a_{2i+1,2} b_{2j+1,2} r^{2t+2i+2j+2} \\ & + \sum_{t=0}^{[(m-1)/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[l/2]} \rho_{11} b_{2t+1,2} a_{2i+1,2} a_{2j,1} r^{2t+2i+2j+2} \\ & + \sum_{t=0}^{[m/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[l/2]} \rho_{12} b_{2t,2} a_{2i+1,2} a_{2j,1} r^{2t+2i+2j+1} \\ & + \sum_{t=0}^{[m/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(l-1)/2]} \rho_{13} b_{2t,2} a_{2i+1,2} a_{2j+1,1} r^{2t+2i+2j+2} \\ & + \sum_{t=0}^{[(m-1)/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(l-1)/2]} \rho_{14} b_{2t+1,2} a_{2i+1,2} a_{2j+1,1} r^{2t+2i+2j+3} \end{aligned}$$

$$\begin{aligned}
& + \sum_{t=0}^{\lfloor l/2 \rfloor} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor m/2 \rfloor} \rho_{15} a_{2t,1} a_{2i+1,2} b_{2j,2} r^{2t+2i+2j+1} \\
& + \sum_{t=0}^{\lfloor (l-1)/2 \rfloor} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor m/2 \rfloor} \rho_{16} a_{2t+1,1} a_{2i+1,2} b_{2j,2} r^{2t+2i+2j+2} \\
& + \sum_{t=0}^{\lfloor (l-1)/2 \rfloor} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} \rho_{17} a_{2t+1,1} a_{2i+1,2} b_{2j+1,2} r^{2t+2i+2j+3} \\
& + \sum_{t=0}^{\lfloor l/2 \rfloor} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor (l-1)/2 \rfloor} \rho_{18} a_{2t,1} a_{2i+1,2} a_{2j+1,1} r^{2t+2i+2j+3} \\
& + \sum_{t=0}^{\lfloor (l-1)/2 \rfloor} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor (l-1)/2 \rfloor} \rho_{19} a_{2t+1,1} a_{2i+1,2} a_{2j+1,1} r^{2t+2i+2j+4},
\end{aligned}$$

for some constants ρ_s for $s = 1, \dots, 19$ which depend on t, i, j . \square

For an explicit expression of the polynomial $F_{30}^3(r)$ we refer the reader to the proof of Lemma 5.

Lemma 5. *The integral $F_{30}^3(r)$ is a polynomial in the variable r of degree less than or equal to λ_2 given by*

$$\begin{aligned}
\lambda_2 = \max\{ & 2(\lfloor (n-1)/2 \rfloor + \lfloor n/2 \rfloor) + 2, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor + \lfloor (m-1)/2 \rfloor) + 1, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor + \lfloor l/2 \rfloor) + 1, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor (m-1)/2 \rfloor + \lfloor (l-1)/2 \rfloor) + 3, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor l/2 \rfloor + \lfloor (l-1)/2 \rfloor) + 3, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor) + m, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor (l-1)/2 \rfloor) + m + 1, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor) + l + 1, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor (l-1)/2 \rfloor) + l + 2\}.
\end{aligned}$$

Proof. Using the expression of $F_2(r, \theta)$ in (12) and $y_1(\theta, r)$ in (11) together with eliminating the integrals that are zero (see the formulae in the Appendix for those integrals) we have that $2\pi F_{30}^3(r)$ is equal to

$$\begin{aligned}
& \sum_{t=0}^{\lfloor (n-1)/2 \rfloor} \sum_{i=0}^n (i+1) a_{2t+1,2} c_{i,2} r^{2t+i+2} \sum_{s=0}^{t+1} \tilde{\gamma}_{t,s} \int_0^{2\pi} \cos^i \theta \sin \theta \sin((2s+1)\theta) d\theta \\
& + \sum_{t=0}^{\lfloor (n-1)/2 \rfloor} \sum_{i=0}^m i a_{2t+1,2} d_{i,2} r^{2t+i+1} \sum_{s=0}^{t+1} \tilde{\gamma}_{t,s} \int_0^{2\pi} \cos^i \theta \sin \theta \sin((2s+1)\theta) d\theta \\
& + \sum_{t=0}^{\lfloor (n-1)/2 \rfloor} \sum_{i=0}^l (i+1) a_{2t+1,2} c_{i,1} r^{2t+i+2} \sum_{s=0}^{t+1} \tilde{\gamma}_{t,s} \int_0^{2\pi} \cos^{i+1} \theta \sin \theta \sin((2s+1)\theta) d\theta \\
& - \sum_{t=0}^{\lfloor (n-1)/2 \rfloor} \sum_{i=0}^n \sum_{j=0}^n (i+j+1) a_{2t+1,2} a_{i,2} a_{j,2} r^{2t+i+j+2} \sum_{s=0}^{t+1} \tilde{\gamma}_{t,s}
\end{aligned}$$

$$\begin{aligned}
& \int_0^{2\pi} \cos^{i+j+1} \theta \sin \theta (1 - \cos^2 \theta) \sin((2s+1)\theta) d\theta \\
& - \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^m \sum_{j=0}^m (i+j-1) a_{2t+1,2} b_{i,2} b_{j,2} r^{2t+i+j} \sum_{s=0}^{t+1} \tilde{\gamma}_{t,s} \\
& \int_0^{2\pi} \cos^{i+j+1} \theta \sin \theta \sin((2s+1)\theta) d\theta \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^m \sum_{j=0}^l (i+j) a_{2t+1,2} b_{i,2} a_{j,1} r^{2t+i+j+1} \sum_{s=0}^{t+1} \tilde{\gamma}_{t,s} \\
& \int_0^{2\pi} \cos^{i+j} \theta (1 - 2\cos^2 \theta) \sin \theta \sin((2s+1)\theta) d\theta \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^l \sum_{j=0}^l (i+j+1) a_{2t+1,2} a_{i,1} a_{j,1} r^{2t+i+j+2} \sum_{s=0}^{t+1} \tilde{\gamma}_{t,s} \\
& \int_0^{2\pi} \cos^{i+j+1} \theta (1 - \cos^2 \theta) \sin \theta \sin((2s+1)\theta) d\theta \\
& - 2 \sum_{t=0}^m \sum_{i=0}^n \sum_{j=0}^m \frac{(i+j) b_{t,2}}{t+1} a_{i,2} b_{j,2} r^{t+i+j-1} \int_0^{2\pi} (1 - \cos^{t+1} \theta) \cos^{i+j+1} \theta (1 - \cos^2 \theta) d\theta \\
& + \sum_{t=0}^m \sum_{i=0}^n \sum_{j=0}^l \frac{(i+j) b_{t,2}}{t+1} a_{i,2} b_{j,1} r^{t+i+j-1} \\
& \int_0^{2\pi} (1 - \cos^{t+1} \theta) \cos^{i+j} \theta (1 - 3\cos^2 \theta + 2\cos^4 \theta) d\theta \\
& - 2 \sum_{t=0}^l \sum_{i=0}^n \sum_{j=0}^m \frac{(i+j) a_{t,1}}{t+2} a_{i,2} b_{j,2} r^{t+i+j} \int_0^{2\pi} (1 - \cos^{t+2} \theta) \cos^{i+j+1} \theta (1 - \cos^2 \theta) d\theta \\
& + \sum_{t=0}^l \sum_{i=0}^n \sum_{j=0}^l \frac{(i+j) a_{t,1}}{t+2} a_{i,2} b_{j,1} r^{t+i+j} \\
& \int_0^{2\pi} (1 - \cos^{t+2} \theta) \cos^{i+j} \theta (1 - 3\cos^2 \theta + 2\cos^4 \theta) d\theta.
\end{aligned}$$

Then, now using the integrals in the appendix which are not zero we conclude that

$$\begin{aligned}
& \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[n/2]} \kappa_1 a_{2t+1,2} c_{2i,2} r^{2t+2i+2} \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[m/2]} \kappa_2 a_{2t+1,2} d_{2i,2} r^{2t+2i+1} \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[(l-1)/2]} \kappa_3 a_{2t+1,2} c_{2i+1,1} r^{2t+2i+3}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{t=0}^{\lfloor (n-1)/2 \rfloor} \sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} \kappa_4 a_{2t+1,2} b_{2i,2} b_{2j+1,2} r^{2t+2i+2j+1} \\
& + \sum_{t=0}^{\lfloor (n-1)/2 \rfloor} \sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{j=0}^{\lfloor l/2 \rfloor} \kappa_5 a_{2t+1,2} b_{2i,2} a_{2j,1} r^{2t+2i+2j+1} \\
& + \sum_{t=0}^{\lfloor (n-1)/2 \rfloor} \sum_{i=0}^{\lfloor (m-1)/2 \rfloor} \sum_{j=0}^{\lfloor (l-1)/2 \rfloor} \kappa_6 a_{2t+1,2} b_{2i+1,2} a_{2j+1,1} r^{2t+2i+2j+3} \\
& + \sum_{t=0}^{\lfloor (n-1)/2 \rfloor} \sum_{i=0}^{\lfloor l/2 \rfloor} \sum_{j=0}^{\lfloor (l-1)/2 \rfloor} \kappa_7 a_{2t+1,2} a_{2i,1} a_{2j+1,1} r^{2t+2i+2j+3} \\
& + \sum_{t=0}^m \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor m/2 \rfloor} \kappa_8 b_{t,2} a_{2i+1,2} b_{2j,2} r^{t+2i+2j} \\
& + \sum_{t=0}^{\lfloor m/2 \rfloor} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} \kappa_9 b_{2t,2} a_{2i+1,2} b_{2j+1,2} r^{2t+2i+2j+1} \\
& + \sum_{t=0}^m \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor (l-1)/2 \rfloor} \kappa_{10} b_{t,2} a_{2i+1,2} b_{2j+1,1} r^{t+2i+2j+1} \\
& + \sum_{t=0}^{\lfloor m/2 \rfloor} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor l/2 \rfloor} \kappa_{11} b_{2t,2} a_{2i+1,2} b_{2j,1} r^{2t+2i+2j} \\
& + \sum_{t=0}^l \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor m/2 \rfloor} \kappa_{12} a_{t,1} a_{2i+1,2} b_{2j,2} r^{t+2i+2j+1} \\
& + \sum_{t=0}^{\lfloor (l-1)/2 \rfloor} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} \kappa_{13} a_{2t+1,1} a_{2i+1,2} b_{2j+1,2} r^{2t+2i+2j+3} \\
& + \sum_{t=0}^l \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor (l-1)/2 \rfloor} \kappa_{14} a_{t,1} a_{2i+1,2} b_{2j+1,1} r^{t+2i+2j+2} \\
& + \sum_{t=0}^{\lfloor (l-1)/2 \rfloor} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor l/2 \rfloor} \kappa_{15} a_{2t+1,1} a_{2i+1,2} b_{2j,1} r^{2t+2i+2j+2},
\end{aligned}$$

for some constants κ_r for $r = 1, \dots, 15$ constants that depend on t, i, j . \square

For an explicit expression of the polynomial $F_{30}^4(r)$ we refer the reader to the proof of Lemma 6.

Lemma 6. *The integral $F_{30}^2(r)$ is a polynomial in the variable r of degree less than or equal to of degree less than or equal to λ_3 given by*

$$\begin{aligned}
\lambda_3 = \max\{ & 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor + \lfloor (m-1)/2 \rfloor) + 1, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor + \lfloor l/2 \rfloor) + 1, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor (m-1)/2 \rfloor + \lfloor (l-1)/2 \rfloor) + 3, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor l/2 \rfloor + \lfloor (l-1)/2 \rfloor) + 3\}.
\end{aligned}$$

Proof. We have that

$$2\pi F_{30}^4 = \int_0^{2\pi} F_3(r, \theta) d\theta = \int_0^{2\pi} C d\theta - \int_0^{2\pi} \frac{BA_1 + AB_1}{r} d\theta + \int_0^{2\pi} \frac{A_1^2 A}{r^2} d\theta$$

and we denote

$$\tilde{F}_{30}^4 = \int_0^{2\pi} C d\theta, \quad \hat{F}_{30}^4 = \int_0^{2\pi} \frac{BA_1 + AB_1}{r} d\theta, \quad \bar{F}_{30}^4 = \int_0^{2\pi} \frac{A_1^2 A}{r^2} d\theta.$$

Using the formulae in the Appendix it is easy to see that

$$\tilde{F}_{30}^4 = \sum_{i=0}^n p_{i,2} r^{i+1} \int_0^{2\pi} \cos^i \theta \sin^2 \theta d\theta = \sum_{i=0}^{[n/2]} W_1 p_{2i,2} r^{2i+1},$$

for some constant W . Now we note that

$$\begin{aligned} \hat{F}_{30}^4 &= 2 \sum_{i=0}^n \sum_{j=0}^m c_{i,2} b_{j,2} r^{i+j} \int_0^{2\pi} \cos^{i+j+1} \theta (1 - \cos^2 \theta) d\theta \\ &+ \sum_{i=0}^n \sum_{j=0}^l c_{i,2} a_{j,1} r^{i+j+1} \int_0^{2\pi} \cos^{i+j+2} \theta (3 \cos^2 \theta - 2 \cos^4 \theta - 1) d\theta \\ &+ 2 \sum_{i=0}^m \sum_{j=0}^n d_{i,2} a_{j,2} r^{i+j} \int_0^{2\pi} \cos^{i+j+1} \theta (1 - \cos^2 \theta) d\theta \\ &- \sum_{i=0}^l \sum_{j=0}^n c_{i,1} a_{j,2} r^{i+j+1} \int_0^{2\pi} \cos^{i+j} \theta (3 \cos^2 \theta - 2 \cos^4 \theta - 1) d\theta. \end{aligned}$$

Using the formulae in the Appendix we obtain

$$\begin{aligned} \hat{F}_{30}^4 &= \sum_{i=0}^{[n/2]} \sum_{j=0}^{[(m-1)/2]} \gamma_1 c_{2i,2} b_{2j+1,2} r^{2i+2j+1} \\ &+ \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[m/2]} \gamma_2 c_{2i+1,2} b_{2j,2} r^{2i+2j+1} \\ &+ \sum_{i=0}^{[n/2]} \sum_{j=0}^{[l/2]} \gamma_3 c_{2i,2} a_{2j,1} r^{2i+2j+1} \\ &+ \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(l-1)/2]} \gamma_4 c_{2i+1,2} a_{2j+1,1} r^{2i+2j+3} \\ &+ \sum_{i=0}^{[m/2]} \sum_{j=0}^{[(n-1)/2]} \gamma_5 d_{2i,2} a_{2j+1,2} r^{2i+2j+1} \\ &+ \sum_{i=0}^{[(l-1)/2]} \sum_{j=0}^{[(n-1)/2]} \gamma_6 c_{2i+1,1} a_{2j+1,2} r^{2i+2j+3}, \end{aligned}$$

for some constants $\gamma_{\kappa,i,j}$ for $\kappa = 1, \dots, 6$.

Now we note that

$$\begin{aligned}
A_1^2 &= \sum_{i=0}^n \sum_{j=0}^n a_{i,2} a_{j,2} r^{i+j+2} \cos^{i+j+2} \theta \sin^2 \theta + 2 \sum_{i=0}^n \sum_{j=0}^m a_{i,2} b_{j,2} r^{i+j+1} \cos^{i+j+2} \theta \sin \theta \\
&\quad - 2 \sum_{i=0}^n \sum_{j=0}^l a_{i,2} a_{j,1} r^{i+j+2} \cos^{i+j+1} \theta \sin^3 \theta + \sum_{i=0}^m \sum_{j=0}^m b_{i,2} b_{j,2} r^{i+j} \cos^{i+j+2} \theta \\
&\quad - 2 \sum_{i=0}^m \sum_{j=0}^l b_{i,2} a_{j,1} r^{i+j+1} \cos^{i+j+1} \theta \sin^2 \theta + \sum_{i=0}^l \sum_{j=0}^l a_{i,1} a_{j,1} r^{i+j+2} \cos^{i+j} \theta \sin^4 \theta.
\end{aligned}$$

Hence,

$$\begin{aligned}
\bar{F}_{30}^4 &= \sum_{t=0}^n \sum_{i=0}^n \sum_{j=0}^n a_{t,2} a_{i,2} a_{j,2} r^{t+i+j+1} \int_0^{2\pi} \cos^{t+i+j+2} \theta (1 - 2 \cos^2 \theta + \cos^4 \theta) d\theta \\
&\quad + \sum_{t=0}^n \sum_{i=0}^m \sum_{j=0}^m a_{t,2} b_{i,2} b_{j,2} r^{t+i+j-1} \int_0^{2\pi} \cos^{t+i+j+2} \theta (1 - \cos^2 \theta) d\theta \\
&\quad - 2 \sum_{t=0}^n \sum_{i=0}^m \sum_{j=0}^l a_{t,2} b_{i,2} a_{j,1} r^{t+i+j} \int_0^{2\pi} \cos^{t+i+j+1} \theta (1 - 2 \cos^2 \theta + \cos^4 \theta) d\theta \\
&\quad + \sum_{t=0}^n \sum_{i=0}^l \sum_{j=0}^l a_{t,2} a_{i,1} a_{j,1} r^{t+i+j+1} \int_0^{2\pi} \cos^{t+i+j} \theta (1 - 3 \cos^2 \theta + 3 \cos^4 \theta - \cos^6 \theta) d\theta \\
&\quad + 2 \sum_{t=0}^m \sum_{i=0}^n \sum_{j=0}^m b_{t,2} a_{i,2} b_{j,2} r^{t+i+j-1} \int_0^{2\pi} \cos^{t+i+j+2} \theta (1 - \cos^2 \theta) d\theta \\
&\quad - 2 \sum_{t=0}^m \sum_{i=0}^n \sum_{j=0}^l b_{t,2} a_{i,2} a_{j,1} r^{t+i+j} \int_0^{2\pi} \cos^{t+i+j+1} \theta (1 - 2 \cos^2 \theta + \cos^4 \theta) d\theta \\
&\quad + 2 \sum_{t=0}^l \sum_{i=0}^n \sum_{j=0}^m a_{t,1} a_{i,2} b_{j,2} r^{t+i+j} \int_0^{2\pi} \cos^{t+i+j+3} \theta (1 - \cos^2 \theta) d\theta \\
&\quad - 2 \sum_{t=0}^l \sum_{i=0}^n \sum_{j=0}^l a_{t,1} a_{i,2} a_{j,1} r^{t+i+j+1} \int_0^{2\pi} \cos^{t+i+j+2} \theta (1 - 2 \cos^2 \theta + \cos^4 \theta) d\theta.
\end{aligned}$$

Using the formulae in the appendix we get

$$\begin{aligned}
\bar{F}_{30}^4 &= \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[m/2]} \sum_{j=0}^{[(m-1)/2]} \omega_1 a_{2t+1,2} b_{2i,2} b_{2j+1,2} r^{2t+2i+2j+1} \\
&\quad + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[m/2]} \sum_{j=0}^{[l/2]} \omega_2 a_{2t+1,2} b_{2i,2} a_{2j,1} r^{2t+2i+2j+1} \\
&\quad + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[(m-1)/2]} \sum_{j=0}^{[(l-1)/2]} \omega_3 a_{2t+1,2} b_{2i+1,2} a_{2j+1,1} r^{2t+2i+2j+3} \\
&\quad + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[l/2]} \sum_{j=0}^{[(l-1)/2]} \omega_4 a_{2t+1,2} a_{2i,1} a_{2j+1,1} r^{2t+2i+2j+3}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{t=0}^{\lfloor m/2 \rfloor} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} \omega_5 b_{2t,2} a_{2i+1,2} b_{2j+1,2} r^{2t+2i+2j+1} \\
& + \sum_{t=0}^{\lfloor m/2 \rfloor} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor l/2 \rfloor} \omega_6 b_{2t,2} a_{2i+1,2} a_{2j,1} r^{2t+2i+2j+1} \\
& + \sum_{t=0}^{\lfloor (m-1)/2 \rfloor} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor (l-1)/2 \rfloor} \omega_7 b_{2t+1,2} a_{2i+1,2} a_{2j+1,1} r^{2t+2i+2j+3} \\
& + \sum_{t=0}^{\lfloor l/2 \rfloor} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor m/2 \rfloor} \omega_8 a_{2t,1} a_{2i+1,2} b_{2j,2} r^{2t+2i+2j+1} \\
& + \sum_{t=0}^{\lfloor (l-1)/2 \rfloor} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} \omega_9 a_{2t+1,1} a_{2i+1,2} b_{2j+1,2} r^{2t+2i+2j+3} \\
& + \sum_{t=0}^{\lfloor l/2 \rfloor} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=0}^{\lfloor (l-1)/2 \rfloor} \omega_{10} a_{2t,1} a_{2i+1,2} a_{2j+1,1} r^{2t+2i+2j+3},
\end{aligned}$$

for some constants $\omega_{\kappa,t,i,j}$ for $\kappa = 1, \dots, 10$. \square

For an explicit expression of the polynomial $F_{30}^2(r)$ we refer the reader to the proof of Lemma 7.

Lemma 7. *The integral $F_{30}^2(r)$ is a polynomial in the variable r of degree less than or equal to λ_4 given by*

$$\begin{aligned}
\lambda_4 = \max\{ & 2(\lfloor (n-1)/2 \rfloor + \lfloor n/2 \rfloor) + 2, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor + \lfloor (m-1)/2 \rfloor) + 1, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor + \lfloor l/2 \rfloor) + 1, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor (m-1)/2 \rfloor + \lfloor (l-1)/2 \rfloor) + 3, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor l/2 \rfloor + \lfloor (l-1)/2 \rfloor) + 3, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor) + m, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor (l-1)/2 \rfloor) + m + 2, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor) + l + 1, \\
& 2(\lfloor (n-1)/2 \rfloor + \lfloor (l-1)/2 \rfloor) + l + 3\}.
\end{aligned}$$

Proof. We note that

$$2\pi F_{30}^2 = \int_0^{2\pi} \frac{\partial F_1}{\partial r}(r, \theta) \int_0^\theta \frac{\partial F_1}{\partial r}(r, \psi) y_1(r, \psi) d\psi + \int_0^{2\pi} \frac{\partial F_1}{\partial r}(r, \theta) \int_0^\theta F_2(r, \psi) d\psi.$$

We denote

$$\begin{aligned}
\bar{F}_{30}^2 &= \int_0^{2\pi} \frac{\partial F_1}{\partial r}(r, \theta) \int_0^\theta F_2(r, \psi) d\psi, \\
\tilde{F}_{30}^2 &= \int_0^{2\pi} \frac{\partial F_1}{\partial r}(r, \theta) \int_0^\theta \frac{\partial F_1}{\partial r}(r, \psi) y_1(r, \psi) d\psi.
\end{aligned}$$

We first compute $\bar{F}_{30}^2(r)$. To do it, we will start computing $\int_0^\theta F_2(r, \psi) d\psi$. It follows from (12) that

$$\begin{aligned}
& \sum_{i=0}^n \frac{c_{i,2}}{i+1} r^{i+1} (1 - \cos^{i+1} \theta) + \sum_{i=0}^m \frac{d_{i,2}}{i+1} r^i (1 - \cos^{i+1} \theta) + \sum_{i=0}^l \frac{c_{i,1}}{i+2} r^{i+1} (1 - \cos^{i+2} \theta) \\
& - \sum_{i=0}^n \sum_{j=0}^n a_{i,2} a_{j,2} r^{i+j+1} \left(\frac{1 - \cos^{i+j+2} \theta}{i+j+2} - \frac{1 - \cos^{i+j+4} \theta}{i+j+4} \right) \\
& - 2 \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[m/2]} a_{2i+1,2} b_{2j,2} r^{2i+2j+1} \left(\tilde{\beta}_{i+j+2} \theta + \sum_{s=1}^{i+j+2} \tilde{\beta}_{i+j+1,s} \sin(2s\theta) \right) \\
& - 2 \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(m-1)/2]} a_{2i+1,2} b_{2j+1,2} r^{2i+2j+2} \sum_{s=0}^{i+j+2} \tilde{\gamma}_{i+j+1,s} \sin((2s+1)\theta) \\
& + \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[l/2]} a_{2i+1,2} a_{2j,1} r^{2i+2j+2} \sum_{s=0}^{i+j+2} \tilde{\gamma}_{i+j+2,s} \sin((2s+1)\theta) \\
& + \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(l-1)/2]} a_{2i+1,2} a_{2j+1,1} r^{2i+2j+3} \left(\bar{\beta}_{i+j+2} \theta + \sum_{s=1}^{i+j+2} \bar{\beta}_{i+j+2,s} \sin(2s\theta) \right) \\
& - \sum_{i=0}^m \sum_{j=0}^m \frac{b_{i,2} b_{j,2}}{i+j+2} r^{i+j-1} (1 - \cos^{i+j+2} \theta) \\
& + \sum_{i=0}^m \sum_{j=0}^l b_{i,2} a_{j,1} r^{i+j} \left(\frac{1 - \cos^{i+j+1} \theta}{i+j+1} - 2 \frac{1 - \cos^{i+j+3} \theta}{i+j+3} \right) \\
& + \sum_{i=0}^l \sum_{j=0}^l a_{i,1} a_{j,1} r^{i+j+1} \left(\frac{1 - \cos^{i+j+2} \theta}{i+j+2} - \frac{1 - \cos^{i+j+4} \theta}{i+j+4} \right)
\end{aligned}$$

for some non-zero constants $\tilde{\beta}_{i+j+1}, \tilde{\beta}_{i+j+1,s}, \tilde{\gamma}_{i+j+1,s}, \bar{\beta}_{i+j+2}, \bar{\beta}_{i+j+2,s}, \tilde{\gamma}_{i+j+2,s}$. Hence we obtain

$$\begin{aligned}
\bar{F}_{30}^2 &= \sum_{t=0}^{[(n-1)/2]} (2t+2) \sum_{i=0}^n a_{2t+1,2} \frac{c_{i,2}}{i+1} r^{2t+i+2} \int_0^{2\pi} \cos^{2t+1} \theta (1 - \cos^2 \theta) (1 - \cos^{i+1} \theta) d\theta \\
&+ \sum_{t=0}^{[(n-1)/2]} (2t+2) \sum_{i=0}^m a_{2t+1,2} \frac{d_{i,2}}{i+1} r^{2t+i+1} \int_0^{2\pi} \cos^{2t+1} \theta (1 - \cos^2 \theta) (1 - \cos^{i+1} \theta) d\theta \\
&+ \sum_{t=0}^{[(n-1)/2]} (2t+2) \sum_{i=0}^l a_{2t+1,2} \frac{c_{i,1}}{i+2} r^{2t+i+2} \int_0^{2\pi} \cos^{2t+1} \theta (1 - \cos^2 \theta) (1 - \cos^{i+2} \theta) d\theta \\
&- \sum_{t=0}^{[(n-1)/2]} (2t+2) \sum_{i=0}^n \sum_{j=0}^n a_{2t+1,2} a_{i,2} a_{j,2} r^{2t+i+j+2} \\
&\int_0^{2\pi} \cos^{2t+1} \theta (1 - \cos^2 \theta) \left(\frac{1 - \cos^{i+j+2} \theta}{i+j+2} - \frac{1 - \cos^{i+j+4} \theta}{i+j+4} \right) d\theta
\end{aligned}$$

$$\begin{aligned}
& - \sum_{t=0}^{[(n-1)/2]} (2t+2) \sum_{i=0}^m \sum_{j=0}^m a_{2t+1,2} \frac{b_{i,2} b_{j,2}}{i+j+2} r^{2t+i+j} \\
& \int_0^{2\pi} \cos^{2t+1} \theta (1 - \cos^2 \theta) (1 - \cos^{i+j+2} \theta) d\theta \\
& + \sum_{t=0}^{[(n-1)/2]} (2t+2) \sum_{i=0}^m \sum_{j=0}^l a_{2t+1,2} b_{i,2} a_{j,1} r^{2t+i+j+1} \\
& \int_0^{2\pi} \cos^{2t+1} \theta (1 - \cos^2 \theta) \left(\frac{1 - \cos^{i+j+1} \theta}{i+j+1} - 2 \frac{1 - \cos^{i+j+3} \theta}{i+j+3} \right) d\theta \\
& + \sum_{t=0}^{[(n-1)/2]} (2t+2) \sum_{i=0}^l \sum_{j=0}^l a_{2t+1,2} a_{i,1} a_{j,1} r^{2t+i+j+2} \\
& \int_0^{2\pi} \cos^{2t+1} \theta (1 - \cos^2 \theta) \left(\frac{1 - \cos^{i+j+2} \theta}{i+j+2} - \frac{1 - \cos^{i+j+4} \theta}{i+j+4} \right) d\theta \\
& - 2 \sum_{t=0}^m t \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[m/2]} b_{t,2} a_{2i+1,2} b_{2j,2} r^{t+2i+2j} \\
& \int_0^{2\pi} \cos^t \theta \sin \theta \left(\tilde{\beta}_{i+j+1} \theta + \sum_{s=1}^{i+j+1} \tilde{\beta}_{i+j+1,s} \sin(2s\theta) \right) d\theta \\
& - 2 \sum_{t=0}^m t \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(m-1)/2]} b_{t,2} a_{2i+1,2} b_{2j+1,2} r^{t+2i+2j+1} \sum_{s=0}^{i+j+1} \tilde{\gamma}_{i+j+1,s} \\
& \int_0^{2\pi} \cos^t \theta \sin \theta \sin((2s+1)\theta) d\theta \\
& + \sum_{t=0}^m t \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[l/2]} b_{t,2} a_{2i+1,2} b_{2j,1} r^{t+2i+2j} \sum_{s=0}^{i+j+2} \tilde{\gamma}_{i+j+2,s} \\
& \int_0^{2\pi} \cos^t \theta \sin \theta \sin((2s+1)\theta) d\theta \\
& + \sum_{t=0}^m t \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(l-1)/2]} b_{t,2} a_{2i+1,2} b_{2j+1,1} r^{t+2i+2j+1} \\
& \int_0^{2\pi} \cos^t \theta \sin \theta \left(\bar{\beta}_{i+j+2} \theta + \sum_{s=1}^{i+j+2} \bar{\beta}_{i+j+2,s} \sin(2s\theta) \right) d\theta \\
& - 2 \sum_{t=0}^l (t+1) \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[m/2]} a_{t,1} a_{2i+1,2} b_{2j,2} r^{t+2i+2j+1} \\
& \int_0^{2\pi} \cos^{t+1} \theta \sin \theta \left(\tilde{\beta}_{i+j+1} \theta + \sum_{s=1}^{i+j+1} \tilde{\beta}_{i+j+1,s} \sin(2s\theta) \right) d\theta
\end{aligned}$$

$$\begin{aligned}
& - 2 \sum_{t=0}^l (t+1) 2 \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(m-1)/2]} a_{t,1} a_{2i+1,2} b_{2j+1,2} r^{t+2i+2j+2} \sum_{s=0}^{i+j+1} \tilde{\gamma}_{i+j+1,s} \\
& \int_0^{2\pi} \cos^{t+1} \theta \sin \theta \sin((2s+1)\theta) d\theta \\
& + \sum_{t=0}^l (t+1) \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[l/2]} a_{t,1} a_{2i+1,2} b_{2j,1} r^{t+2i+2j+1} \sum_{s=0}^{i+j+2} \tilde{\gamma}_{i+j+2,s} \\
& \int_0^{2\pi} \cos^{t+1} \theta \sin \theta \sin((2s+1)\theta) d\theta \\
& + \sum_{t=0}^l (t+1) \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(l-1)/2]} a_{t,1} a_{2i+1,2} b_{2j+1,1} r^{t+2i+2j+2} \\
& \int_0^{2\pi} \cos^{t+1} \theta \sin \theta \left(\bar{\beta}_{i+j+2} \theta + \sum_{s=1}^{i+j+2} \bar{\beta}_{i+j+2,s} \sin(2s\theta) \right) d\theta.
\end{aligned}$$

Therefore

$$\begin{aligned}
\bar{F}_{30}^2 & = \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[n/2]} \nu_1 a_{2t+1,2} c_{2i,2} r^{2t+2i+2} \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[m/2]} \nu_2 a_{2t+1,2} d_{2i,2} r^{2t+2i+1} \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[(l-1)/2]} \nu_3 a_{2t+1,2} c_{2i+1,1} r^{2t+2i+3} \\
& - \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[m/2]} \sum_{j=0}^{[(m-1)/2]} \nu_4 a_{2t+1,2} b_{2i,2} b_{2j+1,2} r^{2t+2i+2j+1} \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[m/2]} \sum_{j=0}^{[l/2]} \nu_5 a_{2t+1,2} b_{2i,2} a_{2j,1} r^{2t+2i+2j+1} \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[(m-1)/2]} \sum_{j=0}^{[(l-1)/2]} \nu_6 a_{2t+1,2} b_{2i+1,2} a_{2j+1,1} r^{2t+2i+2j+3} \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[l/2]} \sum_{j=0}^{[(l-1)/2]} \nu_7 a_{2t+1,2} a_{2i,1} a_{2j+1,1} r^{2t+2i+2j+3} \\
& - \sum_{t=0}^m \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[m/2]} \nu_8 b_{t,2} a_{2i+1,2} b_{2j,2} r^{t+2i+2j} \\
& + \sum_{t=0}^{[m/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(m-1)/2]} \nu_9 b_{2t,2} a_{2i+1,2} b_{2j+1,2} r^{2t+2i+2j+1}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{t=0}^{[m/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[l/2]} \nu_{10} b_{2t,2} a_{2i+1,2} b_{2j,1} r^{2t+2i+2j} \\
& + \sum_{t=0}^m \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(l-1)/2]} \nu_{11} b_{t,2} a_{2i+1,2} b_{2j+1,1} r^{t+2i+2j+1} \\
& - \sum_{t=0}^l \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[m/2]} \nu_{12} a_{t,1} a_{2i+1,2} b_{2j,2} r^{t+2i+2j+1} \\
& - \sum_{t=0}^{[(l-1)/2]} 2 \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(m-1)/2]} \nu_{13} a_{2t+1,1} a_{2i+1,2} b_{2j+1,2} r^{2t+2i+2j+3} \\
& + \sum_{t=0}^{[(l-1)/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[l/2]} \nu_{14} a_{2t+1,1} a_{2i+1,2} b_{2j,1} r^{2t+2i+2j+2} \\
& + \sum_{t=0}^l \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(l-1)/2]} \nu_{15} a_{t,1} a_{2i+1,2} b_{2j+1,1} r^{t+2i+2j+2}
\end{aligned}$$

for some constants ν_κ for $\kappa = 1, \dots, 15$ depending on t , i and j .

On the other hand we have that $y_1(r, \psi) \frac{\partial F_1}{\partial r}(r, \psi)$ is equal to

$$\begin{aligned}
& \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(n-1)/2]} (2j+2) a_{2i+1,2} a_{2j+1,2} r^{2i+2j+3} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \cos^{2j+1} \psi (1 - \cos^2 \psi) \sin((2s+1)\psi) \\
& + \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^m j a_{2i+1,2} b_{j,2} r^{2i+j+1} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \cos^j \psi \sin \psi \sin((2s+1)\psi) \\
& + \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^l (j+1) a_{2i+1,2} a_{j,1} r^{2i+j+2} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \cos^{j+1} \psi \sin \psi \sin((2s+1)\psi) \\
& + \sum_{i=0}^m \sum_{j=0}^{[(n-1)/2]} \frac{(2j+2) b_{i,2}}{i+1} a_{2j+1,2} r^{i+2j+1} (1 - \cos^{i+1} \psi) (1 - \cos^2 \psi) \cos^{2j+1} \psi \\
& + \sum_{i=0}^m \sum_{j=0}^m \frac{j b_{i,2}}{i+1} b_{j,2} r^{i+j-1} (1 - \cos^{i+1} \psi) \sin \psi \cos^j \psi \\
& + \sum_{i=0}^m \sum_{j=0}^l \frac{(j+1) b_{i,2}}{i+1} a_{j,1} r^{i+j} (1 - \cos^{i+1} \psi) \sin \psi \cos^{j+1} \psi \\
& + \sum_{i=0}^l \sum_{j=0}^{[(n-1)/2]} \frac{(2j+2) a_{i,1}}{i+2} a_{2j+1,2} r^{i+2j+2} (1 - \cos^{i+2} \psi) (1 - \cos^2 \psi) \cos^{2j+1} \psi \\
& + \sum_{i=0}^l \sum_{j=0}^m \frac{j a_{i,1}}{i+2} b_{j,2} r^{i+j} (1 - \cos^{i+2} \psi) \sin \psi \cos^j \psi \\
& + \sum_{i=0}^l \sum_{j=0}^l \frac{(j+1) a_{i,1}}{i+2} a_{j,1} r^{i+j+1} (1 - \cos^{i+2} \psi) \sin \psi \cos^{j+1} \psi.
\end{aligned}$$

Now we consider the integrals $\int_0^\theta y_1(r, \psi) \frac{\partial F_1}{\partial r}(r, \psi) d\psi$. We have that $\int_0^\theta y_1(r, \psi) \frac{\partial F_1}{\partial r}(r, \psi) d\psi$ is equal to

$$\begin{aligned}
& \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(n-1)/2]} (2j+2)a_{2i+1,2}a_{2j+1,2}r^{2i+2j+3} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \sum_{r=0}^{j+s+2} \tilde{P}_{j,s,r} \cos(2r\theta) \\
& + \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[m/2]} 2ja_{2i+1,2}b_{2j,2}r^{2i+2j+1} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \left(\tilde{R}_{j,s}\theta + \sum_{r=1}^{j+s+1} \tilde{R}_{j,s,r} \sin(2r\theta) \right) \\
& + \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(m-1)/2]} (2j+1)a_{2i+1,2}b_{2j+1,2}r^{2i+2j+2} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \sum_{r=0}^{j+s+1} \tilde{Q}_{j,s,r} \sin((2r+1)\theta) \\
& + \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[l/2]} (2j+1)a_{2i+1,2}a_{2j,1}r^{2i+2j+2} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \sum_{r=0}^{j+s+1} \tilde{Q}_{j,s,r} \sin((2r+1)\theta) \\
& + \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(l-1)/2]} (2j+2)a_{2i+1,2}a_{2j+1,1}r^{2i+2j+3} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \left(R_{j+1,s}\theta + \sum_{r=1}^{j+s+2} R_{j+1,s,r} \sin(2r\theta) \right) \\
& + \sum_{i=0}^{[m/2]} \sum_{j=0}^{[(n-1)/2]} \frac{(2j+2)b_{2i,2}}{2i+1} a_{2j+1,2}r^{2i+2j+1} \left(\hat{\beta}_{i+j+3,s}\theta + \sum_{s=1}^{i+j+2} \hat{\beta}_{i+j+3,s,r} \sin(2s\theta) \right) \\
& + \sum_{i=0}^{[(m-1)/2]} \sum_{j=0}^{[(n-1)/2]} \frac{(2j+2)b_{2i+1,2}}{2i+2} a_{2j+1,2}r^{2i+2j+2} \sum_{s=0}^{i+j+2} \hat{\gamma}_{i+j+3,s,r} \sin((2s+1)\theta) \\
& + \sum_{i=0}^m \sum_{j=0}^m \frac{jb_{i,2}}{i+1} b_{j,2}r^{i+j-1} \left(\frac{1-\cos^{j+1}\theta}{j+1} - \frac{1-\cos^{i+j+2}\theta}{i+j+2} \right) \\
& + \sum_{i=0}^m \sum_{j=0}^l \frac{(j+1)b_{i,2}}{i+1} a_{j,1}r^{i+j} \left(\frac{1-\cos^{j+2}\theta}{j+2} - \frac{1-\cos^{i+j+3}\theta}{i+j+3} \right) \\
& + \sum_{i=0}^{[l/2]} \sum_{j=0}^{[(n-1)/2]} \frac{(j+1)a_{2i,1}}{i+1} a_{2j+1,2}r^{2i+2j+2} \sum_{s=0}^{i+j+2} \gamma_{i+j+2,s,r}^* \sin((2s+1)\theta) \\
& + \sum_{i=0}^{[(l-1)/2]} \sum_{j=0}^{[(n-1)/2]} \frac{2(j+1)a_{2i+1,1}}{2i+3} a_{2j+1,2}r^{2i+2j+3} \left(\beta_{i+j+2,s}^*\theta + \sum_{s=1}^{i+j+3} \beta_{i+j+2,s,r}^* \sin(2s\theta) \right) \\
& + \sum_{i=0}^l \sum_{j=0}^m \frac{ja_{i,1}}{i+2} b_{j,2}r^{i+j} \left(\frac{1-\cos^{j+1}\theta}{j+1} - \frac{1-\cos^{i+j+3}\theta}{i+j+3} \right) \\
& + \sum_{i=0}^l \sum_{j=0}^l \frac{(j+1)a_{i,1}}{i+2} a_{j,1}r^{i+j+1} \left(\frac{1-\cos^{j+2}\theta}{j+2} - \frac{1-\cos^{i+j+4}\theta}{i+j+4} \right),
\end{aligned}$$

for some non-zero constants $\tilde{P}_{j,s,r}, \tilde{R}_{j,s}, \tilde{R}_{j,s,r}, \tilde{Q}_{j,s,r}, \hat{\beta}_{i+j+3,s}, \hat{\beta}_{i+j+3,s,r}, \hat{\gamma}_{i+j+3,s,r}, \beta_{i+j+2,s}^*, \beta_{i+j+2,s,r}^*, \gamma_{i+j+2,s,r}^*$. Hence using the formulae in the appendix we can write

$$\begin{aligned}
\tilde{F}_{30}^2 = & \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^m \sum_{j=0}^m 2(t+1)a_{2t+1,2} \frac{j b_{i,2}}{i+1} b_{j,2} r^{2t+i+j} \\
& \int_0^{2\pi} \cos^{2t+1} \theta (1 - \cos^2 \theta) \left(\frac{1 - \cos^{j+1} \theta}{j+1} - \frac{1 - \cos^{i+j+2} \theta}{i+j+2} \right) d\theta \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^m \sum_{j=0}^l 2(t+1)a_{2t+1,2} \frac{(j+1)b_{i,2}}{i+1} a_{j,1} r^{2t+i+j+1} \\
& \int_0^{2\pi} \cos^{2t+1} \theta (1 - \cos^2 \theta) \left(\frac{1 - \cos^{j+2} \theta}{j+2} - \frac{1 - \cos^{i+j+3} \theta}{i+j+3} \right) d\theta \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^l \sum_{j=0}^m 2(t+1)a_{2t+1,2} \frac{j a_{i,1}}{i+2} b_{j,2} r^{2t+i+j+1} \\
& \int_0^{2\pi} \cos^{2t+1} \theta (1 - \cos^2 \theta) \left(\frac{1 - \cos^{j+1} \theta}{j+1} - \frac{1 - \cos^{i+j+3} \theta}{i+j+3} \right) d\theta \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^l \sum_{j=0}^l 2(t+1)a_{2t+1,2} \frac{(j+1)a_{i,1}}{i+2} a_{j,1} r^{2t+i+j+2} \\
& \int_0^{2\pi} \cos^{2t+1} \theta (1 - \cos^2 \theta) \left(\frac{1 - \cos^{j+2} \theta}{j+2} - \frac{1 - \cos^{i+j+4} \theta}{i+j+4} \right) d\theta \\
& + \sum_{t=0}^m \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[m/2]} 2j t b_{t,2} a_{2i+1,2} b_{2j,2} r^{t+2i+2j} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \\
& \int_0^{2\pi} \cos^t \theta \sin \theta \left(\tilde{R}_{j,s} \theta + \sum_{r=1}^{j+s+1} \tilde{R}_{j,s,r} \sin(2r\theta) \right) d\theta \\
& + \sum_{t=0}^l \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[m/2]} 2j(t+1)a_{t,1} a_{2i+1,2} b_{2j,2} r^{t+2i+2j+1} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \\
& \int_0^{2\pi} \cos^{t+1} \theta \sin \theta \left(\tilde{R}_{j,s} \theta + \sum_{r=1}^{j+s+1} \tilde{R}_{j,s,r} \sin(2r\theta) \right) d\theta \\
& + \sum_{t=0}^{[m/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(m-1)/2]} 2(2j+1) t b_{2t,2} a_{2i+1,2} b_{2j+1,2} r^{2t+2i+2j+1} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \sum_{r=0}^{j+s+1} \tilde{Q}_{j,s,r} \\
& \int_0^{2\pi} \cos^{2t} \theta \sin \theta \sin((2r+1)\theta) d\theta \\
& + \sum_{t=0}^{[(l-1)/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(m-1)/2]} (2j+1)(t+1)a_{2t+1,1} a_{2i+1,2} b_{2j+1,2} r^{2t+2i+2j+4} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \sum_{r=0}^{j+s+1} \tilde{Q}_{j,s,r} \\
& \int_0^{2\pi} \cos^{2t+2} \theta \sin \theta \sin((2r+1)\theta) d\theta
\end{aligned}$$

$$\begin{aligned}
& + \sum_{t=0}^{[m/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[l/2]} 2(2j+1)tb_{2t,2}a_{2i+1,2}a_{2j,1}r^{2t+2i+2j+1} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \sum_{r=0}^{j+s+1} \tilde{Q}_{j,s,r} \\
& \int_0^{2\pi} \cos^{2t} \theta \sin \theta \sin((2r+1)\theta) d\theta \\
& + \sum_{t=0}^{[(l-1)/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[l/2]} 2(2j+1)(t+1)a_{2t+1,1}a_{2i+1,2}a_{2j,1}r^{2t+2i+2j+3} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \sum_{r=0}^{j+s+1} \tilde{Q}_{j,s,r} \\
& \int_0^{2\pi} \cos^{2t+2} \theta \sin \theta \sin((2r+1)\theta) d\theta \\
& + \sum_{t=0}^m \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(l-1)/2]} (2j+2)tb_{t,2}a_{2i+1,2}a_{2j+1,1}r^{t+2i+2j+2} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \\
& \int_0^{2\pi} \cos^t \theta \sin \theta \left(R_{j+1,s} \theta + \sum_{r=1}^{j+s+2} R_{j+1,s,r} \sin(2r\theta) \right) d\theta \\
& + \sum_{t=0}^l \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(l-1)/2]} (2j+2)(t+1)a_{t,1}a_{2i+1,2}a_{2j+1,1}r^{t+2i+2j+3} \sum_{s=0}^{i+1} \tilde{\gamma}_{i,s} \\
& \int_0^{2\pi} \cos^{t+1} \theta \sin \theta \left(R_{j+1,s} \theta + \sum_{r=1}^{j+s+2} R_{j+1,s,r} \sin(2r\theta) \right) d\theta \\
& + \sum_{t=0}^m \sum_{i=0}^{[m/2]} \sum_{j=0}^{[(n-1)/2]} \frac{(2j+2)tb_{t,2}b_{2i,2}}{2i+1} a_{2j+1,2}r^{t+2i+2j} \\
& \int_0^{2\pi} \cos^t \theta \sin \theta \left(\hat{\beta}_{i+j+3,s} \theta + \sum_{s=1}^{i+j+2} \hat{\beta}_{i+j+3,s,r} \sin(2s\theta) \right) d\theta \\
& + \sum_{t=0}^l \sum_{i=0}^{[m/2]} \sum_{j=0}^{[(n-1)/2]} \frac{(2j+2)(2t+2)a_{t,1}b_{2i,2}}{2i+1} a_{2j+1,2}r^{t+2i+2j+1} \\
& \int_0^{2\pi} \cos^{t+1} \theta \sin \theta \left(\hat{\beta}_{i+j+3,s} \theta + \sum_{s=1}^{i+j+2} \hat{\beta}_{i+j+3,s,r} \sin(2s\theta) \right) d\theta \\
& + \sum_{t=0}^m \sum_{i=0}^{[(m-1)/2]} \sum_{j=0}^{[(n-1)/2]} \frac{(2j+2)tb_{t,2}b_{2i+1,2}}{2i+2} a_{2j+1,2}r^{t+2i+2j+1} \sum_{s=0}^{i+j+2} \hat{\gamma}_{i+j+3,s,r} \\
& \int_0^{2\pi} \cos^t \theta \sin \theta \sin((2s+1)\theta) d\theta \\
& + \sum_{t=0}^l \sum_{i=0}^{[(m-1)/2]} \sum_{j=0}^{[(n-1)/2]} \frac{(2j+2)(t+1)a_{t,1}b_{2i+1,2}}{2i+2} a_{2j+1,2}r^{t+2i+2j+2} \sum_{s=0}^{i+j+2} \hat{\gamma}_{i+j+3,s,r} \\
& \int_0^{2\pi} \cos^{t+1} \theta \sin \theta \sin((2s+1)\theta) d\theta
\end{aligned}$$

$$\begin{aligned}
& + \sum_{t=0}^m \sum_{i=0}^{[l/2]} \sum_{j=0}^{[(n-1)/2]} \frac{(j+1)tb_{t,2}a_{2i,1}}{i+1} a_{2j+1,2} r^{t+2i+2j+1} \sum_{s=0}^{i+j+2} \gamma_{i+j+2,s,r}^* \\
& \int_0^{2\pi} \cos^t \theta \sin \theta \sin((2s+1)\theta) d\theta \\
& + \sum_{t=0}^l \sum_{i=0}^{[l/2]} \sum_{j=0}^{[(n-1)/2]} \frac{(j+1)(t+1)a_{t,1}a_{2i,1}}{i+1} a_{2j+1,2} r^{t+2i+2j+2} \sum_{s=0}^{i+j+2} \gamma_{i+j+2,s,r}^* \\
& \int_0^{2\pi} \cos^{t+1} \theta \sin \theta \sin((2s+1)\theta) d\theta \\
& + \sum_{t=0}^m \sum_{i=0}^{[(l-1)/2]} \sum_{j=0}^{[(n-1)/2]} \frac{2(j+1)tb_{t,2}a_{2i+1,1}}{2i+3} a_{2j+1,2} r^{t+2i+2j+2} \\
& \int_0^{2\pi} \cos^t \theta \sin \theta \left(\beta_{i+j+2,s}^* \theta + \sum_{s=1}^{i+j+3} \beta_{i+j+2,s,r}^* \sin(2s\theta) \right) d\theta \\
& + \sum_{t=0}^l \sum_{i=0}^{[(l-1)/2]} \sum_{j=0}^{[(n-1)/2]} \frac{2(j+1)(t+1)a_{t,1}a_{2i+1,1}}{2i+3} a_{2j+1,2} r^{t+2i+2j+3} \\
& \int_0^{2\pi} \cos^{t+1} \theta \sin \theta \left(\beta_{i+j+2,s}^* \theta + \sum_{s=1}^{i+j+3} \beta_{i+j+2,s,r}^* \sin(2s\theta) \right) d\theta.
\end{aligned}$$

Hence, using now the non-zero integrals in the Appendix we conclude that

$$\begin{aligned}
\tilde{F}_{30}^2 & = \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^m \sum_{j=0}^{[m/2]} \mu_1 a_{2t+1,2} b_{i,2} b_{2j,2} r^{2t+i+2j} \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[m/2]} \sum_{j=0}^{[(m-1)/2]} \mu_2 a_{2t+1,2} b_{2i,2} b_{2j+1,2} r^{2t+2i+2j+1} \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^m \sum_{j=0}^{[(l-1)/2]} \mu_3 a_{2t+1,2} b_{i,2} a_{2j+1,1} r^{2t+i+2j+2} \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[m/2]} \sum_{j=0}^{[l/2]} \mu_4 a_{2t+1,2} b_{2i,2} a_{2j,1} r^{2t+2i+2j+1} \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^l \sum_{j=0}^{[m/2]} \mu_5 a_{2t+1,2} a_{i,1} b_{2j,2} r^{2t+i+2j+1} \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[(l-1)/2]} \sum_{j=0}^{[(m-1)/2]} \mu_6 a_{2t+1,2} a_{2i+1,1} b_{2j+1,2} r^{2t+2i+2j+3} \\
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^l \sum_{j=0}^{[(l-1)/2]} \mu_7 a_{2t+1,2} a_{i,1} a_{2j+1,1} r^{2t+i+2j+3}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{t=0}^{[(n-1)/2]} \sum_{i=0}^{[(l-1)/2]} \sum_{j=0}^{[l/2]} \mu_8 a_{2t+1,2} a_{2i+1,1} a_{2j,1} r^{2t+2i+2j+3} \\
& + \sum_{t=0}^m \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[m/2]} \mu_9 b_{t,2} a_{2i+1,2} b_{2j,2} r^{t+2i+2j} \\
& + \sum_{t=0}^l \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[m/2]} \mu_{10} a_{t,1} a_{2i+1,2} b_{2j,2} r^{t+2i+2j+1} \\
& + \sum_{t=0}^{[m/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(m-1)/2]} \mu_{11} b_{2t,2} a_{2i+1,2} b_{2j+1,2} r^{2t+2i+2j+1} \\
& + \sum_{t=0}^{[(l-1)/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(m-1)/2]} \mu_{12} a_{2t+1,1} a_{2i+1,2} b_{2j+1,2} r^{2t+2i+2j+3} \\
& + \sum_{t=0}^{[m/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[l/2]} \mu_{13} b_{2t,2} a_{2i+1,2} a_{2j,1} r^{2t+2i+2j+1} \\
& + \sum_{t=0}^{[(l-1)/2]} \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[l/2]} \mu_{14} a_{2t+1,1} a_{2i+1,2} a_{2j,1} r^{2t+2i+2j+3} \\
& + \sum_{t=0}^m \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(l-1)/2]} \mu_{15} b_{t,2} a_{2i+1,2} a_{2j+1,1} r^{t+2i+2j+2} \\
& + \sum_{t=0}^l \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(l-1)/2]} \mu_{16} a_{t,1} a_{2i+1,2} a_{2j+1,1} r^{t+2i+2j+3} \\
& + \sum_{t=0}^m \sum_{i=0}^{[m/2]} \sum_{j=0}^{[(n-1)/2]} \mu_{17} b_{t,2} b_{2i,2} a_{2j+1,2} r^{t+2i+2j} \\
& + \sum_{t=0}^l \sum_{i=0}^{[m/2]} \sum_{j=0}^{[(n-1)/2]} \mu_{18} a_{t,1} b_{2i,2} a_{2j+1,2} r^{t+2i+2j+1} \\
& + \sum_{t=0}^{[m/2]} \sum_{i=0}^{[(m-1)/2]} \sum_{j=0}^{[(n-1)/2]} \mu_{19} b_{2t,2} b_{2i+1,2} a_{2j+1,2} r^{2t+2i+2j+1} \\
& + \sum_{t=0}^{[(l-1)/2]} \sum_{i=0}^{[(m-1)/2]} \sum_{j=0}^{[(n-1)/2]} \mu_{20} a_{2t+1,1} b_{2i+1,2} a_{2j+1,2} r^{2t+2i+2j+3} \\
& + \sum_{t=0}^{[m/2]} \sum_{i=0}^{[l/2]} \sum_{j=0}^{[(n-1)/2]} \mu_{21} b_{2t,2} a_{2i,1} a_{2j+1,2} r^{2t+2i+2j+1} \\
& + \sum_{t=0}^{[(l-1)/2]} \sum_{i=0}^{[l/2]} \sum_{j=0}^{[(n-1)/2]} \mu_{22} a_{2t+1,1} a_{2i,1} a_{2j+1,2} r^{2t+2i+2j+3}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{t=0}^m \sum_{i=0}^{[(l-1)/2]} \sum_{j=0}^{[(n-1)/2]} \mu_{23} b_{t,2} a_{2i+1,1} a_{2j+1,2} r^{t+2i+2j+2} \\
& + \sum_{t=0}^l \sum_{i=0}^{[(l-1)/2]} \sum_{j=0}^{[(n-1)/2]} \mu_{24} a_{t,1} a_{2i+1,1} a_{2j+1,2} r^{t+2i+2j+3},
\end{aligned}$$

for some constants μ_κ for $\kappa = 1, \dots, 24$ depending in t, i and j . \square

APPENDIX: FORMULAE

In this appendix we recall some formulae that will be used during the paper, see for more details [1]. For $i \geq 0$ we have

$$\begin{aligned}
\int_0^{2\pi} \cos^{2i+1} \theta \sin^2 \theta d\theta &= \int_0^{2\pi} \cos^i \theta \sin \theta d\theta = \int_0^{2\pi} \cos^{2i+1} \theta d\theta = 0, \\
\frac{1}{2\pi} \int_0^{2\pi} \cos^{2i} \theta d\theta &= \alpha_i
\end{aligned}$$

where α_i is a non-zero constant.

$$\begin{aligned}
\int_0^\theta \cos^{2i} \phi d\phi &= \beta_i \theta + \sum_{l=1}^i \beta_{i,l} \sin(2l\theta), & \int_0^\theta \cos^{2i+1} \phi d\phi &= \sum_{l=0}^i \gamma_{i,l} \sin((2l+1)\theta), \\
\int_0^\theta \cos^i \phi \sin \phi d\phi &= \frac{1}{i+1} (1 - \cos^{i+1} \theta), \\
\int_0^\theta \cos^{2i+1} \psi \sin((2s+1)\psi) d\psi &= \sum_{r=1}^{i+s+1} P_{i,s,r} \cos(2r\theta), \\
\int_0^\theta \cos^{2i+1} \psi \sin \psi \sin((2s+1)\psi) d\psi &= \sum_{r=0}^{i+s+1} Q_{i,s,r} \sin((2r+1)\theta), \\
\int_0^\theta \cos^{2i} \psi \sin \psi \sin((2s+1)\psi) d\psi &= \left(R_{i,s} \theta + \sum_{r=1}^{i+s+1} R_{i,s,r} \sin(2r\theta) \right),
\end{aligned}$$

where $P_{i,s,r}, Q_{i,s,r}, R_{i,j}, R_{i,s,r}$ are non-zero constants.

$$\begin{aligned}
\int_0^{2\pi} \cos^{2i+1} \theta \sin \theta \sin((2l+1)\theta) d\theta &= \int_0^{2\pi} \cos^{2i} \theta \sin \theta \sin(2l\theta) d\theta = 0, \quad l \geq 0 \\
\int_0^{2\pi} \cos^i \theta \sin((2l+1)\theta) d\theta &= \int_0^{2\pi} \cos^i \theta \sin(2l\theta) d\theta = 0, \quad l \geq 0, \\
\frac{1}{2\pi} \int_0^{2\pi} \cos^{2i} \theta \sin \theta \sin((2l+1)\theta) d\theta &= C_{i,l}, \quad l \geq 0, \\
\frac{1}{2\pi} \int_0^{2\pi} \cos^{2i+1} \theta \sin \theta \sin(2l\theta) d\theta &= K_{i,l}, \quad l \geq 1,
\end{aligned}$$

where $C_{i,j}$ and $K_{i,l}$ are nonzero constants.

$$\int_0^{2\pi} \cos^i \theta \sin \theta \sin(r\theta) \sin(s\theta) d\theta = 0, \quad r, s \in \mathbb{N}$$

$$\begin{aligned}
& \int_0^{2\pi} \cos^{2i+1} \theta \sin((2r+1)\theta) \sin((2s+1)\theta) d\theta = 0, \quad r, s \in \mathbb{Z} \\
& \int_0^{2\pi} \cos^{2i+1} \theta \sin(2r\theta) \sin(2s\theta) d\theta = 0, \quad r, s \in \mathbb{Z} \\
& \int_0^{2\pi} \cos^{2i} \theta \sin((2r+1)\theta) \sin(2s\theta) d\theta = 0, \quad r, s \in \mathbb{Z} \\
& \int_0^{2\pi} \cos^i \theta \sin \theta \cos(2s\theta) d\theta = \int_0^{2\pi} \cos^{2i+1} \theta \cos(2s\theta) d\theta = 0, \quad s \in \mathbb{Z}, \\
& \int_0^{2\pi} \theta \cos^{2i+1} \theta d\theta = 0 \\
& \frac{1}{2\pi} \int_0^{2\pi} \cos^{2i} \theta \sin((2r+1)\theta) \sin((2s+1)\theta) d\theta = \Delta_{i,r,s}, \quad r, s \in \mathbb{Z} \\
& \frac{1}{2\pi} \int_0^{2\pi} \cos^{2i} \theta \sin(2r\theta) \sin(2s\theta) d\theta = \Gamma_{i,r,s}, \quad r, s \in \mathbb{Z} \\
& \int_0^{2\pi} \cos^{2i+1} \theta \sin((2r+1)\theta) \sin(2s\theta) d\theta = \Upsilon_{i,r,s}, \quad r, s \in \mathbb{Z} \\
& \int_0^{2\pi} \cos^{2i} \theta \cos(2s\theta) d\theta = U_{i,s}, \quad \int_0^{2\pi} \cos^i \theta \sin \theta d\theta = V_i,
\end{aligned}$$

where $\Delta_{i,r,s}$, $\Gamma_{i,r,s}$, $\Upsilon_{i,r,s}$, $U_{i,s}$ and V_i are non-zero constants.

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