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Analytic integrability of the Bianchi Class B cosmological models

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We follow Bogoyavlensky's approach to deal with Bianchi class B cosmological models. We characterize the analytic integrability of such systems.

Bianchi models are cosmological models that describe space–times which are foliated by homogeneous hypersurfaces of constant time and are divided into two classes, Class A and Class B. There are many studies about the integrability of Class A. Here we study the integrability of Class B. For the homogeneous cosmological models of Class B, Einstein's system of differential equations reduces to a dynamical system of dimension seven according to Bogoyavlensky's approach. We show that in order to study the integrability of such systems it is sufficient to deal with homogeneous polynomial differential systems of dimension six. Concretely, Bianchi V is the simplest model and can be written as a homogeneous polynomial differential system of degree 2. Bianchi IV is dealt as a homogeneous polynomial differential system of degree 3 and the rest of the models, Bianchis III, VI and VII are of degree 5. Due to the fact that all Bianchi class B models have been reduced to homogeneous polynomial differential systems, the study of their analytic integrability reduces to analyze their homogeneous polynomial first integrals. We show that Bianchi model V admits polynomial first integral, and we prove that the corresponding homogeneous polynomial differential systems that represent models Bianchi IV, III, VI and VII do not admit polynomial first integrals.

I. INTRODUCTION AND STATEMENT OF THE RESULTS

Einstein's equations relate the geometry of the space-time with the properties of the matter which occupied it. The matter occupying the space-time is determined by the stress energy tensor of the matter. In our study we follow[3] and we consider the hydrodynamical tensor of the matter. We will work with an equation of state of matter of the form $p = k\varepsilon$, where ε is the energy density of the matter, p is the

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pressure and $0 \leq k < 1$.

For the homogeneous cosmological models of Class B Einstein's system of equations reduces to the following dynamical system in the phase space $p_i, q_i, p_\varphi, \varphi$, $i = 1, 2, 3$,

$$\frac{dq_i}{d\tau} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{d\tau} = -\frac{\partial H}{\partial q_i} - h_i,$$
$$\frac{d\varphi}{d\tau} = \frac{\partial H}{\partial p_\varphi}, \qquad \frac{dp_\varphi}{d\tau} = -\frac{\partial H}{\partial \varphi} - h_\varphi, \tag{1}$$

where the Hamiltonian H is

$$H = \frac{1}{(q_1 q_2 q_3)^{\frac{1-k}{2}}} \left(T(p_i q_i) + V_G(q_i) \right), \tag{2}$$

with

$$T(p_i, q_i, p_\varphi) = 2 \sum_{1\leq i<j\leq 3} p_i p_j q_i q_j - \sum_{i=1}^{3} p_i^2 q_i^2 - \frac{p_\varphi^2 q_1 q_2}{(n_1 q_1 - n_2 q_2)^2},$$
$$V_G(q_i) = -\frac{1}{4}\left(12 a^2 q_1 q_2 + (n_1 q_1 - n_2 q_2)^2\right),$$

and

$$h_1 = \frac{a^2 q_2}{(q_1 q_2 q_3)^{\frac{1-k}{2}}}, \qquad h_2 = \frac{a^2 q_1}{(q_1 q_2 q_3)^{\frac{1-k}{2}}},$$
$$h_3 = \frac{-2a^2 q_1 q_2}{q_3 (q_1 q_2 q_3)^{\frac{1-k}{2}}}, \qquad h_\varphi = \frac{a(n_1 q_1 - n_2 q_2)^2}{(q_1 q_2 q_3)^{\frac{1-k}{2}}}.$$

System (1) is given in the subspace defined by Einstein's equation

$$p_1 q_1 + p_2 q_2 - 2 p_3 q_3 + \frac{p_\varphi}{2a} = 0. \tag{3}$$

The constants $a \neq 0, n_1, n_2$ determine the type of model according to Table I, see also[1–3]. System (1)

Type	III	IV	V	VI	VII
a	1	1	1	$a \neq 1$	a
n_1	1	1	0	1	1
n_2	-1	0	0	-1	1

TABLE I. The classification of Bianchi class B cosmologies.

in an explicit form writes as

$$\begin{aligned}
\frac{dq_1}{d\tau} &= -\frac{2q_1}{(q_1q_2q_3)^{(1-k)/2}}(p_1q_1 + p_2q_2 - p_3q_3),\\
\frac{dq_2}{d\tau} &= -\frac{2q_2}{(q_1q_2q_3)^{(1-k)/2}}(p_1q_1 + p_3q_3 - p_2q_2),\\
\frac{dq_3}{d\tau} &= -\frac{2q_3}{(q_1q_2q_3)^{(1-k)/2}}(p_1q_1 + p_2q_2 - p_3q_3),\\
\frac{dp_1}{d\tau} &= -\frac{1}{(q_1q_2q_3)^{(1-k)/2}}\Big(2p_1(p_2q_2 + p_3q_3 - p_1q_1) + \frac{k-1}{2}\frac{\bar{H}}{q_1} + \frac{p_\varphi^2(n_1q_1+n_2q_2)}{(n_1q_1-n_2q_2)^3}q_2 \\
&\qquad -2a^2q_2 - \frac{1}{2}n_1(n_1q_1 - n_2q_2)\Big)\\
\frac{dp_2}{d\tau} &= -\frac{1}{(q_1q_2q_3)^{(1-k)/2}}\Big(2p_2(p_1q_1 + p_3q_3 - p_2q_2) + \frac{k-1}{2}\frac{\bar{H}}{q_2} - \frac{p_\varphi^2(n_1q_1+n_2q_2)}{(n_1q_1-n_2q_2)^3}q_1 \\
&\qquad -2a^2q_1 - \frac{1}{2}n_2(n_1q_1 - n_2q_2)\Big),\\
\frac{dp_3}{d\tau} &= -\frac{1}{(q_1q_2q_3)^{(1-k)/2}}\Big(2p_3(p_1q_1 + p_2q_2 - p_3q_3) -2a^2\frac{q_1q_2}{q_3} + \frac{k-1}{2}\frac{\bar{H}}{q_3}\Big),\\
\frac{d\varphi}{d\tau} &= -\frac{1}{(q_1q_2q_3)^{(1-k)/2}}\frac{2q_1q_2p_\varphi}{(n_1q_1 - n_2q_2)^2},\\
\frac{dp_\varphi}{d\tau} &= -\frac{1}{(q_1q_2q_3)^{(1-k)/2}}a(n_1q_1 - n_2q_2)^2,
\end{aligned} \tag{4}$$

with $\bar{H} = T + V_G$.

We notice that for Bianchi V we have $p_\varphi = 0$ and $\dot{\varphi} = 0$, so for this model we obtain a simplest model of six equations. After the change of coordinates and time

$$x_i = q_i, \;\; x_{i+3} = p_i q_i, \;\; i = 1, 2, 3, \;\; d\tau_0 = \frac{(q_1 q_2 q_3)^{\frac{k-1}{2}}}{2} d\tau,$$

the Bianchi V model writes as a homogeneous polynomial differential system of degree 2:

$$\begin{aligned}
\dot{x}_1 &= x_1(-x_4 + x_5 + x_6), \\
\dot{x}_2 &= x_2(x_4 - x_5 + x_6), \\
\dot{x}_3 &= x_3(x_4 + x_5 - x_6), \\
\dot{x}_4 &= x_1x_2 + \frac{k-1}{4}(3x_1x_2 + \Lambda), \\
\dot{x}_5 &= x_1x_2 + \frac{k-1}{4}(3x_1x_2 + \Lambda), \\
\dot{x}_6 &= x_1x_2 + \frac{k-1}{4}(3x_1x_2 + \Lambda),
\end{aligned} \tag{5}$$

where

$$\Lambda = x_4^2 + x_5^2 + x_6^2 - 2(x_4x_5 + x_4x_6 + x_5x_6). \tag{6}$$

The condition stated in (3) reduces to $x_4 + x_5 - 2x_6 = 0$ for Bianchi V model. The Hamiltonian (2) becomes the first integral

$$(x_1x_2x_3)^{\frac{k-1}{2}}\left(3x_1x_2 + \Lambda\right).$$

For Bianchi models III, IV, VI and VII we notice that $n_1 = 1$. Moreover system (4) is defined on the subspace (3), hence we can eliminate the momentum p_φ. So we can reduce the dimension of system (4) to seven. Note also that in order to study the integrability of system (4) we do not need to consider the seventh equation. In short, for studying the integrability of system (4) it is sufficient to deal with a differential system of dimension six.

For Bianchi IV after the change of coordinates and time

$$x_i = q_i,\ x_{i+3} = p_iq_i,\ i = 1, 2, 3,\ d\tau_0 = \frac{q_1(q_1q_2q_3)^{\frac{k-1}{2}}}{2}d\tau,$$

and the above considerations we get the six-dimensional homogeneous polynomial differential system of degree 3

$$\begin{aligned}
\dot{x}_1 &= x_1^2(-x_4 + x_5 + x_6), \\
\dot{x}_2 &= x_1x_2(x_4 - x_5 + x_6), \\
\dot{x}_3 &= x_1x_3(x_4 + x_5 - x_6), \\
\dot{x}_4 &= \frac{x_1^3}{4} + x_1^2x_2 - 2x_2(x_4 + x_5 - 2x_6)^2 + \frac{k-1}{4}R, \\
\dot{x}_5 &= x_1^2x_2 + 2x_2(x_4 + x_5 - 2x_6)^2 + \frac{k-1}{4}R, \\
\dot{x}_6 &= x_1^2x_2 + \frac{k-1}{4}R,
\end{aligned} \tag{7}$$

where

$$R = x_1^3/4 + 3x_1^2x_2 + 4x_2(x_4 + x_5 - 2x_6)^2 + x_1\Lambda.$$

The Hamiltonian (2) becomes the first integral

$$(x_1x_2x_3)^{\frac{k-1}{2}}\left(\frac{x_1^2}{4} + 3x_1x_2 + \frac{4x_2(x_4 + x_5 - 2x_6)^2}{x_1} + \Lambda\right).$$

In the rest of the cases i.e. for the Bianchi models III, VI and VII after the change of coordinates and time

$$x_i = q_i,\ x_{i+3} = p_iq_i,\ i = 1, 2, 3,\ d\tau_0 = \frac{N^3(q_1q_2q_3)^{\frac{k-1}{2}}}{2}d\tau,$$

where $N = x_1 - n_2x_2$, and the above considerations we obtain the six-dimensional homogeneous polynomial differential system of degree 5

$$\begin{aligned}
\dot{x}_1 &= x_1N^3(-x_4 + x_5 + x_6), \\
\dot{x}_2 &= x_2N^3(x_4 - x_5 + x_6), \\
\dot{x}_3 &= x_3N^3(x_4 + x_5 - x_6), \\
\dot{x}_4 &= \frac{1}{4}x_1N^4 + a^2x_1x_2N^3 + \frac{k-1}{4}NS, \\
&\quad - 2a^2x_1x_2(x_1 + n_2x_2)(x_4 + x_5 - 2x_6)^2 \\
\dot{x}_5 &= \frac{1}{4}N^5 - \frac{1}{4}x_1N^4 + a^2x_1x_2N^3 + \frac{k-1}{4}NS, \\
&\quad + 2a^2x_1x_2(x_1 + n_2x_2)(x_4 + x_5 - 2x_6)^2 \\
\dot{x}_6 &= a^2x_1x_2N^3 + \frac{k-1}{4}NS,
\end{aligned} \tag{8}$$

where

$$S = N^4/4 + 4a^2x_1x_2(x_4 + x_5 - 2x_6)^2 + N^2(3a^2x_1x_2 + \Lambda).$$

The Hamiltonian (2) becomes the first integral

$$(x_1x_2x_3)^{\frac{k-1}{2}}\left(\frac{N^2}{4}+3a^2x_1x_2+\Lambda + \frac{4a^2x_1x_2(x_4+x_5-2x_6)^2}{N^2}\right).$$

A study of the integrability of Bianchi models using its symmetries has been done in[4], but no information is given there on the analytic integrability of these systems. The analytic integrability of Class A has been studied in previous works by several authors [5–9,12–16]. Here, our aim is to study the analytic integrability of all Bianchi models of class B in the variables $(x_1, x_2, x_3, x_4, x_5, x_6)$. In our study we will use the following result, see[11].

Proposition 1. *Let F be an analytic function and let $F=\sum_i F_i$ be its decomposition into homogeneous polynomials F_i of degree i. Then F is an analytic first integral of a homogeneous differential system if and only if for all i F_i is a homogeneous polynomial first integral of the homogeneous system.*

Due to Proposition 1 and the fact that all Bianchi class B models have been reduced to homogeneous polynomial differential systems, see the systems (5), (7) and (8), the study of their analytic integrability reduces to analyze their homogeneous polynomial first integrals. Our main result is the following.

Theorem 2. *The following statements hold.*

(a) System (5) has the two independent first integrals $x_4 - x_5$ and $x_4 - x_6$. These two first integrals become the same when we restrict system (5) to (3), i.e. $x_4 + x_5 - 2x_6 = 0$, and any other polynomial first integral is a polynomial in the variable $x_4 - x_5$.

(b) System (7) has no polynomial first integrals.

(c) Systems (8) have no polynomial first integrals.

Section II provides some technical lemmas that we will use for the proof of Theorem 2. In Section III we prove the first statement of Theorem 2, namely we study Bianchi V. In Section IV we deal with Bianchi IV. The integrability of Bianchi III, VI and VII is studied in Section V.

II. SOME AUXILIARY RESULTS

Lemma 3 (see[10]). *Let x_k be a one-dimensional variable, $k \in \{1,\ldots,n\}$, $n > 1$ and let $f = f(x_1,\ldots,x_n)$ be a polynomial. For $l \in \{1,\cdots,n\}$ and c_0 a constant let $f_l = f(x_1,\ldots,x_n)|_{x_l=c_0}$. Then there exists a polynomial $g = g(x_1,\ldots,x_n)$ such that $f = f_l + (x_l - c_0)g$.*

The next two lemmas are proved in[7].

Lemma 4. *Let $g = g(x_4, x_5, x_6)$ be a homogeneous polynomial solution of the homogeneous partial differential equation*

$$(a_1x_4+a_2x_5+a_3x_6)g+\frac{k-1}{4}\Lambda\left(\frac{\partial g}{\partial x_4}+\frac{\partial g}{\partial x_5}+\frac{\partial g}{\partial x_6}\right)=0, \quad (9)$$

where $a_1, a_2, a_3 \in \mathbb{R}$ are such that $(a_1-a_2)^2+(a_1-a_3)^2 \neq 0$. Then $g \equiv 0$.

Lemma 5. *Let $g = g(x_4, x_5, x_6)$ and $h_2 = h_2(x_4 - x_5, x_4 - x_6)$ be homogeneous polynomials of respective degrees $n-2$ and n such that*

$$2(x_4-x_5+x_6)g+\frac{k-1}{4}\Lambda\left(\frac{\partial g}{\partial x_4}+\frac{\partial g}{\partial x_5}+\frac{\partial g}{\partial x_6}\right)+\frac{\partial h_2}{\partial x_5}=0. \quad (10)$$

Then $h_2 = h_2(x_4 - x_6)$ and $g \equiv 0$.

III. PROOF OF STATEMENT (A) OF THEOREM 2

It is clear that system (5) has the first integrals $x_4 - x_5$ and $x_4 - x_6$. We shall prove in this section that under the restriction $x_4 + x_5 - 2x_6 = 0$ system (5) has only one independent first integral and it is $x_4 - x_5$.

Suppose that system (5) has a homogeneous polynomial first integral h of degree n. According to Lemma 3 we can write $h = h_1(x_2, x_3, x_4, x_5, x_6) + x_1^j g_1(x_1, x_2, x_3, x_4, x_5, x_6)$, with $x_1 \nmid g_1$ and $j \in \mathbb{N}$. Suppose that $g_1 \not\equiv 0$. System (5) on $x_1 = 0$ writes

$$\begin{aligned} \dot{x}_2 &= x_2(x_4 - x_5 + x_6), \\ \dot{x}_3 &= x_3(x_4 + x_5 - x_6) \\ \dot{x}_4 &= \frac{k-1}{4}\Lambda, \\ \dot{x}_5 &= \frac{k-1}{4}\Lambda, \\ \dot{x}_6 &= \frac{k-1}{4}\Lambda. \end{aligned} \tag{11}$$

Since h is a first integral of system (5), we have that h_1 is a first integral of system (11). System (11) admits the two polynomial first integrals $x_4 - x_5$ and $x_4 - x_6$ and the two non-polynomial first integrals

$$x_2^{\frac{3}{2}(k-1)}\Lambda \left(\frac{x_4 + x_5 + x_6 - 2\sqrt{\Delta}}{x_4 + x_5 + x_6 + 2\sqrt{\Delta}} \right)^{\frac{x_4 - 2x_5 + x_6}{\sqrt{\Delta}}}$$

and

$$x_3^{\frac{3}{2}(k-1)}\Lambda \left(\frac{x_4 + x_5 + x_6 - 2\sqrt{\Delta}}{x_4 + x_5 + x_6 + 2\sqrt{\Delta}} \right)^{\frac{x_4 + x_5 - 2x_6}{\sqrt{\Delta}}},$$

where

$$\Delta = x_4^2 + x_5^2 + x_6^2 - x_4x_5 - x_4x_6 - x_5x_6. \tag{12}$$

As these four first integrals of system (11) are independent and h_1 is a polynomial first integral of (11), we get, taking into account the equality $x_4 + x_5 - 2x_6 = 0$, that $h_1 = C_1(x_4 - x_5)^n$, where $C_1 \in \mathbb{R}$. Now since h and $x_4 - x_5$ are first integrals of system (5) we have that $x_1^j g_1$ is also a first integral of system (5). Therefore

$$\begin{aligned} & j(-x_4 + x_5 + x_6)g_1 + x_1(-x_4 + x_5 + x_6)\frac{\partial g_1}{\partial x_1} \\ & \quad + x_2(x_4 - x_5 + x_6)\frac{\partial g_1}{\partial x_2} + x_3(x_4 + x_5 - x_6)\frac{\partial g_1}{\partial x_3} \\ & \quad + \left(x_1x_2 + \frac{k-1}{4}\Lambda \right) \left(\frac{\partial g_1}{\partial x_4} + \frac{\partial g_1}{\partial x_5} + \frac{\partial g_1}{\partial x_6} \right) = 0. \end{aligned}$$

Let $\bar{g}_1 = g_1|_{x_1=0} \not\equiv 0$. On $x_1 = 0$ we have

$$\begin{aligned} & j(-x_4 + x_5 + x_6)\bar{g}_1 + x_2(x_4 - x_5 + x_6)\frac{\partial \bar{g}_1}{\partial x_2} \\ & + x_3(x_4 + x_5 - x_6)\frac{\partial \bar{g}_1}{\partial x_3} \\ & + \frac{k-1}{4}\Lambda \left(\frac{\partial \bar{g}_1}{\partial x_4} + \frac{\partial \bar{g}_1}{\partial x_5} + \frac{\partial \bar{g}_1}{\partial x_6} \right) = 0. \end{aligned}$$

Write $\bar{g}_1 = x_2^l g_2$, with $l \in \mathbb{N} \cup \{0\}$, $x_2 \nmid g_2$ and $g_2 \not\equiv 0$. Then

$$\begin{aligned} & [j(-x_4 + x_5 + x_6) + l(x_4 - x_5 + x_6)]g_2 \\ & + x_2(x_4 - x_5 + x_6)\frac{\partial g_2}{\partial x_2} + x_3(x_4 + x_5 - x_6)\frac{\partial g_2}{\partial x_3} \\ & + \frac{k-1}{4}\Lambda \left(\frac{\partial g_2}{\partial x_4} + \frac{\partial g_2}{\partial x_5} + \frac{\partial g_2}{\partial x_6} \right) = 0. \end{aligned}$$

Set $\bar{g}_2 = g_2|_{x_2=0} \not\equiv 0$. Then on $x_2 = 0$ we have

$$\begin{aligned} & [j(-x_4 + x_5 + x_6) + l(x_4 - x_5 + x_6)]\bar{g}_2 \\ & + x_3(x_4 + x_5 - x_6)\frac{\partial \bar{g}_2}{\partial x_3} \\ & + \frac{k-1}{4}\Lambda \left(\frac{\partial \bar{g}_2}{\partial x_4} + \frac{\partial \bar{g}_2}{\partial x_5} + \frac{\partial \bar{g}_2}{\partial x_6} \right) = 0. \end{aligned}$$

Now let $\bar{g}_2 = x_3^m g_3$, with $m \in \mathbb{N} \cup \{0\}$, $x_3 \nmid g_3$ and $g_3 \not\equiv 0$. We have

$$\begin{aligned} & [j(-x_4 + x_5 + x_6) + l(x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6)]g_3 \\ & + x_3(x_4 + x_5 - x_6)\frac{\partial g_3}{\partial x_3} \\ & + \frac{k-1}{4}\Lambda \left(\frac{\partial g_3}{\partial x_4} + \frac{\partial g_3}{\partial x_5} + \frac{\partial g_3}{\partial x_6} \right) = 0. \end{aligned}$$

Let $\bar{g}_3 = g_3|_{x_3=0} \not\equiv 0$. On $x_3 = 0$ we obtain

$$\begin{aligned} & [j(-x_4 + x_5 + x_6) + l(x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6)]\bar{g}_3 \\ & + \frac{k-1}{4}\Lambda \left(\frac{\partial \bar{g}_3}{\partial x_4} + \frac{\partial \bar{g}_3}{\partial x_5} + \frac{\partial \bar{g}_3}{\partial x_6} \right) = 0. \end{aligned}$$

We are under the hypotheses of Lemma 4, hence we have $\bar{g}_3 \equiv 0$, which is in contradiction with the assumptions. Therefore $h = C_1(x_4 - x_5)^n$ and the proof follows.

IV. PROOF OF STATEMENT (B) OF THEOREM 2

We study in this section the analytic integrability of Bianchi model IV. We consider system (7). Let $h = h(x_1, \ldots, x_6)$ be a homogeneous polynomial first integral of degree n of system (7). Using Lemma 3 we can write $h = h_2(x_1, x_3, \ldots, x_6) + x_2^l g_2(x_1, \ldots, x_6)$, with $l \in \mathbb{N}$ and h_2 and g_2 homogeneous polynomials such that $x_2 \nmid g_2$. Assume that $g_2 \not\equiv 0$. On $x_2 = 0$ system (7) becomes, after canceling a common factor x_1, doing a change in the independent variable,

$$\begin{aligned} \dot{x}_1 &= x_1(-x_4 + x_5 + x_6), \\ \dot{x}_3 &= x_3(x_4 + x_5 - x_6), \\ \dot{x}_4 &= \frac{x_1^2}{4} + \frac{k-1}{4}\left(\frac{x_1^2}{4} + \Lambda\right), \\ \dot{x}_5 &= \frac{k-1}{4}\left(\frac{x_1^2}{4} + \Lambda\right), \\ \dot{x}_6 &= \frac{k-1}{4}\left(\frac{x_1^2}{4} + \Lambda\right). \end{aligned} \tag{13}$$

We notice that h_2 is a first integral of system (13). We write $h_2 = h_3(x_3, \ldots, x_6) + x_1^j g_3(x_1, x_3, \ldots, x_6)$, with $j \in \mathbb{N}$ and h_3 and g_3 homogeneous polynomials such that $x_1 \nmid g_3$. Assume that $g_3 \not\equiv 0$. System (13) on $x_1 = 0$ writes

$$\begin{aligned} \dot{x}_3 &= x_3(x_4 + x_5 - x_6), \\ \dot{x}_4 &= \frac{k-1}{4}\Lambda, \\ \dot{x}_5 &= \frac{k-1}{4}\Lambda, \\ \dot{x}_6 &= \frac{k-1}{4}\Lambda. \end{aligned} \tag{14}$$

Straightforward computations show that system (14) has the three independent first integrals $x_4 - x_5$, $x_4 - x_6$ and

$$x_3^{\frac{3}{2}(k-1)\Lambda}\left(\frac{x_4 + x_5 + x_6 - 2\sqrt{\Delta}}{x_4 + x_5 + x_6 + 2\sqrt{\Delta}}\right)^{\frac{x_4 + x_5 - 2x_6}{\sqrt{\Delta}}},$$

where Δ is given by relation (12). As h_3 is a polynomial first integral of system (14), we have $h_3 = h_3(x_4 - x_5, x_5 - x_6)$. The following lemma shows that indeed $h = h_3(x_5 - x_6) + x_2^l g_2(x_1, \ldots, x_6)$.

Lemma 6. *We have that $h_3 = h_3(x_5 - x_6)$ and $g_3 \equiv 0$.*

Proof. As $h_2 = h_3 + x_1^j g_3$ is a first integral of system (13), we have

$$x_1^j \Bigg[j(-x_4 + x_5 + x_6)g_3 + x_1(-x_4 + x_5 + x_6)\frac{\partial g_3}{\partial x_1} + x_3(x_4 + x_5 - x_6)\frac{\partial g_3}{\partial x_3} + \frac{x_1^2}{4}\frac{\partial g_3}{\partial x_4} + \frac{k-1}{4}\left(\frac{x_1^2}{4} + \Lambda\right)\left(\frac{\partial g_3}{\partial x_4} + \frac{\partial g_3}{\partial x_5} + \frac{\partial g_3}{\partial x_6}\right)\Bigg] + \frac{x_1^2}{4}\frac{\partial h_3}{\partial x_4} = 0. \tag{15}$$

We distinguish some cases depending on the value of j.

If $j = 1$ then equation (15) becomes

$$(-x_4 + x_5 + x_6)g_3 + x_1(-x_4 + x_5 + x_6)\frac{\partial g_3}{\partial x_1} + x_3(x_4 + x_5 - x_6)\frac{\partial g_3}{\partial x_3} + \frac{x_1^2}{4}\frac{\partial g_3}{\partial x_4} + \frac{k-1}{4}\left(\frac{x_1^2}{4} + \Lambda\right)\left(\frac{\partial g_3}{\partial x_4} + \frac{\partial g_3}{\partial x_5} + \frac{\partial g_3}{\partial x_6}\right) + \frac{x_1}{4}\frac{\partial h_3}{\partial x_4} = 0.$$

Let $\bar{g}_3 = g_3|_{x_1=0} \not\equiv 0$. On $x_1 = 0$ we have

$$(-x_4 + x_5 + x_6)\bar{g}_3 + x_3(x_4 + x_5 - x_6)\frac{\partial \bar{g}_3}{\partial x_3} + \frac{k-1}{4}\Lambda\left(\frac{\partial \bar{g}_3}{\partial x_4} + \frac{\partial \bar{g}_3}{\partial x_5} + \frac{\partial \bar{g}_3}{\partial x_6}\right) = 0.$$

Write $\bar{g}_3 = x_3^m g_4 \not\equiv 0$, with $m \in \mathbb{N} \cup \{0\}$ and $x_3 \nmid g_4$. Then,

$$[(-x_4 + x_5 + x_6) + m(x_4 + x_5 - x_6)]g_4 + x_3(x_4 + x_5 - x_6)\frac{\partial g_4}{\partial x_3} + \frac{k-1}{4}\Lambda\left(\frac{\partial g_4}{\partial x_4} + \frac{\partial g_4}{\partial x_5} + \frac{\partial g_4}{\partial x_6}\right) = 0.$$

Let $\bar{g}_4 = g_4|_{x_3=0} \not\equiv 0$. On $x_3 = 0$ we have

$$[(-x_4 + x_5 + x_6) + m(x_4 + x_5 - x_6)]\bar{g}_4 + \frac{k-1}{4}\Lambda\left(\frac{\partial \bar{g}_4}{\partial x_4} + \frac{\partial \bar{g}_4}{\partial x_5} + \frac{\partial \bar{g}_4}{\partial x_6}\right) = 0.$$

Applying Lemma 4 we obtain $\bar{g}_4 \equiv 0$, which is a contradiction. Hence $g_3 \equiv 0$. Back to equation (15) we have $\partial h_3/\partial x_4 \equiv 0$, which means that $h_3 = h_3(x_5 - x_6)$. Then the lemma follows in the case $j = 1$.

If $j > 2$, then from equation (15) we have that $x_1 \,|(\partial h_3/\partial x_4)$ and thus $\partial h_3/\partial x_4 \equiv 0$. Now we can proceed as in the case $j = 1$ to obtain the equation

$$[j(-x_4 + x_5 + x_6) + m(x_4 + x_5 - x_6)]\bar{g}_4 + \frac{k-1}{4}\Lambda\left(\frac{\partial \bar{g}_4}{\partial x_4} + \frac{\partial \bar{g}_4}{\partial x_5} + \frac{\partial \bar{g}_4}{\partial x_6}\right) = 0.$$

Applying Lemma 4 we arrive to a contradiction and hence $g_3 \equiv 0$. The lemma follows in the case $j > 2$.

If $j = 2$ then equation (15) writes

$$2(-x_4 + x_5 + x_6)g_3 + x_1(-x_4 + x_5 + x_6)\frac{\partial g_3}{\partial x_1} + x_3(x_4 + x_5 - x_6)\frac{\partial g_3}{\partial x_3} + \frac{x_1^2}{4}\frac{\partial g_3}{\partial x_4} + \frac{k-1}{4}\left(\frac{x_1^2}{4} + \Lambda\right)\left(\frac{\partial g_3}{\partial x_4} + \frac{\partial g_3}{\partial x_5} + \frac{\partial g_3}{\partial x_6}\right) + \frac{1}{4}\frac{\partial h_3}{\partial x_4} = 0.$$

Let $\bar{g}_3 = g_3|_{x_1=0} \not\equiv 0$. On $x_1 = 0$ we have

$$2(-x_4 + x_5 + x_6)\bar{g}_3 + x_3(x_4 + x_5 - x_6)\frac{\partial \bar{g}_3}{\partial x_3} + \frac{k-1}{4}\Lambda\left(\frac{\partial \bar{g}_3}{\partial x_4} + \frac{\partial \bar{g}_3}{\partial x_5} + \frac{\partial \bar{g}_3}{\partial x_6}\right) + \frac{1}{4}\frac{\partial h_3}{\partial x_4} = 0.$$

Write $\bar{g}_3 = x_3^m g_4 \not\equiv 0$, with $m \in \mathbb{N} \cup \{0\}$ and $x_3 \nmid g_4$. Then we get

$$x_3^m\left[[2(-x_4 + x_5 + x_6) + m(x_4 + x_5 - x_6)]g_4 + x_3(x_4 + x_5 - x_6)\frac{\partial g_4}{\partial x_3} + \frac{k-1}{4}\Lambda\left(\frac{\partial g_4}{\partial x_4} + \frac{\partial g_4}{\partial x_5} + \frac{\partial g_4}{\partial x_6}\right)\right] + \frac{1}{4}\frac{\partial h_3}{\partial x_4} = 0.$$

If $m > 0$ then $x_3 \,|(\partial h_3/\partial x_4)$. Hence $\partial h_3/\partial x_4 \equiv 0$ and $h_3 = h_3(x_5 - x_6)$. Let $\bar{g}_4 = g_4|_{x_3=0} \not\equiv 0$. On $x_3 = 0$ we obtain

$$[2(-x_4 + x_5 + x_6) + m(x_4 + x_5 - x_6)]\bar{g}_4 + \frac{k-1}{4}\Lambda\left(\frac{\partial \bar{g}_4}{\partial x_4} + \frac{\partial \bar{g}_4}{\partial x_5} + \frac{\partial \bar{g}_4}{\partial x_6}\right) = 0.$$

Applying Lemma 4 we get a contradiction, hence $g_3 \equiv 0$.

If $m = 0$, let $\bar{g}_4 = g_4|_{x_3=0} \not\equiv 0$. On $x_3 = 0$ we obtain

$$2(-x_4+x_5+x_6)\bar{g}_4 + \frac{k-1}{4}\Lambda\left(\frac{\partial \bar{g}_4}{\partial x_4} + \frac{\partial \bar{g}_4}{\partial x_5} + \frac{\partial \bar{g}_4}{\partial x_6}\right) + \frac{1}{4}\frac{\partial h_3}{\partial x_4} = 0$$

We can apply Lemma 5, permuting the variables x_4 by x_5, and we get that $h_3 = h_3(x_5 - x_6)$ so $\partial h_3/\partial x_4 \equiv 0$. Now applying Lemma 4 we obtain $\bar{g}_4 \equiv 0$. Hence the lemma follows in the case $j = 2$. □

After Lemma 6 we have that $h = h_3(x_5 - x_6) + x_2^l g_2(x_1, \ldots, x_6)$, with $l \in \mathbb{N}$ and $x_2 \nmid g_2$. We write $h_3(x_5 - x_6) = C_3(x_5 - x_6)^n$, with $C_3 \in \mathbb{R}$. We recall that h is a first integral of system (7). Thus it satisfies the equation

$$x_2^l\left[lx_1(x_4 - x_5 + x_6)g_2 + x_1^2(-x_4 + x_5 + x_6)\frac{\partial g_2}{\partial x_1} + x_1x_2(x_4 - x_5 + x_6)\frac{\partial g_2}{\partial x_2} + x_1x_3(x_4 + x_5 - x_6)\frac{\partial g_2}{\partial x_3} + \frac{k-1}{4}R\left(\frac{\partial g_2}{\partial x_4} + \frac{\partial g_2}{\partial x_5} + \frac{\partial g_2}{\partial x_6}\right) + \left(\frac{x_1^3}{4} + x_1^2x_2 - 2x_2(x_4 + x_5 - 2x_6)^2\right)\frac{\partial g_2}{\partial x_4} + \left(x_1^2x_2 + 2x_2(x_4 + x_5 - 2x_6)^2\right)\frac{\partial g_2}{\partial x_5} + x_1^2x_2\frac{\partial g_2}{\partial x_6}\right] + 2C_3nx_2(x_4 + x_5 - 2x_6)^2(x_5 - x_6)^{n-1} = 0. \tag{16}$$

The following lemma ends the proof of statement (b) of Theorem 2, as it shows that $h \equiv 0$.

Lemma 7. *We have that $h_3 \equiv 0$ and $g_2 \equiv 0$.*

Proof. Suppose that $g_2 \not\equiv 0$. We distinguish two cases depending on the value of l. If $l > 1$ then from equation (16) we must take $C_3 = 0$ and therefore $h_3 \equiv 0$. Let $\bar{g}_2 = g_2|_{x_2=0} \not\equiv 0$. After simplifying x_2^l, equation (16) on

$x_2 = 0$ becomes, after cancelling a common factor x_1,

$$l(x_4 - x_5 + x_6)\bar{g}_2 + x_1(-x_4 + x_5 + x_6)\frac{\partial \bar{g}_2}{\partial x_1} + x_3(x_4 + x_5 - x_6)\frac{\partial \bar{g}_2}{\partial x_3} + \frac{x_1^2}{4}\frac{\partial \bar{g}_2}{\partial x_4} + \frac{k-1}{4}\left(\frac{x_1^2}{4} + \Lambda\right)\left(\frac{\partial \bar{g}_2}{\partial x_4} + \frac{\partial \bar{g}_2}{\partial x_5} + \frac{\partial \bar{g}_2}{\partial x_6}\right) = 0.$$

Write $\bar{g}_2 = x_1^j g_3 \not\equiv 0$, with $j \in \mathbb{N} \cup \{0\}$ and $x_1 \nmid g_3$. We get

$$[j(-x_4 + x_5 + x_6) + l(x_4 - x_5 + x_6)]g_3 + x_1(-x_4 + x_5 + x_6)\frac{\partial g_3}{\partial x_1} + x_3(x_4 + x_5 - x_6)\frac{\partial g_3}{\partial x_3} + \frac{x_1^2}{4}\frac{\partial g_3}{\partial x_4} + \frac{k-1}{4}\left(\frac{x_1^2}{4} + \Lambda\right)\left(\frac{\partial g_3}{\partial x_4} + \frac{\partial g_3}{\partial x_5} + \frac{\partial g_3}{\partial x_6}\right) = 0.$$

Let $\bar{g}_3 = g_3|_{x_1=0} \not\equiv 0$. Then, on $x_1 = 0$ we have

$$[j(-x_4 + x_5 + x_6) + l(x_4 - x_5 + x_6)]\bar{g}_3 + x_3(x_4 + x_5 - x_6)\frac{\partial \bar{g}_3}{\partial x_3} + \frac{k-1}{4}\Lambda\left(\frac{\partial \bar{g}_3}{\partial x_4} + \frac{\partial \bar{g}_3}{\partial x_5} + \frac{\partial \bar{g}_3}{\partial x_6}\right) = 0.$$

Now write $\bar{g}_3 = x_3^m g_4 \not\equiv 0$, with $m \in \mathbb{N} \cup \{0\}$ and $x_3 \nmid g_4$. We get

$$[j(-x_4 + x_5 + x_6) + l(x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6)]g_4 + x_3(x_4 + x_5 - x_6)\frac{\partial g_4}{\partial x_3} + \frac{k-1}{4}\Lambda\left(\frac{\partial g_4}{\partial x_4} + \frac{\partial g_4}{\partial x_5} + \frac{\partial g_4}{\partial x_6}\right) = 0.$$

Let $\bar{g}_4 = g_4|_{x_3=0} \not\equiv 0$. Then, on $x_3 = 0$ we have

$$[j(-x_4 + x_5 + x_6) + l(x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6)]\bar{g}_4 + \frac{k-1}{4}\Lambda\left(\frac{\partial \bar{g}_4}{\partial x_4} + \frac{\partial \bar{g}_4}{\partial x_5} + \frac{\partial \bar{g}_4}{\partial x_6}\right) = 0.$$

Applying Lemma 4 we obtain $\bar{g}_4 \equiv 0$, a contradiction. Hence $g_2 \equiv 0$ and the lemma follows in the case $l > 1$.

If $l = 1$ then we can cancel a common factor x_2 in equation (16). Let $\bar{g}_2 = g_2|_{x_2=0} \not\equiv 0$. On $x_2 = 0$ equation (16) becomes

$$x_1\Bigg[(x_4 - x_5 + x_6)\bar{g}_2 + x_1(-x_4 + x_5 + x_6)\frac{\partial \bar{g}_2}{\partial x_1} + x_3(x_4 + x_5 - x_6)\frac{\partial \bar{g}_2}{\partial x_3} + \frac{x_1^2}{4}\frac{\partial \bar{g}_2}{\partial x_4} + \frac{k-1}{4}\left(\frac{x_1^2}{4} + \Lambda\right)\left(\frac{\partial \bar{g}_2}{\partial x_4} + \frac{\partial \bar{g}_2}{\partial x_5} + \frac{\partial \bar{g}_2}{\partial x_6}\right)\Bigg] + 2C_3 n(x_4 + x_5 - 2x_6)^2(x_5 - x_6)^{n-1} = 0.$$

Clearly we must take $C_3 = 0$, and hence $h_3 \equiv 0$. Write now $\bar{g}_2 = x_1^j g_3 \not\equiv 0$, with $j \in \mathbb{N} \cup \{0\}$ and $x_1 \nmid g_3$. Then the above equation becomes

$$[j(-x_4 + x_5 + x_6) + (x_4 - x_5 + x_6)]g_3 + x_1(-x_4 + x_5 + x_6)\frac{\partial g_3}{\partial x_1} + \frac{x_1^2}{4}\frac{\partial g_3}{\partial x_4} + x_3(x_4 + x_5 - x_6)\frac{\partial g_3}{\partial x_3} + \frac{k-1}{4}\left(\frac{x_1^2}{4} + \Lambda\right)\left(\frac{\partial g_3}{\partial x_4} + \frac{\partial g_3}{\partial x_5} + \frac{\partial g_3}{\partial x_6}\right) = 0. \tag{17}$$

Let $\bar{g}_3 = g_3|_{x_1=0} \not\equiv 0$. Then, on $x_1 = 0$ we have

$$[j(-x_4 + x_5 + x_6) + (x_4 - x_5 + x_6)]\bar{g}_3 + x_3(x_4 + x_5 - x_6)\frac{\partial \bar{g}_3}{\partial x_3} + \frac{k-1}{4}\Lambda\left(\frac{\partial \bar{g}_3}{\partial x_4} + \frac{\partial \bar{g}_3}{\partial x_5} + \frac{\partial \bar{g}_3}{\partial x_6}\right) = 0.$$

Now write $\bar{g}_3 = x_3^m g_4 \not\equiv 0$, with $m \in \mathbb{N} \cup \{0\}$ and $x_3 \nmid g_4$. We get

$$[j(-x_4 + x_5 + x_6) + (x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6)]g_4 + x_3(x_4 + x_5 - x_6)\frac{\partial g_4}{\partial x_3} + \frac{k-1}{4}\Lambda\left(\frac{\partial g_4}{\partial x_4} + \frac{\partial g_4}{\partial x_5} + \frac{\partial g_4}{\partial x_6}\right) = 0.$$

Let $\bar{g}_4 = g_4|_{x_3=0} \not\equiv 0$. Then, on $x_3 = 0$ we have

$$[j(-x_4 + x_5 + x_6) + (x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6)]\bar{g}_4 + \frac{k-1}{4}\Lambda\left(\frac{\partial \bar{g}_4}{\partial x_4} + \frac{\partial \bar{g}_4}{\partial x_5} + \frac{\partial \bar{g}_4}{\partial x_6}\right) = 0.$$

Applying Lemma 4 we obtain $\bar{g}_4 \equiv 0$, a contradiction. Hence $g_2 \equiv 0$ and the lemma follows also in the case $l = 1$. □

V. PROOF OF STATEMENT (C) OF THEOREM 2

We study in this section the analytic integrability of Bianchi models III, VI and VII. We consider system (8). Let $h = h(x_1, \ldots, x_6)$ be a homogeneous polynomial first integral of degree n of system (8). Using Lemma 3 we can write $h = h_1(x_2, \ldots, x_6) + x_1^j g_1(x_1, \ldots, x_6)$, with $j \in \mathbb{N}$ and h_1 and g_1 homogeneous polynomials such that $x_1 \nmid g_1$. Assume that $g_1 \not\equiv 0$. On $x_1 = 0$ system (8) becomes, after canceling a common factor $-n_2 x_2^3$,

$$\begin{aligned}\dot{x}_2 =& x_2(x_4 - x_5 + x_6),\\ \dot{x}_3 =& x_3(x_4 + x_5 - x_6),\\ \dot{x}_4 =& \frac{k-1}{4}\left(\frac{x_2^2}{4} + \Lambda\right),\\ \dot{x}_5 =& \frac{x_2^2}{4} + \frac{k-1}{4}\left(\frac{x_2^2}{4} + \Lambda\right),\\ \dot{x}_6 =& \frac{k-1}{4}\left(\frac{x_2^2}{4} + \Lambda\right).\end{aligned} \tag{18}$$

We notice that h_1 is a first integral of system (18). From Lemma 3 we write $h_1 = h_2(x_3, \ldots, x_6) + x_2^l g_2(x_2, \ldots, x_6)$, with $l \in \mathbb{N}$ and h_2 and g_2 homogeneous polynomials such that $x_2 \nmid g_2$. Assume that $g_2 \not\equiv 0$. System (18) on $x_2 = 0$ writes

$$\begin{aligned}\dot{x}_3 &= x_3(x_4 + x_5 - x_6),\\ \dot{x}_4 &= \frac{k-1}{4}\Lambda,\\ \dot{x}_5 &= \frac{k-1}{4}\Lambda,\\ \dot{x}_6 &= \frac{k-1}{4}\Lambda.\end{aligned} \tag{19}$$

Straightforward computations show that system (19) has the three independent first integrals $x_4 - x_5$, $x_4 - x_6$ and

$$x_3^{\frac{3}{2}(k-1)}\Lambda\left(\frac{x_4 + x_5 + x_6 - 2\sqrt{\Delta}}{x_4 + x_5 + x_6 + 2\sqrt{\Delta}}\right)^{\frac{x_4 + x_5 - 2x_6}{\sqrt{\Delta}}}.$$

As h_2 is a polynomial first integral of system (19), we have $h_2 = h_2(x_4 - x_5, x_4 - x_6)$. The following lemma shows that indeed $h = h_2(x_4 - x_6) + x_1^j g_1(x_1, \ldots, x_6)$.

Lemma 8. *We have that $h_2 = h_2(x_4 - x_6)$ and $g_2 \equiv 0$.*

Proof. As $h_1 = h_2 + x_2^l g_2$ is a first integral of system (18), we have

$$\begin{aligned}x_2^l\Bigg[&l(x_4 - x_5 + x_6)g_2 + x_2(x_4 - x_5 + x_6)\frac{\partial g_2}{\partial x_2}\\ &+ x_3(x_4 + x_5 - x_6)\frac{\partial g_2}{\partial x_3} + \frac{x_2^2}{4}\frac{\partial g_2}{\partial x_5}\\ &+ \frac{k-1}{4}\left(\frac{x_2^2}{4} + \Lambda\right)\left(\frac{\partial g_2}{\partial x_4} + \frac{\partial g_2}{\partial x_5} + \frac{\partial g_2}{\partial x_6}\right)\Bigg] + \frac{x_2^2}{4}\frac{\partial h_2}{\partial x_5} = 0.\end{aligned} \tag{20}$$

We distinguish the following cases depending on the value of l.

If $l = 1$ then equation (20) becomes

$$\begin{aligned}&(x_4 - x_5 + x_6)g_2 + x_2(x_4 - x_5 + x_6)\frac{\partial g_2}{\partial x_2}\\ &+ x_3(x_4 + x_5 - x_6)\frac{\partial g_2}{\partial x_3} + \frac{x_2^2}{4}\frac{\partial g_2}{\partial x_5}\\ &+ \frac{k-1}{4}\left(\frac{x_2^2}{4} + \Lambda\right)\left(\frac{\partial g_2}{\partial x_4} + \frac{\partial g_2}{\partial x_5} + \frac{\partial g_2}{\partial x_6}\right) + \frac{x_2}{4}\frac{\partial h_2}{\partial x_5} = 0.\end{aligned}$$

Let $\bar{g}_2 = g_2|_{x_2=0} \not\equiv 0$. On $x_2 = 0$ we have

$$\begin{aligned}&(x_4 - x_5 + x_6)\bar{g}_2 + x_3(x_4 + x_5 - x_6)\frac{\partial \bar{g}_2}{\partial x_3}\\ &+ \frac{k-1}{4}\Lambda\left(\frac{\partial \bar{g}_2}{\partial x_4} + \frac{\partial \bar{g}_2}{\partial x_5} + \frac{\partial \bar{g}_2}{\partial x_6}\right) = 0.\end{aligned}$$

Write $\bar{g}_2 = x_3^m g_3 \not\equiv 0$, with $m \in \mathbb{N} \cup \{0\}$ and $x_3 \nmid g_3$. Then

$$\begin{aligned}&[(x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6)]g_3\\ &+ x_3(x_4 + x_5 - x_6)\frac{\partial g_3}{\partial x_3}\\ &+ \frac{k-1}{4}\Lambda\left(\frac{\partial g_3}{\partial x_4} + \frac{\partial g_3}{\partial x_5} + \frac{\partial g_3}{\partial x_6}\right) = 0.\end{aligned}$$

Let $\bar{g}_3 = g_3|_{x_3=0} \not\equiv 0$. On $x_3 = 0$ we have

$$\begin{aligned}&[(x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6)]\bar{g}_3\\ &+ \frac{k-1}{4}\Lambda\left(\frac{\partial \bar{g}_3}{\partial x_4} + \frac{\partial \bar{g}_3}{\partial x_5} + \frac{\partial \bar{g}_3}{\partial x_6}\right) = 0.\end{aligned}$$

Applying Lemma 4 we obtain $\bar{g}_3 \equiv 0$, which is a contradiction. Hence $g_2 \equiv 0$. Back to equation (20) we have $\partial h_2/\partial x_5 \equiv 0$, which means that $h_2 = h_2(x_4 - x_6)$. Then the lemma follows in the case $l = 1$.

If $l > 2$, then from equation (20) we have that $x_2 \,|(\partial h_2/\partial x_5)$ and thus $\partial h_2/\partial x_5 \equiv 0$. Therefore $h_2 = h_2(x_4 - x_6)$. Now we can proceed as in the case $l = 1$ to obtain the equation

$$[l(x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6)]\bar{g}_3 + \frac{k-1}{4}\Lambda\left(\frac{\partial \bar{g}_3}{\partial x_4} + \frac{\partial \bar{g}_3}{\partial x_5} + \frac{\partial \bar{g}_3}{\partial x_6}\right) = 0.$$

Applying Lemma 4 we arrive to a contradiction and hence $g_2 \equiv 0$. The lemma follows in the case $l > 2$.

If $l = 2$ then equation (20) writes

$$2(x_4 - x_5 + x_6)g_2 + x_2(x_4 - x_5 + x_6)\frac{\partial g_2}{\partial x_2} + x_3(x_4 + x_5 - x_6)\frac{\partial g_2}{\partial x_3} + \frac{x_2^2}{4}\frac{\partial g_2}{\partial x_5} + \frac{k-1}{4}\left(\frac{x_2^2}{4} + \Lambda\right)\left(\frac{\partial g_2}{\partial x_4} + \frac{\partial g_2}{\partial x_5} + \frac{\partial g_2}{\partial x_6}\right) + \frac{1}{4}\frac{\partial h_2}{\partial x_5} = 0.$$

Let $\bar{g}_2 = g_2|_{x_2=0} \not\equiv 0$. On $x_2 = 0$ we have

$$2(x_4 - x_5 + x_6)\bar{g}_2 + x_3(x_4 + x_5 - x_6)\frac{\partial \bar{g}_2}{\partial x_3} + \frac{k-1}{4}\Lambda\left(\frac{\partial \bar{g}_2}{\partial x_4} + \frac{\partial \bar{g}_2}{\partial x_5} + \frac{\partial \bar{g}_2}{\partial x_6}\right) + \frac{1}{4}\frac{\partial h_2}{\partial x_5} = 0.$$

Write $\bar{g}_2 = x_3^m g_3 \not\equiv 0$, with $m \in \mathbb{N} \cup \{0\}$ and $x_3 \nmid g_3$. Then we get

$$x_3^m\left[(2(x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6))g_3 + x_3(x_4 + x_5 - x_6)\frac{\partial g_3}{\partial x_3} + \frac{k-1}{4}\Lambda\left(\frac{\partial g_3}{\partial x_4} + \frac{\partial g_3}{\partial x_5} + \frac{\partial g_3}{\partial x_6}\right)\right] + \frac{1}{4}\frac{\partial h_2}{\partial x_5} = 0.$$

If $m > 0$ then $x_3 \,|(\partial h_2/\partial x_5)$. Hence $\partial h_2/\partial x_5 \equiv 0$ and $h_2 = h_2(x_4 - x_6)$. Let $\bar{g}_3 = g_3|_{x_3=0} \not\equiv 0$. On $x_3 = 0$ we obtain

$$[2(x_4 - x_5 + x_6) + m(x_4 + x_5 - x_6)]\bar{g}_3 + \frac{k-1}{4}\Lambda\left(\frac{\partial \bar{g}_3}{\partial x_4} + \frac{\partial \bar{g}_3}{\partial x_5} + \frac{\partial \bar{g}_3}{\partial x_6}\right) = 0.$$

Applying Lemma 4 we get a contradiction; hence $g_2 \equiv 0$.

If $m = 0$, let $\bar{g}_3 = g_3|_{x_3=0} \not\equiv 0$. On $x_3 = 0$ we obtain

$$2(x_4 - x_5 + x_6)\bar{g}_3 + \frac{1}{4}\frac{\partial h_2}{\partial x_5} + \frac{k-1}{4}\Lambda\left(\frac{\partial \bar{g}_3}{\partial x_4} + \frac{\partial \bar{g}_3}{\partial x_5} + \frac{\partial \bar{g}_3}{\partial x_6}\right) = 0.$$

We can apply Lemma 5 and we get $\frac{\partial h_2}{\partial x_5} \equiv 0$ and $\bar{g}_3 \equiv 0$, hence the lemma follows in the case $l = 2$. □

After Lemma 8 we have that $h = h_2(x_4 - x_6) + x_1^j g_1(x_1, \ldots, x_6)$, with $j \in \mathbb{N}$ and $x_1 \nmid g_1$. We recall that h is a first integral of system (8). Thus it satisfies the equation

$$\begin{aligned}
&x_1^j\Bigg[jN^3(-x_4 + x_5 + x_6)g_1 + x_1N^3(-x_4 + x_5 + x_6)\frac{\partial g_1}{\partial x_1} \\
&+ x_2N^3(x_4 - x_5 + x_6)\frac{\partial g_1}{\partial x_2} + x_3N^3(x_4 + x_5 - x_6)\frac{\partial g_1}{\partial x_3} \\
&+ \frac{k-1}{4}NS\left(\frac{\partial g_1}{\partial x_4} + \frac{\partial g_1}{\partial x_5} + \frac{\partial g_1}{\partial x_6}\right) \\
&+ \Big(\frac{1}{4}x_1N^4 + a^2x_1x_2N^3 \\
&\qquad -2a^2x_1x_2(x_1 + n_2x_2)(x_4 + x_5 - 2x_6)^2\Big)\frac{\partial g_1}{\partial x_4} \\
&+ \Big(\frac{1}{4}N^5 - \frac{1}{4}x_1N^4 + a^2x_1x_2N^3 \\
&\qquad +2a^2x_1x_2(x_1 + n_2x_2)(x_4 + x_5 - 2x_6)^2\Big)\frac{\partial g_1}{\partial x_5} \\
&+a^2x_1x_2N^3\frac{\partial g_1}{\partial x_6}\Bigg] + a^2x_1x_2N^3\frac{\partial h_2}{\partial x_6} \\
&+ \Big(\frac{1}{4}x_1N^4 + a^2x_1x_2N^3 \\
&\qquad -2a^2x_1x_2(x_1 + n_2x_2)(x_4 + x_5 - 2x_6)^2\Big)\frac{\partial h_2}{\partial x_4} = 0.
\end{aligned} \tag{21}$$

The following lemma ends the proof of statement (c) of Theorem 2, as it shows that $h \equiv 0$.

Lemma 9. *We have that $h_2 \equiv 0$ and $g_1 \equiv 0$.*

Proof. Write $h_2(x_4 - x_6) = C_2(x_4 - x_6)^n$, with $C_2 \in \mathbb{R}$. Suppose that $g_1 \not\equiv 0$. We distinguish two cases depending on the value of j. If $j>1$ then from equation (21) we have that x_1 divides

$$C_2 n\left(\frac{x_2^4}{4}-2a^2 n_2 x_2^2(x_4+x_5-2x_6)^2\right).$$

Hence we must take $C_2=0$ and therefore $h_2\equiv 0$. Let $\bar{g}_1=g_1|_{x_1=0}\not\equiv 0$. Equation (21) on $x_1=0$ becomes, after cancelling a common factor $-n_2x_2^3$,

$$\begin{aligned}&j(-x_4+x_5+x_6)\bar{g}_1+x_2(x_4-x_5+x_6)\frac{\partial\bar{g}_1}{\partial x_2}\\&+x_3(x_4+x_5-x_6)\frac{\partial\bar{g}_1}{\partial x_3}+\frac{x_2^2}{4}\frac{\partial\bar{g}_1}{\partial x_5}\\&+\frac{k-1}{4}\left(\frac{x_2^2}{4}+\Lambda\right)\left(\frac{\partial\bar{g}_1}{\partial x_4}+\frac{\partial\bar{g}_1}{\partial x_5}+\frac{\partial\bar{g}_1}{\partial x_6}\right)=0.\end{aligned}$$

Write $\bar{g}_1=x_2^lg_2\not\equiv 0$, with $l\in\mathbb{N}\cup\{0\}$ and $x_2\nmid g_2$. We get

$$\begin{aligned}&[j(-x_4+x_5+x_6)+l(x_4-x_5+x_6)]g_2\\&+x_2(x_4-x_5+x_6)\frac{\partial g_2}{\partial x_2}\\&+x_3(x_4+x_5-x_6)\frac{\partial g_2}{\partial x_3}+\frac{x_2^2}{4}\frac{\partial g_2}{\partial x_5}\\&+\frac{k-1}{4}\left(\frac{x_2^2}{4}+\Lambda\right)\left(\frac{\partial g_2}{\partial x_4}+\frac{\partial g_2}{\partial x_5}+\frac{\partial g_2}{\partial x_6}\right)=0.\end{aligned}$$

Let $\bar{g}_2=g_2|_{x_2=0}\not\equiv 0$. Then, on $x_2=0$ we have

$$\begin{aligned}&[j(-x_4+x_5+x_6)+l(x_4-x_5+x_6)]\bar{g}_2\\&+x_3(x_4+x_5-x_6)\frac{\partial\bar{g}_2}{\partial x_3}\\&+\frac{k-1}{4}\Lambda\left(\frac{\partial\bar{g}_2}{\partial x_4}+\frac{\partial\bar{g}_2}{\partial x_5}+\frac{\partial\bar{g}_2}{\partial x_6}\right)=0.\end{aligned}$$

Now write $\bar{g}_2=x_3^mg_3\not\equiv 0$, with $m\in\mathbb{N}\cup\{0\}$ and $x_3\nmid g_3$. We get

$$\begin{aligned}&[j(-x_4+x_5+x_6)+l(x_4-x_5+x_6)\\&+m(x_4+x_5-x_6)]g_3+x_3(x_4+x_5-x_6)\frac{\partial g_3}{\partial x_3}\\&+\frac{k-1}{4}\Lambda\left(\frac{\partial g_3}{\partial x_4}+\frac{\partial g_3}{\partial x_5}+\frac{\partial g_3}{\partial x_6}\right)=0.\end{aligned}$$

Let $\bar{g}_3=g_3|_{x_3=0}\not\equiv 0$. Then, on $x_3=0$ we have

$$\begin{aligned}&[j(-x_4+x_5+x_6)+l(x_4-x_5+x_6)\\&+m(x_4+x_5-x_6)]\bar{g}_3\\&+\frac{k-1}{4}\Lambda\left(\frac{\partial\bar{g}_3}{\partial x_4}+\frac{\partial\bar{g}_3}{\partial x_5}+\frac{\partial\bar{g}_3}{\partial x_6}\right)=0.\end{aligned}$$

Applying Lemma 4 we obtain $\bar{g}_3\equiv 0$, a contradiction. Hence $g_1\equiv 0$ and the lemma follows in the case $j>1$.

If $j=1$ then we can cancel a common factor x_1 in equation (21). Let $\bar{g}_1=g_1|_{x_1=0}\not\equiv 0$. On $x_1=0$ equation (21) becomes, after cancelling a common factor $-n_2x_2^2$,

$$\begin{aligned}&x_2\Bigg[(-x_4+x_5+x_6)\bar{g}_1+x_2(x_4-x_5+x_6)\frac{\partial\bar{g}_1}{\partial x_2}\\&+x_3(x_4+x_5-x_6)\frac{\partial\bar{g}_1}{\partial x_3}+\frac{x_2^2}{4}\frac{\partial\bar{g}_1}{\partial x_5}\\&+\frac{k-1}{4}\left(\frac{x_2^2}{4}+\Lambda\right)\left(\frac{\partial\bar{g}_1}{\partial x_4}+\frac{\partial\bar{g}_1}{\partial x_5}+\frac{\partial\bar{g}_1}{\partial x_6}\right)\Bigg]\\&+C_2n\left(2a^2(x_4+x_5-2x_6)^2-\frac{n_2}{4}x_2^2\right)(x_4-x_6)^{n-1}=0.\end{aligned}$$

Clearly we must take $C_2=0$, and hence $h_2\equiv 0$. Write now $\bar{g}_1=x_2^lg_2\not\equiv 0$, with $l\in\mathbb{N}\cup\{0\}$ and $x_2\nmid g_2$. Then the above equation becomes

$$\begin{aligned}&[(-x_4+x_5+x_6)+l(x_4-x_5+x_6)]g_2\\&+x_2(x_4-x_5+x_6)\frac{\partial g_2}{\partial x_2}\\&+x_3(x_4+x_5-x_6)\frac{\partial g_2}{\partial x_3}+\frac{x_2^2}{4}\frac{\partial g_2}{\partial x_5}\\&+\frac{k-1}{4}\left(\frac{x_2^2}{4}+\Lambda\right)\left(\frac{\partial g_2}{\partial x_4}+\frac{\partial g_2}{\partial x_5}+\frac{\partial g_2}{\partial x_6}\right)=0.\end{aligned}\tag{22}$$

Let $\bar{g}_2=g_2|_{x_2=0}\not\equiv 0$. Then, on $x_2=0$ we have

$$\begin{aligned}&[(-x_4+x_5+x_6)+l(x_4-x_5+x_6)]\bar{g}_2\\&+x_3(x_4+x_5-x_6)\frac{\partial\bar{g}_2}{\partial x_3}\\&+\frac{k-1}{4}\Lambda\left(\frac{\partial\bar{g}_2}{\partial x_4}+\frac{\partial\bar{g}_2}{\partial x_5}+\frac{\partial\bar{g}_2}{\partial x_6}\right)=0.\end{aligned}$$

Now write $\bar{g}_2=x_3^mg_3\not\equiv 0$, with $m\in\mathbb{N}\cup\{0\}$ and $x_3\nmid g_3$. We get

$$\begin{aligned}&[(-x_4+x_5+x_6)+l(x_4-x_5+x_6)\\&+m(x_4+x_5-x_6)]g_3+x_3(x_4+x_5-x_6)\frac{\partial g_3}{\partial x_3}\\&+\frac{k-1}{4}\Lambda\left(\frac{\partial g_3}{\partial x_4}+\frac{\partial g_3}{\partial x_5}+\frac{\partial g_3}{\partial x_6}\right)=0.\end{aligned}$$

Let $\bar{g}_3=g_3|_{x_3=0}\not\equiv 0$. Then, on $x_3=0$ we have

$$\begin{aligned}&[(-x_4+x_5+x_6)+l(x_4-x_5+x_6)+m(x_4+x_5-x_6)]\bar{g}_3\\&+\frac{k-1}{4}\Lambda\left(\frac{\partial\bar{g}_3}{\partial x_4}+\frac{\partial\bar{g}_3}{\partial x_5}+\frac{\partial\bar{g}_3}{\partial x_6}\right)=0.\end{aligned}$$

Applying Lemma 4 we obtain $\bar{g}_3 \equiv 0$, a contradiction. Hence $g_1 \equiv 0$ and the lemma follows in the case $j = 1$. □

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