

DARBOUX INTEGRABILITY OF THE LÜ SYSTEM

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ABSTRACT. We characterize all the values of the parameters of the Lü system, for which it admits a Darboux first integral.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The following real differential system

$$(1) \quad \begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= cy - xz, \\ \dot{z} &= -bz + xy, \end{aligned}$$

where $a, b, c \in \mathbb{R}$ are parameters is known as the *Lü system* [13]. It connects the Lorenz system [11, 17] with the Chen system [18] and represents a transition from one to the another (see [13] for details).

A number of facts related to the *local* analysis of system (1) are known. For results on Hopf bifurcation see [19]; degenerate Hopf bifurcation have been considered in [16, 14]; see also [15] for center conditions on the local center manifold. *Global* dynamics of the Lü system have also been recently considered in [11] and [10], where the authors studied the existence of invariant algebraic surfaces and determined the dynamics on it, including the infinity.

In this work we further consider the global dynamics of system (1) by studying the integrability of system (1). To be more precise, we characterize all the values of the parameters $a, b, c \in \mathbb{R}$, for which the Lü system admits a *Darboux first integral* (see Theorem 3), i.e. a function of the form (2). In particular we identify, the parameter values of the system for which it admits a *rational* first integral (see Theorem 2). Now, we proceed to present more detailed statements of our main results. See section 2 for definitions of concepts used in the rest of the introduction. For a general introduction to the Darboux theory of integrability see [5] and [6].

One of the most basic elements for construction of the Darboux first integral is the so-called *Darboux polynomial*, which in the context of the Lü system have been studied in [12]. We note that paper [12] is not completely correct because the author misses a Darboux polynomial of system (1) with

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nonzero cofactor (see statement (c) of Theorem 1). However the correct statement follows by looking carefully step by step at the proof of [12].

Theorem 1. *When $a \neq 0$, then all the irreducible Darboux polynomials in $\mathbb{R}[x, y, z]$ for the Lü system with non-zero cofactor are the following ones:*

- (a) $x^2 - 2az$ with cofactor $-2a$ if $b = 2a$ and $c \neq -2a$.
- (b) $y^2 + z^2$ with cofactor $2c$ if $b = -c$ and $c \neq -2a$.
- (c) $x^2 - 2az, y^2 + z^2, \sum_{i=0}^n a_i(x^2 - 2az)^{2i}(y^2 + z^2)^{n-i}$, $n \geq 1$, $a_i \in \mathbb{R}$ with cofactor $-2a, -4a$ and $-4an$, respectively, if $b = -c$ and $c = -2a$.
- (d) $x^4 + \frac{4}{3}cx^2z - \frac{4}{9}c^2y^2 - \frac{8}{9}c^2xy$ with cofactor $\frac{4}{3}c$ if $a = -c/3$ and $b = 0$.
- (e) $x^4 + 4cx^2z - 4c^2y^2 + 8c^2xy + 16c^3z$ with cofactor $4c$ if $c = -a$ and $b = 4a$.

It was proved by Llibre and Valls in [7] that when $a = 0$ system (1) is completely integrable. Therefore, in this paper we will only consider the Lü system with $a \neq 0$.

The first result of this paper is the following.

Theorem 2. *When $a \neq 0$ the Lü system has a rational first integral if and only if either $b = c = 0$, or $b = -c = 2a$. In the first case the first integral is a rational function in the variable $y^2 + z^2$, and in the second the first integral is a rational function in the variable $(x^2 - 2az)^2/(y^2 + z^2)$.*

We note that $H_1 = y^2 + z^2$ and $H_2 = (x^2 - 2az)^2/(y^2 + z^2)$ are first integrals of Darboux type. The proof of Theorem 2 is given in section 2.

The next theorem is the main result of this paper, and it shows that the unique first integrals of Darboux type for the Lü systems are Darboux functions of H_1 and H_2 .

Theorem 3. *The unique first integrals of Darboux type for the Lü system with $a \neq 0$ are the following:*

- (a) *Any function of Darboux type in the variable $y^2 + z^2$, if $b = c = 0$.*
- (b) *Any function of Darboux type in the variable $(x^2 - 2az)^2/(y^2 + z^2)$ if $b = -c = 2a$.*

We divided the proof of Theorem 3 into three different propositions mainly due to the fact that the techniques used in the proof of the three propositions are of different nature. In the next two propositions we identify two different sets of values of parameters $a, b, c \in \mathbb{R}$, for which the Lü system admits a Darboux first integral.

Proposition 4. *The unique first integrals of Darboux type for the Lü system with $a \neq 0$ and $b = c = 0$ are functions of Darboux type in the variable $y^2 + z^2$.*

The proof of Proposition 4 is given in section 3.

Proposition 5. *The unique first integrals of Darboux type for the Lü system with $a \neq 0$ and $b = -c = 2a$ are functions of Darboux type in the variable $(x^2 - 2az)^2/(y^2 + z^2)$.*

The proof of Proposition 5 is given in section 4.

Finally the following proposition shows that $a = b = 0$ or $b = -c = 2a$ are the only values of parameters of the Lü system, for which it admits a Darboux first integral and can be regarded as a non-integrability result.

Proposition 6. *The Lü system with $a \neq 0$ and $(b^2 + c^2)[(b+c)^2 + (c+2a)^2] \neq 0$ has no first integrals of Darboux type.*

The proof of Proposition 6 is given in section 5. In section 2 we introduce some auxiliary results that will be used in the proofs of the above propositions.

2. DARBOUX THEORY OF INTEGRABILITY

The vector field associated to (1) is

$$X = a(y - x)\frac{\partial}{\partial x} + (cy - xz)\frac{\partial}{\partial y} + (-bz + xy)\frac{\partial}{\partial z}.$$

Let U be an open subset in \mathbb{R}^3 such that $\mathbb{R}^3 \setminus U$ has zero Lebesgue measure. We say that a non-constant real function $H = H(x, y, z): \mathbb{R}^3 \rightarrow \mathbb{R}$, is a *first integral* if $H(x(t), y(t), z(t))$ is constant on all solutions $(x(t), y(t), z(t))$ of X contained in U , i.e. $XH|_U = 0$. The existence of a first integral for a differential system in \mathbb{R}^3 allows to reduce its study in one dimension. This is the main reason to look for first integrals.

Two functions $f_1, f_2: \mathbb{R}^3 \rightarrow \mathbb{R}$ are said to be *independent* if their gradients are linearly independent vectors for all $(x, y, z) \in \mathbb{R}^3$ except perhaps for a set of zero Lebesgue measure. If the vector field X has two independent first integrals H_1 and H_2 , we say that it is *completely integrable*. In this case, the orbits of X are contained in the curves $\{H_1(x, y, z) = h_1\} \cap \{H_2(x, y, z) = h_2\}$, where h_1, h_2 vary in \mathbb{R} .

One of the best tools to look for first integrals is the *Darboux theory of integrability*. Now we shall introduce its basic notions. Let $\mathbb{C}[x, y, z]$ be the ring of polynomials in the variables x, y, z with coefficients in \mathbb{C} . We say that $f \in \mathbb{C}[x, y, z]$ is a *Darboux polynomial* of the vector field X if there exists a polynomial $K \in \mathbb{C}[x, y, z]$ such that $Xf = Kf$. The polynomial $K = K(x, y, z)$ is called the *cofactor* of f . It is easy to see that since system (1) is quadratic, the cofactor of a Darboux polynomial has degree at most 1. Note that we look for complex Darboux polynomials in real differential systems. The reason is that frequently the complex structure forces the existence of real first integrals.

An *exponential factor* $F(x, y, z)$ of the field X is an exponential function of the form $\exp(g/h)$ with g and h coprime polynomials in $\mathbb{C}[x, y, z]$ and satisfying $XF = LF$ for some $L \in \mathbb{C}[x, y, z]$ with degree one. The exponential factors appear when some Darboux polynomial has multiplicity larger than one, for more details see [3] and [8].

A first integral of system (1) is called of *Darboux type* if it is a first integral of the form

$$(2) \quad f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q}$$

where f_1, \dots, f_p are Darboux polynomials and F_1, \dots, F_q are exponential factors.

In the proof of Theorem 3 we will use the following well known result of the Darboux theory of integrability, see for instance [5, Chapter 8].

Theorem 7 (Darboux theory of integrability). *Suppose that a polynomial vector field X defined in \mathbb{R}^n of degree m admits p Darboux polynomials f_i with cofactors K_i for $i = 1, \dots, p$, and q exponential factors $F_j = \exp(g_j/h_j)$ with cofactors L_j for $j = 1, \dots, q$. If there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that*

$$(3) \quad \sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = 0,$$

then the following real (multivalued) function of Darboux type

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q},$$

substituting $f_i^{\lambda_i}$ by $|f_i|^{\lambda_i}$ if $\lambda_i \in \mathbb{R}$, is a first integral of the vector field X .

For a proof of the next result see [8, 9].

Proposition 8. *The following statements hold.*

- (a) *If $e^{g/h}$ is an exponential factor for the polynomial differential system (1) and h is not a constant polynomial, then $h = 0$ is an invariant algebraic surface.*
- (b) *Eventually e^g can be an exponential factor, coming from the multiplicity of the infinite invariant plane.*

The proof of the next result can be found in [2].

Lemma 9. *Let f be a polynomial and $f = \prod_{j=1}^s f_j^{\alpha_j}$ its decomposition into irreducible factors in $\mathbb{C}[x, y, z]$. Then f is a Darboux polynomial of system (1) if and only if all the f_j are Darboux polynomials of system (1). Moreover, if K and K_j are the cofactors of f and f_j , then $K = \sum_{j=1}^s \alpha_j K_j$.*

We note that in view of Lemma 9 to study the Darboux polynomials of system (1) it is enough to study the irreducible ones.

3. PROOF OF THEOREM 2 AND PROPOSITION 4

In order to prove Theorem 2 we shall need the next result, which was proved in [7].

Theorem 10. *When $a \neq 0$ the Lü system has a polynomial first integral if and only if $b = c = 0$ and in this case the first integral is a polynomial in the variable $y^2 + z^2$.*

Proof of Theorem 2. We note that having the characterization of all Darboux polynomials (either with zero or with nonzero cofactor) of system (1) we can characterize easily the rational first integrals, because they can only be of the form: either a polynomial first integral (i.e. a Darboux polynomial with zero cofactor), or the quotient of two Darboux polynomials with the same non-zero cofactor. Now the rest of the proof follows easily from Theorems 1 and 10. \square

To prove Proposition 4 we shall use a result due to Llibre and Zhang [4]. We recall that a *generalized rational function* is a function which is the quotient of two analytic functions. In particular rational first integrals and analytic first integrals are particular cases of generalized rational first integrals. Clearly a Darboux type function is a generalized rational function.

Theorem 11. *Assume that the differential system (1) has p as a singular point and let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of the linear part of system (1) at p . Then the number of functionally independent generalized rational first integrals of system (1) is at most the dimension of the minimal vector subspace of \mathbb{R}^3 containing the set*

$$\{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = 0, (k_1, k_2, k_3) \neq (0, 0, 0)\}.$$

Proof of Proposition 4. We consider system (1) with $a \neq 0$ and $b = c = 0$, that is

$$(4) \quad \dot{x} = a(y - x), \quad \dot{y} = -xz, \quad \dot{z} = xy.$$

It follows from [7] that this system has a polynomial first integral $H_1 = y^2 + z^2$, which obviously is a generalized rational first integral. To conclude the proof of the proposition we shall show that system (4) has no other first integrals of Darboux type. To prove this we will use Theorem 11. First we note that the singular points of system (4) are of the form $(0, 0, z)$ with $z \in \mathbb{R}$. We compute the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the Jacobian matrix of this system on these singular points and we get

$$\lambda_1 = 0, \quad \lambda_2 = \frac{1}{2}(-a - \sqrt{a(a - 4z)}), \quad \lambda_3 = \frac{1}{2}(-a + \sqrt{a(a - 4z)}).$$

Therefore $k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = 0$ is equivalent to

$$k_3(-\sqrt{a} + \sqrt{a - 4z}) - k_2(\sqrt{a} + \sqrt{a - 4z}) = 0,$$

or in other words

$$(5) \quad \frac{k_3}{k_2} = \frac{\sqrt{a} + \sqrt{a - 4z}}{-\sqrt{a} + \sqrt{a - 4z}}.$$

It is clear that the left-hand side of (5) is a rational number (once that $k_2, k_3 \in \mathbb{Z}$), and that choosing z in a convenient way the right-hand side of

(5) is irrational. Therefore (5) cannot hold for this convenient choice of z . Hence for this special singular point $(0, 0, z)$, the dimension of the minimal vector subspace of \mathbb{R}^3 containing the set

$$\{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = 0, (k_1, k_2, k_3) \neq (0, 0, 0)\}.$$

is clearly one, generated by $(k_1, 0, 0)$. Thus it follows from Theorem 11 that system (4) can only have one generalized rational first integral, which is a function of H_1 . This completes the proof of the proposition. \square

4. PROOF OF PROPOSITION 5

We consider system (1) with $a \neq 0$ and $b = -c = 2a$, that is

$$(6) \quad \dot{x} = a(y - x), \quad \dot{y} = -2ay - xz, \quad \dot{z} = -2az + xy.$$

First we note that by the rescaling $(x, y, z, t) \mapsto (X, Y, Z, T)$ with

$$X = \frac{x}{a}, \quad Y = \frac{y}{a}, \quad Z = \frac{z}{a}, \quad T = at,$$

we get that system (6) becomes

$$(7) \quad x' = y - x, \quad y' = -2y - xz, \quad z' = -2z + xy,$$

where now the prime indicates derivative with respect to the new time T and we have renamed the variables (X, Y, Z) again as (x, y, z) . We note that system (7) has the rational first integral $H_2 = (x^2 - 2z)^2 / (y^2 + z^2)$, which obviously is a first integral of Darboux type.

We have found the second independent first integral

$$\sqrt{\frac{y^2 + z^2}{x^2 - 2z}} \left(1 - 2\sqrt{\frac{2}{A}} F \left(\arcsin \left(\frac{1+i}{2} \sqrt{\frac{A}{y-iz}} \right), \frac{B}{A} \right) \right)$$

which is not of Darboux type, where $A = x^2 - 2z - \sqrt{x^4 - 4zx^2 - 4y^2}$, $B = x^2 - 2z + \sqrt{x^4 - 4zx^2 - 4y^2}$ and $F(\phi, m)$ is the elliptic integral of the first kind, see for more details [1]. This completes the proof of the proposition.

5. PROOF OF PROPOSITION 6

In this section using the Darboux theory of integrability we prove Proposition 6. We need the following definition.

A polynomial $g(x, y, z)$ is said to be *weight homogeneous* if there exists $s = (s_1, s_2, s_3)$ and $l \in \mathbb{N}$ such that for all $\mu \in \mathbb{R} \setminus \{0\}$ we have

$$g(\mu^{s_1}x, \mu^{s_2}y, \mu^{s_3}z) = \mu^l g(x, y, z),$$

where \mathbb{N} is the set of natural numbers. We shall refer to s as the *weight exponent* of g , and to l as the *weight degree* of g with respect to s .

Proposition 12. *For the Lü system with $a \neq 0$ and $(b^2 + c^2)[(b+c)^2 + (c+2a)^2] \neq 0$, the unique exponential factors modulo constants is $\exp(x)$ with cofactor $a(y-x)$.*

For proving Proposition 12 we shall use the ideas from the proof of Theorem 2.1 of [18].

Proof. Let $F = \exp(g/h)$ be an exponential factor of the Lü system with cofactor K , where $g, h \in \mathbb{C}[x, y, z]$ with $(g, h) = 1$. Then from the definition of exponential factor and in view of Proposition 8 we have that either h is a constant that we can take $h = 1$, or h is a Darboux polynomial of system (1).

Case 1. We first assume that F is of the form $F = \exp(g)$ with $g = g(x, y, z) \in \mathbb{C}[x, y, z]$, and cofactor $q = q(x, y, z)$. So, we have

$$(8) \quad a(y-x)\frac{\partial g}{\partial x} + (cy-xz)\frac{\partial g}{\partial y} + (-bz+xy)\frac{\partial g}{\partial z} = q,$$

with $q \in \mathbb{C}[x, y, z]$ of degree at most one, that without loss of generality we write as

$$q = q(x, y, z) = l_0 + l_1x + l_2y + l_3z.$$

In order to simplify the computations we do the rescaling

$$x = \mu^{-1}X, \quad y = \mu^{-2}Y, \quad z = \mu^{-2}Z, \quad t = \mu T,$$

with $\mu \in \mathbb{R} \setminus \{0\}$. Then the Lü system (1) becomes

$$(9) \quad \begin{aligned} X' &= a(Y - \mu X), \\ Y' &= -XZ + \mu cY, \\ Z' &= XY - \mu bZ, \end{aligned}$$

where the prime denotes the derivative of the variables with respect to T .

Now we define

$$(10) \quad G(X, Y, Z) = \mu^l g(\mu^{-1}X, \mu^{-2}Y, \mu^{-2}Z),$$

where l is the highest weight degree of g with respect to the weight exponents $(1, 2, 2)$. We write

$$(11) \quad G(X, Y, Z) = \sum_{i=0}^m \mu^i G_i(X, Y, Z),$$

where G_i is the weight homogeneous part with weight degree $-(l-i)$ of G , and $l \geq m$. We also define

$$(12) \quad Q(X, Y, Z) = \mu^2 q(\mu^{-1}X, \mu^{-2}Y, \mu^{-2}Z) = l_3Z + l_2Y + \mu l_1X + \mu^2 l_0.$$

Equation (8) can be written as

$$(13) \quad \begin{aligned} &a(Y - \mu X) \sum_{i=0}^m \mu^i \frac{\partial G_i}{\partial X} + (-XZ + \mu cY) \sum_{i=0}^m \mu^i \frac{\partial G_i}{\partial Y} \\ &+ (XY - \mu bZ) \sum_{i=0}^m \mu^i \frac{\partial G_i}{\partial Z} = \mu^{l-1} (l_3Z + l_2Y + \mu l_1X + \mu^2 l_0). \end{aligned}$$

We distinguish several subcases.

Subcase 1.1: $l = 1$. An easy computation shows that $G = l_2X + k$ with the cofactor $Q = l_2a(Y - \mu X)$. Hence, the Lü system always has the exponential factors $\exp(l_2x + k/\mu)$ with the cofactor $q = l_2a(y - x)$. Simplifying constants we can take the exponential factor $\exp(x)$ with cofactor $a(y - x)$.

Subcase 1.2: $l = 2$. Comparing the terms with the same degree in μ of (13) we get that

$$(14) \quad \begin{aligned} L[G_0] &= 0, \\ L[G_1] &= aX \frac{\partial G_0}{\partial X} - cY \frac{\partial G_0}{\partial Y} + bZ \frac{\partial G_0}{\partial Z} + l_3Z + l_2Y, \\ L[G_2] &= aX \frac{\partial G_1}{\partial X} - cY \frac{\partial G_1}{\partial Y} + bZ \frac{\partial G_1}{\partial Z} + l_1X, \\ l_0 &= 0, \end{aligned}$$

where L is the linear partial differential operator of the form

$$(15) \quad L = aY \frac{\partial}{\partial X} - XZ \frac{\partial}{\partial Y} + XY \frac{\partial}{\partial Z}.$$

The characteristic equations associated to this linear partial differential operator are

$$\frac{dX}{dZ} = \frac{aY}{XY}, \quad \frac{dY}{dZ} = -\frac{XZ}{XY}.$$

This system has the general solution

$$X^2 - 2aZ = d_1, \quad Y^2 + Z^2 = d_2,$$

where d_1 and d_2 are constants of integration. According to this, we do the change of variables

$$(16) \quad u = X^2 - 2aZ, \quad v = Y^2 + Z^2, \quad w = Z.$$

Its inverse transformation is

$$(17) \quad X = \pm\sqrt{u + 2aw}, \quad Y = \pm\sqrt{v - w^2}, \quad Z = w.$$

In the following, for simplicity, we only consider the case $X = \sqrt{u + 2aw}$, $Y = \sqrt{v - w^2}$, $Z = w$. Under the changes (16) and (17), the differential operator L becomes

$$\sqrt{u + 2aw} \sqrt{v - w^2} \frac{\partial}{\partial w}.$$

Therefore the first equation of (14) becomes the following ordinary differential equation

$$\sqrt{u + 2aw} \sqrt{v - w^2} \frac{\partial \bar{G}_0}{\partial w} = 0,$$

where \bar{G}_k is G_k written in the variables u, v, w . Therefore we have that $\bar{G}_0 = \bar{G}_0(u, v)$. Since G_0 , u and v have weight degrees 2, 2 and 4, respectively, we must have $G_0 = c_0(X^2 - 2aZ)$, where $c_0 \in \mathbb{C}$.

Substituting G_0 into the second equation of (14) we obtain that

$$L[G_1] = 2ac_0X^2 + (2abc_0 + l_3)Z + l_2Y.$$

In coordinates u, v, w we have

$$\sqrt{u+2aw}\sqrt{v-w^2}\frac{\partial\bar{G}_1}{\partial w} = 2ac_0(u+2aw) + (2abc_0+l_3)w + l_2\sqrt{v-w^2}.$$

Integrating this equation we have

$$\begin{aligned}\bar{G}_1 &= 2ac_0 \int \frac{\sqrt{u+2aw}}{\sqrt{v-w^2}} dw + (2abc_0+l_3) \int \frac{w}{\sqrt{u+2aw}\sqrt{v-w^2}} dw \\ &+ l_2 \int \frac{dw}{\sqrt{u+2aw}} + \bar{R}_1(u, v),\end{aligned}$$

where \bar{R}_1 is an arbitrary function in u and v . Since the first two integrals are elliptic functions, in order that G_1 be a weight homogeneous polynomial we should have $c_0 = 0$ and $l_3 = 0$. Therefore $G_0 = 0$. Since G_0 is the component of G with the highest weight-degree, this implies $G = 0$ and the Lü system does not have exponential factors of the form $\exp(g)$ when $l = 2$.

Subcase 1.3: $l = 2n + 1$ and $n \geq 1$. From (13) we get that $L[G_0] = 0$. As in the previous subcase the general solution of $L[G_0] = 0$ is an arbitrary function of the form $G_0 = G_0(X^2 - 2aZ, Y^2 + Z^2)$. Both $X^2 - 2aZ$ and $Y^2 + Z^2$ have even degree, then G_0 must have even degree. Hence $G_0 = 0$ because the weight degree of G_0 is $l = 2n + 1$. Since G_0 is the component of G with the highest weight-degree, this implies $G = 0$. Therefore, we have $g = 0$ and so the Lü system has no exponential factors of the form $\exp(g)$ when $l = 2n + 1$.

Subcase 1.4: $l = 4n - 2$ and $n \geq 2$. Comparing the terms with the same degree in μ in (13) we have

$$\begin{aligned}(18) \quad &L[G_0] = 0, \\ &L[G_j] = aX\frac{\partial G_{j-1}}{\partial X} - cY\frac{\partial G_{j-1}}{\partial Y} + bZ\frac{\partial G_{j-1}}{\partial Z}, \quad j = 1, \dots, l-2 \\ &L[G_{l-1}] = ax\frac{\partial G_{l-2}}{\partial X} - cY\frac{\partial G_{l-2}}{\partial Y} + bZ\frac{\partial G_{l-2}}{\partial Z} + l_2Y + l_3Z, \\ &0 = aX\frac{\partial G_{l-1}}{\partial X} - cY\frac{\partial G_{l-1}}{\partial Y} + bZ\frac{\partial G_{l-1}}{\partial Z} + l_1X, \\ &0 = l_0.\end{aligned}$$

From the first equation in (18) we get that

$$G_0 = \sum_{i=1}^n a_i (X^2 - 2aZ)^{2i-1} (Y^2 + Z^2)^{n-i}.$$

Substituting G_0 into the second equation of (18), we can prove that

$$\begin{aligned} L[G_1] &= \sum_{i=1}^n [2a(2i-1) - 2c(n-i)] a_i (X^2 - 2aZ)^{2i-1} (Y^2 + Z^2)^{n-i} \\ &\quad + \sum_{i=1}^n [(4a^2 - 2ab)(2i-1)] a_i (X^2 - 2aZ)^{2i-2} (Y^2 + Z^2)^{n-i} Z \\ &\quad + \sum_{i=1}^n 2(b+c)(n-i) a_i (X^2 - 2aZ)^{2i-1} (Y^2 + Z^2)^{n-i-1} Z^2. \end{aligned}$$

Using the transformations (16) and (17) and working in a similar way to the subcase 1.2 for solving \tilde{G}_0 we get the following ordinary differential equation

$$\begin{aligned} \sqrt{u+2aw}\sqrt{v-w^2} \frac{\partial \tilde{G}_1}{\partial w} &= \sum_{i=1}^n [2a(2i-1) - 2c(n-i)] a_i u^{2i-1} v^{n-i} \\ &\quad + \sum_{i=1}^n [(4a^2 - 2ab)(2i-1)] a_i u^{2i-2} v^{n-i} w \\ &\quad + \sum_{i=1}^n 2(b+c)(n-i) a_i u^{2i-1} v^{n-i-1} w^2. \end{aligned}$$

Using

$$\frac{d}{dw} (\sqrt{u+2aw}\sqrt{v-w^2}) = \frac{-uw + a(v-3w^2)}{\sqrt{u+2aw}\sqrt{v-w^2}},$$

and

$$\begin{aligned} &\sum_{i=1}^n [2a(2i-1) - 2c(n-i)] a_i u^{2i-1} v^{n-i} \\ &= \frac{1}{a} \sum_{i=1}^n [2a(2i-1) - 2c(n-i)] a_i u^{2i-1} v^{n-i-1} [av - uw - 3aw^2 + uw + 3aw^2] \\ &= \frac{1}{a} \sum_{i=1}^n [2a(2i-1) - 2c(n-i)] a_i u^{2i-1} v^{n-i-1} (-uw + a(v-3w^2)) \\ &\quad + \frac{1}{a} \sum_{i=1}^n [2a(2i-1) - 2c(n-i)] a_i u^{2i} v^{n-i-1} w \\ &\quad + 3 \sum_{i=1}^n [2a(2i-1) - 2c(n-i)] a_i u^{2i-1} v^{n-i-1} w^2, \end{aligned}$$

it is easy to deduce that the integration of the previous equation with respect to w is

$$\begin{aligned}
\bar{G}_1 &= \sum_{i=1}^n \frac{1}{a} [2a(2i-1) - 2c(n-i)] a_i u^{2i-1} v^{n-i-1} \sqrt{u+2aw} \sqrt{v-w^2} \\
&+ \left((4a^2 - 2ab) a_1 v^{n-1} + \left[\sum_{i=1}^{n-1} [(4a^2 - 2ab)(2i+1) a_{i+1}] \right. \right. \\
&+ \left. \left. \frac{1}{a} (2a(2i-1) - 2c(n-i)) a_i \right] u^{2i} v^{n-i-1} \right) + \frac{1}{av} 2a(2n-1) a_n u^{2n} \Big). \\
&\int \frac{w \, dw}{\sqrt{u+2aw} \sqrt{v-w^2}} \\
&+ \sum_{i=1}^n [2(b+c)(n-i) + 3(2a(2i-1) - 2c(n-i))] a_i u^{2i-1} v^{n-i-1}. \\
&\int \frac{w^2 \, dw}{\sqrt{u+2aw} \sqrt{v-w^2}} + \bar{T}_1(u, v),
\end{aligned}$$

where $\bar{T}_1(u, v)$ is an arbitrary smooth function in u and v . Since \bar{G}_1 is a weight homogeneous polynomial of weight degree $4n-3$, we must have $\bar{T}_1(u, v) = 0$ because the weight degrees of u and v are even. Moreover, since the integrals of $w/\sqrt{u+2aw}\sqrt{v-w^2}$ and $w^2/\sqrt{u+2aw}\sqrt{v-w^2}$ involve elliptic integrals in order that $\bar{G}_1(u, v, w)$ be a polynomial in their variables $\sqrt{u+2aw}$, $\sqrt{v-w^2}$ and w we get that

$$\begin{aligned}
(19) \quad &(4a^2 - 2ab) a_1 = 0, \\
&(4a^2 - 2ab)(2i+1) a_{i+1} + \frac{1}{a} (2a(2i-1) - 2c(n-i)) a_i = 0, \\
&\text{for } i = 1, \dots, n-1, \\
&2(2n-1) a_n = 0, \\
&(2(b+c)(n-i) + 3(2a(2i-1) - 2c(n-i))) a_i = 0, \\
&\text{for } i = 1, \dots, n.
\end{aligned}$$

It is easy to prove that since $a \neq 0$, conditions (19) are equivalent to one of the following conditions:

- (i) $b = 2a = -c$, $n = 1/2$ and there exists $i_0 \in \{1, \dots, n-1\}$ such that $a_{i_0} \neq 0$.
- (ii) $b = 2a$, $n = 1/2$ and $G_0 = a_n(x^2 - 2az)^{2n-1}$.
- (iii) $a_0 = a_1 = \dots = a_n = 0$, that is, $G_0 = 0$.

Since n is a natural number the two first cases are not possible and therefore only the third case is possible. Therefore $G_0 = 0$ and since this is the component of G with the highest weight-degree, this implies that $G = 0$. Then $g = 0$ which yields that the Lü system has no exponential factors of the form $\exp(g)$ in this case $l = 4n - 2$ with $n \geq 2$.

Subcase 1.5: $l = 4n$ and $n \geq 1$. From (18) we have

$$G_0 = \sum_{i=0}^n a_i (X^2 - 2aZ)^{2i} (Y^2 + Z^2)^{n-i}.$$

Substituting G_0 into the second equation of (18) we can prove that

$$\begin{aligned} L[G_1] &= \sum_{i=0}^n [4ai - 2c(n-i)] a_i (X^2 - 2aZ)^{2i} (Y^2 + Z^2)^{n-i} \\ &\quad + \sum_{i=0}^n 2[4a^2 - 2ab] i a_i (X^2 - 2aZ)^{2i-1} (Y^2 + Z^2)^{n-i} Z \\ &\quad + \sum_{i=0}^n 2(b+c)(n-i) a_i (X^2 - 2aZ)^{2i} (Y^2 + Z^2)^{n-i-1} Z^2. \end{aligned}$$

Then we have

$$\begin{aligned} \bar{G}_1 &= \sum_{i=0}^n \frac{1}{a} [4ai - 2c(n-i)] a_i u^{2i} v^{n-i-1} \sqrt{u + 2av} \sqrt{v - w^2} \\ &\quad + \left(\sum_{i=0}^{n-1} \left(2(4a^2 - 2ab) i a_{i+1} + \frac{1}{a} (4ai - 2c(n-i)) a_i \right) u^{2i+1} v^{n-i-1} \right. \\ &\quad \left. + \frac{1}{v} 4na_n u^{2n+1} \right) \int \frac{w}{\sqrt{u + 2av} \sqrt{v - w^2}} dw \\ &\quad + \sum_{i=0}^n [2(b+c)(n-i) + 3(4ai - 2c(n-i))] a_i u^{2i} v^{n-i-1} \\ &\quad \int \frac{w^2}{\sqrt{u + 2av} \sqrt{v - w^2}} dw + \bar{T}_2(u, v), \end{aligned}$$

where $\bar{T}_2(u, v)$ is an arbitrary smooth function in u and v . In order that G_1 be a weight homogeneous polynomial of weight degree $4n - 1$, we must have $\bar{T}_2(u, v) = 0$ and

$$\begin{aligned} &2(4a^2 - 2ab) i a_{i+1} + \frac{1}{a} (4ai - 2c(n-i)) a_i = 0, \quad i = 0, 1, \dots, n-1, \\ (20) \quad &4ana_n = 0, \\ &[2(b+c)(n-i) + 3(4ai - 2c(n-i))] a_i = 0, \quad i = 0, 1, \dots, n. \end{aligned}$$

We can easily prove that since $a \neq 0$, conditions (20) are equivalent to one of the following conditions:

- (i) $a_0 = a_1 = \dots = a_n = 0$, and consequently $G_0 = 0$.
- (ii) $a_n = 0$. If $b = 2a$, then $b = -c$, in contradiction with $(b+c)^2 + (c+2a)^2 \neq 0$.
- (iii) $a_n = 0$, $b \neq 2a$ and $b = -c$. Then $[4ai - 2c(n-i)] a_i = 0$. Therefore, from (20), we obtain $c = 0$, in contradiction with $b^2 + c^2 \neq 0$.

(iv) $a_n = 0$, $b \neq 2a$ and $b + c \neq 0$. Then

$$2(b + c)(n - i)a_i - 6a(4a^2 - 2ab)ia_{i+1} = 0.$$

So, for $i = 0$ we have $a_0 = 0$. For $i = n - 1$ we get $a_{n-1} = 0$. Working with $i = n - 2, n - 3, \dots, 1$ we obtain $a_{n-2} = \dots = a_1 = 0$. Hence $G_0 = 0$.

In summary $G_0 = 0$ and since this is the component of G with the highest weight-degree, this implies that $G = 0$, and the Lü system has no exponential factors of the form $\exp(g)$ when $l = 4n$ with $n \geq 1$.

In short we have proved that if F is of the form $F = \exp(g)$ with $g \in \mathbb{C}[x, y, z]$, then except constants it is of the form $\exp(x)$ with cofactor $a(y - x)$.

Case 2. Now we study the exponential factors of the form $\exp(g/h)$ with $(g, h) = 1$ and cofactor $q = q(x, y, z)$. Therefore g and h satisfy

$$a(y - x)\frac{\partial g}{\partial x} + (cy - xz)\frac{\partial g}{\partial y} + (-bz + xy)\frac{\partial g}{\partial z} = kg + qh,$$

where we have simplified by the common factor $\exp(g/h)$ and we have used the fact that h is a Darboux polynomial of system (1) with cofactor $k \in \mathbb{C}$, see Theorem 1. We define K, Q and H as follows.

$$K(X, Y, Z) = k,$$

$$Q(X, Y, Z) = \mu^2 q(\mu^{-1}X, \mu^{-2}Y, \mu^{-2}Z) = l_3Z + l_2Y + \mu l_1X + \mu^2 l_0,$$

$$H(X, Y, Z) = h(\mu^{-1}X, \mu^{-2}Y, \mu^{-2}Z).$$

Using the weight change of variables, from (9)–(12) we get that G satisfies

$$\begin{aligned} & a(Y - \mu X) \sum_{i=0}^m \mu^i \frac{\partial G_i}{\partial X} + (-XZ + \mu cY) \sum_{i=0}^m \mu^i \frac{\partial G_i}{\partial Y} \\ (21) \quad & + (XY - \mu bZ) \sum_{i=0}^m \mu^i \frac{\partial G_i}{\partial Z} = \mu k \sum_{i=0}^m \mu^i G_i \\ & + \mu^{l-1}(l_3Z + l_2Y + \mu l_1X + \mu^2 l_0)H(X, Y, Z). \end{aligned}$$

Since h is a Darboux polynomial in view of Theorems 1 and 10, h can be only of four different types. We separate then in four different cases.

Subcase 2.1: $h = (x^2 - 2az)^n$, $n \geq 1$ with cofactor $k = -2an$, $b = 2a$ and $c \neq -2a$. Now equation (21) becomes

$$\begin{aligned} & a(Y - \mu X) \sum_{i=0}^m \mu^i \frac{\partial G_i}{\partial X} + (-XZ + \mu cY) \sum_{i=0}^m \mu^i \frac{\partial G_i}{\partial Y} \\ (22) \quad & + (XY - 2a\mu Z) \sum_{i=0}^m \mu^i \frac{\partial G_i}{\partial Z} = -2an\mu \sum_{i=0}^m \mu^i G_i \\ & + \mu^{l-1-2n}(l_3Z + l_2Y + \mu l_1X + \mu^2 l_0)(X^2 - 2aZ)^n. \end{aligned}$$

We claim that $l \geq 2n$. In order to prove the claim, first we assume that $l < 2n$. Computing in (22) the terms of μ^j with $j < 0$, we get $l_1 = l_2 = l_3 = 0$. For μ^0 we get either $l_0 = 0$, or $l = 2n - 1$ in which case we obtain

$$aY \frac{\partial G_0}{\partial X} - XZ \frac{\partial G_0}{\partial Y} + XY \frac{\partial G_0}{\partial Z} = l_0(X^2 - 2aZ)^n.$$

Evaluating this equation on $Y = Z = 0$ we get that $l_0 = 0$. Hence, in both cases $l_0 = 0$. In short, we get that $l_0 = l_1 = l_2 = l_3 = 0$ and thus $Q = 0$. This implies that g is a Darboux polynomial of the Lü system with cofactor $-2an$. It follows from Theorem 1 that $g = \alpha(x^2 - 2az)^n$ with $\alpha \in \mathbb{C}$. Then $(g, h) = (x^2 - 2az)^n$ which contradicts the assumption $(g, h) = 1$. Hence the claim is proved.

By the claim we can write $l = 2n + \ell$ for some non-negative integer ℓ . We define the following differential operator

$$D = aX \frac{\partial}{\partial X} - cY \frac{\partial}{\partial Y} + 2aZ \frac{\partial}{\partial Z} - 2an.$$

Then from (15), (22) we obtain that

$$\begin{aligned} L[G_0] &= 0, \\ L[G_{\ell-i}] &= D[G_{\ell-i-1}], \quad i = 1, \dots, \ell - 1, \\ L[G_{\ell-1}] &= D[G_{\ell-2}] + (l_2Y + l_3Z)(X^2 - 2aZ)^n, \\ (23) \quad L[G_\ell] &= D[G_{\ell-1}] + l_1X(X^2 - 2aZ)^n, \\ L[G_{\ell+1}] &= D[G_\ell] + l_0(X^2 - 2aZ)^n, \\ L[G_j] &= D[G_{j-1}], \quad j = \ell + 2, \dots, 2n + \ell, \\ 0 &= D[G_{2n+\ell}], \end{aligned}$$

with $G_i = 0$ for $i < 0$. We write $G_i = G_i^{(0)} + G_i^{(1)}$ such that

$$\begin{aligned} L[G_i^{(0)}] &= D[G_{i-1}^{(0)}], \quad i = 1, \dots, 2n + \ell + 1, \\ L[G_{\ell-i}^{(1)}] &= 0, \quad i = 2, \dots, \ell, \\ (24) \quad L[G_{\ell-1}^{(1)}] &= (l_2Y + l_3Z)(X^2 - 2aZ)^n, \\ L[G_\ell^{(1)}] &= D[G_{\ell-1}^{(1)}] + l_1X(X^2 - 2aZ)^n, \\ L[G_{\ell+1}^{(1)}] &= D[G_\ell^{(1)}] + l_0(X^2 - 2aZ)^n, \\ L[G_j^{(1)}] &= D[G_{j-1}^{(1)}], \quad j = \ell + 2, \dots, 2n + \ell + 1. \end{aligned}$$

System (24) is equivalent to system (23) where we have split the polynomial G_i of weight degree $(l - i)$ in two pieces $G_i^{(0)}$ and $G_i^{(1)}$, satisfying $G_i^{(0)}$ the equation $L[G_i^{(0)}] = D[G_{i-1}^{(0)}]$ for $i = 1, \dots, 2n + \ell + 1$, from the second equation of (23) we have that $G_i^{(1)} = 0$ for $i = 0, \dots, \ell - 2$, $G_i^{(1)}$ for $i = \ell - 1, \ell, \ell + 1$ satisfy the third, fourth and fifth equations of (23). Now

the existence of $G_{\ell+1}^{(1)}$ creates the existence of a $G_{\ell+2}^{(1)}$, and so on satisfying the last equation of (24).

Defining

$$G^{(0)} = \sum_{i=0}^m \mu^i G_i^{(0)} \quad \text{and} \quad G^{(1)} = \sum_{i=0}^m \mu^i G_i^{(1)},$$

we have that $G = G^{(0)} + G^{(1)}$. We note that the first identity in equation (24) is exactly the equation of a Darboux polynomial of the Lü system with cofactor $-2an$ (in the weight variables (X, Y, Z)). In other words, $g^{(0)} = G^{(0)}|_{\mu=1} = \sum_{i=0}^m G_i^{(0)}$ is a Darboux polynomial of the Lü system with cofactor $-2an$. In view of Theorem 1 we get that $g^{(0)} = \alpha(x^2 - 2az)^n$ with $\alpha \in \mathbb{C}$.

Now from the third equation of (24) and introducing the change of variables given in (16) we get that setting $G_{\ell-1}^{(1)}(X, Y, Z) = \bar{G}_{\ell-1}^{(1)}(u, v, w)$ the third equality in (24) becomes

$$\sqrt{u + 2aw}\sqrt{v - w^2} \frac{\partial \bar{G}_{\ell-1}^{(1)}}{\partial w} = u^n (l_2 \sqrt{v - w^2} + l_3 w).$$

Solving it we conclude that

$$\bar{G}_{\ell-1}^{(1)}(u, v, w) = \frac{l_2}{a} u^n \sqrt{u + 2aw} + l_3 u^n \int \frac{w}{\sqrt{u + 2aw}\sqrt{v - w^2}} dw.$$

Since $G_{\ell-1}^{(1)}$ must be a polynomial in its variables, we obtain that

$$l_3 = 0 \quad \text{and} \quad G_{\ell-1}^{(1)} = \frac{l_2}{a} X(X^2 - 2aZ)^n.$$

Substituting $G_{\ell-1}^{(1)}$ into the fourth equation of (24) and proceeding as for the third equation of (24) we can prove that

$$\bar{G}_{\ell}^{(1)}(u, v, w) = (l_1 + l_2) u^n \arctan \left(\frac{w}{\sqrt{v - w^2}} \right).$$

Again, using that $G_{\ell}^{(1)}$ must be a polynomial we conclude that $l_1 + l_2 = 0$ and $G_{\ell}^{(1)} = 0$.

Substituting $G_{\ell}^{(1)}$ into the fifth equation of (24) and proceeding as above, using that $G_{\ell+1}^{(1)}$ is a polynomial we get that $l_0 = 0$ and $G_{\ell+1}^{(1)} = 0$. Hence, proceeding inductively the sixth equation of (24) yields that $G_j^{(1)} = 0$ for $j = \ell + 3, \dots, 2n + \ell$. This proves that

$$g = (G^{(0)} + G^{(1)})|_{\mu=1} = \alpha(x^2 - 2az)^n + \frac{l_2}{a} x(x^2 - 2az)^n.$$

But in this case $(g, h) = (x^2 - 2az)^n$ which contradicts the assumption $(g, h) = 1$. Then yields that the Lü system has no exponential factors of the form $\exp(g/h)$ in this subcase.

Subcase 2.2: We have $h = (y^2 + z^2)^n$ with cofactor $k = 2cn$, $n \geq 1$, $b = -c \neq 0$ and $c \neq -2a$. Working in a similar way to Subcase 2.1, we can prove that $g = \alpha(y^2 + z^2)^n + \frac{b}{a}x(y^2 + z^2)^n$. Then $(g, h) = (y^2 + z^2)^n$ which contradicts the assumption $(g, h) = 1$. Hence the Lü system has no exponential factors of the form $\exp(g/h)$ in this subcase.

Subcase 2.3: We have $h = (x^4 + \frac{4}{3}cx^2z - \frac{4}{9}c^2y^2 - \frac{8}{9}c^2xy)^n$ with cofactor $k = \frac{4}{3}cn$, $n \geq 1$, $a = -c/3$ and $b = 0$. Working in a similar way to Subcase 2.1, we can prove that $g = g^{(0)} = \alpha(x^4 + \frac{4}{3}cx^2z - \frac{4}{9}c^2y^2 - \frac{8}{9}c^2xy)^n$. Then $(g, h) = (x^4 + \frac{4}{3}cx^2z - \frac{4}{9}c^2y^2 - \frac{8}{9}c^2xy)^n$ which contradicts the assumption $(g, h) = 1$. Hence the Lü system has no exponential factors of the form $\exp(g/h)$ in this subcase.

Subcase 2.4: We have $h = (x^4 + 4cx^2z - 4c^2y^2 + 8c^2xy + 16c^3z)^n$ with cofactor $k = 4cn$, $n \geq 1$, $c = -a$ and $b = 4a$. Working in a similar way to Subcase 2.1, we can prove that $g = g^{(0)} = \alpha(x^4 + 4cx^2z - 4c^2y^2 + 8c^2xy + 16c^3z)^n$. Then $(g, h) = (x^4 + 4cx^2z - 4c^2y^2 + 8c^2xy + 16c^3z)^n$ which contradicts the assumption $(g, h) = 1$. Hence the Lü system has no exponential factors of the form $\exp(g/h)$ in this subcase.

In short, in Case 2 we have proved that the Lü system has no exponential factors of the form $\exp(g/h)$ with $(g, h) = 1$.

Now, joining Cases 1 and 2 we have proved Proposition 12. \square

Proof of Proposition 6. It follows from Theorem 7 that the Lü system has a first integral of Darboux type if and only if there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that equation (3) is satisfied where p, q are the numbers of Darboux polynomials and exponential factors, respectively. Furthermore, K_i, L_j are the cofactors of Darboux polynomials and exponential factors, respectively. It follows from Theorem 1 that the cofactor of the Darboux polynomials of system (1) when $a \neq 0$ is a constant that we denote by k . The proof of Proposition 12 shows that the cofactor of the unique exponential factor is $a(y - x)$. So equation (3) is equivalent to

$$\lambda_1 k + \mu_1 a(y - x) = 0.$$

This last relation is equivalent to $\mu_1 = \lambda_1 = 0$. \square

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