

A NOTE ON LIOUVILLIAN FIRST INTEGRALS AND INVARIANT ALGEBRAIC CURVES

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ABSTRACT. In this paper we study the existence and non-existence of finite invariant algebraic curves for complex planar polynomial differential system having a Liouvillian first integral.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In this paper we consider a complex planar polynomial differential system

$$(1) \quad \frac{dx}{dt} = \dot{x} = P(x, y), \quad \frac{dy}{dt} = \dot{y} = Q(x, y),$$

where the dependent variables x and y are complex, and the independent one (the *time*) t can be real or complex, and $P, Q \in \mathbb{C}[x, y]$, where $\mathbb{C}[x, y]$ is the ring of all polynomials in the variables x and y with coefficients in \mathbb{C} . We denote by $m = \max\{\deg P, \deg Q\}$ the *degree* of the polynomial system.

Let $f = f(x, y) = 0$ be an algebraic curve in \mathbb{C}^2 . We say that it is a *finite invariant algebraic curve* for the polynomial differential system (1) if

$$(2) \quad P \frac{\partial f}{\partial x} + Q \frac{\partial f}{\partial y} = kf,$$

for some polynomial $k = k(x, y) \in \mathbb{C}[x, y]$, called the *cofactor* of the algebraic curve $f = 0$. From (2) it is easy to see that the degree of the polynomial k is at most $m - 1$ and that the algebraic curve $f = 0$ is formed by trajectories of the polynomial differential system (1).

Let $h, g \in \mathbb{C}[x, y]$ and assume that h and g are relatively prime in the ring $\mathbb{C}[x, y]$. Then the function $\exp(g/h)$ is called an *exponential factor* of the polynomial differential system (1) if for some polynomial $k \in \mathbb{C}[x, y]$ of degree at most $m - 1$ it satisfies equation

$$(3) \quad P \frac{\partial \exp(g/h)}{\partial x} + Q \frac{\partial \exp(g/h)}{\partial y} = k \exp(g/h).$$

If $\exp(g/h)$ is an exponential factor and h is not a constant, then it is easy to show that $h = 0$ is an invariant algebraic curve. For more details on exponential factors see for instance [3].

Let U be a non-empty open subset of \mathbb{C}^2 . We say that a not locally constant function $H : U \rightarrow \mathbb{C}$ is a *first integral* of the polynomial differential system (1) in U if H is constant

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on the trajectories of the polynomial differential system (1) contained in U , i.e. if

$$\frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q = 0,$$

in the points of U .

We say that a not locally zero function $R : U \rightarrow \mathbb{C}$ is an *integrating factor* of the polynomial differential system (1) in U if R satisfies that

$$\frac{\partial(RP)}{\partial x} + \frac{\partial(RQ)}{\partial y} = 0,$$

in the points $(x, y) \in U$.

In 1992 Singer [13] showed that if a polynomial differential system has a Liouvillian first integral (i.e. roughly speaking a first integral that can be expressed by quadratures of elementary functions), then it has an integrating factor of the form

$$(4) \quad R(x, y) = \exp \left(\int_{(x_0, y_0)}^{(x, y)} U(x, y) dx + V(x, y) dy \right),$$

where U and V are rational functions which verify $\partial U / \partial y = \partial V / \partial x$. Taking into account this result, in 1999 Christopher [2] showed that in fact the integrating factor (4) can be written into the form

$$(5) \quad R = \exp(g/h) \prod f_i^{\lambda_i},$$

where g , h and f_i are polynomials and $\lambda_i \in \mathbb{C}$. This type of integrability is known as *Liouvillian integrability*.

For more details on all these notions mentioned until here see the seminal work [4], the paper [10] and the references quoted there.

In order to find a first integral for system (1) we can also use non-algebraic invariant curves with polynomial cofactor, see [5, 6]. Some generalizations of the Liouvillian integrability theory are given in [8, 9, 12] where a new kind of first integrals, not only the Liouvillian ones, appears.

There was the belief that a Liouvillian integrable polynomial differential system has always a finite invariant algebraic curve in \mathbb{C}^2 . Adding certain hypotheses, this claim was proved in [14]. However, only assuming the Liouvillian integrability of the system, the claim is refuted in [10] where it is proved that there exist Liouvillian integrable polynomial differential systems without any finite invariant algebraic curve.

In [11] it is proved that if a complex differential equation of the form $dy/dx = a_0(x) + a_1(x)y + \dots + a_n(x)y^n$ with $a_i(x)$ polynomials for $i = 0, 1, \dots, n$, $a_n(x) \neq 0$ and $n \geq 2$ has a Liouvillian first integral, then it has a finite invariant algebraic curve. Consequently, this result applies to the Riccati and Abel polynomial differential equations. The result is not true when $n = 1$, i.e. for first-order linear polynomial differential equations.

Our aim in this note is to extend these results to any complex planar polynomial differential system of the form (1). Without loss of generality, we can write system (1) into the following form

$$(6) \quad \begin{aligned} \dot{x} &= b_0(x) + b_1(x)y + \dots + b_\ell(x)y^\ell, \\ \dot{y} &= a_0(x) + a_1(x)y + \dots + a_n(x)y^n, \end{aligned}$$

where, without loss of generality, we have privileged in (6) the variable y with respect to variable x writing the polynomials $P(x, y)$ and $Q(x, y)$ of (1) as polynomials in y

with coefficients polynomials in x . The particular case studied in [11] is system (6) with $P(x, y) = 1$.

The main results of this note are the following ones.

Theorem 1. *Consider a planar complex polynomial differential system (6) with a Liouvilian first integral. If $n > \ell + 1$ then the system has a finite invariant algebraic curve.*

Corollary 2. *Consider a planar complex polynomial differential system (6) with a Liouvilian first integral. If $n > \ell + 1$, then any Darboux integrating factor of the form (5) exhibits finite invariant algebraic curves.*

These two results are proved in section 2.

When $n \leq \ell + 1$ there are examples of polynomial differential systems (6) with a Liouvilian first integral but with no finite invariant algebraic curves. The following proposition gives some examples.

Proposition 3. *Given $k \geq 0$ and $s > 0$ integers with k even, consider the polynomial differential systems*

$$\begin{aligned}
 (7) \quad & \dot{x} = 2xy + y^k, \quad \dot{y} = 1. \\
 (8) \quad & \dot{x} = y^k + (x + y)^{s-1}(-s + 2xy + 2y^2), \quad \dot{y} = s(x + y)^{s-1}. \\
 (9) \quad & \dot{x} = s(x + y)^{s-1}, \quad \dot{y} = x^k + (x + y)^{s-1}(-s + 2x^2 + 2xy). \\
 (10) \quad & \dot{x} = s(x + y)^{s-1}(1 + 2x(x + y)^s), \quad \dot{y} = 1 - s(x + y)^{s-1}(1 + 2x(x + y)^s). \\
 (11) \quad & \dot{x} = 1 - s(x + y)^{s-1}(1 + 2y(x + y)^s), \quad \dot{y} = s(x + y)^{s-1}(1 + 2y(x + y)^s).
 \end{aligned}$$

These systems are Liouvillian integrable and have no finite invariant algebraic curves.

Proposition 3 is proved in section 2.

In this paragraph ℓ and n are the ones defined in (6). We consider system (7). If $k = 0$, then we have $(\ell, n) = (1, 0)$, which is an example of $n = \ell - 1$. If $k > 1$, then $(\ell, n) = (k, 0)$, which is an example of $n = \ell - k$, with $k > 1$ any even integer. In system (8), we only consider the case in which $k \geq s + 1$, thus $(\ell, n) = (k, s - 1)$ which is an example of $n \leq \ell - 2$. In system (9), we have $(\ell, n) = (s - 1, s)$ which provides an example of $n = \ell + 1$. The system (10) has $(\ell, n) = (2s - 1, 2s - 1)$ which is an example of $n = \ell$ both odd integers. Finally, system (11) has $(\ell, n) = (2s, 2s)$ which is an example of $n = \ell$ both even integers.

System (7) with $k = 0$ has been studied in [10] where it was proved that the system has a Liouvillian first integral and no finite invariant algebraic curve. In the proof of Proposition 3 we will generalize this result for any even integer $k \geq 0$.

2. PROOFS OF THEOREM 1 AND COROLLARY 2

Proof of Theorem 1. The proof is by contradiction. We assume that the differential system (6) is Liouvillian integrable, (i.e. has a Liouvillian first integral), and has no finite invariant algebraic curves. By the results of Christopher in [2] we know that if system (6) is Liouvillian integrable, then it has an integrating factor of the form (5). We recall that $f_i = 0$ and $h = 0$ (if h is not a constant) in (5) are invariant algebraic curves and $\exp(g/h)$ is an exponential factor for system (6), for more details see [1]. Therefore if system (6) is a planar Liouvillian integrable polynomial differential system without finite invariant algebraic curves, then it must have an integrating factor of the form $R = \exp(g(x, y))$, where g is a polynomial. We recall here that $g = 0$ does not need to be an invariant algebraic curve of the differential system (6).

We assume that the degree of g with respect to the variable y is m . Then we write g as a polynomial in the variable y with coefficients polynomials in the variable x , i.e.

$$R = \exp(g(x, y)) = \exp(g_0(x) + g_1(x)y + \cdots + g_m(x)y^m).$$

Now we impose that R is an integrating factor of system, i.e.,

$$(12) \quad \frac{\partial R}{\partial x} P(x, y) + \frac{\partial R}{\partial y} Q(x, y) + \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) R = 0,$$

and we obtain the following identity

$$\begin{aligned} & \left(g'_0(x) + g'_1(x)y + \cdots + g'_m(x)y^m \right) \left(b_0(x) + b_1(x)y + \cdots + b_\ell(x)y^\ell \right) + \\ & \left(g_1(x) + 2g_2(x)y + \cdots + mg_m(x)y^{m-1} \right) \left(a_0(x) + a_1(x)y + \cdots + a_n(x)y^n \right) + \\ & b'_0(x) + b'_1(x)y + \cdots + b'_\ell(x)y^\ell + a_1(x) + 2a_2(x)y + \cdots + na_n(x)y^{n-1} = 0, \end{aligned}$$

after dividing by R .

We are assuming that $n > \ell + 1$. So the highest power in y in the previous identity is y^{m+n-1} whose coefficient is $ma_n(x)g_m(x)$. Since $a_n(x)$ is, by definition, the coefficient of the highest power in y in \dot{y} , it cannot be zero and the same reasoning holds for $g_m(x)$. Hence, $ma_n(x)g_m(x) = 0$ implies $m = 0$. Consequently the integrating factor is of the form $R = \exp(g_0(x))$. Therefore, from equation (12) we have that

$$\begin{aligned} & g'_0(x) \left(b_0(x) + b_1(x)y + \cdots + b_\ell(x)y^\ell \right) + b'_0(x) + b'_1(x)y + \cdots + b'_\ell(x)y^\ell \\ & + a_1(x) + 2a_2(x)y + \cdots + na_n(x)y^{n-1} = 0. \end{aligned}$$

Since $n > \ell + 1$ the highest power in y in the previous expression is $n - 1$ and its coefficient is $na_n(x)$. The vanishing of this coefficient leads to a contradiction because $n > 1$ by assumption and $a_n(x)$ cannot be zero. Therefore, in the case $n > \ell + 1$, if system (6) is Liouvillian integrable then it has finite invariant algebraic curves. Hence Theorem 1 is proved. \square

Note that when $\ell = 0$ implies $n \geq 2$ and we obtain again the result given in [11].

Proof of Corollary 2. Inside the proof of Theorem 1 we have also proved that there cannot exist a Darboux integrating factor of the form $\exp(g(x, y))$ which proves Corollary 2. \square

Proof of Proposition 3. We first consider system (7), which has the inverse integrating factor $V(x, y) = e^{y^2}$. The Liouvillian first integral associated to this system is $H(x, y) = xe^{-y^2} + (1/2)\Gamma((k+1)/2, y^2)$, where $\Gamma(\alpha, z) := \int_z^\infty t^{\alpha-1}e^{-t}dt$ is the Euler–Gamma function. We shall prove that this system has no finite invariant algebraic curve. Assume that $f(x, y) = 0$ is a finite invariant algebraic curve. We write it expanded in powers of x :

$$f(x, y) = f_0(y) + f_1(y)x + \cdots + f_{n-1}(y)x^{n-1} + f_n(y)x^n,$$

where $f_n(y)$ is not identically zero and $f_i(y)$ are polynomials in $\mathbb{C}[y]$. We write its cofactor also expanded in powers of x :

$$k(x, y) = k_0(y) + k_1(y)x + \cdots + k_{m-1}(y)x^{m-1} + k_m(y)x^m,$$

where $k_i(y)$ are polynomials in $\mathbb{C}[y]$. Equation (2) writes as

$$\begin{aligned} & \left(f_1(y) + 2f_2(y)x + \dots + (n-1)f_{n-1}(y)x^{n-2} + nf_n(y)x^{n-1} \right) (2xy + y^k) + \left(f'_0(y) \right. \\ & \left. + f'_1(y)x + \dots + f'_{n-1}(y)x^{n-1} + f'_n(y)x^n \right) = \left(k_0(y) + k_1(y)x + \dots + k_{m-1}(y)x^{m-1} \right. \\ & \left. + k_m(y)x^m \right) \left(f_0(y) + f_1(y)x + \dots + f_{n-1}(y)x^{n-1} + f_n(y)x^n \right). \end{aligned}$$

We observe that the highest order of x on the left-hand side is n and the highest order of x on the right-hand side is $n+m$, which implies $m=0$. Now we equate the highest powers in x of both sides, which correspond to the coefficient of x^n :

$$nf_n(y)2y + f'_n(y) = k_0(y)f_n(y).$$

Since $f_n(y)$ is a polynomial in y , we deduce that $k_0(y) = 2ny$ and $f_n(y)$ needs to be a constant which we take equal to 1 without loss of generality. The equation corresponding to the coefficients of x^{n-1} is $(n-1)f_{n-1}(y)2y + ny^k + f'_{n-1}(y) = 2nyf_{n-1}(y)$ which can be rewritten as

$$(13) \quad f'_{n-1}(y) + ny^k = 2yf_{n-1}(y).$$

We develop $f_{n-1}(y)$ in powers of y expressed into the form

$$f_{n-1}(y) = c_r y^r + c_{r-1} y^{r-1} + \dots + c_1 y + c_0,$$

where c_i are constants and r is the degree of $f_{n-1}(y)$ in y . Equating the highest order terms in y of equation (13), we see that $k = r+1$ and $n = 2c_r$. The particular case $k=0$ implies $f_{n-1}(y) = 0$ and this will be analyzed below. The coefficient of y^r in equation (13) gives $c_{r-1} = 0$. Indeed, we see that the following recurrence in the coefficients is satisfied $j c_j = c_{j-2}$ for $j = 2, \dots, r$. Moreover since y divides $f'_{n-1}(y)$ we have that $c_1 = 0$. Therefore, we deduce that all the coefficients of c_j with j odd need to be zero. Since k is even by hypothesis and $k = r+1$, we have that r is odd and hence $c_r = 0$ which is a contradiction because it is the coefficient of highest order. We deduce that $f_{n-1}(y)$ needs to be identically zero. Hence, from (13) we get $n = 0$. This fact would imply that the only possible finite invariant algebraic curve of system (7) would be a function only depending on x , but this is not possible because $P(x, y) = 2xy + y^k$, when k is even, has no divisor only depending on x .

The other four systems (8), (9), (10) and (11) are obtained by rational transformations of variables from system (7) and a rescaling of time, if necessary. Therefore, they all have a Liouvillian first integral which is the transformation of the first integral of system (7). In order to explicitly write down the transformations without confusing the names of the variables, we write system (7)

$$\dot{u} = 2uv + v^k, \quad \dot{v} = 1.$$

We consider the transformation $(u, v) \rightarrow (x, y)$ with $u = (x+y)^s$ and $v = y$ and we get system (8). The transformation $(u, v) \rightarrow (x, y)$ with $u = (x+y)^s$ and $v = x$ provides system (9). Now we take system (7) with $k=0$ and we do the transformation $(u, v) \rightarrow (x, y)$ with $u = x$ and $v = (x+y)^s$ to get system (10). And finally we take again $k=0$ and the transformation $(u, v) \rightarrow (x, y)$ with $u = y$ and $v = (x+y)^s$ which gives system (11).

We can write system (7) as an ordinary differential equation

$$(14) \quad \frac{du}{dv} = 2uv + v^k.$$

We remark that if one of the systems (8), (9), (10) or (11) has a finite invariant algebraic curve $f(x, y) = 0$ by the rational change of variables, then equation (14) has a particular algebraic solution. In particular, for system (8), we would get the particular solution $f(u^{1/s} - v, v) = 0$. And this fact applies analogously to the other rational transformations.

In order to finish the proof of Proposition 3 we recall here Theorem 3.1 of [7] for the sake of completeness.

Theorem 4. *Consider system (1) and the corresponding ordinary differential equation*

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}.$$

Let $g(x)$ be an algebraic particular solution of the former ordinary differential equation and we call $f(x, y)$ the irreducible polynomial satisfying $f(x, g(x)) \equiv 0$. Then, the curve $f(x, y) = 0$ is an invariant algebraic curve of system (1).

In consequence, Theorem 4 implies that equation (14) has a finite invariant algebraic curve that contains the algebraic branch defined by the particular solution $f(u^{1/s} - v, v) = 0$. This is a contradiction because equation (14) has no finite invariant algebraic curve. \square

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