

DARBOUX INTEGRABILITY OF A GENERALIZED FRIEDMANN-ROBERTSON-WALKER HAMILTONIAN SYSTEM

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ABSTRACT. We study the Darboux first integrals of a generalized Friedmann-Robertson-Walker Hamiltonian system.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Given a system of ordinary differential equations depending on parameters in general is very difficult to recognize for which values of the parameters the equations have first integrals because there are no satisfactory methods to answer this question.

In this paper we study the first integrals of a generalized Friedmann-Robertson-Walker Hamiltonian differential system in \mathbb{R}^4

$$(1) \quad \begin{aligned} \dot{x} &= -p_x, \\ \dot{y} &= p_y, \\ \dot{p}_x &= x - ax^3 - bxy^2, \\ \dot{p}_y &= -y - bx^2y, \end{aligned}$$

with $a, b \in \mathbb{R}$ being parameters and the dot denotes derivative with respect to time t .

The Hamiltonian of this system is

$$H_0 = \frac{1}{2}(p_y^2 - p_x^2) + \frac{1}{2}(y^2 - x^2) + \frac{a}{4}x^4 + \frac{b}{2}x^2y^2.$$

When $a = 0$, system (1) coincides with the Friedmann-Robertson-Walker system that modelates a universe, filled by a conformally coupled by a massive real scalar field (see [2] for more details where the authors present analytical and numerical evidence of the existence of chaotic motion). For a more general point of view on the Friedmann-Robertson-Walker system see for instance the works [5, 7] and the references quoted therein.

In this paper we study the Darboux first integrals of a generalized Friedmann-Robertson-Walker system (see (1)). We recall that a *first integral of Darboux type* is a first integral H which is a function of Darboux type (see below (2) for a precise definition). The study of the Darboux first integrals is a classical problem in the integrability theory of differential equations.

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The vector field associated to system (1) is

$$\mathcal{X} = -p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + x(1 - ax^2 - by^2) \frac{\partial}{\partial p_x} - y(1 + bx^2) \frac{\partial}{\partial p_y}.$$

Let $U \subset \mathbb{C}^4$ be an open set. We say that the non-constant function $H: U \rightarrow \mathbb{C}$ is a *first integral* of the polynomial vector field \mathcal{X} on U if $H(x(t), y(t), p_x(t), p_y(t))$ is constant for all values of t for which the solution $(x(t), y(t), p_x(t), p_y(t))$ of \mathcal{X} is defined on U . Clearly H is a first integral of \mathcal{X} on U if and only if

$$\mathcal{X}H = -p_x \frac{\partial H}{\partial x} + p_y \frac{\partial H}{\partial y} + x(1 - ax^2 - by^2) \frac{\partial H}{\partial p_x} - y(1 + bx^2) \frac{\partial H}{\partial p_y} = 0$$

on U .

In this paper we want to study the so-called Darboux first integrals of the Friedman–Robertson–Walker polynomial differential systems (1), using the Darboux theory of integrability (originated in the papers [4]). For a present state of this theory see the Chapter 8 of [6], the paper [8], and the references quoted in them.

We emphasize that the study of the existence of first integrals is a classical problem in the theory of differential systems, because the knowledge of first integrals of a differential system can be very useful in order to understand and simplify the topological structure of their orbits. Thus, their existence or not can also be viewed as a measure of the complexity of a differential system.

We recall that a first integral is of *Darboux* type if it is of the form

$$(2) \quad f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q},$$

where f_1, \dots, f_p are Darboux polynomials (see section 2 for a definition), F_1, \dots, F_q are exponential factors (see section 2 for a definition), and $\lambda_j, \mu_k \in \mathbb{C}$ for all j and k .

The functions of the form (2) are called *Darboux* functions, and they are the base of the Darboux theory of integrability, which looks when these functions are first integrals or integrating factors. In this last case, the first integrals associated to integrating factors given by Darboux functions are the Liouvillian first integrals, see for more details [6, 8].

The Darboux theory of integrability is essentially an algebraic theory of integrability based in the invariant algebraic hypersurfaces that a polynomial differential system has. In fact to every Darboux polynomial there is associated some invariant algebraic hypersurface (see again section 2), and the exponential factors appear when an invariant algebraic surface has multiplicity larger than 1, for more details see [3, 6, 8]. As far as we know is the unique theory of integrability which is developed for studying the first integrals of polynomial differential systems. In general the other theories of integrability do not need that the differential system be polynomial.

We say that the Hamiltonian system (1) is *completely integrable* if it has two independent first integrals H_1 and H_2 in involution. That is, H_1 and H_2 are *independent* if their gradients are linearly independent over a set of full Lebesgue measure in \mathbb{C}^4 . Moreover, H_1 and H_2 are in *involution* if the Poisson bracket $\{H_i, H_0\} = 0$ for $i \in \{1, 2\}$.

When $b = 0$ system (1) is completely integrable. More precisely we have the following result.

Theorem 1. *System (1) with $b = 0$ is completely integrable with the first integrals*

$$H_1 = -p_x^2 - x^2 + \frac{a}{2}x^4 \quad \text{and} \quad H_2 = p_y^2 - y^2.$$

When $a = 0$, system (1) is the well-known simplified Friedman-Robertson-Walker Hamiltonian system. Its Darboux integrability was studied in [10]. In that paper, the authors proved the following theorem.

Theorem 2. *The unique first integrals of Darboux type of system (1) with $a = 0$ and $b \neq 0$ are functions of Darboux type in the variable*

$$H_0 = \frac{1}{2}(p_y^2 - p_x^2) + \frac{1}{2}(y^2 - x^2) + \frac{b}{2}x^2y^2.$$

From now on we will consider the case in which $ac \neq 0$.

The main result of this paper is the following.

Theorem 3. *The unique first integrals of Darboux type of the generalized Friedman-Robertson-Walker Hamiltonian system (1) with $ab \neq 0$ are functions of Darboux type in the variable H_0 .*

We prove Theorem 3 in section 3.

Since the Darboux theory of integrability of a polynomial differential system is based on the existence of Darboux polynomials and their multiplicity, the study of the existence or not of Darboux first integrals needs to look for the Darboux polynomials. These will be done in the next Theorems 9 and 10, whose results are the main steps for proving Theorem 3.

We must mention that any comment on the existence or non-existence of first integrals additional to the Hamiltonian itself appears in the reference [2]. In that paper the authors are mainly concerned with the chaotic behavior of the Friedman-Robertson-Walker polynomial Hamiltonian system. Of course, roughly speaking the chaotic motion is against the existence of first integrals.

Corollary 4. *The generalized Friedman-Robertson-Walker Hamiltonian system (1) is completely integrable with first integrals of Darboux type if and only if $b = 0$.*

The proof is completely clear.

2. BASIC RESULTS

Let $h = h(x, y, p_x, p_y) \in \mathbb{C}[x, y, p_x, p_y] \setminus \mathbb{C}$. As usual $\mathbb{C}[x, y, p_x, p_y]$ denotes the ring of all complex polynomials in the variables x, y, p_x, p_y . We say that h is a *Darboux polynomial* of system (1) if it satisfies

$$\mathcal{X}h = Kh,$$

the polynomial $K = K(x, y, p_x, p_y) \in \mathbb{C}[x, y, p_x, p_y]$ is called *the cofactor* of h and has degree at most two. Every Darboux polynomial h defines an *invariant algebraic hypersurface* $h = 0$, i.e. if a trajectory of system (1) has a point in $h = 0$, then the whole trajectory is contained in $h = 0$, see for more details [6]. When $K = 0$ the Darboux polynomial h is a polynomial first integral.

An *exponential factor* E of system (1) is a function of the form $E = \exp(g/h) \notin \mathbb{C}$ with $g, h \in \mathbb{C}[x, y, p_x, p_y]$ satisfying $(g, h) = 1$ and

$$\mathcal{X}E = LE,$$

for some polynomial $L = L(x, y, p_x, p_y)$ of degree at most 2, called *the cofactor* of E .

A geometrical meaning of the notion of exponential factor is given by the next result.

Proposition 5. *If $E = \exp(g/h)$ is an exponential factor for the polynomial differential system (1) and h is not a constant polynomial, then $h = 0$ is an invariant algebraic hypersurface, and eventually e^g can be exponential factors, coming from the multiplicity of the infinite invariant hyperplane.*

The proof of Proposition 5 can be found in [3, 9]. We explain a little the last part of the statement of Proposition 5. If we extend to the projective space $\mathbb{P}\mathbb{R}^4$ the polynomial differential system (1) defined in the affine space \mathbb{R}^4 , then the hyperplane at infinity always is invariant by the flow of the extended differential system. Moreover, if this invariant hyperplane has multiplicity higher than 1, then it creates exponential factors of the form e^g , see for more details [9].

Theorem 6. *Suppose that the polynomial vector field \mathcal{X} of degree m defined in \mathbb{C}^4 admits p invariant algebraic hypersurfaces $f_i = 0$ with cofactors K_i , for $i = 1, \dots, p$ and q exponential factors $E_j = \exp(g_j/h_j)$ with cofactors L_j , for $j = 1, \dots, q$. Then there exists $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = 0$$

if and only if the function of Darboux type

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} E_1^{\mu_1} \dots E_q^{\mu_q}$$

is a first integral of \mathcal{X} .

Theorem 6 is proved in [6].

From Theorem 6 it follows easily the next well-known result.

Corollary 7. *The existence of a rational first integral for a polynomial differential system implies either the existence of a polynomial first integral or the existence of two Darboux polynomials with the same non-zero cofactor.*

The following result is well-known.

Lemma 8. *Assume that $\exp(g_1/h_1), \dots, \exp(g_r/h_r)$ are exponential factors of some polynomial differential system*

$$(3) \quad x' = P(x, y, p_x, p_y), \quad y' = Q(x, y, p_x, p_y), \quad p'_x = R(x, y, p_x, p_y), \quad p'_y = U(x, y, p_x, p_y)$$

with $P, Q, R, U \in \mathbb{C}[x, y, p_x, p_y]$ with cofactors L_j for $j = 1, \dots, r$. Then

$$\exp(G) = \exp(g_1/h_1 + \dots + g_r/h_r)$$

is also an exponential factor of system (3) with cofactor $L = \sum_{j=1}^r L_j$.

3. PROOF OF THEOREM 3

According with section 2 for proving Theorem 3 we need to characterize the Darboux polynomials of system (1).

The following result characterizes the polynomial first integrals of system (1) with $b \neq 0$, i.e. it characterizes the Darboux polynomial with zero cofactor.

Theorem 9. *The unique polynomial first integrals of system (1) with $ab \neq 0$ are polynomials in the variable H_0 .*

Proof. Doing the change of variables

$$(4) \quad z_1 = p_x - p_y, \quad z_2 = p_x + p_y,$$

system (1) becomes

$$(5) \quad \begin{aligned} \dot{x} &= -\frac{1}{2}(z_1 + z_2), \\ \dot{y} &= -\frac{1}{2}(z_1 - z_2), \\ \dot{z}_1 &= x + y + x(-ax^2 + by(x - y)), \\ \dot{z}_2 &= x - y - x(ax^2 + by(x + y)), \end{aligned}$$

and H_0 writes as

$$H_0 = \frac{1}{2} \left(y^2 - x^2 - z_1 z_2 + bx^2 y^2 + \frac{a}{2} x^4 \right).$$

Now we restrict to $H_0 = 0$. We get that

$$(6) \quad z_2 = \frac{2bx^2 y^2 + 2y^2 - 2x^2 + ax^4}{2z_1}.$$

Then system (5) on $H_0 = 0$ becomes, after the rescaling by $d\tau = z_1 dt$:

$$(7) \quad \begin{aligned} x' &= -\frac{1}{2}z_1^2 - \frac{1}{2}(bx^2 y^2 + y^2 - x^2 + \frac{a}{2}x^4), \\ y' &= -\frac{1}{2}z_1^2 + \frac{1}{2}(bx^2 y^2 + y^2 - x^2 + \frac{a}{2}x^4), \\ z_1' &= z_1(x + y + x(-ax^2 + by(x - y))), \end{aligned}$$

where the prime denotes the derivative with respect to the variable τ . If $f = f(x, y, p_x, p_y)$ is a polynomial first integral of system (1) then, if we denote by $g = g(x, y, z_1, z_2)$ the polynomial first integral f written in the new variables (x, y, z_1, z_2) , we have that g is also a polynomial first integral of the differential system (5). Furthermore, if we denote by $h = h(x, y, z_1)$ the polynomial first integral g restricted to the invariant hypersurface (6), then h is a rational first integral of the differential system (7) in the sense that it is the quotient of a polynomial in the variables (x, y, z_1) and a power of z_1 . Furthermore h satisfies

$$(8) \quad \begin{aligned} &\left(-\frac{1}{2}z_1^2 - \frac{1}{2}(bx^2 y^2 + y^2 - x^2 + \frac{a}{2}x^4) \right) \frac{\partial h}{\partial x} \\ &+ \left(-\frac{1}{2}z_1^2 + \frac{1}{2}(bx^2 y^2 + y^2 - x^2 + \frac{a}{2}x^4) \right) \frac{\partial h}{\partial y} + z_1(x + y + x(-ax^2 + by(x - y))) \frac{\partial h}{\partial z_1} = 0. \end{aligned}$$

We write

$$(9) \quad h = \sum_{j=-n}^n h_j(x, y, z_1)$$

where each h_j is the quotient of a polynomial in the variable (x, y, z_1) with degree j and the denominator a power in the variable z_1 . Here we define the degree of h_j as j . Hence, computing the terms of degree $n + 4$ we have

$$\frac{x^2}{4}(ax^2 + 2by^2) \left(\frac{\partial h_n}{\partial y} - \frac{\partial h_n}{\partial x} \right) + z_1 x(-ax^2 + by(x - y)) \frac{\partial h_n}{\partial z_1} = 0.$$

We write it as

$$(10) \quad L[h_n] = 0, \quad \text{where} \quad L = \frac{x^2}{4}(ax^2 + 2by^2) \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x} \right) + z_1 x(-ax^2 + by(x-y)) \frac{\partial}{\partial z_1}.$$

The characteristic equations associated to the linear partial differential operator L are

$$(11) \quad \frac{dx}{dy} = -1, \quad \frac{dy}{dz_1} = \frac{x(ax^2 + 2by^2)}{4z_1(-ax^2 + by(x-y))}.$$

We consider two cases.

Case 1: $a = -2b$. In this case (11) has the general solution

$$x + y = d_1, \quad \frac{(y-x)^2}{e^{2y/x} z_1} = d_2,$$

where d_1 and d_2 are constants of integration. According to this, we make the change of variables

$$(12) \quad u = x + y, \quad v = \frac{(y-x)^2}{e^{2y/x} z_1}, \quad w = y.$$

Its inverse transformation is

$$(13) \quad x = u - w, \quad z_1 = (2w - u)^2 (e^{2w/(u-w)} v), \quad y = w.$$

Under the change of variables (12) and (13) equation (10) becomes the following ordinary differential equation (for fixed u and v):

$$\frac{b}{2} u(u-2w)(u-w)^2 \frac{\partial \bar{h}_n}{\partial w} = 0$$

where \bar{h}_n is h_n written in the variables u, v and w . Solving it we have

$$\bar{h}_n = \bar{h}_n(u, v) = h_n \left(x + y, \frac{(y-x)^2}{e^{2y/x} z_1} \right).$$

Since the numerator of h_h is a polynomial of degree n we must have

$$(14) \quad h_n = \sum_{k=0}^n \alpha_k (x+y)^k, \quad \alpha_k \in \mathbb{R}.$$

Now using the transformations (12) and (13) and working in a similar manner to solve \bar{h}_n , computing the terms of degree $n+3$, we get

$$(15) \quad h_{n-1} = \sum_{k=0}^{n-1} \beta_k (x+y)^k, \quad \beta_k \in \mathbb{R}.$$

Now, using the transformations (12) and (13) and working in a similar manner to solve \bar{h}_{n-2} , computing the terms of degree $n+2$, we get

$$\frac{b}{2} u(u-w)^2 \frac{\partial \bar{h}_n}{\partial w} = (2w-u)^3 e^{4w/(u-w)} v^2 \sum_{k=1}^n k \alpha_k u^{k-1}.$$

Integrating this equations, we obtain that $\bar{h}_{n-2} = \bar{h}_{n-2}(u, v, w)$ is

$$\frac{1}{2b} \left(\sum_{k=1}^n k \alpha_k u^{k-2} \right) \left(e^{\frac{4w}{u-w}} (33u^2 - 48uw + 16w^2) - \frac{44}{e^4} u^2 v^2 \text{Ei} \left(\frac{4u}{u-w} \right) \right) + k_{n-2}(u, v)$$

where $Ei(z)$ is the exponential integral function, see for instance [1]. Going back to the variables x , y and z_1 we obtain that $h_{n-2}(x, y, z_1)$ is

$$\frac{1}{2b} \left(\sum_{k=1}^n k \alpha_k (x+y)^{k-2} \right) \left(e^{\frac{4y}{x}} (33(x+y)^2 - 48y(x+y) + 16y^2) - \frac{44}{e^4} (x+y)^2 \frac{(x-4)^4}{e^{4y/x} z_1^2} Ei\left(\frac{4(x+y)}{y}\right) \right) + k_{n-2} \left(x+y, \frac{(y-x)^2}{e^{2y/x} z_1} \right).$$

Since the numerator of $h_{n-2}(x, y, z_1)$ is a polynomial of degree $n-2$ we must have

$$\sum_{k=1}^n k \alpha_k (x+y)^{k-1} = 0.$$

So $\alpha_k = 0$ for $k = 1, \dots, n$. Then $h_n = \alpha_0$, which implies that $h = \alpha_0/z_1^m$ for some nonnegative integer m . Then from (8) we get

$$(16) \quad -\frac{m\alpha_0}{z_1^m} (x+y - ax^3 + bx^2y - bxy^2) = 0.$$

Therefore, $m\alpha_0 = 0$ and consequently h is a constant, in contradiction with the fact that h is a first integral. So, this case is not possible.

Case 2: $a \neq -2b$. In this case the general solution of (11) is

$$x+y = d_1, \quad \frac{(ax^2 + 2by^2)^{\sqrt{bx}}}{z_1^{\sqrt{bx}} e^{2y\sqrt{b} + \sqrt{2ax} \arctan\left(\frac{\sqrt{2by}}{\sqrt{ax}}\right)}} = d_2$$

where d_1 and d_2 are constants of integration. According to this, we make the change of variables

$$(17) \quad u = x+y, \quad v = \frac{(ax^2 + 2by^2)^{\sqrt{bx}}}{z_1^{\sqrt{bx}} e^{2y\sqrt{b} + \sqrt{2ax} \arctan\left(\frac{\sqrt{2by}}{\sqrt{ax}}\right)}}, \quad w = y.$$

Its inverse transformation is

$$(18) \quad x = u-w, \quad z_1 = (a(u-w)^2 + 2bw^2) \left(e^{2wv^{1/\sqrt{b}}} \right)^{1/(w-u)} e^{-\sqrt{2a/b} \arctan\left(\frac{\sqrt{2bw}}{\sqrt{a(u-w)}}\right)}, \quad y = w.$$

Under the change of variables (17) and (18) equation (10) becomes the following ordinary differential equation (for fixed u and v):

$$\frac{(u-w)^2}{4} (a(u-w)^2 + 2bw^2) \frac{\partial \bar{h}_n}{\partial w} = 0$$

where \bar{h}_n is h_n written in the variables. Solving it we have

$$\bar{h}_n = \bar{h}_n(u, v) = h_n \left(x+y, \frac{(ax^2 + 2by^2)^{\sqrt{bx}}}{z_1^{\sqrt{bx}} e^{2y\sqrt{b} + \sqrt{2ax} \arctan\left(\frac{\sqrt{2by}}{\sqrt{ax}}\right)}} \right).$$

Since h_n has degree n in the numerator we must have

$$(19) \quad h_n = \sum_{k=0}^n \alpha_k (x+y)^k, \quad \alpha_k \in \mathbb{R}.$$

Now using the transformations (17) and (18) and working in a similar manner to solve \bar{h}_n , computing the terms of degree $n + 3$, we get

$$(20) \quad h_{n-1} = \sum_{k=0}^{n-1} \beta_k (x+y)^k, \quad \beta_k \in \mathbb{R}.$$

Now, using the transformations (17) and (18) and working in a similar manner to solve \bar{h}_{n-2} , computing the terms of degree $n + 2$, we get

$$(21) \quad \frac{(u-w)^2}{4} \frac{\partial \bar{h}_{n-2}}{\partial w} = (a(u-w)^2 + 2bw^2) \left(e^{2w} v^{1/\sqrt{b}} \right)^{2/(w-u)} e^{-2\sqrt{2a/b} \arctan\left(\frac{\sqrt{2bw}}{\sqrt{a(u-w)}}\right)} \sum_{k=1}^n k \alpha_k u^{k-1}.$$

When $u = 0$, if we denote by $\tilde{h}_{n-2} = \tilde{h}_{n-2}(v, w)$ the restriction of \bar{h}_{n-2} on $u = 0$ we have

$$\begin{aligned} \tilde{h}_{n-2} &= 4(a+2b)\alpha_1 e^{-2\sqrt{2a/b} \arctan(-2\sqrt{b/a})} \left(\left(e^{2w} v^{1/\sqrt{b}} \right)^{2/w} w \right. \\ &\quad \left. + 2e^4 Ei\left(-4 + \frac{2\log(e^{2w} v^{1/\sqrt{b}})}{w}\right) \left(2w - \log(e^{2w} v^{1/\sqrt{b}}) \right) + c_{n-2}(v), \right. \end{aligned}$$

where again $Ei(z)$ is the exponential integral function. Going back to the variables x, y and z_1 if we denote by \hat{h}_{n-2} the function \tilde{h}_{n-2} in the variables (x, y, z_1) we get

$$\begin{aligned} \hat{h}_{n-2} &= 4(a+2b)\alpha_1 e^{-2\sqrt{2a/b} \arctan(-2\sqrt{b/a})} \left(y \left(\frac{ax^2 + 2by^2}{z_1} \right)^{2/(xy)} e^{-\frac{2\sqrt{2a}}{\sqrt{bxy}} \arctan\left(\frac{\sqrt{2by}}{\sqrt{ax}}\right)} \right. \\ &\quad \left. + 2e^4 Ei\left(-4 + \frac{2y}{x} \left(\log\left(\frac{ax^2 + 2by^2}{z_1}\right) - \sqrt{\frac{2a}{b}} \arctan\left(\frac{\sqrt{2by}}{\sqrt{ax}}\right)\right) \right) \right. \\ &\quad \left(2y - \frac{1}{x} \log\left(\frac{ax^2 + 2by^2}{z_1}\right) - \sqrt{\frac{2a}{b}} \arctan\left(\frac{\sqrt{2by}}{\sqrt{ax}}\right) \right) \\ &\quad \left. + c_{n-2} \left(\frac{(ax^2 + 2by^2)^{\sqrt{bx}}}{z_1^{\sqrt{bx}} e^{2y\sqrt{b} + \sqrt{2ax} \arctan\left(\frac{\sqrt{2by}}{\sqrt{ax}}\right)}} \right) \right). \end{aligned}$$

Taking into account that \hat{h}_{n-2} must be a polynomial, we must have $c_{n-2} = 0$ and $\alpha_1 = 0$ (because $a + 2b \neq 0$). Therefore, $h_{n-2} = (x+y)g_{n-2}$ for some polynomial g_{n-2} . Going to the variables u, v, w we have that $\bar{h}_{n-2} = u\bar{g}_{n-2}$ where, after simplifying by u , \bar{g}_{n-2} satisfies

$$\frac{(u-w)^2}{4} \frac{\partial \bar{g}_{n-2}}{\partial w} = (a(u-w)^2 + 2bw^2) \left(e^{2w} v^{1/\sqrt{b}} \right)^{2/(w-u)} e^{-2\sqrt{2a/b} \arctan\left(\frac{\sqrt{2bw}}{\sqrt{a(u-w)}}\right)} \sum_{k=2}^n k \alpha_k u^{k-2}.$$

Now proceeding in the same way as we did for h_{n-2} we obtain that $\alpha_2 = 0$ (compare the above equation with (21)). Proceeding inductively we obtain that $\alpha_k = 0$ for $k = 1, \dots, n$. Then $h_n = \alpha_0$, which implies that $h = \alpha_0/z_1^m$ for some nonnegative integer m . Then from (8) we get (16), i.e., $m\alpha_0 = 0$ and consequently h is a constant, in contradiction with the fact that h is a first integral. This concludes the proof of the theorem. \square

Now we characterize the existence of Darboux polynomials.

Theorem 10. *System (1) with $ab \neq 0$ has no Darboux polynomial with non-zero cofactor.*

Proof. Let f be a Darboux polynomial with non-zero cofactor K . Then K has cofactor of degree at most 2. We write it as

$$K = a_0 + a_1x + a_2y + a_3p_x + a_4p_y + a_5x^2 + a_6xy + a_7xp_x + a_8xp_y + a_9y^2 + a_{10}yp_x \\ + a_{11}yp_y + a_{12}p_x^2 + a_{13}p_xp_y + a_{14}p_y^2.$$

Note that system (1) is invariant under the transformation $\sigma: \mathbb{C}^4 \rightarrow \mathbb{C}^4$ with

$$\sigma(x) = -x, \sigma(y) = -y, \sigma(p_x) = -p_x, \sigma(p_y) = -p_y.$$

If f is a Darboux polynomial of system (1) then $g = f \cdot \sigma(f)$ is a Darboux polynomial of system (1) invariant by σ and with cofactor

$$K_1 = K + \sigma(K) = 2(a_0 + a_5x^2 + a_6xy + a_7xp_x + a_8xp_y + a_9y^2 + a_{10}yp_x + a_{11}yp_y \\ + a_{12}p_x^2 + a_{13}p_xp_y + a_{14}p_y^2).$$

Let f be a Darboux polynomial of system (1) with non-zero cofactor K . If f is invariant by σ then $K = K_1$ and if f is not invariant by σ then we consider $g = f \cdot \sigma(f)$ a new Darboux polynomial of system (1) invariant by σ and with cofactor K_1 . We have

$$(22) \quad p_x \frac{\partial g}{\partial x} - p_y \frac{\partial g}{\partial y} + x(1 - ax^2 - by^2) \frac{\partial g}{\partial p_x} - y(1 + bx^2) \frac{\partial g}{\partial p_y} = K_1g.$$

We write g as a polynomial in the variable p_x as

$$g = \sum_{i=0}^k g_i(x, y, p_y) p_x^i,$$

where each g_i is a polynomial in the variables x, y, p_y . Without loss of generality we can assume that $g_k(x, y, p_y) \neq 0$. Then computing the coefficient of p_x^{k+2} in (22) we get

$$2a_{12}g_k = 0 \quad \text{which yields} \quad a_{12} = 0.$$

Now computing the coefficient of p_x^{k+1} in (22) we obtain

$$-\frac{\partial g_k}{\partial x} = 2(a_7x + a_{10}y + a_{13}p_y)g_k.$$

Solving it we get

$$g_k = K_k(y, p_y) e^{a_7x^2 + 2a_{10}yx + 2a_{13}p_yx}.$$

Since g_k must be a polynomial we get $a_7 = a_{10} = a_{13} = 0$.

Now we write g as a polynomial in the variable p_y as

$$g = \sum_{i=0}^m \bar{g}_i(x, y, p_x) p_y^i,$$

where each \bar{g}_i is a polynomial in the variables x, y, p_x . Without loss of generality we can assume that $\bar{g}_m(x, y, p_x) \neq 0$. Then computing the coefficient of p_y^{m+2} in (22) we get

$$2a_{14}\bar{g}_m = 0 \quad \text{which yields} \quad a_{14} = 0.$$

Now computing the coefficient of p_y^{m+1} in (22) we obtain

$$\frac{\partial \bar{g}_m}{\partial y} = -2(a_8x + a_{11}y)\bar{g}_m.$$

Solving it we get

$$\bar{g}_m = K_n(x, p_x) e^{2a_8xy + a_{11}y^2}.$$

Since \bar{g}_m must be a polynomial we get $a_8 = a_{11} = 0$. Hence, $K_1 = 2(a_0 + a_5x^2 + a_6xy + a_9y^2)$. Then

$$-p_x \frac{\partial g}{\partial x} + p_y \frac{\partial g}{\partial y} + x(1 - ax^2 - by^2) \frac{\partial g}{\partial p_x} - y(1 + bx^2) \frac{\partial g}{\partial p_y} = 2(a_0 + a_5x^2 + a_6xy + a_9y^2)g.$$

Now we write $g = \sum_{j=0}^{\ell} g_j(x, y, p_x, p_y)$ is written as a sum of homogeneous polynomials of degree j . Then computing the terms of degree $\ell + 2$ we get

$$-x \left((ax^2 + by^2) \frac{\partial g_{\ell}}{\partial p_x} + bxy \frac{\partial g_{\ell}}{\partial p_y} \right) = 2(a_5x^2 + a_6xy + a_9y^2)g_{\ell}.$$

Solving it we get

$$g_{\ell} = K \left(x, y, \frac{ap_yx^2 - bp_xxy + bp_yy^2}{ax^2 + by^2} \right) e^{-\frac{2px(a_5x^2 + a_6xy + a_9y^2)}{bxy^2}}.$$

Since it must be a polynomial we get $a_5 = a_6 = a_9 = 0$. Then, $K_1 = 2a_0$.

Now proceeding as in the proof of Theorem 9 introducing the change of variables (4) and the restriction $H_0 = 0$ (see (7)) we get that $g(x, y, p_x, p_y) = h(x, y, z_1)$ with

$$(23) \quad \left(-\frac{1}{2}z_1^2 - \frac{1}{2}(bx^2y^2 + y^2 - x^2 + \frac{a}{2}x^4) \right) \frac{\partial h}{\partial x} + \left(-\frac{1}{2}z_1^2 + \frac{1}{2}(bx^2y^2 + y^2 - x^2 + \frac{a}{2}x^4) \right) \frac{\partial h}{\partial y} \\ + z_1(x + y + x(-ax^2 + by(x - y))) \frac{\partial h}{\partial z_1} = 2a_0z_1h.$$

We write h as in (9). Now we consider two cases.

Case 1: $a = -2b$. In this case, we get h_n and h_{n-1} as in (14) and (15). Now, using the transformations (12) and (13) and working in a similar manner to solve \bar{h}_{n-2} , computing the terms of degree $n + 2$ in (23), we get

$$\frac{b}{2}u(u-w)^2 \frac{\partial \bar{h}_{n-2}}{\partial w} = (2w-u)^3 e^{4w/(u-w)} v^2 \sum_{k=1}^n k \alpha_k u^{k-1} \\ + 2a_0(2w-u) e^{2w/(u-w)} v \sum_{k=0}^n \alpha_k u^k.$$

Integrating this equation and going back to the variables x, y and z_1 we get that

$$\bar{h}_{n-2} = -\frac{1}{2b(x+y)} e^{-2+2y/x} \frac{(y-x)^2}{e^{2y/x} z_1} \left(e^2 \left(-4a_0 \sum_{k=0}^n \alpha_k u^k - e^{2y/x} (33(x+y)^2 \right. \right. \\ \left. \left. - 48(x+y)y + 16y^2) \frac{(y-x)^2}{e^{2y/x} z_1} \sum_{k=1}^n k \alpha_k u^{k-1} \right) \right. \\ \left. + 8e^{-2(x+y)/x} \left(2a_0 \sum_{k=0}^n \alpha_k u^k Ei \left(\frac{2(x+y)}{x} \right) \right. \right. \\ \left. \left. + 11 \sum_{k=1}^n k \alpha_k u^{k-1} (x+y)^2 \frac{(y-x)^2}{e^{2y/x} z_1} Ei \left(\frac{4(x+y)}{x} \right) \right) \right) + k_{n-2} \left(x + y, \frac{(y-x)^2}{e^{2y/x} z_1} \right).$$

Since the numerator of h_{n-2} must be a polynomial of degree $n - 2$ we have

$$\sum_{k=1}^n k\alpha_k(x+y)^{k-1} = 0, \quad \text{and} \quad a_0 \sum_{k=0}^n \alpha_k(x+y)^k = 0.$$

This implies that $\alpha_k = 0$ for $k = 1, \dots, n$ and $a_0\alpha_0 = 0$.

Case 2: $a \neq -2b$. In this case, we get h_n and h_{n-1} as in (14) and (15). Now, using the transformations (17) and (18) and working in a similar manner to solve \bar{h}_{n-2} , computing the terms of degree $n + 2$ in (23), we get

$$\begin{aligned} \frac{(u-w)^2}{4} \frac{\partial \bar{h}_{n-2}}{\partial w} &= (a(u-w)^2 + 2bw^2) \left(e^{2wv^{1/\sqrt{b}}} \right)^{2/(w-u)} e^{-2\sqrt{2a/b} \arctan\left(\frac{\sqrt{2bw}}{\sqrt{a(u-w)}}\right)} \sum_{k=1}^n k\alpha_k u^{k-1} \\ &\quad + 2a_0 \left(e^{2wv^{1/\sqrt{b}}} \right)^{1/(w-u)} e^{-\sqrt{2a/b} \arctan\left(\frac{\sqrt{2bw}}{\sqrt{a(u-w)}}\right)} \sum_{k=0}^n \alpha_k u^k. \end{aligned}$$

When $u = 0$ (denoting by \tilde{h}_{n-2} the restriction of \bar{h}_{n-2} to $u = 0$, we get

$$\begin{aligned} \frac{w^2}{4} \frac{\partial \tilde{h}_{n-2}}{\partial w} &= (a + 2b)\alpha_1 w^2 \left(e^{2wv^{1/\sqrt{b}}} \right)^{2/w} e^{-2\sqrt{2a/b} \arctan\left(\frac{\sqrt{2b}}{\sqrt{a}}\right)} \\ &\quad + 2a_0\alpha_0 \left(e^{2wv^{1/\sqrt{b}}} \right)^{1/w} e^{-\sqrt{2a/b} \arctan\left(\frac{\sqrt{2b}}{\sqrt{a}}\right)}. \end{aligned}$$

Integrating this equation, and going back to the variables x, y and z_1 (denoting by \hat{h}_{n-2} the function \tilde{h}_{n-2} in the variables x, y, z_1) we get that

$$\begin{aligned} \hat{h}_{n-2} &= 4e^{-\sqrt{2a/b} \arctan\left(\frac{\sqrt{2b}}{\sqrt{a}}\right)} \left(2\alpha_1(a+2b)e^4 Ei\left(\frac{2}{y} \log\left(\frac{z_1^x}{(a+2b)y^2} e^{-\sqrt{\frac{2a}{b}} \arctan\left(\frac{\sqrt{2by}}{\sqrt{ax}}\right)}\right)\right) - 4 \right) \\ &\quad \left(2y - \log\left(\frac{z_1^x}{(a+2b)y^2} e^{-\sqrt{\frac{2a}{b}} \arctan\left(\frac{\sqrt{2by}}{\sqrt{ax}}\right)}\right) \right) + \left(\frac{z_1^x}{(a+2b)y^2} e^{-\sqrt{\frac{2a}{b}} \arctan\left(\frac{\sqrt{2by}}{\sqrt{ax}}\right)} \right)^{1/y} \\ &\quad \left(\alpha_1(a+2b) \left(\frac{z_1^x}{(a+2b)y^2} e^{-\sqrt{\frac{2a}{b}} \arctan\left(\frac{\sqrt{2by}}{\sqrt{ax}}\right)} \right)^{1/y} y \right. \\ &\quad \left. - \frac{2\alpha_0 a_0}{\log\left(\frac{z_1^x}{(a+2b)y^2} e^{-\sqrt{\frac{2a}{b}} \arctan\left(\frac{\sqrt{2by}}{\sqrt{ax}}\right)}\right) - 2y} \right) + k_{n-2} \left(\frac{(ax^2 + 2by^2)^{\sqrt{bx}}}{z_1^{\sqrt{bx}} e^{2y\sqrt{b} + \sqrt{2ax} \arctan\left(\frac{\sqrt{2by}}{\sqrt{ax}}\right)}} \right). \end{aligned}$$

Since the numerator of \hat{h}_{n-2} must be a polynomial of degree $n - 2$ we have $\alpha_1 = 0$ and $a_0\alpha_0 = 0$. Proceeding inductively, as we did in the proof of Theorem 9 we obtain that $\alpha_k = 0$ for $k = 1, \dots, n$ and $a_0\alpha_0 = 0$.

In summary, in both cases we get that $\alpha_k = 0$ for $k = 1, \dots, n$ and $a_0\alpha_0 = 0$.

If $\alpha_0 = 0$ then $h = 0$, in contradiction that h is a Darboux polynomial. Therefore $a_0 = 0$ and $K_1 = 0$. Thus $g = f \cdot \sigma(f)$ is a polynomial first integral. By Theorem 9 the unique polynomial first integrals are polynomials in the variable H_0 , we must have that f is a polynomial in the variable H_0 which is not possible because f is a Darboux polynomial of system (1) with non-zero cofactor. This concludes the proof of the theorem. \square

Theorem 11. *The unique rational first integrals of system (1) with $b \neq 0$ are rational functions in the variable H_0 .*

Proof. It follows directly from Corollary 7 and Theorems 9 and 10. \square

Now we proceed as in the proof of Theorem 10.

Proof of Theorem 3. It follows from Theorems 6, 9 and 10 and Proposition 5 that in order to have a first integral of Darboux type we must have q exponential factors $E_j = \exp(g_j/h_j(H_0))$ with cofactors L_j such that $\sum_{j=1}^q \mu_j L_j = 0$. Let $G = \sum_{j=1}^q \mu_j g_j/h_j(H_0)$, then $E = \exp(G)$ is an exponential factor of system (1) with cofactor $L = \sum_{j=1}^q \mu_j L_j$ (see Lemma 8) and G satisfies that $\mathcal{X}G = 0$, that is, G must be a rational first integral of system (1). By Theorem 11 it must be a rational function in the variable H_0 . \square

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