

## HOLOMORPHIC NONINTEGRABILITY OF MAGNETIC FIELDS

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**ABSTRACT.** We show that a magnetic field created by a simple planar configuration of three rectilinear wires may not be holomorphically integrable when considered as a vector field in  $\mathbb{C}^3$ . In particular the method of the proof gives an easy way of showing that the corresponding real vector field does not admit a real polynomial first integral. This is also an alternative way of contradicting the Stefanescu conjecture in the polynomial setting.

### 1. INTRODUCTION

Magnetic fields created by current flows appear in several branches of sciences such as electrical engineering [5], spectroscopy [11], medicine [8]. In order to define a magnetic field mathematically consider a smooth curve  $L \subset \mathbb{R}^3$ , parameterized by the map  $l : I \ni \tau \rightarrow l(\tau) \in \mathbb{R}^3$ , where  $I \subset \mathbb{R}$  is an interval,  $L$  represents the electric wire and  $J$  is the current intensity associated with it. Using the *Biot–Savart law* [7] we can compute the magnetic field  $\mathbf{B}$  generated by a steady current associated with a *current distribution*  $(L, J)$  as follows

$$(1) \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0 J}{4\pi} \int_I \frac{l'(\tau) \times (\mathbf{r} - l(\tau))}{|\mathbf{r} - l(\tau)|^3} d\tau,$$

where  $\mu_0$  is a magnetic constant which is the value of the magnetic permeability in a classical vacuum,  $l'(\tau) = dl/d\tau$ ,  $|\cdot|$  represents the Euclidean norm in  $\mathbb{R}^3$  and  $\times$  represents the vector product. A magnetic field  $\mathbf{B}$  created by a configuration  $(L_1, J_1), \dots, (L_n, J_1)$  is obtained via linear superposition, that is  $\mathbf{B} = \mathbf{B}_1 + \dots + \mathbf{B}_n$ , where each  $\mathbf{B}_i$  is obtained from the Biot–Savart law (1). Consequently the resulting vector field  $\mathbf{B}$  is defined everywhere in  $\mathbb{R}^3 \setminus (\bigcup_{i=1}^n L_i)$ .

In this work we are concerned with the integrability of magnetic fields, i.e. the existence of a function which is constant on the magnetic lines. It is known that for certain configurations of wires with the constant current, the resulting magnetic field is integrable. For example the Biot–Savart magnetic field created by a straight line wire perpendicular to the  $z = 0$  plane, has two independent polynomial first integrals:  $F_1(x, y, z) = z$  and  $F_2(x, y, z) = x^2 + y^2$ . Also, a magnetic field created by two rectilinear wires, admits at least one polynomial first integral. Similar observations together with some computations motivated S. Stefanescu to state in 1986 the following conjecture ([12], see also [13, 1]): There exists an *algebraic* first integral for any magnetic field originated by a configuration of piecewise rectilinear wires.

In other words the conjecture is asking to prove that there is always a *polynomial* or a *rational* first integral for a magnetic field generated by a rectilinear configuration of wires. Numerical simulations suggest that for the nonplanar configuration of rectilinear wires the resulting magnetic field can be very difficult, even chaotic [6, 1]. Therefore one does not expect those cases to be integrable. On the other hand magnetic fields created by planar configuration of wires always possess two

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independent smooth first integrals in a sufficiently small tubular neighborhood of each current line, provided that the tubular neighborhood does not enclose any non-regular point [1]. This is only a *local* result, and does not say anything about the existence of the *global* first integral. Thus, the more precise statement of the Stefanescu conjecture could be:

**Stefanescu's conjecture:** There exists an *algebraic* first integral for any magnetic field originated by a *planar* configuration of piecewise rectilinear wires.

The conjecture was motivated by the belief that the magnetic field induced by rectilinear wires cannot be too complicated [6, 9]. It became clear that even quite simple configurations of wires can lead to a very complicated distribution of magnetic lines, which is an obstacle for the existence of a first integral. Based on this observation, the idea is to consider the *simplest possible* configuration of rectilinear wires that would not admit an integrable magnetic field. A very simple configuration was studied in [1, p.56], where the authors proved that the magnetic field induced by a planar rectilinear configuration of only three wires of right angle type may not have a polynomial first integral, contradicting Stefanescu's conjecture in the polynomial setting.

Our main goal is twofold. On one hand we prove that the simple example of a magnetic field mentioned before induced by three planar rectilinear configuration of wires, thought of as a vector field in  $\mathbb{C}^3$ , does not admit any *holomorphic* first integral. More precisely, we show that it does not admit any first integral whose domain contains points  $(\pm i, y, -1)$  with  $y \in \mathbb{C}$ . In particular our proof gives an alternative and simpler way of proving the polynomial nonintegrability of the real vector field. Furthermore, we illustrate how the old theorem of Poincaré can be used to show analytic (in particular polynomial) nonintegrability of a given analytic system. Thus the method of the proof can be easily extended or applied to other situations.

To be more precise the following theorem is our main result:

**Theorem 1.** *Let  $\mathbf{B}$  be the magnetic field associated with the rectilinear wires on the  $x=0$  plane with a unit current flows given by*

$$\begin{aligned} L_1 &= \{x = 0, y = -1\}, & \text{in the positive } z \text{ direction,} \\ L_2 &= \{x = 0, y = 1\}, & \text{in the negative } z \text{ direction,} \\ L_3 &= \{x = 0, z = 0\}, & \text{in the positive } y \text{ direction.} \end{aligned}$$

*Then if  $\mathbf{B}$  is thought of as a vector field on  $\mathbb{C}^3$ , then it does not admit any holomorphic first integral, defined in the neighborhood of  $(\pm i, y, -1)$ , where  $y \in \mathbb{C}$ .*

The result of [1] as well as our main theorem does not contradict the existence of an algebraic but non-polynomial first integral for a magnetic field induced by planar rectilinear configuration of wires.

## 2. PROOF OF THEOREM 1

Before proving our main result we introduce a criterium on the existence of analytic first integrals in a neighbourhood of a singular point that we shall use. Consider

$$(2) \quad \dot{\mathbf{x}} = f(\mathbf{x}), \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{C}^n,$$

where  $f(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_n(\mathbf{x}))$  is an  $n$ -dimensional vector-function such that  $f(\mathbf{0}) = \mathbf{0}$ . We say that a non-constant analytic function  $H: U \rightarrow \mathbb{C}$ , where  $U$  is an

open connected subset of  $\mathbb{C}^n$ , is the *analytic first integral* of (2) if

$$\sum_{i=1}^n f_i(\mathbf{x}) \frac{\partial H}{\partial x_i} \equiv 0.$$

We shall need the following well known result due to Poincaré [10] about the analytic integrability (for a proof see for instance [3]).

**Theorem 2** (Poincaré). *If the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$  of the Jacobian matrix  $\partial f / \partial \mathbf{x}(\mathbf{0})$  of (2) at  $\mathbf{x} = 0$  do not satisfy any resonance condition of the form*

$$\sum_{i=1}^n k_i \lambda_i = 0, \quad k_i \in \mathbb{Z}^+, \quad \sum_{i=1}^n k_i \neq 0,$$

*then system (2) does not have any analytic first integrals in a neighborhood of  $\mathbf{x} = 0$ .*

*Proof of Theorem 1.* Following [1] we shall consider a magnetic field created by  $L_1$ ,  $L_2$  and  $L_3$ , with the unit current flows in the positive  $z$  direction for  $L_1$ , in the negative  $z$  direction in  $L_2$  and positive  $y$  direction in  $L_3$ . Then the Biot-Savart law gives us  $\mathbf{B} = \mu_0/2\pi(B_x, B_y, B_z)$ , where

$$\begin{aligned} B_x &= \frac{-(y+1)}{x^2 + (y+1)^2} + \frac{y-1}{x^2 + (y-1)^2} + \frac{z}{x^2 + z^2}, \\ (3) \quad B_y &= \frac{x}{x^2 + (y+1)^2} - \frac{x}{x^2 + (y-1)^2}, \\ B_z &= \frac{-x}{x^2 + z^2}. \end{aligned}$$

Doing a simple rescaling of the time in the vector field  $\mathbf{B}$ , which does not affect the integrability, and writing this vector field as a system of differential equations we get

$$\begin{aligned} (4) \quad \dot{x} &= 2(x^2 + z^2)(y^2 - x^2 - 1) + z[x^2 + (y+1)^2][x^2 + (y-1)^2], \\ \dot{y} &= -4xy(x^2 + z^2), \\ \dot{z} &= -x[x^2 + (y+1)^2][x^2 + (y-1)^2]. \end{aligned}$$

We shall prove that system (4) does not admit an analytic first integral.

We note that  $y = 0$  is an invariant plane of system (4). If  $h = h(x, y, z)$  is an holomorphic first integral of system (4) it can be written in the form  $h(x, y, z) = h_0(x, z) + yg(x, y, z)$  where  $h_0$  and  $g$  are holomorphic functions in their variables. Furthermore, without loss of generality, we can assume that  $h_0$  is either zero or a holomorphic first integral of system (4) restricted to  $y = 0$ .

Consider the restriction of (4) to  $y = 0$ , that is

$$\dot{x} = -2(x^2 + z^2)(x^2 + 1) + z(x^2 + 1)^2, \quad \dot{z} = -x(x^2 + 1)^2.$$

Rescaling the time variable by  $(x^2 + 1)$  we get

$$(5) \quad \dot{x} = -2(x^2 + z^2) + z(x^2 + 1), \quad \dot{z} = -x(x^2 + 1).$$

This system has six singular points:

$$(0, 0), \quad (0, 1/2), \quad (-i, -1), \quad (i, -1), \quad (-i, 1), \quad (i, 1).$$

The eigenvalues of the Jacobian matrix of system (5) at  $(-i, -1)$  are  $\lambda_1 = 2i$  and  $\lambda_2 = 4i$ . Then we have that

$$k_1 \lambda_1 + k_2 \lambda_2 = 2i(k_1 + 2k_2) \neq 0,$$

for  $(k_1, k_2)$  being integers such that  $k_1 + k_2 > 0$ . Therefore, by Theorem 2 system (5) has no formal series in a neighborhood of  $(-i, -1)$ . Working in a similar manner, using that the eigenvalues of the Jacobian matrix of system (5) at  $(i, -1)$  are  $\lambda_1 =$

$-2i$  and  $\lambda_2 = -4i$  and Theorem 2, we also get that system (5) has no formal series in a neighborhood of  $(i, -1)$ . Therefore, system (5) has no holomorphic first integrals whose domain contains  $(\pm i, -1)$ .

Hence if  $h$  is a holomorphic first integral of system (4) whose domain contains either  $(-i, y, -1)$  or  $(-i, y, 1)$  it must be written in the form  $h(x, y, z) = yg(x, y, z)$  where  $g$  is a holomorphic function. We write  $g$  as a formal series in the variable  $y$  as

$$g(x, y, z) = \sum_{j \geq 0} g_j(x, z)y^j,$$

where each  $g_j$  is a formal series in the variables  $(x, z)$ . Then substituting  $h$  in (4) and simplifying by  $y$  we get that  $g$  satisfies the following relation

$$(6) \quad A(x, y, z) \frac{\partial g}{\partial x} - 4xy(x^2 + z^2) \frac{\partial g}{\partial y} - B(x, y, z) \frac{\partial g}{\partial z} - 4x(x^2 + z^2)g = 0,$$

where

$$\begin{aligned} A(x, y, z) &= 2(x^2 + z^2)(y^2 - x^2 - 1) + z[x^2 + (y + 1)^2](x^2 + (y - 1)^2), \\ B(x, y, z) &= x[x^2 + (y + 1)^2][x^2 + (y - 1)^2]. \end{aligned}$$

We will show that  $g = 0$ . We proceed by contradiction. We assume that  $g \neq 0$  and we consider two complementary cases:

*Case 1:  $g$  is not divisible by  $y$ .* In this case we have that  $g_0 = g_0(x, z) \neq 0$  and  $g_0$  satisfies (6) restricted to  $y = 0$ , that is

$$(7) \quad (x^2 + 1)[z(x^2 + 1) - 2(x^2 + z^2)] \frac{\partial g_0}{\partial x} - x(x^2 + 1)^2 \frac{\partial g_0}{\partial z} - 4x(x^2 + z^2)g_0 = 0.$$

It follows from (7) that  $g_0$  must be divisible by  $(x^2 + 1)$ . Therefore, it is of the form  $g_0 = (x^2 + 1)^m f$ , with  $m \geq 1$  and  $f = f(x, z)$  is a formal series in its variables such that it is not divisible by  $(x^2 + 1)$ . Then, introducing  $g_0$  in (7) and simplifying by  $(x^2 + 1)^m$ , we get that  $f$  satisfies the following equality

$$(8) \quad -(x^2 + 1)[2(x^2 + z^2) - z(x^2 + 1)] \frac{\partial f}{\partial x} - x(x^2 + 1)^2 \frac{\partial f}{\partial z} - E(x, z)f = 0,$$

where

$$E(x, z) = 2mx(1 + x^2)z - 4(1 + m)x(x^2 + z^2).$$

From (8) we deduce that  $f$  must be divisible by  $(x^2 + 1)$  and we arrive to a contradiction.

*Case 2:  $g$  is divisible by  $y$ .* In this case we have that  $g_0 = \dots = g_{m-1} = 0$  and

$$g = \sum_{j \geq m} y^j g_j(x, z) = y^m \sum_{j \geq 0} g_{m+j} y^j = y^m G, \quad m \geq 1,$$

where

$$G = G(x, y, z) = \sum_{j \geq 0} g_{m+j} y^j.$$

We note that  $g_m = G(x, 0, z)$ . Then imposing that  $g$  satisfies (6) and simplifying by  $y^m$  we obtain

$$(9) \quad A(x, y, z) \frac{\partial G}{\partial x} - 4xy(x^2 + z^2) \frac{\partial G}{\partial y} - B(x, y, z) \frac{\partial G}{\partial z} - 4mx(x^2 + z^2)G = 0.$$

Restricting (9) to  $y = 0$  we get that  $g_m$  satisfies

$$(10) \quad (x^2 + 1)[z(x^2 + 1) - 2(x^2 + z^2)] \frac{\partial g_m}{\partial x} - x(x^2 + 1)^2 \frac{\partial g_m}{\partial z} - 4mx(x^2 + z^2)g_m = 0.$$

It follows from (10) that  $g_m$  must be divisible by  $(x^2 + 1)$ . Therefore, it is of the form  $g_m = (x^2 + 1)^l f$ , with  $l \geq 1$  and  $f = f(x, z)$  is a formal series in its variables

such that it is not divisible by  $(x^2 + 1)$ . Introducing  $g_m$  in (10) and simplifying by  $(x^2 + 1)^l$  we get that  $f$  satisfies the following equality

$$(11) \quad -(x^2 + 1)[2(x^2 + z^2) - z(x^2 + 1)]\frac{\partial f}{\partial x} - x(x^2 + 1)^2\frac{\partial f}{\partial z} + F(x, z)f = 0,$$

where

$$F(x, z) = 2mx(1 + x^2)z - 4(1 + m + l)x(x^2 + z^2).$$

Then from (11) we deduce that  $f$  must be divisible by  $(x^2 + 1)$  and we have a contradiction. This concludes the proof of the theorem.  $\square$

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