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LIMIT CYCLES FOR A CLASS OF DISCONTINUOUS GENERALIZED LIENARD POLYNOMIAL DIFFERENTIAL EQUATIONS

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Abstract. We divide $\mathbb{R}^2$ in l sectors $S_1, ..., S_l$, with $l > 1$ even. We define in $\mathbb{R}^2$ a discontinuous differential system such that in each sector S_k, for $k = 1, ..., l$, is defined a smooth generalized Lienard polynomial differential equation $\ddot{x} + f_i(x)\dot{x} + g_i(x) = 0$, $i = 1, 2$ alternatively, where f_i and g_i are polynomials of degree $n-1$ and m respectively. We apply the averaging theory of first order for discontinuous differential systems to this class of non-smooth generalized Lienard polynomial differential systems and we show that for any n and m there are such non-smooth Lienard polynomial equations having at least $\max\{n, m\}$ limit cycles. Note that this number is independent of l.

Roughly speaking this result shows that the non-smooth classical ($m = 1$) Lienard polynomial differential systems can have at least the double number of limit cycles than the smooth ones, and that the non-smooth generalized Lienard polynomial differential systems can have at least one more limit cycle than the smooth ones. Of course, these comparisons are done with the present known results.

1. Introduction

A large number of problems from mechanics and electrical engineering, theory of automatic control, economy, impact systems among others cannot be described with smooth dynamical systems. This fact has motivated many researchers to the study of qualitative aspects of the phase space of non-smooth dynamical systems.

One of the main problems in the qualitative theory of real planar continuous and discontinuous differential systems is the determination of their limit cycles. The non-existence, existence, uniqueness and other properties of limit cycles have been studied extensively by mathematicians and physicists, and more recently also by chemists, biologists, economists, etc (see for instance the books [1, 3, 22]). This problem restricted to continuous planar polynomial differential equations is the well known 16th Hilbert's problem [8]. Since this Hilbert's problem turned out a strongly difficult one Smale [21] particularized it to Lienard polynomial differential equations in his list of problems for the present century.

The classical Lienard polynomial differential equations

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

where $f(x)$ and $g(x) = x$ goes back to [9]. The dot denotes differentiation with respect to the time t. This second order differential equation (1) can be written as

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the following first order differential system in $\mathbb{R}^2$

$$\begin{aligned} \dot{x} &= \quad y - F(x), \\ \dot{y} &= \quad -g(x), \end{aligned} \qquad \text{where} \quad F(x) = \int_0^x f(s)ds.$$

Many results on the number of limit cycles has been obtained for the continuous generalized polynomial differential equations (1) being $f(x)$ and $g(x)$ polynomials in the variable x of degrees $n-1$ and m respectively. The continuous classical Lienard polynomial differential equations (1) were studied in 1977 by Lins, de Melo and Pugh [10] who stated the following conjecture: *if $f(x)$ has degree $n-1>0$ and $g(x)=x$, then* (1) *has at most $[(n-1)/2]$ limit cycles.* Here $[z]$ denotes the integer part function of $z \in \mathbb{R}$. They also proved the conjecture for $n=2,3$. For $n=4$ this conjecture has been proved in 2012 (see [11]). For $n \geq 7$ Dumortier, Panazzolo and Roussarie proved that this conjecture is not true in [5], they show that these differential equations can have $[(n-1)/2]+1$ limit cycles. Recently De Maesschalck and Dumortier proved in [20] that the classical Lienard equation of degree $n \geq 6$ can have $[(n-1)/2]+2$ limit cycles. The conjecture for $n=5$ is still open.

Results on the number of limit cycles for continuous generalized Lienard polynomial differential equations can be found in [12] where the authors show that there are differential equations (1) having at least $[(n+m-2)/2]$ limit cycles. See also [4, 6, 16, 17, 18, 19, 23].

The objective of this work is to star the study of the number of limit cycles for a kind of discontinuous generalized Lienard polynomial differential systems. Here we shall play with many straight lines of discontinuities through the origin of coordinates and with two different continuous generalized Lienard polynomial differential systems located alternatively in the sectors defined by these straight lines.

A similar work but with only one classical Lienard polynomial differential system and only one straight line of discontinuity was studied in [14] obtaining $[n/2]$ limit cycles, instead of the $[(n-1)/2]$ of the continuous classical Lienard polynomial differential equation obtained in [10].

Now we shall define the discontinuous generalized Lienard polynomial differential system that we will study. We consider the function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$h(x,y) = \prod_{k=0}^{\frac{l}{2}-1} \left(y - \tan\left(\alpha + \frac{2k\pi}{l}\right) x \right),$$

where $l > 1$ even. The set

$$h^{-1}(0) = \bigcup_{k=0}^{\frac{l}{2}-1} \left\{ (x,y) : y = \tan\left(\alpha + \frac{2k\pi}{l}\right) x \right\},$$

divides $\mathbb{R}^2$ into l sectors, $S_1, S_2, ..., S_l$, i.e. $h^{-1}(0)$ is the product of $l/2$ straight lines passing through the origin of coordinates dividing the plane in sectors of angle $2\pi/l$.

In this work we study the maximum number of limit cycles given by the averaging theory of first order, which can bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, perturbed inside the following class of discontinuous Lienard polynomial differential systems:

$$\dot{X} = Z(x, y) = \begin{cases} Y_1(x, y) & \text{if } h(x, y) > 0, \\ Y_2(x, y) & \text{if } h(x, y) < 0, \end{cases} \tag{2}$$

where

$$Y_1(x, y) = \begin{pmatrix} y - \varepsilon F_1(x) \\ -x - \varepsilon g_1(x) \end{pmatrix}, \qquad Y_2(x, y) = \begin{pmatrix} y - \varepsilon F_2(x) \\ -x - \varepsilon g_2(x) \end{pmatrix}, \tag{3}$$

ε is a small parameter, and $F_i(x)$ and $g_i(x)$, for $i = 1, 2$ are polynomials in the variable x of degrees n and m respectively. System (2) can be written using the sign function as

$$\dot{X} = Z(x, y) = G_1(x, y) + \operatorname{sign}(h(x, y))G_2(x, y), \tag{4}$$

where $G_1(x, y) = \dfrac{1}{2}(Y_1(x, y) + Y_2(x, y))$ and $G_2(x, y) = \dfrac{1}{2}(Y_1(x, y) - Y_2(x, y))$.

Our main result is the following one.

Theorem 1. *Assume that for $i = 1, 2$ the polynomials $F_i(x)$ and $g_i(x)$ have degree $n \geq 1$ and $m \geq 1$ respectively, and that $l > 1$ is even. Then for $|\varepsilon|$ sufficiently small there are discontinuous Lienard polynomial differential systems* (2) *having at least* $\max\{n, m\}$ *limit cycles bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$.*

In short, taking into account Theorem 1 we can say roughly speaking that the non-smooth classical Lienard polynomial differential systems can have at least $\max\{n, 1\}$ limit cycles, i.e. roughly speaking the double number of limit cycles than the smooth ones which at least have $[(n-1)/2] + 2$ for $n \geq 6$. Comparing the mentioned result from [12], that smooth generalized Lienard polynomial differential systems have at least $[(n + m - 1)/2]$ limit cycles with Theorem 1, we can say that the non-smooth generalized Lienard polynomial differential systems can have at least one more limit cycle than the smooth ones. Of course all these comparisons are done with the present known results.

2. Averaging theory of first order for discontinuous differential systems

The first-order averaging theory developed for discontinuous differential systems in [13] is presented in this section. It is summarized as follows.

Theorem 2. *We consider the following discontinuous differential system*

$$x'(t) = \varepsilon F(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{5}$$

with

$$F(t, x) = F_1(t, x) + \operatorname{sign}(h(t, x))F_2(t, x),$$
$$R(t, x, \varepsilon) = R_1(t, x, \varepsilon) + \operatorname{sign}(h(t, x))R_2(t, x, \varepsilon),$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ and $h : \mathbb{R} \times D \to \mathbb{R}$ are continuous functions, T–periodic in the variable t and D is an open subset of $\mathbb{R}^n$. We also suppose that h is a C^1 function having 0 as a regular value. Denote by $\mathcal{M} = h^{-1}(0)$, by $\Sigma = \{0\} \times D \nsubseteq \mathcal{M}$, by $\Sigma_0 = \Sigma \backslash \mathcal{M} \neq \varnothing$, and its elements by $z \equiv (0, z) \notin \mathcal{M}$.

Define the averaged function $f : D \to \mathbb{R}^n$ as

$$f(x) = \int_0^T F(t,x)dt.$$

We assume the following three conditions.

(i) *F_1, F_2, R_1, R_2 and h are locally L–Lipschitz with respect to x;*
(ii) *for $a \in \Sigma_0$ with $f(a) = 0$, there exist a neighborhood V of a such that $f(z) \neq 0$ for all $z \in \overline{V} \backslash \{a\}$ and $d_B(f, V, a) \neq 0$, (i.e. the Brouwer degree of f at a is not zero).*
(iii) *If $\partial h/\partial t(t_0, z_0) = 0$ for some $(t_0, z_0) \in \mathcal{M}$, then $\big(\langle \nabla_x h, F_1\rangle^2 - \langle \nabla_x h, F_2\rangle^2\big)(t_0, z_0) > 0$.*

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a T–periodic solution $x(\cdot, \varepsilon)$ of system (5) such that $x(t, \varepsilon) \to a$ as $\varepsilon \to 0$.

Remark 1. *We note that if the function $f(z)$ is C^1 and the Jacobian of f at a is not zero, then $d_B(f, V, a) \neq 0$. For more details on the Brouwer degree see* [2] *and* [15].

3. Proof of Theorem 1

The discontinuous Lienard differential systems (2) in polar coordinates (r, θ) become

$$\begin{aligned}\dot r &= -\varepsilon \left(\cos\theta\ F_i(r\cos\theta) + \sin\theta\ g_i(r\cos\theta)\right),\\ \dot\theta &= -1 + \frac{\varepsilon}{r}\left(\sin\theta\ F_i(r\cos\theta) - \cos\theta\ g_i(r\cos\theta)\right),\end{aligned}$$

with $i = 1$ if $\operatorname{sign}(h(r\cos\theta, r\sin\theta)) > 0$ and $i = 2$ if $\operatorname{sign}(h(r\cos\theta, r\sin\theta)) < 0$. Taking the angle θ as new independent variable the discontinuous differential systems become

$$\dot r = \varepsilon \left(\cos\theta\ F_i(r\cos\theta) + \sin\theta\ g_i(r\cos\theta)\right) + O(\varepsilon^2). \tag{6}$$

The discontinuous differential system (6) is under the assumptions of Theorem 2 taking

$t = \theta,\ \ T = 2\pi,\ \ x = r,\ \ \mathcal{M} = h^{-1}(0) = \displaystyle\bigcup_{k=0}^{\frac{l}{2}-1}\left\{(\theta, r) : \theta = \alpha + \frac{2k\pi}{l}, r > 0\right\}$. So according with Theorem 2 we must study the zeros of the averaged function

$$\begin{aligned} f(r) = \sum_{k=1}^{l}\ \Bigg[&\int_{\alpha+\frac{2(k-1)\pi}{l}}^{\alpha+\frac{(2k-1)\pi}{l}} \left(\cos\theta\ F_1(r\cos\theta) + \sin\theta\ g_1(r\cos\theta)\right) d\theta + \\ &\int_{\alpha+\frac{(2k-1)\pi}{l}}^{\alpha+\frac{2k\pi}{l}} \left(\cos\theta\ F_2(r\cos\theta) + \sin\theta\ g_2(r\cos\theta)\right) d\theta\Bigg]. \end{aligned} \tag{7}$$

If we denote by $F_1(x)=\sum_{i=0}^{n} a_i x^i$, $F_2(x)=\sum_{i=0}^{n} b_i x^i$, $g_1(x)=\sum_{i=0}^{m} c_i x^i$ and $g_2(x)=\sum_{i=0}^{m} d_i x^i$ we have

$$
\begin{aligned}
f(r)=\sum_{k=1}^{l} &\left[\int_{\alpha+\frac{2(k-1)\pi}{l}}^{\alpha+\frac{(2k-1)\pi}{l}}\left(\sum_{i=0}^{n} a_i r^i \cos^{i+1}\theta+\sum_{i=0}^{m} c_i r^i \cos^i\theta\sin\theta\right)d\theta+\right.\\
&\left.\int_{\alpha+\frac{(2k-1)\pi}{l}}^{\alpha+\frac{2k\pi}{l}}\left(\sum_{i=0}^{n} b_i r^i \cos^{i+1}\theta+\sum_{i=0}^{m} d_i r^i \cos^i\theta\sin\theta\right)d\theta\right].
\end{aligned}
$$

In order to calculate the exact expression of $f(r)$ we use the following formulae [2.513 3] and [2.513 4] proved in [7]:

$$
\begin{aligned}
\int \cos^{2m}\theta\, d\theta &= \frac{1}{2^{2m}}\binom{2m}{m}\theta+\frac{1}{2^{2m-1}}\sum_{j=0}^{m-1}\binom{2m}{j}\frac{\sin(2m-2j)\theta}{2m-2j},\\
\int \cos^{2m+1}\theta\, d\theta &= \frac{1}{2^{2m}}\sum_{j=0}^{m}\binom{2m+1}{j}\frac{\sin(2m-2j+1)\theta}{2m-2j+1}.
\end{aligned}
$$

Thus we have

$$
f(r)=f_1(r)+f_2(r)+f_3(r)+f_4(r),
$$

where

$$
\begin{aligned}
f_1(r)= &\sum_{k=1}^{l}\int_{\alpha+\frac{2(k-1)\pi}{l}}^{\alpha+\frac{(2k-1)\pi}{l}}\left(\sum_{i=0}^{n} a_i r^i\cos^{i+1}\theta\right)d\theta=\\
&\sum_{k=1}^{l}\left[\sum_{\substack{i=1\\ i\text{ odd}}}^{n} a_i r^i\left[\frac{1}{2^{i+1}}\binom{i+1}{(i+1)/2}\frac{\pi}{l}+\frac{1}{2^i}\sum_{j=0}^{\frac{i-1}{2}}\binom{i+1}{j}\varphi_{i,j,k}\right]+\right.\\
&\left.\sum_{\substack{i=0\\ i\text{ even}}}^{n}\frac{a_i r^i}{2^i}\sum_{j=0}^{\frac{i}{2}}\binom{i+1}{j}\varphi_{i,j,k}\right]=\\
&\sum_{k=1}^{l}\left[\sum_{\substack{i=1\\ i\text{ odd}}}^{n}\frac{a_i r^i}{2^{i+1}}\binom{i+1}{(i+1)/2}\frac{\pi}{l}+\sum_{i=0}^{n}\frac{a_i r^i}{2^i}\sum_{j=0}^{\left[\frac{i}{2}\right]}\binom{i+1}{j}\varphi_{i,j,k}\right],
\end{aligned}
$$

with

$$
\varphi_{i,j,k}=\frac{\sin\left((i-2j+1)\left(\alpha+\frac{(2k-1)\pi}{l}\right)\right)-\sin\left((i-2j+1)\left(\alpha+\frac{2(k-1)\pi}{l}\right)\right)}{i-2j+1}\neq 0;
$$

$$f_2(r) = \sum_{k=1}^{l} \int_{\alpha+\frac{2(k-1)\pi}{l}}^{\alpha+\frac{(2k-1)\pi}{l}} \left(\sum_{i=0}^{m} c_i r^i \cos^i\theta \sin\theta \right) d\theta = -\sum_{k=1}^{l}\sum_{i=0}^{m} \frac{c_i r^i}{i+1}\phi_{i,k},$$

with

$$\phi_{i,k} = \cos^{i+1}\left(\alpha + \frac{(2k-1)\pi}{l}\right) - \cos^{i+1}\left(\alpha + \frac{2(k-1)\pi}{l}\right) \neq 0;$$

$$f_3(r) = \sum_{k=1}^{l} \int_{\alpha+\frac{2(k-1)\pi}{l}}^{\alpha+\frac{(2k-1)\pi}{l}} \left(\sum_{i=0}^{n} b_i r^i \cos^{i+1}\theta \right) d\theta =$$

$$\sum_{k=1}^{l} \left[\sum_{\substack{i=1 \\ i \text{ odd}}}^{n} \frac{b_i r^i}{2^{i+1}} \binom{i+1}{(i+1)/2} \frac{\pi}{l} + \sum_{i=0}^{n} \frac{b_i r^i}{2^i} \sum_{j=0}^{\left[\frac{i}{2}\right]} \binom{i+1}{j} \psi_{i,j,k} \right],$$

with

$$\psi_{i,j,k} = \frac{\sin\left((i-2j+1)\left(\alpha+\frac{2k\pi}{l}\right)\right) - \sin\left((i-2j+1)\left(\alpha+\frac{(2k-1)\pi}{l}\right)\right)}{i-2j+1} \neq 0;$$

$$f_4(r) = \sum_{k=1}^{l} \int_{\alpha+\frac{(2k-1)\pi}{l}}^{\alpha+\frac{2k\pi}{l}} \left(\sum_{i=0}^{m} d_i r^i \cos^i\theta \sin\theta \right) d\theta = -\sum_{k=1}^{l}\sum_{i=0}^{m} \frac{d_i r^i}{i+1}\zeta_{i,k},$$

with

$$\zeta_{i,k} = \cos^{i+1}\left(\alpha + \frac{2k\pi}{l}\right) - \cos^{i+1}\left(\alpha + \frac{(2k-1)\pi}{l}\right) \neq 0.$$

Thus

$$f(r) = \sum_{k=1}^{l} \left[\sum_{\substack{i=1 \\ i \text{ odd}}}^{n} \frac{r^i}{2^{i+1}} \binom{i+1}{(i+1)/2} \frac{\pi}{l}(a_i+b_i) + \right.$$

$$\left. \sum_{i=0}^{n} \frac{r^i}{2^i} \sum_{j=0}^{\left[\frac{i}{2}\right]} \binom{i+1}{j} (a_i\varphi_{i,j,k} + b_i\psi_{i,j,k}) - \sum_{i=0}^{m} \frac{r^i}{i+1}(c_i\phi_{i,k} + d_i\zeta_{i,k}) \right].$$

The function $f(r)$ is a polynomial in the variable r of degree $\max\{n,m\}$ therefore $f(r)$ has at most $\max\{n,m\}$ positive roots. If r^* is a simple zero of $f(r)$, i.e. $f(r^*) = 0$ and $\left.\dfrac{df}{dr}\right|_{r=r^*} \neq 0$, then the Brouwer degree $d_B(f, V, r^*) \neq 0$ being V a convenient open neighborhood of r^* (see Remark 1). We can choose the coefficients a_i, b_i, c_i e d_i in such a way that $f(r)$ has exactly $\max\{n,m\}$ simple positive roots. Hence Theorem 1 is proved.

4. Examples

In this section we illustrate Theorem 1 by studying the existence of 2π-periodic solutions for two non-smooth Lienard polynomial differential systems.

Example 1. We consider $l = 2$ and $\alpha = 0$. Thus the function $h : \mathbb{R}^2 \to \mathbb{R}$ is defined by $h(x, y) = y$ and $h^{-1}(0) = \{(x, y) \in \mathbb{R}^2 : y = 0\}$. System (2) becomes

$$\dot{X} = Z(x,y) = \begin{cases} Y_1(x,y) & \text{if } y > 0, \\ Y_2(x,y) & \text{if } y < 0, \end{cases} \tag{8}$$

where $F_1 = 1 + x + x^2 + \left(\frac{1}{9\pi} - 1\right) x^3$, $F_2 = 1 + \left(\frac{11}{12\pi} - 1\right) x + x^2 + x^3$, $g_1 = \frac{7}{8} + x + \frac{5}{8}x^2$ and $g_2 = 1 + x + x^2$. Thus we have

$$Y_1(x,y) = \begin{pmatrix} y - \varepsilon\left(1 + x + x^2 + \left(\frac{1}{9\pi} - 1\right) x^3\right) \\ -x - \varepsilon\left(\frac{7}{8} + x + \frac{5}{8}x^2\right) \end{pmatrix},$$

$$Y_2(x,y) = \begin{pmatrix} y - \varepsilon\left(1 + \left(\frac{11}{12\pi} - 1\right) x + x^2 + x^3\right) \\ -x - \varepsilon\left(1 + x + x^2\right) \end{pmatrix}.$$

The averaging function (7) is given by

$$\begin{aligned} f(r) = & \int_0^{\pi} \left(\cos\theta\ F_1(r\cos\theta) + \sin\theta\ g_1(r\cos\theta)\right) d\theta + \\ & \int_{\pi}^{2\pi} \left(\cos\theta\ F_2(r\cos\theta) + \sin\theta\ g_2(r\cos\theta)\right) d\theta = \\ & -6 + 11r - 6r^2 + r^3. \end{aligned}$$

The zeros of $f(r)$ are $r = 1$, $r = 2$ and $r = 3$, and they are simple. Hence, by Theorem 1 it follows that for $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (8) has three periodic solutions.

Example 2. We consider $l = 4$ and $\alpha = \pi/4$. Thus the function $h : \mathbb{R}^2 \to \mathbb{R}$ is defined by $h(x, y) = (y-x)(y+x)$ and $h^{-1}(0) = \{(x, y) : y = x\} \cup \{(x, y) : y = -x\}$. System (2) becomes

$$\dot{X} = Z(x,y) = \begin{cases} Y_1(x,y) & \text{if } (y-x)(y+x) > 0, \\ Y_2(x,y) & \text{if } (y-x)(y+x) < 0, \end{cases} \tag{9}$$

where $F_1 = x^2$, $F_2 = 12\sqrt{2}\pi + \frac{72\sqrt{2}}{5}x^2$, $g_1 = 1+x+x^2+x^3$ and $g_2 = -88\pi x - \frac{32\pi}{3}x^3$. Thus we have

$$Y_1(x,y) = \begin{pmatrix} y - \varepsilon x^2 \\ -x - \varepsilon\left(1 + x + x^2 + x^3\right) \end{pmatrix}$$

$$Y_2(x,y) = \begin{pmatrix} y - \varepsilon\left(12\sqrt{2}\pi + \frac{72\sqrt{2}}{5}x^2\right) \\ -x - \varepsilon\left(-88\pi x - \frac{32\pi}{3}x^3\right) \end{pmatrix}.$$

The averaging function (7) is given by

$$
\begin{aligned}
f(r) = & \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\cos\theta\ F_1(r\cos\theta) + \sin\theta\ g_1(r\cos\theta))\, d\theta + \\
& \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} (\cos\theta\ F_2(r\cos\theta) + \sin\theta\ g_2(r\cos\theta))\, d\theta + \\
& \int_{\frac{5\pi}{4}}^{\frac{7\pi}{4}} (\cos\theta\ F_1(r\cos\theta) + \sin\theta\ g_1(r\cos\theta))\, d\theta + \\
& \int_{\frac{7\pi}{4}}^{\frac{9\pi}{4}} (\cos\theta\ F_2(r\cos\theta) + \sin\theta\ g_2(r\cos\theta))\, d\theta = \\
& -6 + 11r - 6r^2 + r^3.
\end{aligned}
$$

The zeros of $f(r)$ are $r = 1$, $r = 2$ and $r = 3$, and they are simple. Hence, by Theorem 1 it follows that for $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (9) has three periodic solutions.

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