This is a preprint of: "Polynomial and rational first integrals for planar quasi-homogeneous polynomial differential sytems", Jaume Giné, Maite Grau, Jaume Llibre, *Discrete Contin. Dyn. Syst.*, vol. 33(10), 4531–4547, 2013.
DOI: [10.3934/dcds.2013.33.4531]

POLYNOMIAL AND RATIONAL FIRST INTEGRALS FOR PLANAR QUASI–HOMOGENEOUS POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. In this paper we find necessary and sufficient conditions in order that a planar quasi–homogeneous polynomial differential system has a polynomial or a rational first integral. We also prove that any planar quasi–homogeneous polynomial differential system can be transformed into a differential system of the form $\dot{u} = uf(v)$, $\dot{v} = g(v)$ with $f(v)$ and $g(v)$ polynomials, and vice versa.

1. Introduction

The characterization of polynomial or rational integrability of a differential system goes back to Poincaré, see [20, 21, 22] and has attracted the attention of many authors, see for instance [2, 3, 8, 14, 17, 18, 19, 23, 24] and references therein. For quasi–homogeneous polynomial differential systems if we control the polynomial first integrals we are controlling all analytical first integrals of the system, see [15, 17].

We assume that there exists an analytic first integral H for an analytic differential system of the form $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$. The analytic functions H, P and Q can be decomposed in sum of quasi–homogeneous polynomials of the same weight degree, i.e. $H = H_m + H_{m+1} + \ldots$, $P = P_r + P_{r+1} + \ldots$ and $Q = Q_r + Q_{r+1} + \ldots$. Then, the quasi–homogeneous polynomial of the lowest weight degree H_m must be a first integral of the quasi–homogeneous differential system $\dot{x} = P_r(x, y)$, $\dot{y} = Q_r(x, y)$, see [12, 16]. So the study of the integrability of the quasi–homogeneous polynomial differential systems is a good first step for studying the integrability of more general differential systems, see for instance [1, 16].

Some links between Kowalevskaya exponents of quasi–homogeneous polynomial differential systems and the degree of their quasi–homogeneous polynomial first integrals are established in [6, 9, 17, 24].

2010 *Mathematics Subject Classification.* 34C05, 34A34, 34C20.
Key words and phrases. quasi–homogeneous polynomial differential equations, integrability problem, polynomial first integral, rational first integral.

Recently in [10] the polynomial and rational integrability of homogeneous polynomial differential systems has been characterized. In the present paper we give the characterization of polynomial or rational integrability for quasi–homogeneous polynomial differential systems. In [2] the rational and polynomial integrability of a quasi–homogeneous polynomial differential systems is characterized in terms of conservative dissipative decompositions of the corresponding vector field. In our work, the characterization is different, shorter and easier because essentially is based in the fact that any quasi–homogeneous polynomial differential system can be transformed in a differential system of the form $\dot{u} = uf(v)$, $\dot{v} = g(v)$, with $f(v)$ and $g(v)$ polynomials. Moreover the converse is also true. Note that this last differential system is of separable variables.

As usual $\mathbb{C}$ denotes the set of the complex numbers and $\mathbb{C}[x, y]$ denotes the ring of all polynomials in the variables x and y with coefficients in $\mathbb{C}$. Let $\mathbb{C}(x, y)$ be its quotient field, that is, the field of rational functions in the variables x and y with coefficients in $\mathbb{C}$. In this work we deal with polynomial differential systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where $P(x, y), Q(x, y) \in \mathbb{C}[x, y]$. The dot denotes derivative with respect to an independent variable t real or complex. We say that the degree of the system is $n = \max\{\deg P, \deg Q\}$. For the sake of simplicity, we assume for the rest of the paper that system (1) is not linear, that is, $n > 1$. The linear case is included in the results of [10] where all the homogeneous systems were studied.

Let $\mathbb{N}$ denote the set of positive integers. The polynomial differential system (1) is *quasi–homogeneous* if there exists $s_1, s_2, d \in \mathbb{N}$ such that for arbitrary $\alpha \in \mathbb{C}$,

$$P(\alpha^{s_1} x, \alpha^{s_2} y) = \alpha^{s_1 - 1 + d} P(x, y), \ Q(\alpha^{s_1} x, \alpha^{s_2} y) = \alpha^{s_2 - 1 + d} Q(x, y), \tag{2}$$

where s_1 and s_2 are called the *weight exponents* of system (1), and d the *weight degree* with respect to the weight exponents s_1 and s_2. We say that system (1) satisfying conditions (2) is a quasi–homogeneous system of weight (s_1, s_2, d). In the particular case that $s_1 = s_2 = 1$, then system (1) is the classical homogeneous polynomial differential system of degree d. Our aim is to characterize the quasi–homogeneous systems (1) of weight (s_1, s_2, d) which have a polynomial or a rational first integral.

We recall that given a planar polynomial differential system (1), we say that a function $H : \mathcal{U} \subseteq \mathbb{C}^2 \to \mathbb{C}$, with $\mathcal{U}$ an open set, is a *first integral* of system (1) if H is continuous, not locally constant and constant on each trajectory of the system contained in $\mathcal{U}$. We note that if H is of class at least $\mathcal{C}^1$ in $\mathcal{U}$, then H is a first integral if it is not locally constant and

$$P(x,y)\,\frac{\partial H}{\partial x} \,+\, Q(x,y)\,\frac{\partial H}{\partial y} \,\equiv\, 0$$

in $\mathcal{U}$. We call the *integrability problem* the problem of finding such a first integral and the functional class where it belongs. We say that the system has a *polynomial first integral* if there exists a first integral $H(x,y) \in \mathbb{C}[x,y]$. Analogously, we say that the system has a *rational first integral* if there exists a first integral $H(x,y) \in \mathbb{C}(x,y)$.

We say that a function $V : \mathcal{W} \subseteq \mathbb{C}^2 \to \mathbb{C}$, with $\mathcal{W}$ an open set, is an *inverse integrating factor* of system (1) if V is of class $\mathcal{C}^1$, not locally zero and satisfies the following linear partial differential equation

$$P(x,y)\,\frac{\partial V}{\partial x} \,+\, Q(x,y)\,\frac{\partial V}{\partial y} \,=\, \left(\frac{\partial P}{\partial x} \,+\, \frac{\partial Q}{\partial y}\right) V(x,y) \tag{3}$$

in $\mathcal{W}$. The knowledge of an inverse integrating factor defined in $\mathcal{W}$ allows the computation of a first integral in $\mathcal{U} = \mathcal{W} \setminus \{V = 0\}$ doing the line integral

$$H(x,y) \,=\, \int_{(x_0,y_0)}^{(x,y)} \frac{P(x,y)dy \,-\, Q(x,y)dx}{V(x,y)},$$

where $(x_0, y_0) \in \mathcal{U}$ is any point. An easy computation shows that the polynomial $V(x,y) = s_2 y P(x,y) - s_1 x Q(x,y)$ is an inverse integrating factor for the quasi–homogeneous system (1) of weight (s_1, s_2, d).

A proof of this well-known fact can be found in Proposition 15 of [8]. This result generalizes the particular case for homogeneous systems that can be found in Lemma 3 of [4]. Therefore the quasi–homogeneous systems (and in particular the homogeneous ones) have always a polynomial inverse integrating factor. In [5] necessary conditions were given in order to have a polynomial inverse integrating factor for general polynomial differential systems. To see the relation between the functional classes of the inverse integrating factors and their associated first integrals see Theorem 3 of [11].

Our main results are stated in Lemma 2, Theorem 9 and Proposition 11. Theorem 9 characterizes when a quasi–homogeneous polynomial differential system (1) has either a polynomial, or a rational first integral. In Lemma 2 and Proposition 11 we show that any planar quasi–homogeneous polynomial differential system can be transformed into a differential system of the form $\dot{u} = uf(v)$, $\dot{v} = g(v)$ with $f(v)$ and $g(v)$ polynomials, and vice versa. The rest of the paper is organized as follows. In section 2 we present some lemmas (including Lemma 2) which will allow us to prove Theorem 9 and Proposition 11 in section 3.

2. Preliminary results

Lemma 1. *Given a quasi–homogeneous system* (1) *of weight* (s_1, s_2, d)*, we can suppose without restriction that* s_1 *and* s_2 *are coprime.*

Proof. Let r be the maximum common divisor of s_1 and s_2. Then $s_1 = rs_1^*$ and $s_2 = rs_2^*$ with s_1^* and s_2^* coprime. Let $x^{i_p}y^{j_p}$ be a monomial with nonzero coefficient of $P(x, y)$. We know its existence because P and Q are coprime. We also know that

$$P(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_1-1+d}P(x, y) \quad \text{for all} \quad \alpha \in \mathbb{C}.$$

Therefore we have $(\alpha^{s_1}x)^{i_p}(\alpha^{s_2}y)^{j_p} = \alpha^{s_1 i_p}\alpha^{s_2 j_p}x^{i_p}y^{j_p} = \alpha^{s_1-1+d}x^{i_p}y^{j_p}$ which implies $s_1 i_p + s_2 j_p = s_1 - 1 + d$, or equivalently $s_1(i_p - 1) + s_2 j_p = d - 1$. Consequently $d - 1$ is divisible by r except if $(i_p, j_p) = (1, 0)$. If $P(x, y) = x$, then we consider a monomial $x^{i_q}y^{j_q}$ with nonzero coefficient of $Q(x, y)$. Taking into account that

$$Q(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_2-1+d}Q(x, y) \quad \text{for all} \quad \alpha \in \mathbb{C},$$

we obtain $s_1\, i_q + s_2(j_q - 1) = d - 1$. From here we deduce that $d - 1$ is divisible by r except if $(i_q, j_q) = (0, 1)$. Therefore the unique case to study is $P(x, y) = x$ and $Q(x, y) = y$, i.e. $\dot{x} = x$ and $\dot{y} = y$, which is a homogenous system of degree 1 with $(s_1, s_2, d) = (1, 1, 1)$. We have excluded linear systems from consideration. In any other case we have that $d - 1$ is divisible by r and we can write $d - 1 = r^*(d^* - 1)$.

In short, we claim that system (1) is (s_1^*, s_2^*, d^*) quasi–homogeneous with s_1^* and s_2^* coprime. Indeed, we have that the monomial $x^{i_p}y^{j_p}$ of $P(x, y)$ must verify that $s_1(i_p - 1) + s_2\, j_p = d - 1$, but substituting $s_1 = rs_1^*$, $s_2 = rs_2^*$ and $d - 1 = r^*(d^* - 1)$ we obtain $s_1^*(i_p - 1) + s_2^* j_p = d^* - 1$. In a similar way for any monomial $x^{i_q}y^{j_q}$ of $Q(x, y)$ we obtain $s_1^*\, i_q + s_2^*(j_q - 1) = d^* - 1$. Hence we obtain a quasi–homogeneous system of weight (s_1^*, s_2^*, d^*). □

Lemma 2. *The change of variables*

$$x = u^{\frac{1}{s_2}}, \quad y = (uv)^{\frac{1}{s_1}} \text{ whose inverse is } u = x^{s_2}, \quad v = y^{s_1}/x^{s_2}, \tag{4}$$

and the rescaling of time given by $u^{-\frac{d-1}{s_1 s_2}}v^{-\frac{s_1-1-m}{s_1}}$*, with* $m \in \mathbb{N} \cup \{0\}$*, transforms a quasi–homogeneous system* (1) *of weight* (s_1, s_2, d) *into a polynomial system of the form*

$$\dot{u} = uf(v), \qquad \dot{v} = g(v). \tag{5}$$

Moreover we can choose m in such a way that the polynomials $f(v)$ and $g(v)$ are coprime.

Proof. The change of variables (4) transforms the quasi–homogeneous system (1) of weight (s_1, s_2, d) into the form

(6)
$$\begin{array}{rcl}\dot u &=& s_2 u^{\frac{s_2-1}{s_2}} P(u^{\frac{1}{s_2}}, (uv)^{\frac{1}{s_1}}),\\ \dot v &=& u^{-\frac{s_1+s_2}{s_1 s_2}} v^{\frac{s_1-1}{s_1}} \big[- s_2 u^{\frac{1}{s_1}} v^{\frac{1}{s_1}} P(u^{\frac{1}{s_2}}, (uv)^{\frac{1}{s_1}}) + s_1 u^{\frac{1}{s_2}} Q(u^{\frac{1}{s_2}}, (uv)^{\frac{1}{s_1}})\big].\end{array}$$

Notice that if we take $\alpha = u^{\frac{1}{s_1 s_2}}$ we have $P(\alpha^{s_1}, \alpha^{s_2} v^{\frac{1}{s_1}}) = P(u^{\frac{1}{s_2}}, (uv)^{\frac{1}{s_1}})$ and by the quasi–homogeneity we have

$$\alpha^{s_1-1+d} P(1, v^{\frac{1}{s_1}}) = u^{\frac{s_1-1+d}{s_1 s_2}} P(1, v^{\frac{1}{s_1}}).$$

In a similar way and also by quasi–homogeneity we have

$$Q(u^{\frac{1}{s_2}}, (uv)^{\frac{1}{s_1}}) = u^{\frac{s_2-1+d}{s_1 s_2}} Q(1, v^{\frac{1}{s_1}}).$$

Hence system (6) becomes

$$\begin{array}{rcl}\dot u &=& s_2 u^{1+\frac{d-1}{s_1 s_2}} P(1, v^{\frac{1}{s_1}}),\\ \dot v &=& u^{\frac{d-1}{s_1 s_2}} v \big[- s_2 P(1, v^{\frac{1}{s_1}}) + s_1 v^{-\frac{1}{s_1}} Q(1, v^{\frac{1}{s_1}})\big].\end{array} \tag{7}$$

Now we divide the system by $u^{\frac{d-1}{s_1 s_2}} v^{\frac{s_1-1-m}{s_1}}$ and we have

$$\begin{array}{rcl}\dot u &=& s_2 u v^{\frac{m+1-s_1}{s_1}} P(1, v^{\frac{1}{s_1}}),\\ \dot v &=& v^{\frac{m}{s_1}} \big[- s_2 v^{\frac{1}{s_1}} P(1, v^{\frac{1}{s_1}}) + s_1 Q(1, v^{\frac{1}{s_1}})\big].\end{array} \tag{8}$$

We know that the monomials $x^{i_p} y^{j_p}$ of $P(x, y)$ and the monomials $x^{i_q} y^{j_q}$ of $Q(x, y)$ satisfy

$$s_1(i_p - 1) + s_2 j_p = d - 1, \qquad s_1\, i_q + s_2(j_q - 1) = d - 1. \tag{9}$$

Hence $\dot u$ has the monomials in v of the form $v^{\frac{m+1+j_p-s_1}{s_1}}$, and $\dot v$ has the monomials in v of the form

$$v^{\frac{m+1+j_p}{s_1}} \text{ or } v^{\frac{m+j_q}{s_1}}.$$

The identities (9) modulo s_1 are of the form

$$s_2 j_p = d - 1 \pmod{s_1}, \qquad s_2(j_q - 1) = d - 1 \pmod{s_1}. \tag{10}$$

If s_1 and s_2 are coprime, i.e. $(s_1, s_2) = 1$, then there exists $s_2^{-1} \in \mathbb{Z}$ such that $s_2^{-1} s_2 = 1 \pmod{s_1}$. Since s_1 and s_2 are coprime, by the Bézout identity there exist two integers x and y such that $s_1 x + s_2 y = 1$. Hence, modulo s_1 we have $s_2 y = 1$ and we take $s_2^{-1} = y$. Therefore the identities (10) can be simplified multiplying by s_2^{-1} and we obtain

$$j_p = s_2^{-1}(d - 1) \pmod{s_1}, \qquad j_q - 1 = s_2^{-1}(d - 1) \pmod{s_1},$$

and consequently $j_p \equiv j_q - 1 \pmod{s_1}$ for all j_p and j_q. We define $m \in \mathbb{N} \cup \{0\}$ the smaller value such that $m+1+j_p \equiv m+j_q \equiv 0 \pmod{s_1}$, and with this choice all the monomials that appear in system (8) have nonnegative integer exponents and consequently system (8) is polynomial and of the form $\dot{u} = uf(v)$ and $\dot{v} = g(v)$. We need to see that the obtained system (8) is coprime.

Note that for the choice of m, being the smallest one, it cannot happen that $v|f(v)$ and $v|g(v)$, because in this case we could choose m even smaller.

Let $\xi \in \mathbb{C}$ such that $P(1, \xi^{\frac{1}{s_1}}) = 0$, i.e. ξ is a root of $f(v)$. We are going to see that this root ξ is not a root of $g(v)$. If $g(\xi) = 0$ then we have that $-s_2 \xi^{\frac{1}{s_1}} P(1, \xi^{\frac{1}{s_1}}) + s_1 Q(1, \xi^{\frac{1}{s_1}}) = 0$ which implies $Q(1, \xi^{\frac{1}{s_1}}) = 0$. Hence we have $P(1, \xi^{\frac{1}{s_1}}) = Q(1, \xi^{\frac{1}{s_1}}) = 0$. Now we consider the algebraic curve $y^{s_1} - \xi x^{s_2} = 0$, and we parameterize it as follows $x = \alpha^{s_1}$, $y = \alpha^{s_2} \xi^{1/s_1}$, with $\alpha \in \mathbb{C}$. By quasi–homogeneity and taking into account that $P(1, \xi^{\frac{1}{s_1}}) = Q(1, \xi^{\frac{1}{s_1}}) = 0$ we have

$$\begin{array}{rcccl} P(\alpha^{s_1}, \alpha^{s_2} \xi^{\frac{1}{s_1}}) & = & \alpha^{s_1-1+d} P(1, \xi^{\frac{1}{s_1}}) & = & 0, \\ Q(\alpha^{s_1}, \alpha^{s_2} \xi^{\frac{1}{s_1}}) & = & \alpha^{s_2-1+d} Q(1, \xi^{\frac{1}{s_1}}) & = & 0. \end{array}$$

Hence, $P(x, y)$ and $Q(x, y)$ vanish at the same place. Therefore by the Bézout theorem, P and Q would have a common factor, in contradiction with the initial hypothesis that P and Q were coprime. We observe that this argument does not hold in the case that $f(v)$ is a constant different from zero and $g(v) \equiv 0$. In this case, system (1) is linear, see Lemma 3, and linear systems have been excluded from consideration. $\square$

One immediate consequence of Lemma 2 is that if system (5) has $H(u, v)$ as a first integral, then the quasi–homogeneous system (1) of weight (s_1, s_2, d) has a first integral of the form $H(x^{s_2}, y^{s_1}/x^{s_2})$. On the other hand, if $\tilde{H}(x, y)$ is a first integral of the quasi–homogeneous system (1) of weight (s_1, s_2, d) then $\tilde{H}(u^{1/s_2}, (u, v)^{1/s_1})$ is a first integral of system (5).

Lemma 3. *Consider system* (1) *with P and Q coprime. The particular case when $f(v)$ is constant different from zero and $g(v) \equiv 0$ corresponds to the system $\dot{x} = s_1 x$ and $\dot{y} = s_2 y$.*

Proof. The condition $g(v) \equiv 0$ implies that the inverse integrating factor $V(x, y) \equiv 0$, i.e. $s_2 y P(x, y) - s_1 x Q(x, y) \equiv 0$. In this case $P(x, y)$ is divisible by x due to P and Q are coprime. Hence, we can find a polynomial $M(x, y)$ such that $P(x, y) = s_1 x M(x, y)$ and consequently

$Q(x,y) = s_2 y M(x,y)$. However as P and Q are coprime we have that $M(x,y)$ must be a constant different from zero, and with a time–rescaling we obtain the system $\dot{x} = s_1 x$ and $\dot{y} = s_2 y$. This linear system has the rational first integral $H = x^{s_2} y^{-s_1}$ and has not any polynomial first integral. $\square$

Now we are going to study the polynomial and rational integrability of system (5).

Lemma 4. *Consider a polynomial system of the form* (5)*, where $f(v)$ and $g(v)$ are coprime.*

(i) *The polynomial $V(u,v) = ug(v)$ is an inverse integrating factor of system* (5)*.*
(ii) *If system* (5) *has a rational (resp. polynomial) first integral, then it has a first integral of the form $H(u,v) = h(v)u^s$ where $h(v)$ is a non constant rational (resp. polynomial) function and $s \in \mathbb{N}$.*
(iii) *If system* (5) *has a a rational first integral, then* $\deg f < \deg g$*.*
(iv) *If system* (5) *has a a rational first integral, then $g(v)$ (equivalently $V(u,v)$) is square-free.*
(v) *If $g(v)$ is square-free and its factorization in $\mathbb{C}[v]$ is given by $g(v) = c(v-\alpha_1)(v-\alpha_2)\cdots(v-\alpha_k)$ where $c \in \mathbb{C}\backslash\{0\}$, $\alpha_i \in \mathbb{C}$ for $i = 1,2,\ldots,k$, we have that $f(\alpha_i)g'(\alpha_i) \neq 0$ for $i = 1,2,\ldots,k$.*
(vi) *If $V(u,v)$ is square-free and degf $<$ $\deg g$, then a first integral of system* (5) *is $H(u,v) = u^{-1}(v-\alpha_1)^{\gamma_1}(v-\alpha_2)^{\gamma_2}\cdots(v-\alpha_k)^{\gamma_k}$ with $\gamma_i = f(\alpha_i)/g'(\alpha_i)$ for $i = 1,2,\ldots,k$.*
(vii) *System* (5) *has a rational first integral if and only if $g(v)$ is square-free,* $\deg f < \deg g$ *and $\gamma_i \in \mathbb{Q}$ for $i = 1,2,\ldots,k$.*
(viii) *System* (5) *has a polynomial first integral if and only if $g(v)$ is square-free,* $\deg f < \deg g$ *and $\gamma_i \in \mathbb{Q}^-$ for $i = 1,2,\ldots,k$.*

Proof. (i) It is easy to see that the function $V(u,v) = ug(v)$ satisfies the partial differential equation (3), i.e.

$$uf(v)\,\frac{\partial V}{\partial u} + g(v)\,\frac{\partial V}{\partial v} = (f(v) + g'(v))\,V. \tag{11}$$

(ii) Assume that system (5) has a rational or polynomial first integral of the form

$$H(u,v) = \frac{A(u,v)}{B(u,v)},$$

where A and B are coprime polynomials that can be written in powers of u into the form

$$\begin{aligned} A(u,v) &= a_0(v) + a_1(v)u + \cdots + a_n(v)u^n, \\ B(u,v) &= b_0(v) + b_1(v)u + \cdots + b_m(v)u^m. \end{aligned}$$

with $n, m \in \mathbb{N} \cup \{0\}$ and $a_n(v)\, b_m(v) \neq 0$. The polynomial case corresponds to $m = 0$ and $b_0(v) = 1$.

In the particular case that $n = m$ and if there exists $k \in \mathbb{C}$ such that $a_n(v) = kb_m(v)$ we take as a first integral $H(u,v) - k$ instead of $H(u,v)$ and we have

$$H(u,v) - k = \frac{(a_0 - kb_0) + (a_1 - kb_1)u + \cdots (a_{n-1} - kb_{n-1})u^{n-1}}{b_0(v) + b_1(v)u + \cdots + b_n(v)u^n}.$$

In the polynomial case, if $n = m = 0$ then we will have that $H(u,v) = a_0(v)$ which implies $\dot{v} = 0$, and this gives a contradiction because $f(v)$ and $g(v)$ are coprime.

We can assume that $n \geq m$, because if H is a first integral of system (5) then $1/H$ is also a first integral, and for the case $n = m$, we can assume that $a_n(v)/b_n(v)$ is not constant as we have shown in the previous paragraph. If H is a first integral of system (5) then it must satisfy

$$uf(v)\frac{\partial H}{\partial u} + g(v)\frac{\partial H}{\partial v} = 0. \tag{12}$$

Substituting $H = A/B$ in (12) and multiplying by B^2 we have

$$\Big(uf(v)\frac{\partial A}{\partial u} + g(v)\frac{\partial A}{\partial v}\Big)B - \Big(uf(v)\frac{\partial B}{\partial u} + g(v)\frac{\partial B}{\partial v}\Big)A = 0.$$

We now consider the highest degree coefficient in this expression, which corresponds to u^{n+m} and we obtain that

$$\begin{aligned} u^{n+m} \Big[&\Big(nf(v)a_n(v) + g(v)a_n'(v)\Big)b_m(v) \\ &- \Big(mf(v)b_m(v) + g(v)b_m'(v)\Big)a_n(v)\Big] = 0. \end{aligned}$$

This identity says that $u^{n-m}a_n(v)/b_m(v)$ is a first integral of system (5). Therefore we have a first integral of the form $u^s h(v)$ where $s = n - m \in \mathbb{N}$, $h(v)$ is rational when H is rational, and is polynomial when H is polynomial. Note that $h(v)$ cannot be a constant and s must be different from zero because both cases are in contradiction with the hypothesis that $f(v)$ and $g(v)$ are coprime. Hence we have proved statement (ii).

(iii) We assume that system (5) has a rational first integral and we take the first integral of the form $u^s h(v)$. If $u^s h(v)$ is a first integral of system (5) we have $uf(v)\,su^{s-1}h(v) + g(v)u^s h'(v) = 0$. Simplifying u^s we have $s\,f(v)h(v) + g(v)h'(v) = 0$, and we have the following differential equation

$$\frac{h'(v)}{h(v)} = \frac{-s\,f(v)}{g(v)}. \tag{13}$$

Now we assume that $\deg f \geq \deg g$ and we consider the Euclidean division of $-sf(v)$ by $g(v)$, so we have

$$-sf(v) = q(v)g(v) + \psi(v),$$

where $\psi(v)$ cannot be zero taking into account that f and g are coprime and $\deg \psi < \deg g$. Hence equation (13) takes the form

$$\frac{h'(v)}{h(v)} = q(v) + \frac{\psi(v)}{g(v)}. \tag{14}$$

Integrating this differential equation we have

$$h(v) = Ce^{\tilde{q}(v)}e^{\int \frac{\psi(v)}{g(v)}dv} \tag{15}$$

where C is a constant of integration and $\tilde{q}\,'(v) = q(v)$. Therefore the first factor of (15) cannot cancel with the second factor of (15) and this gives a contradiction with the fact that $h(v)$ is a rational function. Hence, we conclude that $\deg f < \deg g$.

(iv) From the proof of statement (iii) we have obtained the differential equation (13). We assume now that $V(u,v)$ is not square-free. Using an affine transformation of the form $v \to v + \alpha$ with $\alpha \in \mathbb{C}$ if it is necessary, we can assume that v is a multiple factor of $V(u,v)$ with multiplicity $\mu > 1$. Then we have that $V(u,v) = ug(u) = uv^{\mu}r(v)$ with $r(0) \neq 0$. We know that $f(0) \neq 0$ because f and g are coprime. Now we develop the right–hand side of (13) in simple fractions of v, that is,

$$\frac{-sf(v)}{g(v)} = \frac{c_\mu}{v^\mu} + \frac{c_{\mu-1}}{v^{\mu-1}} + \cdots + \frac{c_1}{v} + \frac{\alpha_1(v)}{r(v)} + \alpha_0(v),$$

where $\alpha_0(v)$ and $\alpha_1(v)$ are polynomials with $\deg \alpha_1(v) < \deg r(v)$ and $c_i \in \mathbb{C}$, for $i = 1, 2, \ldots, \mu$. Equating both expressions, we get that $c_\mu = -s\,f(0)/r(0) \neq 0$. Moreover as a consequence of statement (iii) we know that $\alpha_0(v) \equiv 0$. Therefore equation (13) becomes

$$\frac{h'(v)}{h(v)} = \frac{c_\mu}{v^\mu} + \frac{c_{\mu-1}}{v^{\mu-1}} + \cdots + \frac{c_1}{v} + \frac{\alpha_1(v)}{r(v)},$$

with $c_\mu \neq 0$. Now if we integrate this expression we get

$$\begin{aligned} h(v) &= C \exp\left[\frac{c_\mu}{1-\mu}\frac{1}{v^{\mu-1}}\right] \cdot \\ & \exp\left[\int\left(\frac{c_{\mu-1}}{v^{\mu-1}} + \ldots + \frac{c_1}{v} + \frac{\alpha_1(v)}{r(v)}\right) dv\right], \end{aligned}$$

where C is a constant of integration. The first exponential factor cannot be simplified with any part of the second exponential factor. Thus, we get a contradiction with the fact that $h(v)$ is a rational function. Therefore we conclude that $V(u,v)$ and $g(v)$ are square-free. This proves statement (iv).

(v) Recalling that f and g are coprime and that α_i is a root of g it is clear that $f(\alpha_i) \neq 0$ for $i = 1, 2, \ldots, k$. As g is square-free, then $g'(\alpha_i) \neq 0$ for $i = 1, 2, \ldots, k$.

(vi) We must prove that $H(u,v) = u^{-1}\varphi(v)$ where $\varphi(v) = (v - \alpha_1)^{\gamma_1}(v-\alpha_2)^{\gamma_2} \cdots (v-\alpha_k)^{\gamma_k}$ is a first integral of system (5), i.e.

$$uf(v)\frac{\partial H}{\partial u} + g(v)\frac{\partial H}{\partial v} = u^{-1}(-f(v)\varphi(v) + g(v)\varphi'(v)) = 0.$$

To see that this last expression is identically zero is equivalent to see that $\varphi'(v)/\varphi(v) = f(v)/g(v)$. Recalling the expression of $\varphi(v)$ we have

$$\frac{\varphi'(v)}{\varphi(v)} = \frac{\gamma_1}{v-\alpha_1} + \frac{\gamma_2}{v-\alpha_2} + \cdots + \frac{\gamma_k}{v-\alpha_k}.$$

Taking common denominator and recalling that $g(v) = c(v-\alpha_1)(v-\alpha_2)(v-\alpha_3)\cdots(v-\alpha_k)$ we obtain

$$\frac{\varphi'(v)}{\varphi(v)} = \frac{1}{g(v)}\sum_{i=1}^{k} c\,\gamma_i \prod_{j=1, j\neq i}^{k} (v-\alpha_j).$$

Now substituting the values of $\gamma_i = f(\alpha_i)/g'(\alpha_i)$ and taking into account that

$$g'(\alpha_i) = c \prod_{j=1, j\neq i}^{k} (\alpha_i - \alpha_j),$$

we obtain

$$\frac{\varphi'(v)}{\varphi(v)} = \frac{1}{g(v)}\sum_{i=1}^{k} c\, f(\alpha_i) \prod_{j=1, j\neq i}^{k} \frac{v-\alpha_j}{\alpha_i - \alpha_j} = \frac{f(v)}{g(v)}. \tag{16}$$

Since $\deg f < \deg g$, the expression in the sum is the Lagrange polynomial which interpolates the k points $(\alpha_i, f(\alpha_i))$, for $i = 1, 2, \ldots, k$, see for more details [13]. Therefore, this polynomial is $f(v)$ and we conclude that expression (16) is identically satisfied.

(vii) We assume that system (5) has a rational first integral, then by statement (ii) it has one of the form $h(v)u^s$ where $h(v)$ satisfies identity (13). Using statements (iii), (iv) and (vi) we have that

$$\frac{f(v)}{g(v)} = \frac{\gamma_1}{v-\alpha_1} + \frac{\gamma_2}{v-\alpha_2} + \cdots + \frac{\gamma_k}{v-\alpha_k}. \tag{17}$$

Hence, integrating identity (13) we obtain

$$h(v) = C\,\left((v-\alpha_1)^{\gamma_1}(v-\alpha_2)^{\gamma_2}\cdots(v-\alpha_k)^{\gamma_k}\right)^{-s}. \tag{18}$$

Then, as $h(v)$ must be a rational function and $s \in \mathbb{N}$ we have that $\gamma_i \in \mathbb{Q}$ for $i = 1, 2, \ldots, k$. Conversely, if $\gamma_i \in \mathbb{Q}$ for $i = 1, 2, \ldots, k$ the first integral described in statement (vi) elevated to certain natural power leads to a rational first integral of system (5).

(viii) If system (5) has a polynomial first integral, then reasoning as in the previous statement we arrive to the same expression for $h(v)$ given by (18) which is polynomial if and only if $\gamma_i \in \mathbb{Q}^-$ for $i = 1, 2, \ldots, k$. Conversely, if $\gamma_i \in \mathbb{Q}^-$ for $i = 1, 2, \ldots, k$ the multiplicative inverse of the first integral described in statement (vi) gives a polynomial first integral of system (5). This completes the proof of the lemma. □

In order to relate the first integrals of the quasi–homogeneous system (1) of weight (s_1, s_2, d) with the first integrals of system (5), we must see how the change of variables described in Lemma 2 affects to the rationality/polinomiality of a first integral of system (1). First we need an auxiliary lemma about quasi–homogeneous polynomials.

Lemma 5. *The following statements hold.*

(i) *Let $A_\ell(x, y)$ be a quasi–homogeneous polynomial of weight exponents (s_1, s_2) and of weight degree ℓ, then $\partial A_\ell/\partial x$ is a quasi–homogeneous polynomial of weight exponents (s_1, s_2) and of weight degree $\ell - s_1$, and $\partial A_\ell/\partial y$ is a quasi–homogeneous polynomial of weight exponents (s_1, s_2) and of weight degree $\ell - s_2$.*

(ii) *Let $A_\ell(x, y)$ and $B_m(x, y)$ be two quasi–homogeneous polynomials of weight exponents (s_1, s_2) and of weight degrees ℓ and m, respectively, then their product is a quasi–homogeneous polynomial of weight exponents (s_1, s_2) and of weight degree $\ell + m$ and their quotient is a quasi–homogeneous function of weight exponents (s_1, s_2) and of weight degree $\ell - m$.*

(iii) *Let $A(x, y)$ be a polynomial. Then $A(x, y)$ can be written as an ordered finite sum of quasi–homogeneous polynomials of weight* *exponents* (s_1, s_2) *and of weight positive degrees, i.e.* $A(x, y) = A_0(x, y) + A_1(x, y) + \cdots + A_\ell(x, y)$ *where* $A_i(x, y)$ *is a quasi–homogeneous polynomials of weight exponents* (s_1, s_2) *and of weight degree* i*.*

Proof. Although the proof is well-known, we give it here for sake of completeness.

(i) First we recall the definition of quasi–homogeneous polynomial of weight exponents (s_1, s_2) given in the introduction. We have that $A_\ell(x, y)$ is a quasi–homogeneous polynomial of weight exponents (s_1, s_2) and of weight degree ℓ if

$$A_\ell(\alpha^{s_1} x, \alpha^{s_2} y) = \alpha^\ell A_\ell(x, y), \qquad \forall \alpha \in \mathbb{C}.$$

Hence, if $x^i y^j$ is a monomial with a nonzero coefficient of $A_\ell(x, y)$ we have $\alpha^{s_1 i} x^i \alpha^{s_2 j} y^j = \alpha^\ell x^i y^j$ from here we obtain $s_1 i + s_2 j = \ell$ for all i, j. The polynomial $\partial A_\ell/\partial x$ has the monomial $i x^{i-1} y^j$ (except for the case $i = 0$) and using the same reasoning we will obtain the relation $s_1(i - 1) + s_2 j$. In fact taking into account that $s_1 i + s_2 j = \ell$ we get $s_1(i - 1) + s_2 j = \ell - s_1$ which implies that the polynomial $\partial A_\ell/\partial x$ is a quasi–homogeneous polynomial of weight exponents (s_1, s_2) and of weight degree $\ell - s_1$. The proof for $\partial A_\ell/\partial y$ is analogous.

(ii) We consider the product polynomial and we evaluate at $(\alpha^{s_1} x, \alpha^{s_2} y)$ and we have

$$A_\ell(\alpha^{s_1} x, \alpha^{s_2} y) B_m(\alpha^{s_1} x, \alpha^{s_2} y) = \alpha^{\ell+m} A_\ell(x, y) B_m(x, y) \qquad \forall \alpha \in \mathbb{C}.$$

The same reasoning is valid for the quotient.

(iii) Let $x^i y^j$ be a monomial with a nonzero coefficient of $A(x, y)$. It is clear that belongs to a quasi–homogeneous polynomial of weight exponents (s_1, s_2) and of weight degree $s_1 i + s_2 j$. Since $i, j \geq 0$ we have that this degree is not negative. As the number of monomials of $A(x, y)$ is finite we obtain a finite number of quasi–homogeneous polynomials of weight exponents (s_1, s_2) and the weight degrees are ordered. □

Lemma 6. *If system* (1) *has a rational (resp. polynomial) first integral then system* (1) *has has a rational (resp. polynomial) first integral which is a quasi–homogeneous function of weight exponents* (s_1, s_2) *and of weight degree* $m \geq 0$*.*

Proof. The polynomial case could be deduced from the Proposition 1 in [17].

We consider $H(x, y) = A(x, y)/B(x, y)$ the rational or polynomial first integral of system (1) and, using Lemma 5 statement (iii) we write

$A(x, y)$ and $B(x, y)$ as a sum of quasi–homogeneous polynomials of weight exponents (s_1, s_2)

$$\begin{aligned} A(x,y) &= A_0(x,y) + \cdots + A_a(x,y), \\ B(x,y) &= B_0(x,y) + \cdots + B_b(x,y), \end{aligned}$$

where $a, b \in \mathbb{N} \cup \{0\}$ and $A_a(x, y)B_b(x, y) \neq 0$. Note that the polynomial case is $b = 0$ and $B_0(x, y) = 1$. In the particular case $a = b$ and if there exists $k \in \mathbb{C}$ such that $A_a(x, y) = kB_a(x, y)$ we take $H(x, y) - k$ instead of $H(x, y)$ in order to have a first integral where the degree of the numerator is less than the degree of the denominator. We remark that if H is a first integral then $1/H$ is also a first integral. Therefore we can assume that $a \geq b$ and in the particular case that $a = b$ we can assume that the quotient $A_a(x, y)/B_a(x, y)$ is not constant. If $H = A/B$ is first integral of system (1) it must satisfy

$$P(x,y)\frac{\partial H}{\partial x} + Q(x,y)\frac{\partial H}{\partial y} = 0.$$

Substituting $H = A/B$ and multiplying by B^2 we have

$$\left(P\frac{\partial A}{\partial x} + Q\frac{\partial A}{\partial y}\right)B - \left(P\frac{\partial B}{\partial x} + Q\frac{\partial B}{\partial y}\right)A = 0. \tag{19}$$

Taking into account Lemma 5 we have that P is a quasi–homogeneous polynomial of weight exponents (s_1, s_2) and of weight degree $s_1 - 1 + d$ and Q is a quasi–homogeneous polynomial of weight exponents (s_1, s_2) and of weight degree $s_2 - 1 + d$, and the polynomial $P\partial A_i/\partial x + Q\partial A_i/\partial y$ is a quasi–homogeneous polynomial of weight exponents (s_1, s_2) and of weight degree $d-1+i$ for $i = 0, 1, 2, \ldots, a$ and the same happens for the polynomial $P\partial B_i/\partial x + Q\partial B_i/\partial y$ for $i = 0, 1, 2, \ldots, b$. Taking the terms of highest degree in (19), which correspond to the quasi–homogeneous polynomial of weight exponents (s_1, s_2) and of weight degree $d-1+a+b$ we obtain

$$\left(P\frac{\partial A_a}{\partial x} + Q\frac{\partial A_a}{\partial y}\right)B_b - \left(P\frac{\partial B_b}{\partial x} + Q\frac{\partial B_b}{\partial y}\right)A_a = 0. \tag{20}$$

This equality says that $A_a(x, y)/B_b(x, y)$ is a first integral of system (1). The quotient of two quasi–homogeneous polynomials of weight exponents (s_1, s_2) is a quasi–homogeneous function of weight exponents (s_1, s_2) and of weight degree the difference between the weight degrees of the two polynomials (see statement (ii) of Lemma 5). Hence, the rational function $A_a(x, y)/B_b(x, y)$ is a quasi–homogeneous function of weight exponents (s_1, s_2) and of weight degree $a - b \geq 0$. $\square$

The following lemma shows the relationship between first integrals of system (1) and the ones of system (5).

Lemma 7. *The following statements hold.*

(i) *System* (1) *has a rational first integral if and only if system* (5) *has a rational first integral.*
(ii) *If system* (1) *has a polynomial first integral, then system* (5) *has a polynomial first integral.*
(iii) *If system* (5) *has a polynomial first integral, then system* (1) *has a first integral of the form* $x^\ell p(x,y)$ *where* $p(x,y)$ *is a* (s_1,s_2) *quasi–homogeneous polynomial and* $\ell \in \mathbb{Z}$.

Proof. In all this proof we assume that s_1 and s_2 are coprime using Lemma 1.

(i) We assume that system (1) has a rational first integral, then using Lemma 6, we know that it has a first integral which is a quasi–homogeneous function of weight exponents (s_1,s_2) and of the form $A_a(x,y)/B_b(x,y)$ where $A_a(x,y)$ is a quasi–homogeneous polynomial of weight exponents (s_1,s_2) and of weight degree a and $B_b(x,y)$ is a quasi–homogeneous polynomial of weight exponents (s_1,s_2) and of weight degree b. We consider the first integral

$$H(x,y) = \left(\frac{A_a(x,y)}{B_b(x,y)}\right)^L = \frac{A_a(x,y)^L}{B_b(x,y)^L},$$

where $L \in \mathbb{N}$ is such that the degree of the numerator and the denominator is a multiple of $s_1 s_2$. Hence, we can assume that we have a rational first integral of system (1) of the form $A(x,y)/B(x,y)$ where $A(x,y)$ and $B(x,y)$ are quasi–homogeneous polynomials of weight exponents (s_1,s_2) and of weight degree a and b respectively where $a = s_1 s_2 \tilde{a}$ and $b = s_1 s_2 \tilde{b}$ with $\tilde{a}$ and $\tilde{b} \in \mathbb{N}$. Let $x^i y^j$ be a monomial with a nonzero coefficient of $A(x,y)$ (or $B(x,y)$). By quasi–homogeneity we have $s_1 i + s_2 j = a$ which implies $s_1 i + s_2 j = s_1 s_2 \tilde{a}$, that is, $s_2 j = s_1(s_2\tilde{a} - i)$. As s_1 and s_2 are coprime, we deduce that j is a multiple of s_1, that is, $j = s_1 \tilde{j}$ with $\tilde{j} \in \mathbb{N} \cup \{0\}$. The change of variables described in Lemma 2 tells us that if $H = A(x,y)/B(x,y)$ is a first integral of system (1), then $H(u^{\frac{1}{s_2}}, (uv)^{\frac{1}{s_1}})$ is a first integral of system (5). Moreover we recall that $H(x,y)$ is a quasi–homogeneous function of weight exponents (s_1,s_2) and of weight degree $a - b = s_1 s_2(\tilde{a} - \tilde{b})$, and we shall denote by $\tilde{m} = \tilde{a} - \tilde{b}$. We take $\alpha = u^{\frac{1}{s_1 s_2}}$ and by quasi–homogeneity we have

$$H(u^{\frac{1}{s_2}}, (uv)^{\frac{1}{s_1}}) = H(\alpha^{s_1}, \alpha^{s_2} v^{\frac{1}{s_1}}) = \alpha^{s_2 s_2 \tilde{m}} H(1, v^{\frac{1}{s_1}}) = u^{\tilde{m}} H(1, v^{\frac{1}{s_1}}),$$

where $\tilde{m} \in \mathbb{Z}$. As all the monomials $x^i y^j$ that appear in $H(1, v^{\frac{1}{s_1}})$ have j multiple of s_1, we have that $H(1, v^{\frac{1}{s_1}})$ is a rational function of v. Therefore we have a first integral of system (5) of the form $u^{\tilde{m}} h(v)$ with $h(v)$ a rational function and $\tilde{m} \in \mathbb{Z}$.

Conversely, we assume that system (5) has a rational first integral $H(u, v)$. By the change of variables described in Lemma 2 we have that $H(x^{s_2}, y^{s_1}/x^{s_2})$ is a first integral of system (1) which clearly is a rational function.

(ii) We assume that system (1) has a polynomial first integral. By similar arguments used in statement (i) we can assume that system (1) has a polynomial first integral $H(x, y)$ which is a quasi–homogeneous polynomial of weight exponents (s_1, s_2) and of weight degree $m = s_1 s_2 \tilde{m}$ with $\tilde{m} \in \mathbb{N}$. Analogously as in the previous section, we can see that all the monomials $x^i y^j$ that appear in $H(x, y)$ have j multiple of s_1, i.e. $j = \tilde{j} s_1$ with $\tilde{j} \in \mathbb{N} \cup \{0\}$. Moreover, we have that $H(u^{\frac{1}{s_2}}, (uv)^{\frac{1}{s_1}})$ is a first integral of system (5) and, by quasi–homogeneity, as before, we can see that

$$H(u^{\frac{1}{s_2}}, (uv)^{\frac{1}{s_1}}) = u^{\tilde{m}} H(1, v^{\frac{1}{s_1}}).$$

As $H(x, y)$ is a polynomial and all the monomials $x^i y^j$ satisfy that j is a multiple of s_1, we have that $h(v) = H(1, v^{\frac{1}{s_1}})$ is a polynomial in v. Therefore, we have that $u^{\tilde{m}} h(v)$ is a polynomial first integral of system (5).

(iii) If system (5) has a polynomial first integral, using statement (ii) of Lemma 4, we know that it has a first integral of the form $u^s h(v)$ with $s \in \mathbb{N}$ and $h(v)$ a polynomial. Therefore by the change of variables described in Lemma 2 we have that $x^{s_2 s} h(y^{s_1}/x^{s_2})$ is a first integral of system (1). As $h(v)$ is a polynomial we can write it into the form $h(v) = c_0 + c_1 v + \cdots + c_k v^k$ where $c_i \in \mathbb{C}$ and we have

$$h\left(\frac{y^{s_1}}{x^{s_2}}\right) = \frac{c_0 x^{s_2 k} + c_1 y^{s_1} x^{s_2(k-1)} + \cdots + c_k y^{s_1 k}}{x^{s_2 k}}.$$

We denote by

$$r(x, y) = c_0 x^{s_2 k} + c_1 y^{s_1} x^{s_2(k-1)} + \cdots + c_k y^{s_1 k} = x^{s_2 k} h\left(\frac{y^{s_1}}{x^{s_2}}\right),$$

and we have that $x^{s_2(s-k)} r(x, y)$ is a rational first integral of system (1). We note that $r(x, y)$ is a quasi–homogeneous polynomial of weight exponents (s_1, s_2) due to

$$r(\alpha^{s_1}x, \alpha^{s_2}y) = (\alpha^{s_1}x)^{s_2k} h\left(\frac{(\alpha^{s_2}y)^{s_1}}{(\alpha^{s_1}x)^{s_2}}\right) = \alpha^{s_1s_2k}x^{s_2k}h\left(\frac{y^{s_1}}{x^{s_2}}\right),$$

which implies $r(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_1s_2k}r(x, y)$. □

The following example shows that there are systems (5) having polynomial first integrals and such that their corresponding systems (1) have no polynomial first integral.

Example 8. *Consider the following differential system*

$$\dot{x} = x(3y^2 - x^2), \qquad \dot{y} = y(x^2 + y^2). \tag{21}$$

Its associated system (5) *has a polynomial first integral, but system* (21) *has a rational first integral and has no polynomial first integrals.*

Proof. System (21) is a quasi–homogeneous system of weight exponents $(1, 1)$ and of weight degree 3. Applying the change of variables described in Lemma 2 given in this case by $x = u$ and $y = uv$ we obtain the system

$$\dot{u} = u(3v^2 - 1), \qquad \dot{v} = 2v(1 - v)(1 + v). \tag{22}$$

Comparing with system (5) we have that $f(v) = 3v^2 - 1$ and $g(v) = 2v(1 - v)(1 + v)$. System (22) has the polynomial first integral $\tilde{H} = u^2v(v - 1)(v + 1)$ and undoing the change we obtain that system (21) has the rational first integral

$$H(x, y) = \frac{y(y - x)(y + x)}{x}.$$

Now we prove that system (21) has no polynomial first integrals. We assume that it has a polynomial first integral and by Lemma 6 it must have a homogeneous polynomial first integral $H_m(x, y)$ of degree $m \geq 0$. Moreover as $H_m(x, y)$ is a first integral we have $m > 0$. Using the change of variables of Lemma 2 and applying the results of Lemma 7, system (21) would have a first integral of the form $u^m h(v)$ where $h(v)$ will be a polynomial. We know that $h(v)$, as we have seen in the proof of statement (iii) of Lemma 4, must satisfy the differential equation (13) and in this case we have

$$\frac{h'(v)}{h(v)} = \frac{-mf(v)}{g(v)} = \frac{-m(3v^2 - 1)}{2v(1 - v)(v + 1)} = -m\left(\frac{-\frac{1}{2}}{v} + \frac{-\frac{1}{2}}{v - 1} + \frac{-\frac{1}{2}}{v + 1}\right),$$

Solving this equation we obtain $h(v) = C(v(v - 1)(v + 1))^{m/2}$ where C is a constant of integration that must be different from zero that we take equal to 1 without loss of generality. Undoing the change $x = u$

and $y = uv$, we obtain that $H_m(x,y) = x^{-m/2}y^{m/2}(y^2 - x^2)^{m/2}$. Since $m > 0$ we obtain that $H_m(x,y)$ cannot be a polynomial and this gives a contradiction with the existence of a polynomial first integral for system (21). □

3. The main results

The following theorem is the main result of this paper and characterizes when system (1) has a polynomial or a rational first integral. As usual $\mathbb{Q}$ denotes the set of rational numbers, and $\mathbb{Q}^+$ (resp. $\mathbb{Q}^-$) the set of positive (resp. negative) rational numbers.

Theorem 9. *Consider system* (1) *which can be transformed by the change defined in Lemma* 2 *in system* (5)*. Using the same notation than in previous lemmas, the following statements hold.*

(a) *System* (1) *has a rational first integral if and only if $g(v)$ is square-free,* $\deg f < \deg g$ *and* $\gamma_i \in \mathbb{Q}$ *for* $i = 1, 2, \ldots, k$.
(b) *System* (1) *has a polynomial first integral if and only if $g(v)$ is square-free,* $\deg f < \deg g$, $\gamma_i \in \mathbb{Q}^-$ *for* $i = 1, 2, \ldots, k$ *and* $1 + \gamma_1 + \gamma_2 + \cdots + \gamma_k \geq 0$.

Proof. (a) By statement (i) of Lemma 7 system (1) has a rational first integral if and only if, system (5) has a rational first integral. By statement (vii) of Lemma 4 system (5) has a rational first integral if and only if, $g(v)$ is square-free, $\deg f < \deg g$ and $\gamma_i \in \mathbb{Q}$ for $i = 1, 2, \ldots, k$.

(b) We assume that system (1) has a polynomial first integral. By statement (ii) of Lemma 7 we have that system (5) has a polynomial first integral. By statement (viii) of Lemma 4 we have that $g(v)$ is square-free, $\deg f < \deg g$ and $\gamma_i \in \mathbb{Q}^-$ for $i = 1, 2, \ldots, k$. We need to see that $1 + \gamma_1 + \gamma_2 + \cdots + \gamma_k \geq 0$.

We consider a polynomial first integral of system (1). By Lemma 6, we can suppose that this first integral is a quasi–homogeneous polynomial of weight exponents (s_1, s_2) and of weight degree $m \geq 0$ and as it is a nonconstant polynomial we have that $m > 0$. We denote this first integral by $H_m(x,y)$. As we have seen in the proof of statement (ii) of Lemma 7, we have a polynomial first integral of system (5) of the form $u^{\tilde{m}}h(v)$ with $\tilde{m} > 0$ and $h(v)$ a polynomial. Moreover, as we have seen in the proof of statement (iii) of Lemma 4, the polynomial $h(v)$ has the form

$$h(v) = C\ \left((v - \alpha_1)^{\gamma_1}(v - \alpha_2)^{\gamma_2} \cdots (v - \alpha_k)^{\gamma_k}\right)^{-\tilde{m}},$$

where C is a constant of integration that we can take without loss of generality equal to 1. In fact, due to the change of variables defined in

Lemma 2 we obtain that the first integral $u^{\tilde{m}}h(v)$ is transformed to

$$\begin{aligned}\tilde{H}(x,y) &= (x^{s_2})^{\tilde{m}} h\left(\frac{y^{s_1}}{x^{s_2}}\right)\\ &= x^{s_2\tilde{m}}\left(\left(\frac{y^{s_1}}{x^{s_2}}-\alpha_1\right)^{\gamma_1}\cdots,\left(\frac{y^{s_1}}{x^{s_2}}-\alpha_k\right)^{\gamma_k}\right)^{-\tilde{m}}\\ &= x^{s_2\tilde{m}(1+\gamma_1+\cdots+\gamma_k)}(y^{s_1}-\alpha_1x^{s_2})^{-\gamma_1\tilde{m}}\cdots(y^{s_1}-\alpha_1x^{s_2})^{-\gamma_k\tilde{m}},\end{aligned}$$

and we observe that a certain power of $\tilde{H}$ gives a polynomial first integral, taking into account that $\gamma_i\in\mathbb{Q}^-$ for $i=1,2,\ldots,k$, if and only if $1+\gamma_1+\gamma_2+\cdots+\gamma_k\geq 0$.

Conversely, if we assume that $g(v)$ is square-free, $\deg f<\deg g$, $\gamma_i\in\mathbb{Q}^-$ for $i=1,2,\ldots,k$ and $1+\gamma_1+\gamma_2+\cdots+\gamma_k\geq 0$, by statement (iii) of Lemma 4 we have that system (5) has a polynomial first integral. As $\gamma_i\in\mathbb{Q}^-$, then there exist $N,n_1,n_2,\ldots,n_k\in\mathbb{N}$ such that $\gamma_i=-n_i/N$ for $i=1,2,\ldots,k$. By statement (vi) of Lemma 4 we have that

$$H(u,v)=u^{-1}(v-\alpha_1)^{-\frac{n_1}{N}}\cdots(v-\alpha_k)^{-\frac{n_k}{N}},$$

is a first integral of system (5), and consequently

$$\tilde{H}(u,v)=u^N(v-\alpha_1)^{n_1}\cdots(v-\alpha_k)^{n_k},$$

is a first integral of system (5). Recalling that $1+\gamma_1+\gamma_2+\cdots+\gamma_k\geq 0$ we have $1-n_1/N-n_2/N-\cdots-n_k/N\geq 0$ which implies $N-n_1-n_2-\cdots-n_k\geq 0$. Undoing the change of variables given in Lemma 2 we obtain

$$\begin{aligned}\tilde{H}\left(x^{s_2},\frac{y^{s_1}}{x^{s_2}}\right) &= x^{s_2N}\left(\frac{y^{s_1}}{x^{s_2}}-\alpha_1\right)^{n_1}\cdots\left(\frac{y^{s_1}}{x^{s_2}}-\alpha_k\right)^{n_k}\\ &= x^{s_2(N-n_1-\cdots-n_k)}(y^{s_1}-\alpha_1x^{s_2})^{n_1}\cdots(y^{s_1}-\alpha_kx^{s_2})^{n_k},\end{aligned}$$

which is a polynomial first integral of system (1). □

One small improvement of statement (b) in Theorem 9 is the next result.

Lemma 10. *If $g(v)$ is square-free,* $\deg f<\deg g$ *and $P(x,y)$ has not x as a divisor, then* $1+\gamma_1+\gamma_2+\cdots+\gamma_k=0$.

Proof. We recall that $P(x,y)$ is a quasi–homogeneous polynomial of weight exponents (s_1,s_2) and of weight degree s_1-1+d and that $P(0,0)=0$. Since $P(x,y)$ has not x as a divisor, it has a monomial with nonzero coefficient of the form $y^{\tilde{j}_p}$. By quasi–homogeneity this implies that $s_2\tilde{j}_p=s_1-1+d$. Taking into account that for any monomial $x^{i_p}y^{j_p}$ of $P(x,y)$ we have $s_1i_p+s_2j_p=s_1-1+d$, it follows that $\tilde{j}_p > j_p$ for any other monomial because $s_1 i_p > 0$. Consequently the polynomial

$$f(v) = s_2 v^{\frac{m+1-s_1}{s_1}} P(1, v^{\frac{1}{s_1}})$$

is of degree

$$\frac{m+1-s1+\tilde{j}_p}{s_1} = \tilde{k}-1, \quad \text{where} \quad \tilde{k} = \frac{m+1+\tilde{j}_p}{s_1}.$$

Moreover we observe that if $\tilde{a}$ is the coefficient of the monomial y^{j_p} of $P(x,y)$ then we have $f(v) = s_2\tilde{a}v^{\tilde{k}-1} + \cdots$, where the dots mean terms in v of lower degree. Let $\tilde{j}_q$ be the largest natural such that $x^{\tilde{i}_q} y^{\tilde{j}_q}$ is a monomial with nonzero coefficient of $Q(x,y)$. We denote by $\tilde{b}$ the coefficient of this monomial. By the quasi–homogeneity of $Q(x,y)$ we have $s_1\tilde{i}_q + s_2\tilde{j}_q = s_2 - 1 + d$. Therefore $-s_1 + s_2\tilde{j}_p = d-1$ and $s_1\tilde{i}_q + s_2(\tilde{j}_q - 1) = d-1$, which implies

$$-s_1 + s_2\tilde{j}_p = s_1\tilde{i}_q + s_2(\tilde{j}_q - 1),$$

or equivalently

$$s_2\tilde{j}_p = s_1(\tilde{i}_q + 1) + s_2(\tilde{j}_q - 1).$$

Recalling that $\tilde{i}_q \geq 0$ and $s_1, s_2 > 0$ we obtain $\tilde{j}_p > \tilde{j}_q - 1$, or equivalently $\tilde{j}_p + 1 > \tilde{j}_q$. Hence $g(v)$ takes takes form

$$g(v) = v^{\frac{m}{s_1}}(-s_2\tilde{a}v^{\frac{\tilde{j}_p+1}{s_1}} + \cdots + s_1\tilde{b}v^{\frac{\tilde{j}_q}{s_1}} + \cdots).$$

where the dots mean terms in v of lower degree. Hence $g(v)$ has degree $\tilde{k}$ in v, i.e. $g(v) = -s_2\tilde{a}v^{\tilde{k}} + \cdots$.

Taking the common denominator in equation (17) of statement (vii) of Lemma 4 we obtain

$$\begin{aligned} \frac{f(v)}{g(v)} &= \frac{\gamma_1}{v-\alpha_1} + \frac{\gamma_2}{v-\alpha_2} + \cdots + \frac{\gamma_k}{v-\alpha_k} \\ &= \frac{(\gamma_1+\gamma_2+\cdots+\gamma_k)v^{k-1}+\cdots}{v^k+\cdots}. \end{aligned}$$

Multiplying both sides by $(v-\alpha_1)(v-\alpha_2)\cdots(v-\alpha_k)$ the equality becomes

$$\frac{f(v)}{-s_2\tilde{a}} = \frac{s_2\tilde{a}v^{k-1}+\cdots}{-s_2\tilde{a}} = (\gamma_1+\gamma_2+\cdots+\gamma_k)v^{k-1}+\cdots,$$

and equating coefficients we get $\gamma_1 + \gamma_2 + \cdots + \gamma_k = -1$. □

In Lemma 2 we have shown that any quasi–homogeneous polynomial differential system after a convenient change of variables can be transformed into a polynomial differential system of the form $\dot{u} = uf(v)$,

$\dot{v} = g(v)$ with $f(v)$ and $g(v)$ polynomials. The next result shows that the converse also works.

Proposition 11. *Consider the differential system $\dot{u} = uf(v)$, $\dot{v} = g(v)$ with $f(v)$ and $g(v)$ polynomials. Then there exists a change of variables and a rescaling of the independent variable such that in the new variables the differential system is quasi–homogeneous.*

Proof. Given s_1 and s_2 two positive integers, we consider the change of variables described in (4) and we get the following system:

$$\dot{x} = \frac{x}{s_2}\, f\left(\frac{y^{s_1}}{x^{s_2}}\right), \quad \dot{y} = \frac{y}{s_1}\left[f\left(\frac{y^{s_1}}{x^{s_2}}\right) + \frac{x^{s_2}}{y^{s_1}}\, g\left(\frac{y^{s_1}}{x^{s_2}}\right)\right].$$

In order to get a polynomial system we multiply the system by $x^{s_2\tilde{m}}y^{s_1-1}$, where $\tilde{m} = \max\{\deg f, \deg g - 1\}$, and we get the following polynomial system:

$$\begin{aligned} \dot{x} &= \frac{x}{s_2}\, x^{s_2\tilde{m}}y^{s_1-1}\, f\left(\frac{y^{s_1}}{x^{s_2}}\right) \\ \dot{y} &= \frac{y}{s_1}\, x^{s_2\tilde{m}}y^{s_1-1}\left[f\left(\frac{y^{s_1}}{x^{s_2}}\right) + \frac{x^{s_2}}{y^{s_1}}\, g\left(\frac{y^{s_1}}{x^{s_2}}\right)\right]. \end{aligned} \tag{23}$$

We only need to prove that system (23) is quasi–homogeneous with weight exponents (s_1, s_2). We denote by $P(x,y)$ and $Q(x,y)$ the polynomials such that $\dot{x} = P(x,y)$ and $\dot{y} = Q(x,y)$ and we want to show that identities (2) are satisfied for arbitrary $\alpha \in \mathbb{C}$. We define $d = 1 + s_1 s_2 \tilde{m} + s_2(s_1 - 1)$ and have that

$$\begin{aligned} P(\alpha^{s_1}x, \alpha^{s_2}y) &= \frac{\alpha^{s_1}x}{s_2}\, \alpha^{s_1 s_2\tilde{m}}x^{s_2\tilde{m}}\alpha^{s_2(s_1-1)}y^{s_1-1}\, f\left(\frac{\alpha^{s_2 s_1}y^{s_1}}{\alpha^{s_1 s_2}x^{s_2}}\right) \\ &= \alpha^{s_1+d-1}\, \frac{x}{s_2}\, x^{s_2\tilde{m}}y^{s_1-1}\, f\left(\frac{y^{s_1}}{x^{s_2}}\right) \\ &= \alpha^{s_1+d-1}P(x,y), \\ Q(\alpha^{s_1}x, \alpha^{s_2}y) &= \frac{\alpha^{s_2}y}{s_1}\, \alpha^{s_1 s_2\tilde{m}}x^{s_2\tilde{m}}\alpha^{s_2(s_1-1)}y^{s_1-1}\left[f\left(\frac{\alpha^{s_2 s_1}y^{s_1}}{\alpha^{s_1 s_2}x^{s_2}}\right)\right. \\ &\qquad \left. + \frac{\alpha^{s_1 s_2}x^{s_2}}{\alpha^{s_2 s_1}y^{s_1}}\, g\left(\frac{\alpha^{s_2 s_1}y^{s_1}}{\alpha^{s_1 s_2}x^{s_2}}\right)\right] \\ &= \alpha^{s_2+d-1}\, \frac{y}{s_1}\, x^{s_2\tilde{m}}y^{s_1-1}\left[f\left(\frac{y^{s_1}}{x^{s_2}}\right) + \frac{x^{s_2}}{y^{s_1}}\, g\left(\frac{y^{s_1}}{x^{s_2}}\right)\right] \\ &= \alpha^{s_2+d-1}Q(x,y). \end{aligned}$$

Therefore, system (23) is a quasi–homogeneous polynomial differential system of degree (s_1, s_2, d). $\square$

4. Acknowledgements

The first and second authors are partially supported by a MICINN/ FEDER grant number MTM2011-22877 and by a AGAUR (Generalitat de Catalunya) grant number 2009SGR 381. The third author is partially supported by a MICINN/ FEDER grant number MTM2008-03437, by a AGAUR grant number 2009SGR 410, by ICREA Academia and by FP7-PEOPLE-2012-IRSES-316338.

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