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ON THE DARBOUX INTEGRABILITY OF BLASIUS AND FALKNER–SKAN EQUATION

JAUME LLIBRE[1], AND CLAUDIA VALLS[2]

ABSTRACT. We study the Darboux integrability of the celebrated Falkner–Skan equation $f''' + ff'' + \lambda(1 - f'^2) = 0$, where λ is a parameter. When $\lambda = 0$ this equation is known as Blasius equation. We show that both differential systems have no first integrals of Darboux type. Additionally we compute all the Darboux polynomials and all the exponential factors of these differential equations.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The *Falkner–Skan equation* is

$$f''' + ff'' + \lambda(1 - f'^2) = 0, \tag{1}$$

where $\lambda \in \mathbb{R}$ is a parameter. This equation was first derived in [6] as a model of the steady two–dimensional flow of a slightly viscous incompressible fluid past a wedge. The special case $\lambda = 0$, in which the wedge reduces to a flat plate, is called *Blasius equation* and was considered for a first time in [2].

Both equations are the subject of an extensive literature. For the derivation of this equation see [1]. For the existence and uniqueness of the solutions

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see, for example, [16], [19], [5], [15], [3] and [10] and references therein. Recently there has been also a renewed interest in the mathematical aspects of the Falkner–Skan equation. The dynamic features of this equation such as the existence of oscillating and periodic orbits have been studied in [7], [8], [9] and [13]; and for more recent works on the bifurcations in this equation see [11], [18] and [17].

In MathSciNet appears in this moment 214 articles related with the Falkner–Skan equation, but any of these papers analyze the integrability or non–integrability of this equation. In this work we are interested in the Darboux integrability of Blasius and Falkner–Skan equation. Before we state our main result (Theorem 1) we need to introduce some definitions and auxiliary results.

We can express (1) as a system of differential equations

$$\dot{x} = y, \quad \dot{y} = z \quad \dot{z} = -xz - \lambda(1 - y^2), \tag{2}$$

and the associated vector field is

$$\mathcal{X} = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + [-xz - \lambda(1 - y^2)]\frac{\partial}{\partial z}. \tag{3}$$

Let $U \subset \mathbb{R}^3$ be an open subset. We say that the non–constant function $H\colon U \to \mathbb{R}$ is the *first integral* of the polynomial vector field (3) on U associated to system (2), if $H(x(t), y(t), z(t)) =$ constant for all values of t for which the solution $(x(t), y(t), z(t))$ of $\mathcal{X}$ is defined on U. Clearly H

is a first integral of $\mathcal{X}$ on U if and only if $\mathcal{X}H = 0$ on U. When H is a polynomial we say that H is a *polynomial first integral*.

For proving our main results concerning the existence of first integrals of Darboux type we shall use the invariant algebraic surfaces of system (2). This is the basis of the Darboux theory of integrability. The Darboux theory of integrability works for real or complex polynomial ordinary differential equations. The study of complex invariant algebraic curves is necessary for obtaining all the real first integrals of a real polynomial differential equation, for more details see [12].

Let $h = h(x, y, z) \in \mathbb{C}[x, y, z]$ be a non–constant polynomial. We say that $h = 0$ is an *invariant algebraic surface* of the vector field $\mathcal{X}$ in (3) if it satisfies $\mathcal{X}h = Kh$, for some polynomial $K = K(x, y, z) \in \mathbb{C}[x, y, z]$, called the *cofactor* of h. Note that K has degree at most 1. The polynomial h is called a *Darboux polynomial*, and we also say that K is the *cofactor* of the Darboux polynomial h. We note that a Darboux polynomial with zero cofactor is a polynomial first integral.

Let $g, h \in \mathbb{C}[x, y, z]$ be coprime. We say that a non–constant function $E = e^{h/g}$ is an *exponential factor* of the vector field $\mathcal{X}$ given in (3) if it satisfies $\mathcal{X}E = LE$, for some polynomial $L = L(x, y, z) \in \mathbb{C}[x, y, z]$, called the *cofactor* of E and having degree at most 1. This relation is equivalent to

$$y\frac{\partial(g/h)}{\partial x} + z\frac{\partial(g/h)}{\partial y} + [-xz - \lambda(1 - y^2)]\frac{\partial(g/h)}{\partial z} = K. \tag{4}$$

For a geometrical and algebraic meaning of the exponential factors see [4].

A first integral G of system (2) is called of *Darboux type* if it is of the form

$$G = f_1^{\lambda_1} \cdots f_p^{\lambda_p} E_1^{\mu_1} \cdots E_q^{\mu_q},$$

where $f_1, \ldots, f_p$ are Darboux polynomials, $E_1, \ldots, E_q$ are exponential factors and $\lambda_j, \mu_k \in \mathbb{C}$ for $j = 1, \ldots, p$, $k = 1, \ldots, q$. For more information on the Darboux theory of integrability see, for instance, [12, 14] and the references quoted there.

The main result of this paper is the following:

Theorem 1. *For the Falkner–Skan and Blasius system the following statements hold.*

(a) *Both systems have no polynomial first integrals.*

(b) *The unique irreducible Darboux polynomial with nonzero cofactor of the Blasius system is z; the unique Darboux polynomial of the Falkner–Skan system is $1 - y^2 + 2xz$ if $\lambda = 1/2$.*

(c) *The unique exponential factors of both systems are e^x and e^y, except if $\lambda = -1$ then we have the additional exponential factor e^{z+xy}.*

(d) *Both systems have no first integrals of Darboux type.*

Theorem 1 is proved in the next section.

2. Proof of Theorem 1

We separate the proof of Theorem 1 in four propositions, one for every statement.

Proposition 2. *System* (2) *has no polynomial first integrals.*

Proof. Let h be a polynomial first integral of system (2). Then it satisfies

$$y\frac{\partial h}{\partial x}+z\frac{\partial h}{\partial y}+[-xz-\lambda(1-y^2)]\frac{\partial h}{\partial z}=0. \tag{5}$$

Without loss of generality we can write $h=\sum_{j=1}^{n}h_j(x,y,z)$, where each $h_j=h_j(x,y,z)$ is a homogeneous polynomial of degree j, and $h_n\neq 0$.

Computing the terms of degree $n+1$ in (5) we get

$$[-xz+\lambda y^2]\frac{\partial h_n}{\partial z}=0.$$

Therefore $h_n=h_n(x,y)$.

Computing the terms of degree n in (5) we get that

$$y\frac{\partial h_n}{\partial x}+z\frac{\partial h_n}{\partial y}+[-xz+\lambda y^2]\frac{\partial h_{n-1}}{\partial z}=0,$$

that is

$$h_{n-1}=g_{n-1}(x,y)+\frac{z}{x}\frac{\partial h_n}{\partial y}+\frac{y}{x^2}\log(xz-\lambda y^2)\left[\lambda y\frac{\partial h_n}{\partial y}+x\frac{\partial h_n}{\partial x}\right],$$

where $g_{n-1}(x,y)$ is a function in the variables x and y. Since h_{n-1} is a polynomial, we have

$$\lambda y \frac{\partial h_n}{\partial y} + x \frac{\partial h_n}{\partial x} = 0.$$

Therefore $h_n = h_n(x^{-\lambda}y)$. Since $h_n \neq 0$ is a homogeneous polynomial of degree $n \geq 1$, we have $\lambda = -p/q$ with p and q integers such that $p \geq 0$, $q \geq 1$, $p+q = n$ and $h_n = \alpha_n x^p y^q$, where $\alpha_n \in \mathbb{C} \setminus \{0\}$. Of course in the case of the Blasius system $p = 0$. In short we get that $h_n = \alpha_n x^p y^q$.

Now we have that

$$h_{n-1}(x,y,z) = \alpha_n q x^{p-1} y^{q-1} z + g_{n-1}(x,y).$$

So for the Blasius system ($p = 0$) we have a contradiction with the fact that $h_{n-1}(x,y,z)$ is a homogeneous polynomial of degree $n-1$. Thus the proposition is proved for the Blasius system. In what follows we assume that $p \geq 1$, i.e. we restrict our attention to the Falkner–Skan system.

Computing the terms of degree $n-1$ in (5) we get that

$$y \frac{\partial h_{n-1}}{\partial x} + z \frac{\partial h_{n-1}}{\partial y} + [-xz + \lambda y^2] \frac{\partial h_{n-2}}{\partial z} = 0,$$

that is

$$\begin{aligned} h_{n-2} =& g_{n-2}(x,y) + \frac{1}{2}\alpha_n \left[2(p-q)y^2 + (q-1)qxz\right] x^{p-3} y^{q-2} z + \\ & \frac{z}{x}\frac{\partial g_{n-1}}{\partial y} + \frac{y}{qx^4} \log\left(py^2 + qxz\right) A, \end{aligned}$$

where

$$A = -\alpha_n p(p-q)x^p y^{q+1} + qx^3\frac{\partial g_{n-1}}{\partial x} - px^2y\frac{\partial g_{n-1}}{\partial y}.$$

Since h_{n-2} is a homogeneous polynomial of degree $n-2$ we have that $A = 0$. Solving this linear partial differential equation we get that

$$g_{n-1}(x, y, z) = -\frac{1}{p+2q}\alpha_n p(p-q)x^{p-2}y^{q+1} + G\big(x^{p/q}y\big).$$

Since $p + q = n$ and g_{n-1} is a homogeneous polynomial of degree $n-1$ we have that

$$g_{n-1}(x, y, z) = -\frac{1}{p+2q}\alpha_n p(p-q)x^{p-2}y^{q+1}.$$

Therefore

$$\begin{aligned} h_{n-1} =& \alpha_n q x^{p-1}y^{q-1}z - \frac{1}{p+2q}\alpha_n p(p-q)x^{p-2}y^{q+1}, \\ h_{n-2} =& g_{n-2}(x,y) + \frac{1}{2}\alpha_n\left[2(p-q)y^2 + (q-1)qxz\right]x^{p-3}y^{q-2}z- \\ & \frac{1}{p+2q}\alpha_n p(p-q)(q+1)x^{p-3}y^q z. \end{aligned}$$

Note that h_{n-1} is a polynomial if $p \geq 1$ and $q \geq 1$, so $n \geq 2$; and that h_{n-2} is a polynomial if $p \geq 3$ and $q \geq 2$, so $n \geq 5$.

Working in a similar way with the terms of degree $n-2$ in (5) we get that

$$\begin{aligned} g_{n-2} =& \frac{\alpha_n x^{p-4}y^{q-2}}{2q(p+2q)^2}\big[-q^2(p+2q)^2x^4+ \\ & p(p^3(q-1) - 6q^3 + pq^2(8+q) - p^2q(1+2q))y^4\big] \end{aligned}$$

and

$$h_{n-3} = \frac{1}{6(p+2q)^2}\Big[6(p+2q)^2 g_{n-3}(x,y) + \alpha_n x^{p-5} y^{q-3} z\big(3(2-q)q(p+2q)^2x^4 +$$
$$3(p-4)(p-q)(6q^2-p^2-2pq+p^2q-pq^2)y^4 +$$
$$3(-p+q)(p+2q)(p+2q-pq-4q^2+pq^2)xy^2z +$$
$$(q-2)(q-1)q(p+2q)^2x^2z^2\big)\Big],$$

where g_{n-3} is an arbitrary polynomial of degree $n-3$. Since h_{n-3} is a polynomial we must have that $p \geq 5$ and $q \geq 3$, so $n \geq 8$.

In short if there exists a polynomial first integral of degree n, then h_{n-s} is a polynomial if $p \geq 2s-1$ and $q \geq s$, so $n \geq 3s-1$. Thus for $s = n-1$ we get that h_1 is a polynomial if $n \geq 3(n-1)-1 = 3n-4$, i.e. if and only if $4 \geq 2n$, or equivalently if $n = 1, 2$. But direct computations shows that system (2) has no polynomial first integrals of degree 1 and 2. This completes the proof of the proposition. □

The following result on the Darboux polynomials is well known, see for instance [12]. This result shows that it is sufficient to know the irreducible Darboux polynomials.

Proposition 3. *Let f be a polynomial and $f = \prod_{j=1}^{s} f_j^{\alpha_j}$ its decomposition into irreducible factors in $\mathbb{C}[x,y,z]$. Then f is a Darboux polynomial if and only if all the f_j are Darboux polynomials. Moreover, if K and K_j are the cofactors of f and f_j, then $K = \sum_{j=1}^{s} \alpha_j K_j$.*

Proposition 4. *The following statements hold.*

(a) *If $\lambda = 0$ then the unique irreducible Darboux polynomial with nonzero cofactor of system* (2) *is z.*

(b) *If $\lambda \neq 0$ then system* (2) *has no Darboux polynomials, except if $\lambda = 1/2$ then $xz - y^2/2 + 1/2$ is the unique irreducible Darboux polynomial.*

Proof. Let h be an irreducible Darboux polynomial of system (2) with a nonzero cofactor K. Then $K = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 z$, with $\alpha_i \in \mathbb{C}$ for $i = 0, 1, 2, 3$ not all zero.

It is easy to see by direct computations that if h has degree 1, then $h = z$ and $\lambda = 0$.

Now we assume that h has degree greater ≥ 2. Then it satisfies

$$y\frac{\partial h}{\partial x} + z\frac{\partial h}{\partial y} - [xz + \lambda(1 - y^2)]\frac{\partial h}{\partial y} = (\alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 z)h. \tag{6}$$

We write $h = \sum_{j=0}^{n} h_j(x, y, z)$, where each $h_j = h_j(x, y, z)$ is a homogeneous polynomial of degree j and $h_n \neq 0$.

Computing the terms of degree $n + 1$ in (6) we get that

$$(\lambda y^2 - xz)\frac{\partial h_n}{\partial z} = (\alpha_1 x + \alpha_2 y + \alpha_3 z)h_n.$$

Solving this linear differential equation we obtain

$$h_n = g_n(x, y)e^{-\alpha_3 z/x}(xz - \lambda y^2)^{-\alpha_1 - y(\alpha_2 x + \alpha_3 \lambda y)/x^2},$$

where $g_n(x,y)$ is a function in the variables x and y. Since h_n is a homogeneous polynomial of degree n we must have $\alpha_3 = \alpha_2 = 0$ and $\alpha_1 = -m$ for some integer m with $0 \le m \le n/2$ and $g_n \ne 0$. That is $h_n = g_n(x,y)(xz - \lambda y^2)^m$ with g_n a homogeneous polynomial of degree $n - 2m$.

Assume $\lambda = 0$. Then, using that $z = 0$ is invariant for system (2), we can write $h = \bar{h}(x,y) + z h_1(x,y,z)$. Then $\bar{h}$ satisfies (6) restricted to $z = 0$ (and with $\alpha_1 = -m$, $\alpha_2 = \alpha_3 = 0$). Then $\bar{h} \ne 0$ (otherwise h would be reducible) satisfies

$$y\frac{\partial \bar{h}}{\partial x} = (\alpha_0 - mx)\bar{h}.$$

Solving this linear differential equation we get that

$$\bar{h} = e^{-\alpha_0 x/y - mx^2/(2y)}\bar{g}(y),$$

where $\bar{g}(y)$ is a function in the variable y. Using that $\bar{h}$ is a polynomial we have $\alpha_0 = m = 0$. But then $K = 0$, a contradiction with the fact that h is a Darboux polynomial with nonzero cofactor. This concludes the proof of statement (a).

Now assume that $\lambda = 1/2$. Then, it is easy to check that if h has degree 2, then $h = T = xz - y^2 - 1/2 = 0$ is invariant for system (2). Therefore we assume that the degree of h is at least 3. Then if we rewrite system (2) in the variables (x, y, T) with $z = (T + y^2/2 - 1/2)/x$ we obtain the system

$$\dot{x} = y, \quad \dot{y} = \frac{T + y^2/2 - 1/2}{x}, \quad \dot{T} = -xT.$$

Then we have that $h^*(x,y,T) = h(x,y,z)$ and it can be written as $h^*(x,y,T) = \tilde{h}(x,y) + T\tilde{h}_1(x,y,T)$, where $\tilde{h}(x,y)$ is a polynomial in the variable y (not in the variable x) and $\tilde{h}_1$ is a polynomial in the variables x and T. Now equation (6) written in the variables x, y and T, and restricted to $T = 0$ becomes

$$y\frac{\partial \tilde{h}}{\partial x} + \frac{y^2-1}{2x}\frac{\partial \tilde{h}}{\partial y} = (\alpha_0 - mx)\tilde{h}.$$

Solving this partial differential equation we get that

$$\tilde{h} = e^{2xy\left[mx(y^2-3)-3\alpha_0(y^2-1)\right]/\left(3(y^2-1)^2\right)}\tilde{g}\left[\frac{1}{2}\log\left(\frac{y^2-1}{x}\right)\right],$$

where $\tilde{g}(\cdot)$ is a function in its variable. Using that $\tilde{h}$ is a polynomial in the variable y we have $\alpha_0 = m = 0$. But then $K = 0$, a contradiction with the fact that h is a Darboux polynomial with nonzero cofactor. This concludes the proof of statement (b) when $\lambda = 1/2$.

Suppose h is an irreducible Darboux polynomial of system (2) with $\lambda \notin \{0, 1/2\}$ and with nonzero cofactor K. Then $K = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 z$, with $\alpha_i \in \mathbb{C}$ for $i = 0, 1, 2, 3$ not all zero.

We shall assume that the polynomial h has degree ≥ 2, because it is easy to see by direct computations that system (2) with $\lambda \notin \{0, 1/2\}$ has no Darboux polynomials of degree 1.

Clearly a Darboux polynomial h satisfies

$$(7)\qquad y\frac{\partial h}{\partial x} + z\frac{\partial h}{\partial y} + [-xz - \lambda(1-y^2)]\frac{\partial h}{\partial z} = (\alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 z)h.$$

We write h as $h = \sum_{j=0}^{n} h_j(x, y, z)$, where each $h_j = h_j(x, y, z)$ is a homogeneous polynomial of degree j and $h_n \neq 0$ with $n > 2$.

Computing the terms of degree $n + 1$ in (7) we get that

$$[-xz + \lambda y^2]\frac{\partial h_n}{\partial z} = (\alpha_1 x + \alpha_2 y + \alpha_3 z)h_n.$$

Solving this linear differential equation we obtain

$$h_n = g_n(x, y)e^{-\alpha_3 z/x}(xz - \lambda y^2)^{-\alpha_3\lambda y^2/x^2 - \alpha_2 y/x - \alpha_1},$$

where $g_n = g_n(x, y)$ is a function in the variables x, y. Since h_n is a homogeneous polynomial of degree n we must have $\alpha_3 = \alpha_2 = 0$ and $\alpha_1 = -m$ for some integer m with $0 \leq 2m \leq n$ and g_n is a homogeneous polynomial of degree $n - 2m$ in x and y. Then

$$h_n = (xz - \lambda y^2)^m g_n,$$

for some integer m such that $0 \leq 2m \leq n$.

Computing the terms of degree n in (7) we get that

$$y\frac{\partial h_n}{\partial x} + z\frac{\partial h_n}{\partial y} + [-xz + \lambda y^2]\frac{\partial h_{n-1}}{\partial z} - \alpha_0 h_n + mxh_{n-1} = 0.$$

Solving it we obtain

$$h_{n-1} = \left(xz - \lambda y^2\right)^m \Bigg[g_{n-1}(x,y) + \frac{1}{x^2}\left(\frac{m\lambda(2\lambda-1)y^3 g_n}{xz - y^2\lambda} + \left(xz - \lambda y^2\right)\frac{\partial g_n}{\partial y} + \right.$$
$$\left. \log\left(xz - \lambda y^2\right)\left[(my(1-2\lambda) - \alpha_0 x)g_n + \lambda y^2 \frac{\partial g_n}{\partial y} + xy\frac{\partial g_n}{\partial x}\right]\right)\Bigg],$$

where $g_{n-1} = g_{n-1}(x,y)$ is a function in the variables x, y. Since h_{n-1} is a homogeneous polynomial of degree $n-1$ we have that

$$[my(1-2\lambda) - \alpha_0 x]g_n + y\left(\lambda y \frac{\partial g_n}{\partial y} + x\frac{\partial g_n}{\partial x}\right) = 0.$$

From here and if $\lambda \neq 1$ we get that

$$g_n = e^{\alpha_0 x/((1-\lambda)y)} x^{(2\lambda-1)m} G\left(x^{-\lambda} y\right), \tag{8}$$

where G is an arbitrary function in the variable $x^{-\lambda}y$. If $\lambda = 1$ then

$$g_n = x^{m+\alpha_0 x/y} G\left(\frac{y}{x}\right), \tag{9}$$

again G is an arbitrary function in the variable y/x. Since g_n is a polynomial of degree $n-2m$ we have that $\alpha_0 = 0$. Since the cofactor of h cannot be zero, we have that $\alpha_1 = -m \neq 0$. So $m > 0$. Note that once $\alpha_0 = 0$ the expression (8) with $\lambda = 1$ coincides with the expression (9). In order that g_n be a polynomial we need that $G(s) = s^r$ with r some non–negative integer.

Therefore, since the degree of g_n is $n-2m$ we have

$$(2\lambda-1)m+r-\lambda r=n-2m, \quad \text{or equivalently} \quad \lambda=\frac{n-m-r}{2m-r}.$$

Thus $2m-r\neq 0$. Then

$$h_n=\left(xz+\frac{m-n+r}{2m-r}y^2\right)^m x^{n-2m-r}y^r,$$

with $n\geq 2m+r$ and

$$h_{n-1}=\left(xz+\frac{m-n+r}{2m-r}y^2\right)^m\Big[2(2m-r)r(m-n+r)xy^2z+$$
$$r(r-2m)^2x^2z^2+2(2m-r)r(m-n+r)xy^2z+g_{n-1}+$$
$$\frac{(m-n+r)\left(4m^2+(2m+r)(r-n)\right)x^{n-2m-r-2}y^{r+3}}{(2m-r)\left((m-n+r)y^2+(2m-r)xz\right)}\Big],$$

where g_{n-1} is an arbitrary homogeneous polynomial of degree $n-2m$.

Computing the terms of degree $n-1$ in (7) we get

$$y\frac{\partial h_{n-1}}{\partial x}+z\frac{\partial h_{n-1}}{\partial y}-\left(xz+\frac{m-n+r}{2m-r}y^2\right)\frac{\partial h_{n-2}}{\partial z}+\frac{m-n+r}{2m-r}\frac{\partial h_n}{\partial z}+mxh_{n-2}=0.$$

Clearly

$$f(x,y,z)=y\frac{\partial h_{n-1}}{\partial x}+z\frac{\partial h_{n-1}}{\partial y}+\frac{m-n+r}{2m-r}\frac{\partial h_n}{\partial z}$$

is a homogeneous polynomial of degree $n-1$. Solving the ordinary differential equation

$$-\left(xz+\frac{m-n+r}{2m-r}y^2\right)\frac{\partial h_{n-2}}{\partial z}+mxh_{n-2}+f(x,y,z)=0,$$

we obtain that

$$h_{n-2}(x,y,z) = \Big[(m-n+r)y^2+(2m-r)xz\Big]^m \Big[K+(2m-r) \int \Big[(m-n+r)y^2+(2m-r)xz\Big]^{-m-1} f(x,y,z)\,dz\Big],$$

where K is a constant of integration. Since the integrals

$$\Big[(m-n+r)y^2+(2m-r)xz\Big]^m \int \Big[(m-n+r)y^2+(2m-r)xz\Big]^{-m-1} z^k\,dz$$

for $k = 0, 1, \ldots$ are rational functions of the form

$$\frac{p_{2k}(x,y,z)}{x^{k+1}} = \frac{-k!\,(m-n+r)^k y^{2k} + \cdots}{C_k x^{k+1}},$$

where $p_{2k}(x,y,z)$ is a homogeneous polynomial of degree $2k$ coprime with x and C_k is a constant. Recall that $m-n+r \neq 0$ because $\lambda \neq 0$. Therefore $h_{n-2}(x,y,z)$ can never be a polynomial. So there are no Darboux polynomials $h(x,y,z)$ of degree $n > 2$.

□

For a proof of the next proposition see [4].

Proposition 5. *The following statements hold.*

(a) *If $E = e^{g/h}$ is an exponential factor for the polynomial system* (2) *and h is not a constant polynomial, then $h = 0$ is an invariant algebraic curve.*

(b) *Eventually e^g can be an exponential factor, coming from the multiplicity of the infinite invariant straight line.*

Proposition 6. *The unique exponential factors E of system* (2) *are e^x and e^y, except if $\lambda = -1$ then we have the additional exponential factor e^{z+xy}.*

Proof. First we prove the proposition for the Blasius system, i.e. for (2) with $\lambda = 0$. It follows from Propositions 5 and 4(a) that we can write $E = e^{g/z^n}$ with n being a non–negative integer. Clearly, after simplifying by z^n equation (4), g satisfies

$$y\frac{\partial g}{\partial x} + z\frac{\partial g}{\partial y} - xz\frac{\partial g}{\partial z} + nxg = (\beta_0 + \beta_1 x + \beta_2 y + \beta_3 z)z^n, \tag{10}$$

with $\beta_i \in \mathbb{C}$ for $i = 0, 1, 2, 3$ not all zero, otherwise g would be a Darboux polynomial coprime with z for the Blasius system, and by Proposition 4(a) we know that the Blasius system has no such Darboux polynomials.

Denoting $\bar{g} = \bar{g}(x, y) = g(x, y, 0)$ we get that $\bar{g} \neq 0$ (otherwise g and z^n would not be coprime) satisfies (10) restricted to $z = 0$, i.e. it satisfies

$$y\frac{\partial \bar{g}}{\partial x} + nx\bar{g} = 0.$$

Solving this linear equation we obtain $\bar{g} = e^{-nx^2/(2y)}g_1(y)$, where g_1 is a function in y. Since $\bar{g}$ must be a polynomial we get that $n = 0$. Therefore we have that $E = e^g$ and it satisfies

$$y\frac{\partial g}{\partial x} + z\frac{\partial g}{\partial y} - xz\frac{\partial g}{\partial z} = \beta_0 + \beta_1 x + \beta_2 y + \beta_3 z.$$

We consider $F = Ee^{-\beta_2 x - \beta_3 y}$. Then we can write $F = e^h$ with $h = g - \beta_2 x - \beta_3 y \in \mathbb{C}[x, y, z]$ that satisfies

$$y\frac{\partial h}{\partial x} + z\frac{\partial h}{\partial y} - xz\frac{\partial h}{\partial z} = \beta_0 + \beta_1 x. \tag{11}$$

Restricting (11) to $z = 0$ and setting $\bar{h} = h(x, y, 0)$ we get that

$$y\frac{\partial \bar{h}}{\partial x} = \beta_0 + \beta_1 x,$$

and hence

$$\bar{h} = f(y) + \frac{\beta_0 x}{y} + \frac{\beta_1 x^2}{2y},$$

where $f(y)$ is a function in the variable y. Since $\bar{h}$ is a polynomial we obtain that $\beta_0 = \beta_1 = 0$. This implies from (11) that h must be zero or a polynomial first integral of the Blasius system. From Proposition 2 this last case is not possible, then $h = 0$. Therefore $E = e^{\beta_2 x + \beta_3 y}$. This concludes the proof of the proposition for the Blasius system.

Now we consider the Falkner–Skan system. It follows from Proposition 5 that we can write $E = e^g$ and from (4) g satisfies

$$y\frac{\partial g}{\partial x} + z\frac{\partial g}{\partial y} + [-xz - \lambda(1 - y^2)]\frac{\partial g}{\partial z} = \beta_0 + \beta_1 x + \beta_2 y + \beta_3 z,$$

with $\beta_i \in \mathbb{C}$, $i = 0, 1, 2, 3$ not all zero, otherwise g would be a polynomial first integral of the Falkner–Skan system, and by Proposition 2 such first integrals do not exist.

We consider $F = Ee^{-\beta_2 x - \beta_3 y}$. Then we can write $F = e^h$ with $h = g - \beta_2 x - \beta_3 y \in \mathbb{C}[x, y, z]$ that satisfies

$$y\frac{\partial h}{\partial x} + z\frac{\partial h}{\partial y} + [-xz - \lambda(1 - y^2)]\frac{\partial h}{\partial z} = \beta_0 + \beta_1 x, \tag{12}$$

We write $h = \sum_{j=0}^{n} h_j$ where each $h_j = h_j(x, y, z)$ is a homogeneous polynomial of degree j, $h_n \neq 0$ and $n \geq 1$.

It is easy to check by direct computations that for $n = 1$ equation (12) has no solution, and for $n = 2$ it has only one solution $h(x, y, z) = z + xy$ when $\lambda = -1$, $\beta_0 = 1$ and $\beta_1 = 0$. So in what follows we consider $n > 2$.

We get that the terms of degree $n + 1$ in (12) satisfy

$$[-xz + \lambda y^2]\frac{\partial h_n}{\partial z} = 0.$$

Thus $h_n = h_n(x, y)$ a homogeneous polynomial of degree n in the variables (x, y). Computing the terms of degree n in (12) we get that

$$y\frac{\partial h_n}{\partial x} + z\frac{\partial h_n}{\partial y} + [-xz + \lambda y^2]\frac{\partial h_{n-1}}{\partial z} = 0.$$

Its solution is

$$h_{n-1} = \tilde{h}_{n-1}(x, y) + \frac{z}{x}\frac{\partial h_n}{\partial y} + \frac{y}{x^2}\log(xz - \lambda y^2)\left[\lambda y \frac{\partial h_n}{\partial y} + x\frac{\partial h_n}{\partial x}\right],$$

where $\tilde{h}_{n-1} = \tilde{h}_{n-1}(x, y)$ is an arbitrary function in the variables x and y. Since h_{n-1} is a polynomial we obtain

$$\lambda y \frac{\partial h_n}{\partial y} + x\frac{\partial h_n}{\partial x} = 0,$$

and hence $h_n = h_n(x^{-\lambda} y)$. Since $h_n \neq 0$ is a homogeneous polynomial of degree $n > 1$, we have $\lambda = -p/q$ with p and q integers such that $p \geq 1$, $q \geq 1$, $p + q = n$ and $h_n = \alpha_n x^p y^q$ where $\alpha_n \in \mathbb{C} \setminus \{0\}$. Of course $p \geq 1$ because $\lambda \neq 0$. In short we have that $h_n = \alpha_n x^p y^q$. Therefore

$$h_{n-1} = \tilde{h}_{n-1}(x, y) + \alpha_n q x^{p-1} y^{q-1} z.$$

Computing the terms of degree $n - 1$ in (12) we get that

$$y\frac{\partial h_{n-1}}{\partial x} + z\frac{\partial h_{n-1}}{\partial y} + [-xz + \lambda y^2]\frac{\partial h_{n-2}}{\partial z} = 0,$$

that is

$$\begin{aligned} h_{n-2} =& \tilde{h}_{n-2}(x, y) + \frac{1}{2}\alpha_n\left[2(p-q)y^2 + (q-1)qxz\right]x^{p-3}y^{q-2}z + \\ & \frac{z}{x}\frac{\partial \tilde{h}_{n-1}}{\partial y} + \frac{y}{qx^4}\log\left(py^2 + qxz\right)A, \end{aligned}$$

where

$$A = -\alpha_n p(p-q)x^p y^{q+1} + qx^3\frac{\partial \tilde{h}_{n-1}}{\partial x} - px^2y\frac{\partial \tilde{h}_{n-1}}{\partial y}.$$

Since h_{n-2} must be a homogeneous polynomial of degree $n-2$ we must have that $A = 0$. Solving this linear partial differential equation we get that

$$\tilde{h}_{n-1}(x,y,z) = -\frac{1}{p+2q}\alpha_n p(p-q)x^{p-2}y^{q+1} + G\big(x^{p/q}y\big).$$

Since $p+q=n$ and $\tilde{h}_{n-1}$ is a homogeneous polynomial of degree $n-1$ we obtain

$$\tilde{h}_{n-1}(x,y,z) = -\frac{1}{p+2q}\alpha_n p(p-q)x^{p-2}y^{q+1}.$$

Therefore

$$\begin{aligned}
h_n &= \alpha_n x^p y^q,\\
h_{n-1} &= \alpha_n q x^{p-1}y^{q-1}z - \frac{1}{p+2q}\alpha_n p(p-q)x^{p-2}y^{q+1},\\
h_{n-2} &= \tilde{h}_{n-2}(x,y) + \frac{1}{2}\alpha_n\left[2(p-q)y^2 + (q-1)qxz\right]x^{p-3}y^{q-2}z -\\
&\quad \frac{1}{p+2q}\alpha_n p(p-q)(q+1)x^{p-3}y^q z.
\end{aligned}$$

Note that h_{n-1} is a polynomial if $p \geq 1$ and $q \geq 1$, so $n \geq 2$; and that h_{n-2} is a polynomial if $p \geq 3$ and $q \geq 2$, so $n \geq 5$.

Working in a similar way with the terms of degree $n-2$ in (5) we get that

$$\begin{aligned}
\tilde{h}_{n-2} =& \frac{\alpha_n x^{p-4}y^{q-2}}{2q(p+2q)^2}[-q^2(p+2q)^2x^4 +\\
& p(p^3(q-1) - 6q^3 + pq^2(8+q) - p^2q(1+2q))y^4]
\end{aligned}$$

and

$$h_{n-3} = \frac{1}{6(p+2q)^2}\Big[6(p+2q)^2\tilde{h}_{n-3}(x,y) + \alpha_n x^{p-5}y^{q-3}z\big(3(2-q)q(p+2q)^2x^4 +$$
$$3(p-4)(p-q)(6q^2-p^2-2pq+p^2q-pq^2)y^4 +$$
$$3(-p+q)(p+2q)(p+2q-pq-4q^2+pq^2)xy^2z +$$
$$(q-2)(q-1)q(p+2q)^2x^2z^2\big)\Big],$$

where $\tilde{h}_{n-3}$ is an arbitrary polynomial of degree $n-3$. Since h_{n-3} is a polynomial we have that $p \geq 5$ and $q \geq 3$, so $n \geq 8$.

In short if there exists an exponential factor e^h satisfying (12), with h a polynomial of degree n, then h_{n-s} is a polynomial if $p \geq 2s-1$ and $q \geq s$, so $n \geq 3s-1$. Thus for $s = n-1$ we get that h_1 is a polynomial if $n \geq 3(n-1)-1 = 3n-4$, i.e. if and only if $4 \geq 2n$, or equivalently if $n = 1, 2$. But as we have mentioned by direct computations the unique exponential factors e^h with h of degree 1 and 2 are the ones described in the statement of the proposition. This completes the proof of the proposition. □

Finally the last statement of Theorem 1 follows from the next result.

Proposition 7. *The Falkner–Skan system and Blasius system have no first integrals of Darboux type.*

Proof. Let $H = H(x, y, z)$ be a first integral of system (2), then it must satisfy that

$$y\frac{\partial H}{\partial x} + z\frac{\partial H}{\partial y} - [xz + \lambda(1 - y^2)]\frac{\partial H}{\partial z} = 0. \tag{13}$$

Assume $\lambda = 0$. Then by Propositions 4 and 5 the only one Darboux polynomial of system (2) is z and its only exponential factors are e^x and e^y. So if system (2) has a Darboux first integral it must be of the form $H = z^a e^{b_1 x + b_2 y}$. Thus equation (13) becomes

$$z^a e^{b_1 x + b_2 y}(-ax + b_1 y + b_2 z) = 0,$$

or equivalently

$$-ax + b_1 y + b_2 z = 0. \tag{14}$$

Assume $\lambda = 1/2$. Then by Propositions 4 and 5 the only Darboux polynomial of system (2) is $1 - y^2 + 2xz$ and its only exponential factors are e^x and e^y. So if system (2) has a Darboux first integral it must be of the form $H = (1 - y^2 + 2xz)^a e^{b_1 x + b_2 y}$, and equation (13) becomes

$$\left(1 - y^2 + 2xz\right)^a e^{b_1 x + b_2 y}(-ax + b_1 y + b_2 z) = 0,$$

or equivalently

$$-ax + b_1 y + b_2 z = 0. \tag{15}$$

Assume $\lambda = -1$. Then by Propositions 4 and 5 system (2) does not have Darboux polynomials and its only exponential factors are e^x, e^y and e^{z+xy}. So if system (2) has a Darboux first integral it must be of the form $H = e^{b_1 x + b_2 y + b_3(z+xy)}$, and equation (13) becomes

$$e^{b_1 x + b_2 y + b_3(xy+z)}(b_3 + b_1 y + b_2 z) = 0,$$

or equivalently

$$b_3 + b_1 y + b_2 z = 0. \tag{16}$$

Finally suppose that $\lambda \notin \{-1, 0, 1/2\}$. Then by Propositions 4 and 5 system (2) has no Darboux polynomials and only has the exponential factors e^x and e^y. So if system (2) has a Darboux first integral it must be of the form $H = e^{b_1 x + b_2 y}$, and equation (13) becomes

$$e^{b_1 x + b_2 y}(b_1 y + b_2 z) = 0,$$

or equivalently

$$b_1 y + b_2 z = 0. \tag{17}$$

From equations (13)–(17) it follows that $a = 0$ and $b_k = 0$ for $k = 1, 2, 3$ when these coefficients appears in some of the candidates to have a Darboux first integral. Hence system (2) does not admit a Darboux first integral. □

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[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat

[4] Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais 1049-001, Lisboa, Portugal

E-mail address: cvalls@math.ist.utl.pt