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LIMIT CYCLES OF POLYNOMIAL DIFFERENTIAL EQUATIONS WITH QUINTIC HOMOGENOUS NONLINEARITIES

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Abstract. In this paper we mainly study the number of limit cycles which can bifurcate from the periodic orbits of the two centers

$$\begin{array}{ll} \dot{x} = -y, & \dot{y} = x; \\ \dot{x} = -y(1-(x^2+y^2)^2), & \dot{y} = x(1-(x^2+y^2)^2); \end{array}$$

when they are perturbed inside the class of all polynomial differential systems with quintic homogenous nonlinearities. We do this study using the averaging theory of first, second and third order.

1. Introduction and statement of the results

In this paper we only consider differential equations in $\mathbb{R}^2$ of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where P and Q are polynomials of degree at most 5 with only homogeneous nonlinearities. We recall that a *limit cycle* of the differential equation (1) is a periodic orbit of this equation isolated in the set of all periodic orbits of equation (1).

The definition of limit cycle appeared in the years 1891 and 1897 in the works of Poincaré [15]. Almost immediately, in 1990, they become the main object to be studied in the statement of the second part of the 16–th Hilbert's problem [9]. Later on Van der Pol [16] in 1926, Liénard [11] in 1928 and Andronov [1] in 1929 shown that the periodic solution of a self–sustained oscillation of a circuit in a vacuum tube was a limit cycle in the sense defined by Poincaré. After this first observation of the existence of limit cycles in the nature, the existence, non–existence, uniqueness and other properties of the limit cycles has been intensively studied first by the mathematicians and the physicists, and more recently by the chemists, biologists, economists, ... Nowadays the study of the limit cycles of the planar differential systems has been one of the main problems of the qualitative theory of the differential equations. See for instance the recent papers [18, 19, 20] and the references quoted there.

A *center* is a singular point of a differential system (1) for which there exists a neighborhood such that all the orbits in that neighborhood are periodic, with the exception of the singular point.

A good way of producing limit cycles is by perturbing the periodic orbits of a center. This technique has been studied intensively perturbing the periodic orbits of the centers, mainly of centers of the quadratic polynomial differential systems, see for instance the book of Christopher and Li [6], and the references there in.

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The techniques used for studying the limit cycles that can bifurcate from the periodic orbits of a center, are mainly three: Abelian integrals (see [6]), Melnikov functions (see [10]), and averaging theory (see [3]). In the plane at same order all these techniques are equivalent (see [8]), they produce the same results, but the computations can change with the different technique.

In this note we shall consider polynomial differential systems of the form

$$\dot{x} = P_1(x,y) + P_n(x,y), \qquad \dot{y} = Q_1(x,y) + Q_n(x,y), \tag{2}$$

where $P_k(x,y)$ and $Q_k(x,y)$ are homogeneous polynomials of degree k, i.e. we consider polynomial differential systems with homogeneous nonlinearities. For $n = 2$ we have the class of all quadratic polynomial differential systems, whose centers have been completely classified, and there are hundreds of papers studying how many limit cycles can bifurcate from the periodic orbits of these centers, see again [6]. For the general cubic polynomial differential systems the centers are not completely classified, but for the particular class of systems (2) with $n = 3$ their centers have been classified, see [14, 17]. The study of the limit cycles which can bifurcate from the periodic orbits of some centers of this last class ($n = 3$) were made in [12], and for $n = 4$, see [4] . In this work we will study the class $n = 5$.

The easiest center is the linear differential center $\dot{x} = -y$, $\dot{y} = x$. In fact, Iliev [10] proved that the perturbation of this center inside the class of all polynomial differential systems of degree n, using the Melnikov function at order k, produces at most $[k(n-1)/2]$ limit cycles, where $[z]$ denotes the integer part function of $z \in \mathbb{R}$. Another easy center is the degenerate center $\dot{x} = -y((x^2 + y^2)/2)^m$, $\dot{y} = x((x^2 + y^2)/2)^m$ with $m \geq 1$. In [2] the authors improve the bound of Iliev perturbing the mentioned degenerate center. Thus, for $n = 5$ the Iliev result is at most 2, 4 and 6 limit cycles at first, second and third order, respectively. While if we perturb the center $\dot{x} = -y((x^2 + y^2)/2)^2$, $\dot{y} = x((x^2 + y^2)/2)^2$ of degree 5 we get at most 2, 4 and 7 limit cycles at first, second and third order, respectively (see Theorem 1.1 of [2]). We remark that these results are not optimal for the following polynomial differential system

$$\begin{aligned} \dot{x} &= -y + x\sum_{s=1}^{3}\varepsilon^s\lambda_s + \sum_{s=1}^{3}\varepsilon^s\sum_{i=0}^{5}a_{i,s}x^iy^{5-i}, \\ \dot{y} &= \quad x + y\sum_{s=1}^{3}\varepsilon^s\lambda_s + \sum_{s=1}^{3}\varepsilon^s\sum_{i=0}^{5}b_{i,s}x^iy^{5-i}, \end{aligned} \tag{3}$$

of degree 5, because systems (3) does not have terms of degree 2, 3 and 4, only have linear terms and homogeneous nonlinearities of degree 5.

The first objective of our work will be to provide the optimal upper bounds for the number of limit cycles which can be obtained perturbing the centers $\dot{x} = -y$, $\dot{y} = x$ with linear terms and homogeneous nonlinearities of degree 5 by using the averaging theory of first, second and third order, see for more details section 3. In other words, what is the maximum number of limit cycles of systems (3) for $\varepsilon \neq 0$ sufficiently small which bifurcate from the periodic orbits of the centers $\dot{x} = -y$, $\dot{y} = x$ using averaging theory of first, second and third order, respectively? The answers to this question is given in Theorem 1.

Our second objective will be give the optimal upper bounds for the number of limit cycles which can be obtained perturbing the center $\dot{x} = -y(1 - (x^2 + y^2)^2)$,

$\dot{y} = x(1-(x^2+y^2)^2)$ with linear terms and homogeneous nonlinearities of degree 5 by using the averaging theory of first, second and third order. More precisely, what is the maximum number of limit cycles of the system

$$\begin{aligned} \dot{x} &= -y(1-(x^2+y^2)^2) + x\sum_{s=1}^{3}\varepsilon^s\lambda_s + \sum_{s=1}^{3}\varepsilon^s\sum_{i=0}^{5}a_{i,s}x^iy^{5-i}, \\ \dot{y} &= \quad x(1-(x^2+y^2)^2) + y\sum_{s=1}^{3}\varepsilon^s\lambda_s + \sum_{s=1}^{3}\varepsilon^s\sum_{i=0}^{5}b_{i,s}x^iy^{5-i}, \end{aligned} \tag{4}$$

which bifurcate from the periodic orbits of the center $\dot{x} = -y(1-(x^2+y^2)^2)$, $\dot{y} = x(1-(x^2+y^2)^2)$ using averaging theory of first, second and third order? The answer to this question is provided in Theorem 2.

We note that $H = x^2 + y^2$ is a first integral of the two differential systems (3) and (4) when $\varepsilon = 0$. Therefore, such systems when $\varepsilon = 0$ have a center at the origin, and the periodic solutions surrounding them are circles.

Our main results are the following.

Theorem 1. *For $\varepsilon \neq 0$ sufficiently small and for $k = 1, 2, 3$ the maximum number of limit cycles of system* (3) *which bifurcate from the periodic orbits of the center $\dot{x} = -y$, $\dot{y} = x$ using averaging theory of k–th order is k.*

Theorem 2. *For $k = 1, 2$ and 3 the maximum number of limit cycles of system* (4) *which bifurcate from the periodic orbits of the center $\dot{x} = -y(1-(x^2+y^2)^2)$, $\dot{y} = x(1-(x^2+y^2)^2)$ using averaging theory of k –th order is* 1, 2 *and* 4, *respectively.*

Theorems 1 and 2 are proved in section 4.

On the other hand we should consider polynomial differential systems with homogeneous nonlinearities of degree 5 as the ones of systems (3) and (4) such that when $\varepsilon = 0$ the system reduces to

$$\begin{aligned} \dot{x} &= -\frac{1}{4}y(x^2+y^2)^2, \\ \dot{y} &= \quad \frac{1}{4}x(x^2+y^2)^2. \end{aligned}$$

We shall see that those systems have at most 1 limit cycle. But this result is more general, because the mentioned polynomial differential systems with homogeneous nonlinearities of degree 5 are a particular case of the following polynomial differential systems

$$\begin{aligned} \dot{x} &= -\frac{1}{n-1}y(x^2+y^2)^{(n-1)/2} + \lambda x + P_n(x,y), \\ \dot{y} &= \quad \frac{1}{n-1}x(x^2+y^2)^{(n-1)/2} + \lambda y + Q_n(x,y), \end{aligned} \tag{5}$$

when n is odd. The next result shows that systems (5) has at most one limit cycle.

Theorem 3. *The polynomial differential systems* (5) *with linear terms and nonlinear terms of degree n odd at most have one limit cycle, and when it exists is hyperbolic.*

Theorem 3 is also proved in section 4. In fact this theorem implicity could be obtained from the computations of the paper [7].

2. Polar coordinates and Cherkas transformation

We shall use the following result due to Cherkas [5].

Lemma 4. *The differential equation*

$$\frac{dr}{d\theta} = \frac{\lambda r + a(\theta) r^n}{1 + b(\theta) r^{n-1}},$$

doing the change of variables $r \to \rho$ given by $\rho = r^{n-1}/(1 + b(\theta) r^{n-1})$, becomes the Abel differential equation

$$\begin{aligned} \frac{d\rho}{d\theta} &= (n-1)b(\theta)(\lambda b(\theta) - a(\theta))\rho^3 + \\ &\quad [(n-1)(a(\theta) - 2\lambda b(\theta)) - b'(\theta)]\,\rho^2 + (n-1)\lambda\rho. \end{aligned}$$

Combining Lemma 4 with the polar change of coordinates $(x, y) \to (r, \theta)$ where $x = r\cos\theta$ and $y = r\sin\theta$ we shall obtain the following result.

Corollary 5. *Let $P(x, y)$ and $Q(x, y)$ be homogenous polynomials of degree n. Then the differential system*

$$\dot{x} = \lambda x - y + P_n(x, y), \qquad \dot{y} = x + \lambda y + Q_n(x, y), \tag{6}$$

can be transformed into the Abel equation

$$\begin{aligned} \frac{d\rho}{d\theta} = \;& (n-1)B(\theta)(\lambda B(\theta) - A(\theta))\rho^3 + \\ & [(n-1)(A(\theta) - 2\lambda B(\theta)) - B'(\theta)]\,\rho^2 + (n-1)\lambda\rho. \end{aligned} \tag{7}$$

where

$$\begin{aligned} A(\theta) &= \cos\theta P_n(\cos\theta, \sin\theta) + \sin\theta Q_n(\sin\theta, \cos\theta), \\ B(\theta) &= \cos\theta Q_n(\cos\theta, \sin\theta) - \sin\theta P_n(\sin\theta, \cos\theta). \end{aligned}$$

Proof. System (6) expressed in polar coordinates becomes

$$\dot{r} = \lambda r + A(\theta) r^n, \qquad \dot{\theta} = 1 + B(\theta) r^n.$$

Dividing $\dot{r}$ by $\dot{\theta}$ and using Lemma 4 the corollary follows. □

3. Averaging theory

In this section we present the basic results from the averaging theory that we shall need for proving the main results of this paper. The averaging theory up to third order for studying specifically periodic orbits was developed in [3]. It is summarized as follows.

Consider the differential system

$$\dot{x}(t) = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 F_3(t, x) + \varepsilon^4 R(t, x, \varepsilon), \tag{8}$$

where $F_1, F_2, F_3 : \mathbb{R} \times D \to \mathbb{R}$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ are continuous functions, T–periodic in the first variable, and D is an open subset of $\mathbb{R}$. Assume that the following hypotheses (i) and (ii) hold.

(i) $F_1(t, \cdot) \in C^2(D)$, $F_2(t, \cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, F_3, R, $D_x^2 F_1$,$D_x F_2$ are locally Lipschitz with respect to x, and R is twice differentiable with respect to ε.

We define $F_{k0} : D \to \mathbb{R}$ for $k = 1, 2, 3$ as

$$
\begin{aligned}
F_{10}(x) = &\ \frac{1}{T}\int_0^T F_1(s,x)ds,\\
F_{20}(x) = &\ \frac{1}{T}\int_0^T \left[\frac{\partial F_1}{\partial x}(s,x)\cdot y_1(s,x) + F_2(s,x)\right] ds,\\
F_{30}(x) = &\ \frac{1}{T}\int_0^T \Big[\frac{1}{2}\frac{\partial^2 F_1}{\partial x^2}(s,x)y_1(s,x)^2 + \frac{1}{2}\frac{\partial F_1}{\partial x}(s,x)y_2(s,x)\\
&+\frac{\partial F_2}{\partial x}(s,x)y_1(s,x) + F_3(s,x)\Big] ds,
\end{aligned}
$$

where

$$
\begin{aligned}
y_1(s,x) &= \int_0^s F_1(t,x)dt,\\
y_2(s,x) &= 2\int_0^s \left[\frac{\partial F_1}{\partial x}(t,x)y_1(t,x) + F_2(t,x)\right] dt.
\end{aligned}
$$

(ii) For $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, there exists $a \in V$ such that $(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30})(a) = 0$ and

$$\frac{d(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30})}{dx}(a) \neq 0.$$

Then for $|\varepsilon| > 0$ sufficiently small there exists a T–periodic solution $x(t, \varepsilon)$ of the system such that $x(0, \varepsilon) \to a$ as $\varepsilon \to 0$.

If F_{10} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are mainly the zeros of F_{10} for ε sufficiently small. In this case the previous result provides the *averaging theory of first order*.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case the previous result provides the *averaging theory of second order*.

If F_{10} and F_{20} are identically zero and F_{30} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are the zeros of F_{30} for ε sufficiently small. In this case the previous result provides the *averaging theory of third order*.

4. Proofs

Proof of Theorem 1. Using Corollary 5 we write the polynomial differential system (3) as the Abel differential equation (7) with $n = 5$ and where

$$
\begin{aligned}
A(\theta) = \;& \varepsilon\big((a_{51} + \varepsilon a_{52} + \varepsilon^2 a_{53})\cos^6\theta \\
&+(a_{41} + b_{51} + \varepsilon(a_{42} + b_{52}) + \varepsilon^2(a_{43} + b_{53}))\cos^5\theta\sin\theta \\
&+(a_{31} + b_{41} + \varepsilon(a_{32} + b_{42}) + \varepsilon^2(a_{33} + b_{43}))\cos^4\theta\sin^2\theta \\
&+(a_{21} + b_{31} + \varepsilon(a_{22} + b_{32}) + \varepsilon^2(a_{23} + b_{33}))\cos^3\theta\sin^3\theta \\
&+(a_{11} + b_{21} + \varepsilon(a_{12} + b_{22}) + \varepsilon^2(a_{13} + b_{23}))\cos^2\theta\sin^4\theta \\
&+(a_{01} + b_{11} + \varepsilon(a_{02} + b_{12}] + \varepsilon^2(a_{03} + b_{13}))\cos\theta\sin^5\theta \\
&+(b_{01} + \varepsilon b_{02} + \varepsilon^2 b_{03})\sin^6\theta\big), \\
B(\theta) = \;& \varepsilon\big((b_{51} + b_{52}\varepsilon + b_{53}\varepsilon^2)\cos^6\theta \\
&-(a_{51} - b_{41} + \varepsilon(a_{52} - b_{42}) + \varepsilon^2(a_{53} - b_{43}))\cos^5\theta\sin\theta \\
&-(a_{41} - b_{31} + \varepsilon(a_{42} - b_{32}) + \varepsilon^2(a_{43} - b_{33}))\cos^4\theta\sin^2\theta \\
&-(a_{31} - b_{21} + \varepsilon(a_{32} - b_{22}) + \varepsilon^2(a_{33} - b_{23}))\cos^3\theta\sin^3\theta \\
&-(a_{21} - b_{11} + \varepsilon(a_{22} - b_{12}) + \varepsilon^2(a_{23} - b_{13}))\cos^2\theta\sin^4\theta \\
&-(a_{11} - b_{01} + \varepsilon(a_{12} - b_{02}) + \varepsilon^2(a_{13} - b_{03}))\cos\theta\sin^5\theta \\
&-(a_{01} + \varepsilon^2 a_{02} + \varepsilon^3 a_{03})\sin^6\theta\big).
\end{aligned} \tag{9}
$$

Now the Abel differential equation (7) is the normal form (8) for applying the averaging theory up to third order in ε, where in (8) we have now $x = \rho$, $t = \theta$ and $F_k(\theta, \rho)$ is the coefficient of ε^k in $d\rho/d\theta$ for $k = 1, 2, 3$, we do not write their huge expressions, easy to compute and manipulate with an algebraic manipulator as mathematica or mapple.

We compute the function $F_{10}(\rho)$ defined in section 3, and we get

$$F_{10}(\rho) = \frac{1}{4}\rho\big(16\lambda_1 + (5b_{01} + a_{11} + a_{31} + b_{21} + 5a_{51} + b_{41})\,\rho\big).$$

Clearly the polynomial $F_{10}(\rho)$ can have at most one positive real root, and there are polynomial differential systems (4) for which they have such a positive real root. So, from section 3 the proof of the theorem follows for $k = 1$.

For applying the averaging theory of second order we need that $F_{10}(\rho) \equiv 0$. So we take

$$\lambda_1 = 0, \qquad b_{41} = -5b_{01} - a_{11} - a_{31} - b_{21} - 5a_{51}. \tag{10}$$

Computing the function $F_{20}(\rho)$ defined in section 3, we obtain

$$F_{20}(\rho) = \frac{1}{64}\rho\left(256\lambda_2 + 16R_1\rho + R_2\rho^2\right),$$

where

$$
\begin{aligned}
R_1 = \;& 5b_{02} + a_{12} + a_{32} + b_{22} + 5a_{52} + b_{42}, \\
R_2 = \;& 50a_{01}b_{01} + 10a_{01}a_{11} + 3a_{01}a_{31} + 3a_{01}b_{21} + 6b_{01}a_{21} \\
&+2b_{01}b_{11} + 4a_{11}a_{21} + 6a_{11}b_{11} + 18b_{01}a_{41} + 14b_{01}b_{31} \\
&+6a_{11}a_{41} + 4a_{11}b_{31} + 3a_{21}a_{31} + a_{21}b_{21} + 3b_{11}a_{31} \\
&-b_{11}b_{21} + 50b_{01}b_{51} + 10a_{11}b_{51} + 6a_{21}a_{51} + 12b_{11}a_{51} \\
&+7a_{31}a_{41} + 3a_{31}b_{31} + 3b_{21}a_{41} + b_{21}b_{31} + 7a_{31}b_{51} \\
&+7b_{21}b_{51} + 28a_{41}a_{51} + 14b_{31}a_{51}.
\end{aligned} \tag{11}
$$

Clearly the coefficients λ_2, R_1 and R_2 are independent, so the polynomial $F_{20}(\rho)$ can have at most two positive real roots, and there are polynomial differential systems (3) for which they have such two positive real roots. Hence the theorem is proved for $k = 2$.

In order to apply the averaging theory of third order we need that $F_{20}(\rho) \equiv 0$. So we take

$$
\begin{aligned}
\lambda_2 =&\ \ 0,\\
b_{42} =&\ \ -5b_{02} - a_{12} - a_{32} - b_{22} - 5a_{52},\\
b_{51} =&\ \ \frac{1}{50b_{01} + 10a_{11} + 7a_{31} + 7b_{21}}(-50a_{01}b_{01} - 10a_{01}a_{11} - 3a_{01}a_{31}\\
&-3a_{01}b_{21} - 6b_{01}a_{21} - 4a_{11}a_{21} - 6a_{11}b_{11} - 18b_{01}a_{41} - 14b_{31}a_{51}\\
&-14b_{01}b_{31} - 6a_{11}a_{41} - 4a_{11}b_{31} - 3a_{21}a_{31} - a_{21}b_{21} + b_{11}b_{21}\\
&-6a_{21}a_{51} - 12b_{11}a_{51} - 2b_{01}b_{11} - 3b_{11}a_{31} - 7a_{31}a_{41} - 3a_{31}b_{31}\\
&-3b_{21}a_{41} - b_{21}b_{31} - 28a_{41}a_{51}).
\end{aligned}
\tag{12}
$$

Of course, we assume that the denominator $50b_{01}+10a_{11}+7a_{31}+7b_{21} \neq 0$, otherwise we isolated another coefficient. Computing the function $F_{30}(\rho)$ defined in section 3, we obtain

$$
F_{30}(\rho) = \frac{1}{393216(50b_{01} + 10a_{11} + 7a_{31} + 7b_{21})^2}\rho(k_0 + k_1\rho + k_2\rho^2 + k_3\rho^3).
$$

We do not provide the big expressions of the coefficients k_j for $j = 0, 1, 2, 3$, because some of them needs more than one page.

In view of the expression of the polynomial $F_{30}(\rho)$ it follows immediately that $F_{30}(\rho)$ can have at most three positive real roots, and that there are systems (3) for which they have 0, 1, 2 or 3 positive real roots. This is due to the fact that in every coefficient of the polynomial $F_{30}(\rho)$ appears some coefficient of the initial polynomial differential system (3) which not appear in the other coefficients. For instance, for the system

$$
\begin{aligned}
\dot{x} &= -y + \varepsilon^2\frac{183653y^5}{3584} - \varepsilon^3\frac{183653x\left(88y^4 - 3\right)}{49152} + \varepsilon\left(x^5 + x^4y + x^3y^2 + x^2y^3 + y^5\right),\\
\dot{y} &= x + \varepsilon^3\frac{183653y}{16384} + \varepsilon\left(-\frac{11}{4}x^5 - 12x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5\right),
\end{aligned}
$$

the polynomials $F_{10}(\rho) \equiv F_{20}(\rho) \equiv 0$ and

$$
F_{30}(\rho) = -\frac{183653}{24576}(\rho - 3)(\rho - 2)(\rho - 1)\rho.
$$

So this system has three positive real roots. This completes the proof of the theorem. □

Proof of Theorem 2. From Corollary 5 it follows that the polynomial differential system (4) becomes the Abel differential equation (7) with $n = 5$ and with $A(\theta)$ exactly the one given in (9) and $B(\theta)$ equal to the one given in (9) minus 1. Again the Abel differential equation (7) is written in the normal form (8) for applying the averaging theory, where $x = \rho$, $t = \theta$ and $F_k(\theta, \rho)$ is the coefficient ε^k of $d\rho/d\theta$ for $k = 1, 2, 3$.

We compute the function $F_{10}(\rho)$ defined in section 3, and we get

$$
F_{10}(\rho) = \frac{1}{4}\rho(\rho + 1)(16\lambda_1 + (5b_{01} + a_{11} + a_{31} + b_{21} + 5a_{51} + b_{41} + 16\lambda_1)\rho).
$$

Clearly the polynomial $F_{10}(\rho)$ can have at most one positive real root, and there are polynomial differential systems (4) for which they have such a positive real root. So, from section 3 the proof of the theorem follows for $k = 1$.

Again the values (10) imply that $F_{10}(\rho) \equiv 0$. Computing the function $F_{20}(\rho)$ we obtain

$$F_{20}(\rho) = \frac{1}{64}\rho(\rho+1)(256\lambda_2 + 16(R_1 + 16\lambda_2)\rho + R_2\rho^2),$$

where R_1 and R_2 are given in (11). The rest of the proof of the theorem for $k = 2$ follows as in the proof of Theorem 1.

Taking the values (12) we obtain that $F_{20}(\rho) \equiv 0$. We calculate the function $F_{30}(\rho)$ and we obtain

$$F_{30}(\rho) = \frac{1}{3072(50b_{01} + 10a_{11} + 7a_{31} + 7b_{21})}\rho(\rho+1)(m_0+m_1\rho+m_2\rho^2+m_3\rho^3+m_4\rho^4).$$

We do not give the big expressions of the coefficients m_j for $j = 0, 1, 2, 3, 4$. In view of the expression of the polynomial $F_{30}(\rho)$ it follows easily that $F_{30}(\rho)$ can have at most four positive real roots, and that there are systems (4) for which they have 0, 1, 2, 3 or 4 positive real roots. Again this is due to the fact that in every coefficient of the polynomial $F_{30}(\rho)$ appears some coefficient of the initial polynomial differential system (4) which not appear in the other coefficients.

Finally, a tedious but easy computation shows that for the particular polynomial differential system (4) of the form

$$\begin{aligned}
\dot{x} &= -y\left(1 - \left(x^2+y^2\right)^2\right) + \varepsilon\left(xy^4 - \frac{60447}{220960}y^5\right) + \varepsilon^2\frac{71x^2y^3}{441920} - \varepsilon^3\frac{240xy^4 - x}{28282880},\\
\dot{y} &= x\left(1 - \left(x^2+y^2\right)^2\right) + \varepsilon\left(-\frac{27937}{220960}x^5 - x^4y + x^3y^2\right) + \varepsilon^3\frac{y}{28282880},
\end{aligned}$$

we obtain $F_{10}(\rho) \equiv F_{20}(\rho) \equiv 0$ and

$$F_{30}(\rho) = \frac{1}{7070720}\rho(\rho+1)(2\rho-1)(3\rho-1)(4\rho-1)(5\rho-1).$$

So for this system four limit cycles bifurcate from the periodic orbits of the center $\dot{x} = -y(1-(x^2+y^2)^2)$, $\dot{y} = x(1-(x^2+y^2)^2)$. Moreover, in coordinates (ρ, θ), the periodic orbits that bifurcate are $\rho = 1/R$ with $R = 2, 3, 4, 5$. This completes the proof of the theorem when $k = 3$. $\square$

For proving Theorem 3 we shall use the following result

Proposition 6 (see [13]). *If $h(z)$ is the displacement function associated to the differential equation $d\rho/d\theta = S(\rho, \theta)$, then*

$$h''(z) = e^{\int_0^{2\pi}\frac{\partial S}{\partial \rho}(\rho(\theta,y),\theta)d\theta}\left[\int_0^{2\pi}\frac{\partial^2 S}{\partial \rho^2}(\rho(\theta,y),\theta)e^{\int_0^{\theta}\frac{\partial S}{\partial \rho}(\rho(s,y),s)ds}d\theta\right],$$

where $\rho(\theta, y)$ is the solution of the differential equation such that $\rho(0, y) = y$.

Proof of Theorem 3. The polynomial differential system (5) in polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ becomes

$$\dot{r} = \lambda r + a(\theta)r^n, \qquad \dot{\theta} = r^{n-1}b(\theta), \tag{13}$$

where

$$a(\theta) = \cos\theta P_n(\cos\theta, \sin\theta) + \sin\theta Q_n(\cos\theta, \sin\theta),$$
$$b(\theta) = \cos\theta Q_n(\cos\theta, \sin\theta) - \sin\theta P_n(\cos\theta, \sin\theta).$$

Note that if $b(\theta^*) = 0$ for some $\theta^* \in [0, 2\pi)$, then the straight line through the origin of slope $\tan\theta^*$ is invariant and consequently, the system cannot have limit cycles surrounding the origin, or any other equilibrium (r^*, θ^*) of the system. So we can assume that $b(\theta) \neq 0$ for all $\theta \in [0, 2\pi)$.

Doing the change $\rho = r^{3-n}$ and taking θ as new independent variable we get that system (13) is equivalent to the differential equation

$$\frac{d\rho}{d\theta} = (3-n)\left(\frac{a(\theta)}{b(\theta)}\rho + \frac{\lambda}{b(\theta)}\right) = S(\rho, \theta).$$

Applying Proposition 6 to this differential equation we have that the displacement function $h(z)$ satisfies that $h''(z) = 0$, so $h(z) = \alpha z + \beta$ with $\alpha, \beta \in \mathbb{R}$. So $h(z)$ has at most one simple zero, and consequently our polynomial differential system has at most one hyperbolic limit cycle. $\square$

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