This is a preprint of: “Planar quasi-homogeneous polynomial differential systems and their integrability”, Belen García, Jaume Llibre, Jesús Suárez Pérez del Río, *J. Differential Equations*, vol. 255(10), 3185–3204, 2013.
DOI: [10.1016/j.jde.2013.07.032]

Planar quasi–homogeneous polynomial differential systems and their integrability

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Abstract

In this paper we study the quasi homogeneous polynomial differential systems and provide an algorithm for obtaining all these systems with a given degree. Using this algorithm we obtain all quasi homogeneous vector fields of degree 2 and 3.

The quasi homogeneous polynomial differential systems are Liouvillian integrable. In particular, we characterize all the quasi homogeneous vector fields of degree 2 and 3 having a polynomial, rational or global analytical first integral.

Keywords: polynomial first integral, analytic first integral, quasi homogeneous polynomial differential system, quasi homogeneous polynomial vector field
2000 MSC: 37J35, 34A05, 34C14

1. Introduction

We deal with polynomial differential systems of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

with $P, Q \in \mathbb{C}[x, y]$. As usual $\mathbb{C}$ denotes the set of complex numbers and $\mathbb{C}[x, y]$ denotes the ring of all polynomials in the variables x and y with coefficients in $\mathbb{C}$. The dot denotes derivative with respect to an independent variable t, which can be real or complex. We denote by $X = (P, Q)$ the polynomial vector field associated to system (1) and we say that the degree of the system or of the vector field is $n = \max\{\deg(P), \deg(Q)\}$.

Let $\mathbb{N}$ denote the set of positive integers. The polynomial differential system (1) or the vector field X is *quasi homogeneous* if there exist s_1, $s_2, d \in \mathbb{N}$ such that for arbitrary $\alpha \in \mathbb{R}^+ = \{a \in \mathbb{R},\, a > 0\}$,

$$P(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_1-1+d}P(x, y), \quad Q(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_2-1+d}Q(x, y), \qquad (2)$$

We call s_1 and s_2 the *weight exponents* of system (1), and d the *weight degree* with respect to the weight exponents s_1 and s_2. In the particular case that $s_1 = s_2 = 1$, then system (1) is the classical *homogeneous polynomial differential system of degree d*.

Suppose that system (1) is quasi-homogeneous and with weight exponents s_1 and s_2 and with weight degree d. In what follows we denote by $\mathbf{w} = (s_1, s_2, d)$ the *weight vector* formed with the weight exponents and the weight degree of the system. We say that weight vector $\mathbf{w}_m = (s_1^*, s_2^*, d^*)$ is a *minimal weight vector* of the polynomial differential system (1) if any other weight vector $\mathbf{w} = (s_1, s_2, d)$ of system (1) verifies $s_1^* \leq s_1$, $s_2^* \leq s_2$ and $d^* \leq d$.

The homogeneous polynomial differential systems has been studied by several authors. Thus, the quadratic homogeneous ones by [12, 18, 24, 25, 26, 27, 29]; the cubic homogeneous ones by [9]; the homogeneous systems of arbitrary degree by [7, 9, 10, 19], and others. In these previous papers is described an algorithm for studying the phase portraits of homogeneous polynomial vector fields for all degree, the classification of all phase portraits of homogeneous polynomial vector fields of degree 2 and 3, the algebraic classifications of homogeneous polynomial vector fields and the characterization of structurally stable homogeneous polynomial vector fields.

The quasi–homogeneous (and in general non–homogeneous) polynomial differential systems have been studied from many different points of view, mainly for their integrability [3, 4, 14, 15, 16, 17, 22], for their rational integrability [5, 30, 31, 32], for their polynomial integrability [8, 23, 28], for their centers [1, 2, 20], for their normal forms [6], for their limit cycles [21], ... But up to now there was not an algorithm for constructing all the quasi–homogeneous polynomial differential systems of a given degree.

In section 2 we study the basic properties of the quasi–homogeneous polynomial differential systems with weight vector (s_1, s_2, d). These properties will be used in section 3 for providing an algorithm which allows to compute all the quasi–homogeneous polynomial differential systems of a given degree. In particular, using this algorithm we compute all the quasi–homogeneous polynomial differential systems of degree 2 and 3. This algorithm is our main result.

In section 4 first we recall that all the quasi–homogeneous polynomial differential systems are Liouvillian integrable. After we show the existence of an easy inverse integrable factor, which later on is used for computing the explicit Liouvillian first integrals for all the quasi–homogeneous polynomial differential systems of degree 2 and 3, see sections 5 and 6.

Finally in section 7 we provide all the quasi–homogeneous polynomial differential systems of degrees 2 and 3 having a polynomial, rational or a global analytical first integral.

2. Properties of the quasi–homogeneous polynomial vector fields

The next result give us information about the possible weight vectors of a given quasi–homogeneous polynomial differential system.

Lemma 1. *If* $\mathbf{w}_m = (s_1^*, s_2^*, d^*)$ *is a minimal weight vector of* (1)*, then all the vectors of the form* $(rs_1^*, rs_2^*, r(d^*-1)+1)$*, where* r *is a positive integer number, are weight vectors of* (1).

Proof. The proof is easy from the definition of the weight vector. □

Note that the converse is not true. If we consider the system $\dot{x} = x^2, \dot{y} = xy$, we can observe that $w_m = (1,1,2)$ and nevertheless $w = (2,1,3)$ is other weight vector of this system.

Now we shall obtain some properties of the coefficients of the quasi–homogeneous polynomial vector fields. We write the polynomials P and Q of system (1) in its homogeneous parts

$$P(x,y) = \sum_{j=0}^{l} P_j(x,y), \quad \text{where} \quad P_j(x,y) = \sum_{i=0}^{j} a_{i,j-i} x^i y^{j-i}, \tag{3}$$

and

$$Q(x,y) = \sum_{j=0}^{m} Q_j(x,y), \quad \text{where} \quad Q_j(x,y) = \sum_{i=0}^{j} b_{i,j-i} x^i y^{j-i}. \tag{4}$$

From (2), (3) and (4) we deduce that the coefficients of a quasi–homogeneous polynomial vector field satisfy

$$a_{i,j-i}\alpha^{(i-1)s_1+(j-i)s_2-(d-1)} = a_{i,j-i} \tag{5}$$

and

$$b_{i,j-i}\alpha^{is_1+(j-i-1)s_2-(d-1)} = b_{i,j-i}. \tag{6}$$

Proposition 2. *If the polynomial differential system* (1) *is quasi–homogeneous with weight vector* $\mathbf{w} = (s_1, s_2, d)$*, then* $P_0 = Q_0 = 0$*. Furthermore, if* $d > 1$ *then* $b_{01} = a_{10} = 0$.

Proof. It is an easy consequence of (5) and (6). □

Corollary 3. *If the polynomial differential system* (1) *is quasi–homogeneous and* $m = l = 1$*, then system* (1) *becomes a homogeneous linear differential system.*

Proof. From Proposition 2 the corollary follows. □

Proposition 4. *Consider the quasi–homogeneous polynomial differential system* (1) *with weight vector* $\mathbf{w} = (s_1, s_2, d)$.

(a) If $s_1 \neq s_2$ then there is a unique $q \in \{0, 1, ..., m\}$ such that $b_{q,m-q} \neq 0$ and a unique $p \in \{0, 1, ..., l\}$ such that $a_{p,l-p} \neq 0$, and

$$(p - q - 1)(s_1 - s_2) = (m - l)s_2. \tag{7}$$

(b) If $s_1 = s_2$ then $m = l$ and $(m - 1)s_1 = d - 1$.

Proof. Since $\deg(Q) = m$ we obtain that there exists at least a value of q such that $b_{q,m-q} \neq 0$ and from (6) we deduce that

$$qs_1 + (m - q - 1)s_2 = d - 1. \tag{8}$$

Suppose that there exist two values q_1 and q_2 verifying (8). Then, using the expressions of (8) for q_1 and q_2, we get that $(q_1 - q_2)(s_1 - s_2) = 0$. Therefore, if $s_1 \neq s_2$ we have a contradiction and the value of q is unique.

Since $\deg(P) = l$ we obtain that there exists at least a value of p such that $a_{p,l-p} \neq 0$ and from (5) we deduce that

$$(p - 1)s_1 + (l - p)s_2 = d - 1. \tag{9}$$

In a similar way to the equality (8), we can prove that if $s_1 \neq s_2$ the value of p in equality (9) is unique.

The proof of statement (a) is completed if we observe that if $s_1 \neq s_2$, the equality (7) is obtained from (8) and (9).

On the other hand, if $s_1 = s_2$ from (8) and (9) we obtain statement (b) and the proposition follows. □

Remark 5. *In the following, in the case $s_1 \neq s_2$, we can suppose without loss of generality that $s_1 > s_2$, because in the other case we could interchange the variables x and y.*

2.1. Quasi–homogeneous polynomial differential systems with $s_1 > s_2$

By Corollary 3 we can suppose that the degree of the system is $n > 1$.

Proposition 6. *If the polynomial differential system (1) is quasi–homogeneous with weight vector $\mathbf{w} = (s_1, s_2, d)$ and $s_1 > s_2$, then each homogeneous part of P and Q has at most one monomial different from 0.*

Proof. We assume in (3) that there exist $j \in \{0, 1, ..., l\}$ and two different values p_1 and p_2 such that $a_{p_i,j-p_i} \neq 0$. Then by substituting in (5) i by p_1 and p_2 an combining these two expressions we obtain that $(p_1 - p_2)(s_1 - s_2) = 0$, hence $p_1 = p_2$. Using similar arguments with the polynomial Q the proposition follows. □

Now we analyze the relationship between the nonzero monomials of P_j and Q_j.

Proposition 7. *Consider the quasi–homogeneous polynomial differential system* (1) *with weight vector* $\mathbf{w} = (s_1, s_2, d)$ *and* $s_1 > s_2$. *Let* $j \in \{1, \ldots, \min\{l, m\}\}$ *and* $p, q \in \mathbb{N}$ *such that* $0 < p \leq j$ *and* $0 \leq q < j$.

(a) If $b_{q,j-q} \neq 0$, *then* $a_{i,j-i} = 0$ *if* $i \neq q + 1$ *and* $i \leq j$.

(b) If $a_{p,j-p} \neq 0$, *then* $b_{i,j-i} = 0$ *if* $i \neq p - 1$ *and* $i \leq j$.

Proof. We observe that $a_{i,j-i}$ is defined if and only if $i \leq j$. Under the assumptions of statement (a) if $a_{i,j-i} \neq 0$, from (5) and (6), we obtain that $(q-i+1)(s_1-s_2) = 0$, or equivalently $i = q+1$ and statement (a) follows. The proof of statement (b) is similar. □

Assume that the polynomial differential system (1) is quasi–homogeneous with weight vector $\mathbf{w} = (s_1, s_2, d)$ and $s_1 > s_2$. Then, from Proposition 6 we can establish a bijection between each nonzero homogeneous part P_j of P and the corresponding coefficient $a_{i,j-i} \neq 0$ associated to the unique nonzero monomial. Furthermore, if the degree of system (1) is n, we can denote the homogeneous parts of P as P_{n-t} for $t = 0, 1, \ldots, n$, and in the following we identify $P_{n-t} \not\equiv 0$ with the coefficient $a_{i^*,n-t-i^*}$ where i^* is the unique nonnegative integer $i^* \leq n-t$ such that verify $a_{i^*,n-t-i^*} \neq 0$. In a similar way we can identify $Q_{n-t} \not\equiv 0$ with $b_{j^*,n-t-j^*}$, where j^* is the unique nonnegative integer $j^* \leq n-t$ such that $b_{j^*,n-t-j^*} \neq 0$.

In short, if $X = (P, Q)$ is the vector field associated to the quasi–homogeneous polynomial differential system (1), and we denote by X_{n-t} the homogeneous part of X of degree $n-t$, then X_{n-t} satisfies one of the following statements:

(i) $X_{n-t} \equiv 0$.

(ii) If $X_{n-t} \not\equiv 0$, then X_{n-t} is one of the following

- (ii.1) $X_{n-t} = (a_{i^*,n-t-i^*} x^{i^*} y^{n-t-i^*}, 0)$,
- (ii.2) $X_{n-t} = (0, b_{j^*,n-t-j^*} x^{j^*} y^{n-t-j^*})$,
- (ii.3) $X_{n-t} = (a_{i^*,n-t-i^*} x^{i^*} y^{n-t-i^*}, b_{j^*,n-t-j^*} x^{j^*} y^{n-t-j^*})$.

Furthermore, from Proposition 7 if $a_{i^*,n-t-i^*} b_{j^*,n-t-j^*} \neq 0$, then $j^* = i^* - 1$ and we have proved the following result.

Corollary 8. *Consider the quasi–homogeneous polynomial differential system* (1) *of degree* n *with weight vector* $\mathbf{w} = (s_1, s_2, d)$ *and* $s_1 > s_2$. *Assume* $t < n$. *Then* $X_{n-t} \not\equiv 0$ *if and only if there is a unique* $i \in \{0, 1, \ldots, n-t+1\}$ *such that*

$$a_{i,n-t-i}^2 + b_{i-1,n-t-i+1}^2 \neq 0 \quad \textit{and} \quad (i-1)s_1 + (n-t-i)s_2 = d - 1. \qquad (10)$$

Of course, if $a_{0,n-t}$ *or* $b_{n-t,0}$ *are nonzero, we define* $a_{n-t+1,-1} = b_{-1,n-i+1} = 0$.

We observe that if X is a quasi–homogeneous but non–homogeneous vector field of degree n, then there exists at least $\tilde{t}$ $(0 < \tilde{t} < n)$ such that $X_n X_{n-\tilde{t}} \neq 0$.

2.2. Quasi–homogeneous but non–homogeneous polynomial differential systems with $d = 1$

In this subsection we characterize the quasi–homogeneous but non–homogeneous differential systems with $d = 1$. We can assume that $s_1 > s_2$ because if $s_1 = s_2$ and $d = 1$, then, from Proposition 4 (d), system is homogeneous of degree 1.

Proposition 9. *Consider the quasi–homogeneous but non–homogeneous polynomial differential system* (1) *with weight vector* $\mathbf{w} = (s_1, s_2, 1)$*,* $s_1 > s_2$*. Then the following statements hold.*

(a) $l > m = 1$*,* $s_1 = ls_2$ *and* $\mathbf{w}_m = (l, 1, 1)$*.*

(b) The only coefficients of the system that can be different from zero are $a_{1,0}$*,* $a_{0,l}$ *and* $b_{0,1}$*.*

Proof. From Proposition 4 one has that there is at most a unique $q \in \{0, 1, ..., m\}$ such that $b_{q,m-q} \neq 0$ and a unique $p \in \{0, 1, ..., l\}$ such that $a_{p,l-p} \neq 0$, and from (8) and (9) with $d = 1$ we have that

$$qs_1 + (m - q - 1)s_2 = 0, \tag{11}$$

and

$$(p - 1)s_1 + (l - p)s_2 = 0, \tag{12}$$

respectively. Since $q \geq 0$ in order to verify (11) it is necessary that $m-q-1 \leq 0$, that is, $q = m - 1$ or $q = m$. If $q = m - 1$, using (11) one has that $q = 0$ and $m = 1$, and the only coefficient that can be different form zero is b_{01}. In the case $q = m$, again from (11), one has that $s_2 = ms_1$ in contradiction with $s_1 > s_2$.

Since $p \leq l$ from (12) we obtain that $p = 0$ or $p = 1$. If $p = 1$, again from (12) one has that $l = 1$ and we get $l = m = 1$ and the system is homogeneous in contradiction with the assumptions. If $p = 0$, using (12), one has that $s_1 = ls_2$, and since $s_1 > s_2$ then $l > 1$. If we suppose that furthermore there is a coefficient $a_{x,l-t-x}$ with $x \neq 0$, $t > 0$ and $t \leq l - x$ that can be different from zero, using (5) and $ls_2 = s_1$ we obtain that $(l-1)x = t$ and hence $l-1 \leq t$. Since $t \leq l-x$ we obtain that $x = 1$ and $t = l - 1$ and the corresponding coefficient is $a_{1,0}$ and (b) holds. The proof of (a) concludes if we observe that $w = (ls_2, s_2, 1)$ implies, from the definition of minimal weight vector that $\mathbf{w}_m = (l, 1, 1)$. Consequently statements (a) and (b) are proved. □

3. Determining quasi–homogeneous but non–homogeneous vector fields with $d > 1$

In this section we provide an algorithm for determining quasi–homogeneous but non–homogeneous differential systems and with weight degree $d > 1$. Using this algorithm we can get all the quasi-homogeneous vector fields if we add the case $d = 1$ (using Proposition 9) and all the homogeneous vector fields.

Proposition 10. *Consider the quasi–homogeneous but non–homogeneous vector field of degree $n \geq 2$ with weight vector $\mathbf{w} = (s_1, s_2, d)$, $s_1 > s_2$ and $d > 1$. For each $\tilde{t}$ $(0 < \tilde{t} < n)$ such that $X_n X_{n-\tilde{t}} \neq 0$ let us $p \in \{0, 1, \ldots, n+1\}$ and $q \in \{0, 1, \ldots, n - \tilde{t} + 1\}$ the only values verifying* (10) *for $t = 0$ and $t = \tilde{t}$ respectively. Then the following statements hold.*

(a) $k = q - p \geq 1$ *and* $k \leq n - \tilde{t} - p + 1$*;*

(b) $s_1 = (\tilde{t}+k)(d-1)/D$ *and* $s_2 = k(d-1)/D$*, where* $D = (p-1)\tilde{t} + (n-1)k > 0$*;*

(c) the minimal weight vector of system (1) *is* $\mathbf{w}_m = ((\tilde{t}+k)/s, k/s, 1 + D/s)$ *where s is the greatest common divisor of $\tilde{t}$ and k.*

Proof. . Since (10) holds for $t = 0$ and $i = p$, and for $t = \tilde{t}$ and $i = q$, we have

$$\begin{aligned} (p-1)s_1 + (n-p)s_2 &= d-1, \\ (q-1)s_1 + (n-\tilde{t}-q)s_2 &= d-1. \end{aligned} \tag{13}$$

Then $(q-p)(s_1 - s_2) = \tilde{t}s_2$. Taking in account that $s_1 - s_2$, s_2 and $\tilde{t}$ are positive, we obtain that $q > p$, and statement (a) follows.

By substituting q by $p + k$ in (13) we obtain

$$\begin{aligned} (p-1)s_1 + (n-p)s_2 &= d-1, \\ (p+k-1)s_1 + (n-\tilde{t}-p-k)s_2 &= d-1. \end{aligned} \tag{14}$$

If we consider s_1 and s_2 as unknowns, then (14) is a compatible and determined linear system, if and only if $D = (p-1)\tilde{t} + (n-1)k \neq 0$.

We claim that $D > 0$. If $p > 0$, since $n \geq 2$, it follows immediately that $D > 0$. If $p = 0$ we assume that $D \leq 0$. Then, we have that $\tilde{t} \geq (n-1)k$. Since $\tilde{t} < n$, we obtain that $k = 1$ and $\tilde{t} = n - 1$. Then, system (14) becomes

$$\begin{aligned} -s_1 + ns_2 &= d-1, \\ 0 &= d-1, \end{aligned}$$

and $D = 0$, a contradiction. Hence the claim is proved. Finally, solving the linear system (14) with respect to s_1 and s_2 it follows statement (b).

Let s be the greatest common divisor of $\tilde{t}$ and k. From the expression of D given in (b), we can write $D = su$, and consequently $d - 1 = ru$, $s_1 = r(\tilde{t}+k)/s$ and $s_2 = rk/s$. Now, if we take $r = 1$, we obtain statement (c). □

Under the assumptions of Proposition 10 and its proof (see (14)) there are integers $p \in \{0, 1, \ldots, n-1\}$, $t \in \{1, \ldots, n-p\}$ and $k \in \{1, \ldots, n-t-p+1\}$ satisfying the equations

$$\begin{aligned} e_p^0[0] &\equiv (p-1)s_1 + (n-p)s_2 + 1 - d = 0, \\ e_p^t[k] &\equiv (p+k-1)s_1 + (n-t-p-k)s_2 + 1 - d = 0. \end{aligned} \tag{15}$$

We remark that since $t > 0$, it is necessary that $p < n$.

For every $p \in \{0, 1, \ldots, n-1\}$ and $t \in \{1, \ldots, n-p\}$ we define the set of equations

$$A_p(t) = \{e_p^t[k] : k = 1, \ldots, n-t-p+1\}.$$

In what follows we fix $p \in \{0, 1, \ldots, n-1\}$ and we consider the linear system defined by the set of equations

$$E_p = \{e_p^0[0]\} \cup A_p(1) \cup \ldots \cup A_p(n-p). \tag{16}$$

Our goal is to obtain sets of linear equations that contains the equation $e_p^0[0]$ and at most one equation of each set of equations $A_p(t)$ and such that the set of all these equations define a compatible linear system being s_1 and s_2 the unknowns, and satisfying that if we add some other equation the increased linear system be incompatible. We denote such linear systems the *maximal linear systems* associated to (16). Every one of these maximal linear systems will provide a quasi–homogeneous but non–homogeneous differential system of degree $n \geq 2$ with weight vector $\mathbf{w} = (s_1, s_2, d)$, $s_1 > s_2$ and $d > 1$.

Remark 11. *The linear system* (16) *when* $p = 0$ *has two equations that we can omit because never they are satisfied. From the equations* $A_0(n)$ *the equation* $e_0^n[1]$ (*i.e.* $-s_2 = d-1$ *because* $s_2 > 0$ *and* $d > 1$)*, and from the equations* $A_0(n-1)$ *the equation* $e_0^{n-1}[1]$ (*i.e.* $0 = d-1$ *because* $d > 1$)*. In what follows*

$$\mathcal{E}_p = E_p \textit{ if } p > 0, \textit{ and } \mathcal{E}_0 = E_0 \setminus \{e_0^n[1], e_0^{n-1}[1]\}.$$

Now we shall study when X_{n-t_1} and X_{n-t_2} can be simultaneously nonzero. Now we fix t_1 and t_2 with $t_1 \neq t_2$, and we study the compatibility of the linear system defined by the equation $e_p^0[0]$ and the equations $e_p^{t_1}[k_1]$ and $e_p^{t_2}[k_2]$.

Proposition 12. *Consider the quasi–homogeneous but non–homogeneous differential system* (1) *of degree* $n \geq 2$ *with weight vector* $\mathbf{w} = (s_1, s_2, d)$*,* $s_1 > s_2$ *and* $d > 1$*. Let* $p \in \{0, 1, \ldots, n-1\}$ *and* $t_1, t_2 \in \{1, \ldots, n-1\}$*. Then the linear system defined by the three equations* $e_p^0[0]$*,* $e_p^{t_1}[k_1]$ *and* $e_p^{t_2}[k_2]$ *is compatible only if*

$$k_1 t_2 = k_2 t_1. \tag{17}$$

Proof. Consider the linear system

$$\begin{aligned} (p-1)s_1 + (n-p)s_2 - (d-1) &= 0, \\ (p+k_1-1)s_1 + (n-t_1-p-k_1)s_2 - (d-1) &= 0, \\ (p+k_2-1)s_1 + (n-t_2-p-k_2)s_2 - (d-1) &= 0, \end{aligned} \tag{18}$$

with unknowns s_1, s_2 and $(d-1)$. In order to obtain a solution with positive values for s_1, s_2 and d it is necessary that the determinant $k_1 t_2 - k_2 t_1$ of the matrix of system (18) be zero. So the proposition is proved. $\square$

3.1. **The algorithm**

Fixed $n \geq 2$ and $p \in \{0, 1, \ldots, n-1\}$ the following algorithm allows to determine all the compatible *maximal linear systems* associated to the set of equations $\mathcal{E}_p$.

(step 1). We choose the equation $e_p^0[0]$ as the first equation of the future maximal linear system.

(step 2). We fix $t \in \{1, \ldots, n-p\}$ and an equation of $A_p(t) \cap \mathcal{E}_p$, i.e. an equation of the form $e_p^t[k]$ with a $k \in \{1, \ldots, n-t-p+1\}$. From Proposition 10 the resolution of the linear system defined by $e_p^0[0]$ and $e_p^t[k]$, allows to obtain the values of s_1 and s_2, and furthermore the minimal weight vector $\mathbf{w}_m$.

(step 3). For each $t^* \in \{1, \ldots, n-p\}$ with $t \neq t^*$ we determine the value $k_{t^*} \in \{1, \ldots, n-t^*-p+1\}$, if exists, of the equation $e_p^{t^*}[k_{t^*}]$ satisfying (17) with $t_1 = t$, $k_1 = k$, $t_2 = t^*$ and $k_2 = k_{t^*}$.

(step 4). We consider all the equations

$$\mathcal{E}_{p,t,k} = \bigcup_{t^* \in \{1,\ldots,n-p\} \setminus \{t\}} \{e_p^{t^*}[k_{t^*}] : k_{t^*} t = k t^*\} \cup \{e_p^t[k], e_p^0[0]\}, \tag{19}$$

that we have obtained in the steps 1, 2 and 3. Each linear equation $e_p^{t^*}[k_{t^*}]$ contributes, using (10), to the homogeneous part X_{n-t^*} of X with the term $X_{n-t^*}^{t^*,k_{t^*}}$ defined as

$$(a_{p+k_{t^*}, n-t^*-p-k_{t^*}} x^{p+k_{t^*}} y^{n-t^*-p-k_{t^*}}, b_{p+k_{t^*}-1, n-t^*-p-k_{t^*}+1} x^{p+k_{t^*}-1} y^{n-t^*-p-k_{t^*}+1}).$$

The equation $e_p^t[k]$ determines the homogeneous part X_{n-t} equals

$$X_{n-t}^{t,k} = \left(a_{p+k,n-t-p-k} x^{p+k} y^{n-t-p-k}, b_{p+k-1,n-t-p-k+1} x^{p+k-1} y^{n-t-p-k+1}\right),$$

and the equation $e_p^0[0]$ determines the homogeneous part of greatest degree

$$X_n = \left(a_{p,n-p} x^p y^{n-p}, b_{p-1,n-p+1} x^{p-1} y^{n-p+1}\right).$$

In short, the quasi–homogeneous but non–homogeneous differential system (1) of degree $n \geq 2$ with weight vector $\mathbf{w} = (s_1, s_2, d)$, $s_1 = (t+k)(d-1)/D$, $s_2 = k(d-1)/D$, $D = (p-1)t + (n-1)k > 0$ and $d > 1$ corresponding to the set of equations $\mathcal{E}_{p,t,k}$ is

$$X_{p,t,k} = X_n + X_{n-t}^{t,k} + \sum_{t^* \in \{1,\ldots,n-p\} \setminus \{t\} \text{ and } k_{t^*} t = k t^*} X_{n-t^*}^{t^*,k_{t^*}}, \tag{20}$$

where we must consider conditions (see (10)) in the coefficients of the homogeneous parts such that X_n and at least other homogeneous part of X are nonzero. We observe that if X_n is zero, the degree of the vector field is not n and if X_n is the only nonzero homogeneous part, then system is homogeneous.

(step 5). We remove from $\mathcal{E}_p$ the equations $\mathcal{E}_{p,t,k} \setminus \{e_p^0[0]\}$.

(step 6). We go back to (step 1) and we repeat this process as many times it is possible, i.e. until there are no equations to work in (step 1).

Note that all the quasi–homogeneous but non–homogeneous differential systems $X_{p,t,k}$ are associated to the equation $e_p^0[0]$ that we fixed in the step 0. Finally, if we consider all the equations $e_p^0[0]$ for each value of $p = 0, 1, ..., n-1$ the algorithm described provides all quasi–homogeneous but non–homogeneous polynomial differential systems of degree n and weight degree $d > 1$.

3.2. Quasi–homogeneous but non–homogeneous differential systems of degree 2 *and* 3 *with* $d > 1$

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In this subsection we apply the algorithm described in the previous subsection for obtaining all the quasi–homogeneous polynomial differential systems that are not homogeneous and have degree 2 and 3 with $d > 1$.

The case $n = 3$. In this case we must consider the values of $p = 0, 1, 2$. For each value of p we shall construct a matrix R_p^n, that contains in the first and second columns the values of $t = 1, \ldots, n-p$ and $k = 1, \ldots, n-t-p+1$, and in the other columns appear all the equations of $\mathcal{E}_p$ and their contributions to the corresponding homogeneous parts X_{n-t}.

(i) $p = 0$**.** In this case the matrix R_0^3 is defined as

$$R_0^3 = \begin{pmatrix} t & k & \mathcal{E}_0 & & X_{n-t} \\ - & - & e_0^0[0]: & -s_1+3s_2 = d-1 & X_3 = (a_{0,3}y^3, 0) \\ 1 & 1 & e_0^1[1]: & s_2 = d-1 & X_2^{1,1} = (a_{1,1}xy, b_{0,2}y^2) \\ 1 & 2 & e_0^1[2]: & s_1 = d-1 & X_2^{1,2} = (a_{2,0}x^2, b_{1,1}xy) \\ 1 & 3 & e_0^1[3]: & 2s_1 - s_2 = d-1 & X_2^{1,3} = (0, b_{2,0}x^2) \\ 2 & 2 & e_0^2[2]: & s_1 - s_2 = d-1 & X_1^{2,2} = (0, b_{1,0}x) \end{pmatrix}.$$

Now we apply the algorithm to the set of equations $\mathcal{E}_0$ given in the matrix R_0^3.

Of course in (step 1) we choose the equation $e_0^0[0]$.

In (step 2) we choose $t = 1$ and $k = 1$, i.e. the equation $e_0^1[1]$. Applying Proposition 10 and using the notation of this theorem to the system $e_0^0[0]$ and $e_0^1[1]$, we obtain $s_1 = 2(d-1)$, $s_2 = d-1$, $s = D = 1$ and $\mathbf{w}_m = (2, 1, 2)$.

In (step 3) we are forced to choose the equation $e_p^{t^*}[k_{t^*}] = e_0^2[2]$, which satisfies condition (17).

Now from (step 4) we get that $\mathcal{E}_{0,1,1} = \{e_0^0[0], e_0^1[1], e_0^2[2]\}$, and consequently from (20) the quasi–homogeneous but non–homogeneous vector field of degree $n = 3$ with minimal weight vector $\mathbf{w} = (2, 1, 2)$ is

$$X_{0,1,1} = X_3 + X_2^{1,1} + X_1^{2,2}.$$

Therefore, its corresponding quasi–homogeneous but non–homogeneous differential system is

$$\dot{x} = a_{0,3}y^3 + a_{1,1}xy, \qquad \dot{y} = b_{0,2}y^2 + b_{1,0}x. \tag{21}$$

Note that since this differential system must have degree 3 we have that the coefficient $a_{0,3}$ cannot be zero, and since the system is non-homogeneous we must consider conditions in the coefficients of the homogeneous parts such that at least other homogeneous part of X are nonzero, that is, $a_{11}^2 + b_{02}^2 + b_{10}^2 \neq 0$. In the next systems we omit to comment these obvious conditions.

Doing (step 5), i.e. removing from $\mathcal{E}_0$ the equations $\mathcal{E}_{0,1,1} \setminus \{e_0^0[0]\}$, we obtain the new matrix

$$\left(\begin{array}{ccccc} t & k & \mathcal{E}_0 & & X_{n-t} \\ - & - & e_0^0[0]: & -s_1 + 3s_2 = d-1 & X_3 = (a_{0,3}y^3, 0) \\ 1 & 2 & e_0^1[2]: & s_1 = d-1 & X_2^{1,2} = (a_{2,0}x^2, b_{1,1}xy) \\ 1 & 3 & e_0^1[3]: & 2s_1 - s_2 = d-1 & X_2^{1,3} = (0, b_{2,0}x^2) \end{array} \right).$$

Again in (step 1) we choose the equation $e_0^0[0]$.

In (step 2) we choose $t = 1$ and $k = 2$, i.e. the equation $e_0^1[2]$. Applying Proposition 10 to the system $e_0^0[0]$ and $e_0^1[2]$, we obtain $s_1 = d-1$, $s_2 = 2(d-1)/3$, $s = 1$, $D = 3$ and $\mathbf{w}_m = (3, 2, 4)$.

In (step 3) we cannot whose the equation $e_0^1[3]$ because it does not satisfy condition (17).

From (step 4) we get that $\mathcal{E}_{0,1,2} = \{e_0^0[0], e_0^1[2]\}$, and consequently from (20) the quasi–homogeneous but non–homogeneous vector field of degree $n = 3$ with minimal weight vector $\mathbf{w} = (3, 2, 4)$ is

$$X_{0,1,2} = X_3 + X_2^{1,2}.$$

So, its corresponding quasi–homogeneous but non–homogeneous differential system is

$$\dot{x} = a_{0,3}y^3 + a_{2,0}x^2, \qquad \dot{y} = b_{1,1}xy. \tag{22}$$

By (step 5) we remove from $\mathcal{E}_0$ the equations $(\mathcal{E}_{0,1,1} \cup \mathcal{E}_{0,1,2}) \setminus \{e_0^0[0]\}$ and we obtain the new matrix

$$\left(\begin{array}{ccccc} t & k & \mathcal{E}_0 & & X_{n-t} \\ - & - & e_0^0[0]: & -s_1 + 3s_2 = d-1 & X_3 = (a_{0,3}y^3, 0) \\ 1 & 3 & e_0^1[3]: & 2s_1 - s_2 = d-1 & X_2^{1,3} = (0, b_{2,0}x^2) \end{array} \right).$$

We start again with the (step 1) choosing the equation $e_0^0[0]$.

In (step 2) we only can choose $t = 1$ and $k = 3$, i.e. the equation $e_0^1[3]$. Applying Proposition 10 to the system $e_0^0[0]$ and $e_0^1[3]$, we have $s_1 = 4(d-1)/5$, $s_2 = 3(d-1)/5$, $s = 1$, $D = 5$ and $\mathbf{w}_m = (4, 3, 6)$.

In (step 3) we cannot choose any additional equation.

By (step 4) we get that $\mathcal{E}_{0,1,3} = \{e_0^0[0], e_0^1[3]\}$, and consequently from (20) the quasi–homogeneous but non–homogeneous vector field of degree $n = 3$ with minimal weight vector $\mathbf{w} = (4, 3, 6)$ is

$$X_{0,1,3} = X_3 + X_2^{1,3}.$$

So, its corresponding quasi–homogeneous but non–homogeneous differential system is

$$\dot{x} = a_{0,3}y^3, \qquad \dot{y} = b_{2,0}x^2. \tag{23}$$

Finally, by (step 5) we remove from $\mathcal{E}_0$ the equations $(\mathcal{E}_{0,1,1} \cup \mathcal{E}_{0,1,2} \cup \mathcal{E}_{0,1,3}) \setminus \{e_0^0[0]\}$ and we get $\{e_0^0[0]\}$. Since only remains the equation $e_0^0[0]$ the process has finished for $p = 0$.

(ii) $p = 1$**.** In this case we have the matrix

$$R_1^3 = \left(\begin{array}{cccrl} t & k & \mathcal{E}_1 & & X_{n-t} \\ - & - & e_1^0[0]: & 2s_2 = d-1 & X_3 = (a_{1,2}xy^2, b_{0,3}y^3) \\ 1 & 1 & e_1^1[1]: & s_1 = d-1 & X_2^{1,1} = (a_{2,0}x^2, b_{1,1}xy) \\ 1 & 2 & e_1^1[2]: & 2s_1 - s_2 = d-1 & X_2^{1,2} = (0, b_{2,0}x^2) \\ 2 & 1 & e_1^2[1]: & s_1 - s_2 = d-1 & X_1^{2,1} = (0, b_{1,0}x) \end{array} \right).$$

We apply the algorithm to the set of equations $\mathcal{E}_1$ given in the matrix R_1^3.

Of course in (step 1) we choose the equation $e_1^0[0]$.

In (step 2) we choose $t = 1$ and $k = 1$, i.e. the equation $e_1^1[1]$. Applying Proposition 10 and using the notation of this theorem to the system $e_1^0[0]$ and $e_1^1[1]$, we obtain $s_1 = d - 1$, $s_2 = (d-1)/2$, $s = 1$, $D = 2$ and $\mathbf{w}_m = (2, 1, 3)$.

In (step 3) we cannot choose any equation because the unique candidate is the equation $e_1^2[1]$, which does not satisfy condition (17).

Now from (step 4) we get that $\mathcal{E}_{1,1,1} = \{e_1^0[0], e_1^1[1]\}$, and consequently from (20) the quasi–homogeneous but non–homogeneous field of degree $n = 3$ with minimal weight vector $\mathbf{w} = (2, 1, 3)$ is

$$X_{1,1,1} = X_3 + X_2^{1,1}.$$

Therefore, its corresponding quasi–homogeneous but non–homogeneous differential system is

$$\dot{x} = a_{1,2}xy^2 + a_{2,0}x^2, \qquad \dot{y} = b_{0,3}y^3 + b_{1,1}xy. \tag{24}$$

We do (step 5) and remove from $\mathcal{E}_1$ the equations $\mathcal{E}_{1,1,1} \setminus \{e_1^0[0]\}$, we obtain the new matrix

$$\left(\begin{array}{cccrl} t & k & \mathcal{E}_1 & & X_{n-t} \\ - & - & e_1^0[0]: & 2s_2 = d-1 & X_3 = (a_{1,2}xy^2, b_{0,3}y^3) \\ 1 & 2 & e_1^1[2]: & 2s_1 - s_2 = d-1 & X_2^{1,2} = (0, b_{2,0}x^2) \\ 2 & 1 & e_1^2[1]: & s_1 - s_2 = d-1 & X_1^{2,1} = (0, b_{1,0}x) \end{array} \right).$$

Again in (step 1) we choose the equation $e_1^0[0]$.

In (step 2) we choose $t = 1$ and $k = 2$, i.e. the equation $e_1^1[2]$. Applying Proposition 10 to the system $e_1^0[0]$ and $e_1^1[2]$, we obtain $s_1 = 3(d-1)/4$, $s_2 = (d-1)/2$, $s = 1$, $D = 4$ and $\mathbf{w}_m = (3, 2, 5)$.

In (step 3) we cannot whose the equation $e_1^2[1]$ because it does not satisfy condition (17).

From (step 4) we get that $\mathcal{E}_{1,1,2} = \{e_1^0[0], e_2^1[2]\}$, and consequently from (20) quasi–homogeneous but non–homogeneous vector field of degree $n = 3$ with minimal weight vector $\mathbf{w} = (3, 2, 5)$ is

$$X_{1,1,2} = X_3 + X_2^{1,2}.$$

So, its corresponding quasi–homogeneous but non–homogeneous differential system is

$$\dot{x} = a_{1,2}xy^2, \qquad \dot{y} = b_{0,3}y^3 + b_{2,0}x^2. \tag{25}$$

By (step 5) we remove from $\mathcal{E}_1$ the equations $(\mathcal{E}_{1,1,1} \cup \mathcal{E}_{1,1,2}) \setminus \{e_0^0[0]\}$ and we obtain the new matrix

$$\begin{pmatrix} t & k & \mathcal{E}_1 & & X_{n-t} \\ - & - & e_1^0[0]: & 2s_2 = d-1 & X_3 = (a_{1,2}xy^2, b_{0,3}y^3) \\ 2 & 1 & e_1^2[1]: & s_1 - s_2 = d-1 & X_1^{2,1} = (0, b_{1,0}x) \end{pmatrix}.$$

We start again with the (step 1) choosing the equation $e_1^0[0]$.

In (step 2) we only can choose $t = 2$ and $k = 1$, i.e. the equation $e_1^2[1]$. Applying Proposition 10 to the system $e_1^0[0]$ and $e_1^2[1]$, we have $s_1 = 3(d-1)/2$, $s_2 = (d-1)/2$, $s = 1$, $D = 2$ and $\mathbf{w}_m = (3, 1, 3)$.

In (step 3) we cannot choose any additional equation.

By (step 4) we get that $\mathcal{E}_{1,2,1} = \{e_1^0[0], e_1^2[1]\}$, and consequently from (20) the quasi–homogeneous but non–homogeneous vector field of degree $n = 3$ with minimal weight vector $\mathbf{w} = (3, 1, 3)$ is

$$X_{1,2,1} = X_3 + X_1^{2,1}.$$

So, its corresponding quasi–homogeneous but non–homogeneous differential system is

$$\dot{x} = a_{1,2}xy^2, \qquad \dot{y} = b_{0,3}y^3 + b_{1,0}x. \tag{26}$$

Finally, by (step 5) we remove from $\mathcal{E}_1$ the equations $(\mathcal{E}_{1,1,1} \cup \mathcal{E}_{1,1,2} \cup \mathcal{E}_{1,2,1}) \setminus \{e_1^0[0]\}$ and we get $\{e_1^0[0]\}$. Since only remains the equation $e_1^0[0]$ the process has finished for $p = 1$.

(iii) $p = 2$**.** In this case we have the matrix

$$\begin{pmatrix} t & k & \mathcal{E}_2 & & X_{n-t} \\ - & - & e_2^0[0]: & s_1 + s_2 = d-1 & X_3 = (a_{2,1}x^2y, b_{1,2}xy^2) \\ 1 & 1 & e_2^1[1]: & 2s_1 - s_2 = d-1 & X_2^{1,1} = (0, b_{2,0}x^2) \end{pmatrix}.$$

We start with the (step 1) choosing the equation $e_2^0[0]$.

In (step 2) we only can choose $t = 1$ and $k = 1$, i.e. the equation $e_2^1[1]$. Applying Proposition 10 to the system $e_2^0[0]$ and $e_2^1[1]$, we have $s_1 = (d-1)/3$, $s_2 = 2(d-1)/3$, $s = 1$, $D = 3$ and $\mathbf{w}_m = (2, 1, 4)$.

In (step 3) we cannot choose any additional equation.

By (step 4) we get that $\mathcal{E}_{2,1,1} = \{e_2^0[0], e_2^1[1]\}$, and consequently from (20) the quasi–homogeneous but non–homogeneous vector field of degree $n = 3$ with minimal weight vector $\mathbf{w} = (2, 1, 4)$ is

$$X_{2,1,1} = X_3 + X_2^{1,1}.$$

So, its corresponding quasi–homogeneous but non–homogeneous differential system is

$$\dot{x} = a_{2,1}x^2y, \qquad \dot{y} = b_{1,2}xy^2 + b_{2,0}x^2. \tag{27}$$

Finally, by (step 5) we remove from $\mathcal{E}_2$ the equations $\mathcal{E}_{2,1,1} \setminus \{e_2^0[0]\}$ and we get $\{e_2^0[0]\}$. Since only remains the equation $e_2^0[0]$ the process has finished for $p = 2$.

In summary, putting together the 7 forms of quasi–homogeneous but non–homogeneous polynomial differential systems of degree 3 that we have found we have proved the following result.

Proposition 13. *All quasi–homogeneous but non–homogeneous polynomial differential systems of degree* 3*, with* $d > 1$ *can be written in one of the forms* (21)*,* (22)*,* (23)*,* (24)*,* (25)*,* (26) *and* (27)*.*

The case $n = 2$. In this case we must consider the values of $p = 0, 1$.

(i) $p = 0$**.** In this case we have the matrix

$$\begin{pmatrix} t & k & \mathcal{E}_0 & & X_{n-t} \\ - & - & e_0^0[0]: & -s_1 + 2s_2 = d - 1 & X_2 = (a_{0,2}y^2, 0) \\ 1 & 2 & e_0^1[2]: & s_1 - s_2 = d - 1 & X_1^{1,2} = (0, b_{1,0}x) \end{pmatrix}.$$

We start with the (step 1) choosing the equation $e_0^0[0]$.

In (step 2) we only can choose $t = 1$ and $k = 2$, i.e. the equation $e_0^1[2]$. Applying Proposition 10 to the system $e_0^0[0]$ and $e_0^1[2]$, we have $s_1 = 3(d-1)$, $s_2 = 2(d-1)$, $s = D = 1$ and $\mathbf{w}_m = (3, 2, 2)$.

In (step 3) we cannot choose any additional equation.

By (step 4) we get that $\mathcal{E}_{0,1,2} = \{e_0^0[0], e_0^1[2]\}$, and consequently from (20) the quasi–homogeneous but non–homogeneous vector field of degree $n = 2$ with minimal weight vector $\mathbf{w} = (3, 2, 2)$ is

$$X_{0,1,2} = X_2 + X_1^{1,2}.$$

So, its corresponding quasi–homogeneous but non–homogeneous differential system is

$$\dot{x} = a_{0,2}y^2, \qquad \dot{y} = b_{1,0}x. \tag{28}$$

Finally, by (step 5) we remove from $\mathcal{E}_0$ the equations $\mathcal{E}_{0,1,2} \setminus \{e_0^0[0]\}$ and we get $\{e_0^0[0]\}$. Since only remains the equation $e_0^0[0]$ the process has finished for $p = 0$.

(ii) $p = 1$**.** In this case we have the matrix

$$\begin{pmatrix} t & k & \mathcal{E}_1 & & X_{n-t} \\ - & - & e_1^0[0]: & s_2 = d-1 & X_2 = (a_{1,1}xy, b_{0,2}y^2) \\ 1 & 1 & e_1^1[1]: & s_1 - s_2 = d-1 & X_1^{1,1} = (0, b_{1,0}x) \end{pmatrix}.$$

We start with the (step 1) choosing the equation $e_1^0[0]$.

In (step 2) we only can choose $t = 1$ and $k = 1$, i.e. the equation $e_1^1[1]$. Applying Proposition 10 to the system $e_1^0[0]$ and $e_1^1[1]$, we have $s_1 = 2(d-1)$, $s_2 = d-1$, $s = D = 1$ and $\mathbf{w}_m = (2, 1, 2)$.

In (step 3) we cannot choose any additional equation.

By (step 4) we get that $\mathcal{E}_{1,1,1} = \{e_1^0[0], e_1^1[1]\}$, and consequently from (20) the quasi–homogeneous but non–homogeneous vector field of degree $n = 2$ with minimal weight vector $\mathbf{w} = (2, 1, 2)$ is

$$X_{0,1,2} = X_2 + X_1^{1,1}.$$

So, its corresponding quasi–homogeneous but non–homogeneous differential system is

$$\dot{x} = a_{1,1}xy, \qquad \dot{y} = b_{0,2}y^2 + b_{1,0}x. \tag{29}$$

Finally, by (step 5) we remove from $\mathcal{E}_1$ the equations $\mathcal{E}_{1,1,1} \setminus \{e_1^0[0]\}$ and we get $\{e_1^0[0]\}$. Since only remains the equation $e_1^0[0]$ the process has finished for $p = 1$.

In short, putting together the 2 forms of quasi–homogeneous but non–homogeneous differential systems of degree 2 that we have found we have proved the following result.

Proposition 14. *All quasi–homogeneous but non–homogeneous polynomial differential systems of degree* 2*, with* $d > 1$ *can be written in one of the forms* (28) *and* (29)*.*

4. Liouvillian integrability of the quasi–homogeneous polynomial vector fields

All quasi–homogeneous vector fields are integrable. This fact was probably know by Liapunov, and recently some authors have proved it in different ways. For example, García [14] gaves an inverse integrating factor for all quasi–homogeneous polynomial vector fields. Li, Llibre, Yang and Zhang (see [21]) gave another inverse integrating factor and an explicit first integral. Yanxia Hu [17] gaves an algorithm in order to determine an inverse integrating factor for an m–dimensional quasi–homogeneous vector field.

First we need a generalization of the Euler formula for the quasi–homogeneous functions.

Lemma 15 (generalized Euler formula)**.** *If F is a quasi–homogeneous function verifying that*
$$F(\alpha^p x, \alpha^q y) = \alpha^{p+d-1} F(x, y) \tag{30}$$
then
$$pxF_x + qyF_y = (p + d - 1)F. \tag{31}$$

Proof. Derivating (30) with respect to α we obtain
$$p\alpha^{p-1} xF_x(\alpha^p x, \alpha^q y) + q\alpha^{q-1} F_y(\alpha^p x, \alpha^q y) = (p + d - 1)\alpha^{p+d-2} F.$$
Therefore, taking $\alpha = 1$ in the above equality, (31) holds. $\square$

Proposition 16. *Suppose that system* (1) *is quasi–homogeneous of weight exponents s_1 and s_2 and weight degree d. Then $V = s_1 xQ - s_2 yP$ is an inverse of integrating factor of system* (1)*.*

Proof. We recall that V an inverse of integrating factor if and only if $M = V_x P + V_y Q - V(P_x + Q_y) \equiv 0$, see for more details section 8.3 of [13]. Taking into account the expression of V we get
$$M = (s_1 Q + s_1 xQ_x - s_2 yP_x)P + (s_1 xQ_y - s_2 P - s_2 yP_y)Q - (s_1 xQ - s_2 yP)(P_x + Q_y).$$
Now, from Lemma 15, one has that $s_1 xP_x + s_2 yP_y = (s_1 + d - 1)P$ and $s_1 xQ_x + s_2 yQ_y = (s_2 + d - 1)Q$. Therefore
$$M = ((s_1 + s_2 + d - 1)Q)P - (s_1 + s_2 + d - 1)P)Q = 0,$$
and proposition follows. $\square$

In fact the inverse of integrating factor of Proposition 16 can be deduce from the results of [21] because the product of their explicit first integral with their inverse of integrating factor is the V of the proposition, for more details see for instance Proposition 8.1 of [13].

Roughly speaking a *Liouvillian function* is a function which can be expressed by quadratures of elementary functions. If a polynomial differential system has a Liouvillian first integral then we say that it is *Liouvillian integrable*, for more details see Chapter 8 of [13] and the references quoted there. Note that since the inverse integrating factor V of Proposition 16 is polynomial, a Liouvillian first integral of a quasi–homogeneous polynomial differential system can be obtained integrating a rational function.

5. Canonical forms for the quasi–homogeneous polynomial vector fields of degree 2 and 3 without common factors

In this section we assume that the polynomials P and Q of the differential system (1) are coprime, otherwise the system can be reduced to one of lower degree with P and Q coprime doing a rescaling of the independent variable t of the system.

First we provide the canonical forms for the quasi–homogeneous polynomial differential systems without common factors of degree 2.

Proposition 17. *A quasi–homogeneous but non–homogeneous quadratic polynomial differential system after a rescaling of the variables can be written in one of the following forms:*

(a) $x' = y^2$*,* $y' = x$*, with minimal weight vector* $(3, 2, 2)$*.*

(b) $x' = axy$*,* $y' = x + y^2$*, with* $a \neq 0$ *and minimal weight vector* $(2, 1, 2)$*.*

(c) $x' = x + y^2$*,* $y' = ay$*, with* $a \neq 0$ *and minimal weight vector* $(2, 1, 1)$*.*

Proof. By Proposition 14 the quasi–homogeneous but non–homogeneous quadratic polynomial differential systems (1) with $s_1 > s_2$ and $d > 1$ are the systems (28) and (29) with minimal weight vectors $(3, 2, 2)$ and $(2, 1, 2)$, respectively.

Since in (28) we have that $a_{0,2}b_{1,0} \neq 0$, doing the rescaling of the variables $(X, Y, T) = (x, a_{0,2}^{1/3} y/b_{1,0}^{1/3}, a_{0,2}^{1/3} b_{1,0}^{2/3} t)$ system (28) becomes the system of statement (a) with (X, Y, T) instead of (x, y, t).

Now, in system (29) we have that $(a_{1,1}^2 + b_{0,2}^2)b_{1,0} \neq 0$ and since P and Q are coprime we have that $a_{1,1}b_{1,0}b_{0,2} \neq 0$. Doing the rescaling of the variables $(X, Y, T) = (b_{1,0}x/b_{0,2}, y, bt)$ system (29) becomes the system of statement (b) with (X, Y, T) instead of (x, y, t) and $a = a_{1,1}/b_{0,2}$.

Finally, if $d = 1$ by Proposition 9 we obtain the quasi–homogeneous but non–homogeneous differential system

$$x' = a_{1,0}x + a_{0,2}y^2, \qquad y' = b_{0,1}y, \tag{32}$$

with minimal vector degree $(2, 1, 1)$. Since we have that $a_{1,0}b_{0,1}a_{0,2} \neq 0$ we do the rescaling of the variables $(X, Y, T) = (a_{1,0}x/a_{0,2}, y, a_{1,0}t)$, then system (32) becomes the system of statement (c) with (X, Y, T) instead of (x, y, t) and $a = b_{0,1}/a_{1,0}$. This completes the proof of the proposition. □

Now we shall study the quadratic homogeneous polynomial differential systems. We recall the canonical forms for these systems obtained in [11].

Proposition 18. *Every quadratic homogeneous polynomial differential system* (1) *after a linear transformation and a rescaling of independent variable can be written in one of the following forms:*

(a) $x' = -2xy + P_2, \quad y' = -x^2 + y^2 + Q_2,$

(b) $x' = -2xy + P_2, \quad y' = x^2 + y^2 + Q_2,$

(c) $x' = -x^2 + P_2, \quad y' = 2xy + Q_2,$

(d) $x' = P_2, \quad y' = x^2 + Q_2,$

Here $P_2 = 2x(p_1x + p_2y)/3$ *and* $Q_2 = 2y(p_1x + p_2y)/3$*.*

Now we provide the canonical forms for the quasi–homogeneous polynomial differential systems without common factors of degree 3.

Proposition 19. *A quasi–homogeneous but non–homogeneous cubic differential system* (1) *after a rescaling of the variables can be written in one of the following forms:*

(a) $x' = y(ax + by^2)$, $y' = x + y^2$, *with* $a \neq b$, *or* $x' = y(ax \pm y^2)$, $y' = x$, *and both with minimal weight vector* $(2, 1, 2)$.

(b) $x' = x^2 + y^3$, $y' = axy$, *with* $a \neq 0$ *and minimal weight vector* $(3, 2, 4)$.

(c) $x' = y^3$, $y' = x^2$, *with minimal weight vector* $(4, 3, 6)$.

(d) $x' = x(x + ay^2)$, $y' = y(bx + y^2)$, *with* $(a, b) \neq (1, 1)$ *and minimal weight vector* $(2, 1, 3)$.

(e) $x' = axy^2$, $y' = \pm x^2 + y^3$, *with* $a \neq 0$ *and minimal weight vector* $(3, 2, 5)$.

(f) $x' = axy^2$, $y' = x + y^3$, *with* $a \neq 0$ *and minimal weight vector* $(3, 1, 3)$.

(g) $x' = ax + y^3$, $y' = y$, *with* $a \neq 0$ *and minimal weight vector is* $(3, 1, 1)$.

Proof. By Proposition 13 the quasi–homogeneous but non–homogeneous cubic polynomial differential systems (1) with $s_1 > s_2$ and $d > 1$ are the systems (21), (22), (23), (24), (25), (26) and (27) with minimal weight vectors $(2, 1, 2)$, $(3, 2, 4)$, $(4, 3, 6)$, $(2, 1, 3)$, $(3, 2, 5)$, $(3, 1, 3)$, and $(2, 1, 4)$, respectively. We remove system (27) because its polynomials P and Q are not coprime.

We note that in system (21) we have that $b_{1,0} \neq 0$ and $a_{0,3} \neq 0$. If $b_{0,2} \neq 0$ the rescaling of the variables $(X, Y, T) = (b_{1,0} b_{0,2} x, b_{0,2} y, t)$ writes system (21) into the first system of statement (a) with (X, Y, T) instead of (x, y, t). If $b_{0,2} = 0$ the rescaling of the variables $(X, Y, T) = (|a_{0,3} b_{1,0}^3|^{1/2} x, |a_{0,3} b_{1,0}|^{1/2} y, t)$ writes system (21) into the second system of statement (a) with (X, Y, T) instead of (x, y, t).

Since in system (22) we have that $a_{2,0} a_{0,3} b_{1,1} \neq 0$. Then the rescaling of the variables $(X, Y, T) = (a_{2,0} x, (a_{2,0} a_{0,3})^{1/3} y, t)$ writes system (22) into the system of statement (b) with (X, Y, T) instead of (x, y, t) and $a = b_{1,1}/a_{2,0}$.

For system (23) we have that $a_{0,3} b_{2,0} \neq 0$. Then the rescaling of the variables $(X, Y, T) = (a_{0,3}^{1/5} b_{2,0}^{3/5} x, a_{0,3}^{2/5} b_{2,0}^{1/5} y, t)$ writes system (23) into the system of statement (c) with (X, Y, T) instead of (x, y, t).

For system (24) we have that $a_{2,0} b_{0,3} \neq 0$. Then the rescaling of the variables $(X, Y, T) = (a_{2,0} x, |b_{0,3}|^{1/2} y, t)$ writes system (24) into the system of statement (d) with (X, Y, T) instead of (x, y, t), $a = a_{1,2}/|b_{0,3}|$ and $b = b_{1,1}/a_{2,0}$.

For system (25) we have that $a_{1,2} b_{2,0} b_{0,3} \neq 0$. Then the rescaling of the variables $(X, Y, T) = (|b_{0,3}|^{1/4} |b_{0,2}|^{1/2} x, |b_{0,3}|^{1/2} y, t)$ writes system (25) into the system of statement (e) with (X, Y, T) instead of (x, y, t),with the sign $+$ if $b_{2,0} > 0$, and sign $-$ if $b_{2,0} < 0$ and $a = a_{1,2}/|b_{0,3}|$.

For system (26) we have that $a_{1,2} b_{1,0} b_{0,3} \neq 0$. Then the rescaling of the variables $(X, Y, T) = (x, (b_{0,3}/b_{1,0})^{1/3} y, b_{1,0}^{2/3} b_{0,3}^{1/3} t)$ writes system (26) into the system of statement (f) with (X, Y, T) instead of (x, y, t) and $a = a_{1,2}/b_{0,3}$.

Finally, if $d = 1$ by Proposición 9 we obtain the quasi–homogeneous but non–homogeneous differential system

$$x' = a_{1,0}x + a_{0,3}y^3, \qquad y' = b_{0,1}y, \tag{33}$$

with minimal vector degree $(3, 1, 1)$. Since we have that $a_{1,0}b_{0,1}a_{0,3} \neq 0$, we do the rescaling of the variables $(X, Y, T) = (b_{0,1}x/a_{0,3}, y, b_{0,1}t)$, then system (33) becomes the first system of statement (g) with (X, Y, T) instead of (x, y, t) and $a = a_{1,0}/b_{0,1}$ This completes the proof of the proposition. $\square$

Now we present the canonical forms of the cubic homogeneous vector fields, which where obtained in [9].

Proposition 20. *Every cubic homogeneous polynomial differential system* (1) *after a linear transformation and a rescaling of independent variable can be written in one of the following forms:*

(a) $x' = 3(1 + \mu^4)x^2y - 6\mu^2y^3 + P_3$, $y' = 6\mu^2x^3 - 3(1 + \mu^4)xy^2 + Q_3$, *with* $\mu > 1$.

(b) $x' = -\alpha x^2y/2 + \alpha y^3 + P_3$, $y' = \alpha xy^2/2 + Q_3$, *with* $\alpha = \pm 1$.

(c) $x' = \mu y^3 + P_3$, $y' = \mu x^3 + Q_3$, *with* $\mu \neq 0$.

(d) $x' = 3\alpha x^2y - 6\alpha xy^2 - 6\alpha y^3 + P_3$, $y' = -3\alpha xy^2 + 6\alpha y^3 + Q_3$, *with* $\alpha = \pm 1$.

(e) $x' = 2xy^2 - 4y^3 + P_3$, $y' = -2y^3 + Q_3$.

(f) $x' = -3\alpha x^2y - 6y^3 + P_3$, $y' = 3\alpha xy^2 + Q_3$, *with* $\alpha = \pm 1$.

(g) $x' = -\alpha y^3 + P_3$, $y' = Q_3$, *with* $\alpha = \pm 1$.

(h) $x' = -3\alpha\mu x^2y - \alpha y^3 + P_3$, $y' = \alpha x^3 + 3\alpha\mu xy^2 + Q_3$, *with* $\alpha = \pm 1$, $\mu > -1/3$ *and* $\mu \neq 1/3$.

(i) $x' = -\alpha x^2y - \alpha y^3 + P_3$, $y' = \alpha x^3 + \alpha xy^2 + Q_3$, *with* $\alpha = \pm 1$.

Here $P_3 = p_1x^3 + p_2x^2y + p_3xy^2$ *and* $Q_3 = p_1x^2y + p_2xy^2 + p_3y^3$.

6. First integrals of the quasi–homogeneous polynomial vector fields of degree 2 and 3 without common factors

First we provide the first integrals of all quasi–homogeneous but non–homogeneous vector fields of Propositions 17 and 19.

Proposition 21. *The first integrals of Proposition 17's systems are:*

(a) $H(x, y) = 3x^2 - 2y^3$;

(b) $H(x, y) = \left((a - 2)y^2 - 2x\right)^a / x^2$ *if* $a \neq 2$, *and* $xe^{-y^2/x}$ *if* $a = 2$;

(c) $H(x,y) = y\left(y^2 - 2ax + x\right)^{-a}$ *if* $a \neq 1/2$, *and* $e^{-x/(2y^2)}y$ *if* $a = 1/2$.

Proof. Since $V = s_1xQ - s_2yP$ is an inverse integrating factor for a quasi–homogeneous vector field, using it we get the first integrals described in the statements (a), (b) and (c). $\square$

Proposition 22. *(a) For the first system of Proposition 19(a) the first integral is*

$$\log(by^4 + (a-2)xy^2 - 2x^2) - \frac{2(a+2)}{\sqrt{(a-2)^2+8b}}\operatorname{arctanh}\left(\frac{2by^2+(a-2)x}{\sqrt{(a-2)^2+8b}\,x}\right)$$

if $(a-2)^2 + 8b \neq 0$;

$$\log\left(4x - (a-2)y^2\right) - \frac{4(a+2)x}{(a-2)(4x-(a-2)y^2)}$$

if $b = -(a-2)^2/8$ *and* $a \neq 2$; *and* $e^{-y^2/x}x$ *if* $a = 2$ *and* $b = 0$. *For the second system of Proposition 19(a) the first integral is*

$$\log(y^4 + axy^2 - 2x^2) - \frac{2a}{\sqrt{a^2+8}}\operatorname{arctanh}\frac{2y^2+ax}{\sqrt{a^2+8}x};$$

and for the third system of Proposition 19(a) the first integral is

$$\log(y^4 - axy^2 + 2x^2) + \frac{2a}{\sqrt{a^2-8}}\operatorname{arctanh}\frac{2y^2-ax}{\sqrt{a^2-8}x}.$$

(b) *For the system of Proposition 19(b) the first integral is* $y^2\left(2y^3 + (2-3a)x^2\right)^{-a}$ *if* $a \neq 2/3$, *and* $e^{-\frac{x^2}{3y^3}}y$ *if* $a = 2/3$.

(c) *For the system of Proposition 19(c) the first integral is* $-4x^3 + 3y^4$.

(d) *For the system of Proposition 19(d) the first integral is*

$$((2-a)x)^{1-2b}y^{2-a}\left((a-2)y^2 - 2bx + x\right)^{ab-1}$$

if $(a,b) \neq (2,1/2)$; $y^2/x - \log(((1-2b)x)^b/y)$ *if* $a = 2$; *and* $x/y^2 + 2\log((2-a)xy^{-a})$ *if* $a \neq 2$ *and* $b = 1/2$.

(e) *For the first system of Proposition 19(e) the first integral is* $x(3x^2 + (3-2a)y^3)^{-a/3}$ *if* $a \neq 3/2$; *and* $y^3/x^2 - 2\log x$ *if* $a = 3/2$. *For the second system the first integral is* $x\left(3x^2 - (3-2a)y^3\right)^{-a/3}$; *and* $y^3/x^2 + 2\log x$ *if* $a = 3/2$.

(f) *For the system of Proposition 19(f) the first integral is* $((a-3)y^3 - 3x)^a/x^3$ *if* $a \neq 3$; *and* $y^3/x - \log x$ *if* $a = 3$.

(g) *For the first system of Proposition 19(g) the first integral is* $y^{-a}\left(y^3+(a-3)x\right)$ *if* $a \neq 3$*; and* $x/y^3 - \log y$ *if* $a = 3$.

Proof. The proof follows as the proof of Proposition 21. $\square$

Now we provide the first integrals of all homogeneous polynomial vector fields of Propositions 18 and 20.

Proposition 23. *A first integral of the homogeneous polynomial vector fields of Proposition 18 statement:*

(a) *is* $x^{-3-2p_2}(3y^2-x^2)^{p_2-3}\exp(-2\sqrt{3}p_1\mathrm{arctanh}(x/\sqrt{3}y))$*;*

(b) *is* $x^{-3-2p_2}(3y^2+x^2)^{p_2-3}\exp(-2\sqrt{3}p_1\arctan(x/\sqrt{3}y))$*;*

(c) *is* $x^{-2(3+p_1)}y^{2p_1-3}\exp(2p_2y/x)$*;*

(d) *is* $x\exp(-y(2p_1x+p_2y)/(3x^2))$.

Proof. Using that $V = xP - yQ$ is an inverse of integrating factor for the homogeneous polynomial differential system $\dot{x} = P(x,y)$ and $\dot{y} = Q(x,y)$, the first integrals are computed easily. $\square$

Proposition 24. *A first integral of the homogeneous polynomial vector fields of Proposition 20 statement:*

(a) *is* $(-\mu x+y)^a(\mu x+y)^b(-x+\mu y)^c(x+\mu y)^d$,
where $a = -3\mu^5+p_3\mu^2+p_1+(3+p_2)\mu$, $b = -3\mu^5-p_3\mu^2-p_1+(3+p_2)\mu$, $c = -3\mu^5-p_3-\mu^2p_1+(3-p_2)\mu$, $d = -3\mu^5+p_3+\mu^2p_1+(3-p_2)\mu$*;*

(b) *is* $(x+y)^a(x-y)^by^c\exp(-4p_1x/y)$,
where $a = 2(p_1-p_2+p_3)-\alpha$, $b = -2(p_1+p_2+p_3)-\alpha$, $c = 2(2p_2-\alpha)$*;*

(c) *is* $(-x+y)^a(x+y)^b(x^2+y^2)^c\exp(2(p_1-p_3)\arctan(y/x))$,
where $a = -(p_1+p_2+p_3+\mu)$, $b = p_1-p_2+p_3-\mu$, $c = p_2-\mu$*;*

(d) *is* $(x^2-2xy-y^2)^ay^b\exp(2p_1x/y-\sqrt{2}c\mathrm{arctanh}((\mathrm{x}+\mathrm{y})/(\sqrt{2}\mathrm{x}))$,
where $a = 2p_1+p_2-3\alpha$, $b = -4p_1-2p_2-6\alpha$, $c = 3p_1+p_2+p_3+3\alpha$*;*

(e) *is* $(x-y)^ay^b\exp((p_1+p_2)x/y+p_1x^2/(2y^2))$,
where $a = -2+p_1+p_2+p_3$, $b = -(2+p_1+p_2+p_3)$*;*

(f) *is* $(\alpha x^2+y^2)^ay^b\exp(-2p_1x/y-c\arctan(y/(\sqrt{\alpha}x))$,
where $a = -p_2-3\alpha$, $b = 2p_2-6\alpha$, $c = 2(p_1-\alpha p_3)/(\sqrt{\alpha})$*;*

(g) *is* $p_1x^3/(3\alpha y^3)+p_2x^2/(2\alpha y^2)+p_3x/(\alpha y)+\log y$*;*

(h) *is* $(r_1x^2+y^2)^a(r_2x^2+y^2)^b\exp(k_1\arctan(y/(\sqrt{r_1}x)+k_2\mathrm{sign}(x)\arctan(y/(\sqrt{r_2}x))$,
where $k = \sqrt{9\mu^2-1}$, $r_1 = 3\mu+k$, $r_2 = 3\mu-k$, $a = (\alpha k+p_2)/2$, $b = (\alpha k-p_2)/2$, $k_1 = (p_1-r_1p_3)/\sqrt{r_1}$, $k_2 = (r_2p_3-p_1)/\sqrt{r_2}$.

(i) *is* $(x^2+y^2)^\alpha\exp\left((p_2x+(p_3-p_1)y)x/(x^2+y^2)-(p_1+p_3)\arctan(y/x)\right)$.

Proof. The proof follows the proof of Proposition 23. $\square$

7. Polynomial, rational and global analytic first integrals of quasi–homogenous polynomial vector fields of degree 2 and 3

In the previous section we have obtained first integrals of the quasi–homogeneous quadratic and cubic vector fields without common factors. In this section we identify between those systems the ones having a polynomial, rational or analytic first integral.

As usual we denote by $\mathbb{Q}$ the set of all rational numbers, and by $\mathbb{Q}^+$ (respectively $\mathbb{Q}^-$) the set of all positive (respectively negative) rational numbers.

Corollary 25. *A first integral of the quasi–homogeneous but non–homogeneous vector fields of Proposition 17 statement:*

(a) is polynomial;

(b) is rational if and only if $a \in \mathbb{Q} \setminus \{2\}$, and is polynomial if and only if $a \in \mathbb{Q}^-$;

(c) is rational if and only if $a \in \mathbb{Q} \setminus \{1/2\}$, and is polynomial if and only if $a \in \mathbb{Q}^-$.

Proof. It follows easily from Proposition 21. □

Corollary 26. *Consider the quasi–homogeneous but non–homogeneous polynomial vector field of Proposition 19.*

(a) For the first system of Proposition 19(a) a first integral is polynomial (and rational) if and only if either $b = 0$ and $a \neq 2$, or $a = -2$. For the second and third systems of Proposition 19(a) a first integral is polynomial (and rational) if and only if $a = 0$.

(b) For the system of Proposition 19(b) a first integral is rational if and only if $a \in \mathbb{Q} \setminus \{a = 2/3\}$, and a first integral is polynomial if and only if $a \in \mathbb{Q}^-$.

(c) For the system of Proposition 19(c) a first integral is polynomial.

(d) For the system of Proposition 19(d) a first integral is rational if and only if $a, b \in \mathbb{Q}$ and $a \neq 2$ and $b \neq 1/2$, and a first integral is polynomial if and only if $\big\{(a, b) \in \mathbb{Q}^2 : a < 2, b < 1/2, ab \geq 1\big\}$.

(e) For the two systems of Proposition 19(e) a first integral is rational if and only if $a \in \mathbb{Q} \setminus \{a = 3/2\}$, and a first integral is polynomial if and only if $a \in \mathbb{Q}^-$.

(f) For the system of Proposition 19(f) a first integral is rational if and only if $a \in \mathbb{Q} \setminus \{a = 3\}$, and a first integral is polynomial if and only if $a \in \mathbb{Q}^-$.

(g) For the first system of Proposition 19(g) a first integral is rational if and only if $a \in \mathbb{Q} \setminus \{a = 3\}$, and a first integral is polynomial if and only if $a \in \mathbb{Q}^-$. For the second system a first integral is rational if and only if $a \in \mathbb{Q}$, and a first integral is polynomial if and only if $a \in \mathbb{Q}^-$.

Proof. It follows easily from Proposition 22. □

Now we provide the polynomial or rational first integrals of all homogeneous polynomial vector fields of Propositions 18 and 20.

Corollary 27. *A first integral of the homogeneous polynomial vector fields of Proposition 18 statement:*

(a) *is rational if and only if* $p_1 = 0$ *and* $p_2 \in \mathbb{Q}$*, and is polynomial if and only if additionally* $-3/2 < p_2 < 3$*;*

(b) *is rational if and only if* $p_1 = 0$ *and* $p_2 \in \mathbb{Q}$*, and is polynomial if and only if additionally* $-3/2 < p_2 < 3$*;*

(c) *is rational if and only if* $p_2 = 0$ *and* $p_1 \in \mathbb{Q}$*, and is polynomial if and only if additionally* $-3 < p_1 < 3/2$*;*

(d) *is polynomial if and only if* $p_1 = p_2 = 0$*.*

Proof. It follows easily from Proposition 23. □

Corollary 28. *A first integral of the homogeneous polynomial vector fields of Proposition 20 statement:*

(a) *is rational if and only if the values* a*,* b*,* c *and* d *of Proposition 24(a) are rational, and polynomial if and only if additionally* a*,* b*,* c *and* d *have the same sign;*

(b) *is rational if and only if the values* a*,* b *and* c *of Proposition 24(b) are rational and* $p_1 = 0$*, and polynomial if and only if additionally* a*,* b *and* c *have the same sign;*

(c) *is rational if and only if the values* a*,* b *and* c *of Proposition 24(c) are rational and* $p_1 = p_3$*, and polynomial if and only if additionally* a*,* b *and* c *have the same sign;*

(d) *is rational if and only if the values* a*,* b *and* c *of Proposition 24(d) satisfy that* a *and* b *are rational and* $p_1 = c = 0$*, and polynomial if and only if additionally* a *and* b *have the same sign;*

(e) *is rational if and only if the values* a *and* b *of Proposition 24(e) are rational and* $p_1 = p_2 = 0$*, and polynomial if and only if additionally* a *and* b *have the same sign;*

(f) *is rational if and only if the values* a*,* b *and* c *of Proposition 24(f) satisfy that* a *and* b *are rational and* $p_1 = c = 0$*, and polynomial if and only if additionally* a *and* b *have the same sign;*

(g) *is never rational;*

(h) *is rational if and only if the values* a, b, k_1 *and* k_2 *of Proposition 24(h) satisfy that* a *and* b *are rational and* $k_1 = k_2 = 0$, *and polynomial if and only if additionally* a *and* b *have the same sign;*

(i) *is polynomial (and rational) if and only if* $p_1 = p_2 = p_3 = 0$.

Proof. It follows easily from Proposition 24. □

In order to determine when a quasi–homogeneous polynomial differential system has a global analytical first integral (i.e. an analytic first integral defined in the whole $\mathbb{C}^2$), we observe that given an analytic function H we can split it into the form $H = \sum_i H^i$, where H^i is a quasi–homogeneous polynomial of weight degree i with respect to the weight exponents s_1 and s_2; i.e. $H^i(\alpha^{s_1}x_1, \alpha^{s_2}x_n) = \alpha^i H^i(x_1, x_2)$. The following result is well known, see for instance Proposition 1 of [23].

Proposition 29. *Let* H *be an analytic function and let* $H = \sum_i H^i$ *be its decomposition into quasi–homogeneous polynomials of weight degree* i *with respect to the weight exponents* s_1 *and* s_2. *Then* H *is an analytic first integral of the quasi–homogeneous polynomial differential system with weight exponents* s_1 *and* s_2 *if and only if each quasi–homogeneous part* H^i *is a first integral of system for all* i.

From Proposition 29 it follows immediately the next result.

Corollary 30. *A quasi–homogenous polynomial differential system has a global analytic first integral if and only if it has a polynomial first integral.*

acknowledgements

The first and the third authors are partially supported by a MTM2011-22956. The second author is partially supported by a MICINN/FEDER grant number MTM2008–03437, by an AGAUR grant 2009SGR 410, and by ICREA Academia.

[1] A. Algaba, E. Freire, E. Gamero and C. García, *Monodromy, center–focus and integrability problems for quasi–homogeneous polynomial systems*, Nonlinear Anal. **72** (2010), 1726–1736.

[2] A. Algaba, N. Fuentes and C. García, *Centers of quasi–homogeneous polynomial planar systems*, Nonlinear Anal. Real World Appl. **13** (2012), 419–431.

[3] A. Algaba, E. Gamero and C. García, *The integrability problem for a class of planar systems*, Nonlinearity **22** (2009), 395–420.

[4] A. Algaba, C. García and M. Reyes, *Integrability of two dimensional quasi–homogeneous polynomial differential systems*, Rocky Mountain J. Math. **41** (2011), 1–22.

[5] A. Algaba, C. García and M. Reyes, *Rational integrability of two–dimensional quasi–homogeneous polynomial differential systems*, Nonlinear Anal. **73** (2010), 1318–1327.

[6] a. Algaba, C. García and M.A. Teixeira, *Reversibility and quasi–homogeneous normal forms of vector fields*, Nonlinear Anal. **73** (2010), 510–525.

[7] J. Argemí, *Sur les points singuliers multiples de systèmes dynamiques dans* $\mathbb{R}^2$, Annali di Matematica Pura ed Applicata, Serie IV **79** (1968), 35–70.

[8] L. Cairó and J. Llibre, *Polynomial first integrals for weight–homogeneous planar polynomial differential systems of weight degree* 3, J. Math. Anal. Appl. **331** (2007), 1284–1298.

[9] A. Cima and J. Llibre, *Algebraic and topological classification of the homogeneous cubic systems in the plane*, J. Math. Anal. Appl. **147** (1990), 420–448.

[10] C. B. Collins, *Algebraic classification of homogeneous polynomial vector fields in the plane*, Japan J. Indust. Appl. Maths **13** (1996), 63–91.

[11] T. Date and M. Lai, *Canonical forms of real homogeneous quadratic transformations*, J. Math. Anal. Appl. **56** (1976), 650–682.

[12] T. Date, *Classification and analysis of two-.dimensional homogeneous quadratic differential equations systems*, J. of Diff. Equations **32** (1979), 311–334.

[13] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative theory of planar differential systems*, UniversiText, Springer–Verlag, New York, 2006.

[14] I. García, *On the integrability of quasihomogeneous and related planar vector fields*, Int. J. Bifurcation and Chaos **13** (2003), 995–1002.

[15] A. Goriely, *Integrability, partial integrability, and nonintegrability for systems of ordinary differential equations*, J. Math. Phys. **37** (1996), 1871–1893.

[16] A. Goriely, *Integrability and nonintegrability of dynamical systems*, Advanced Series in Nonlinear Dynamics **19**, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[17] Y. Hu, *On the integrability of quasihomogeneous systems and quasidegenerate infinity systems*, Adv. Difference Equ. **2007**, Art ID. 98427, 10 pp.

[18] N. A. Korol, *The integral curves of a certain differential equation*, (Russian), Minsk. Gos. Ped. Inst. Minsk (1973), 47–51.

[19] J. Llibre, J. S. Pérez del Río and J. A. Rodríguez, *Structural stability of planar homogeneous polynomial vector fields. Applications to critical points and to infinity*, J. Differential Equations **125** (1996), 490–520.

[20] J. Llibre and C. Pessoa, *On the centers of the weight–homogeneous polynomial vector fields on the plane*, J. Math. Anal. Appl. **359** (2009), 722–730.

[21] W. Li, J. Llibre, J. Yang and Z. Zhang, *Limit cycles bifurcating from the period annulus of quasi–homogeneous centers*, J. Dyn. Diff. Equat. **21** (2009), 133–152.

[22] M.H. Liu and K.Y. Guan, *Reduction of quasi–homogeneous autonomous systems and reduced Kovalevskaya exponent*, (Chinese) Acta Math. Appl. Sin. **31** (2008), 729–743.

[23] J. Llibre and X. Zhang, *Polynomial first integrals for quasi–homogeneous polynomial differential systems*, Nonlinearity **15** (2002), 1269–1280.

[24] L. S. Lyagina, *The integral curves of the equation* $y' = (ax^2 + bxy + cy^2)/(dx^2 + Exy + fy^2)$ (Russian), Usp. Mat. Nauk, **6-2(42)** (1951), 171–183.

[25] L. Markus, *Quadratic differential equations and non–associative algebras*, Annals of Mathematics Studies, Vol. **45**, Princeton University Press, 1960, pp 185–213.

[26] T. A. Newton, *Two dimensional homogeneous quadratic differential systems*, SIAM Review **20** (1978), 120–138.

[27] K. S. Sibirskii and N. I. Vulpe, *Geometric classification of quadratic differential systems*, Differential Equations **13** (1977), 548–556.

[28] A. Tsygvintsev, *On the existence of polynomial first integrals of quadratic homogeneous systems of ordinary differential equations*, J. Phys. A: Math. Gen. **34** (2001), 2185–2193.

[29] E. V. Vdovina, *Classification of singular points of the equation* $y' = (a_0x^2 + a_1xy + a_2y^2)/(b_0x^2 + b_1xy + b_2y^2)$ *by Forster's method* (Russian), Diff. Uravn. **20** (1984), 1809–1813.

[30] H. Yoshida, *Necessary conditions for existence of algebraic first integrals I and II*, Celestial Mech. **31** (1983), 363–379, 381–399.

[31] H. Yoshida, *A note on Kowalevski exponents and the non–existence of an additional analytic integral*, Celestial Mech. **44** (1988), 313–316.

[32] H. Yoshida, *A criterion for the non–existence of an additional analytic integral in Hamiltonian systems with n degrees of freedom*, Phys. Lett. A **141** (1989), 108–112.