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GENERALIZED WEIERSTRASS INTEGRABILITY FOR THE COMPLEX DIFFERENTIAL EQUATIONS $\dfrac{dy}{dx} = a(x)y^4 + b(x)y^3 + c(x)y^2 + d(x)y + e(x)$

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Abstract. We characterize the differential equations of the form

$$\frac{dy}{dx} = a(x)y^4 + b(x)y^3 + c(x)y^2 + d(x)y + e(x),$$

where a, b, c, d, e are meromorphic functions in the variable x, that admits either a generalized Weierstrass first integral or a generalized Weierstrass inverse integrating factor.

1. Introduction and statement of the main results

Let x and y be complex variables. In this paper we study the differential equations of the form

$$\frac{dy}{dx} = a(x)y^4 + b(x)y^3 + c(x)y^2 + d(x)y + e(x), \tag{1}$$

where a, b, c, d, e are meromorphic functions in the variable x. In fact, when $a(x) \equiv 0$ and $b(x) \not\equiv 0$ the differential equation (1) is an *Abel differential equation*; when $a(x) = b(x) \equiv 0$ and $c(x) \not\equiv 0$ is a *Riccati differential equation*; when $a(x) = b(x) = c(x) \equiv 0$ and $d(x) \not\equiv 0$ is a *linear differential equation*.

When $a(x) \equiv 0$, equations (1) where studied by the first time by Abel in his analysis on the elliptic functions (see [1]). Abel equations appear in the reduction of order of many second and higher order families of differential equations. Hence are frequently found in the modeling of real problems in several areas. Thus, for instance Abel differential equations appear in cosmology (see [11]), in control theory of electrical circuits (see [7]), in ecology (see [6]), ...

In what follows instead of working with the differential equation (1) we shall work with the equivalent differential system

$$\dot{x} = 1, \quad \dot{y} = a(x)y^4 + b(x)y^3 + c(x)y^2 + d(x)y + e(x), \tag{2}$$

where the dot denotes derivative with respect to the time t, real or complex.

The goal of this paper is to study the integrability of the differential equations (1) restricted to a special kind of first integrals. For such systems the notion of integrability is based on the existence of a first integral, and we want to characterize when the differential equations (1) have either a generalized Weierstrass first integral or a generalized Weierstrass inverse integrating factor.

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When one studies the integrability of a differential system, the easiest class of functions to look for first integral is the polynomial ones. After the polynomial first integrals, we look for the analytic first integrals. Usually, this is a hard problem and instead of this, one studies the first integrals that can be described by formal series. The use of formal series is a classical tool in the study of differential equations (see for instance [5], where the author used them for proving the Dulac's conjecture). Here, since the equations (1) are polynomial in the variable y, we study the first integrals that are polynomials in the variable y and formal series in the variable x, called *formal Weierstrass first integrals.*

The integrability of the Abel differential equations has been studied by many authors, see for instance [3, 4, 9, 12]. In [3], [4] and [9], the authors give a list of integrable Abel differential equations with a, b, c and d rational functions.

As usual $\mathbb{C}[[x]]$ is the ring of formal power series in the variable x with coefficients in $\mathbb{C}$, and $\mathbb{C}[y]$ is the ring of polynomials in the variable y with coefficients in $\mathbb{C}$. A polynomial of the form

$$\sum_{i=0}^{n} a_i(x)y^i \in \mathbb{C}[[x]][y], \tag{3}$$

is called a *formal generalized Weierstrass polynomial* in y of degree n if and only if $a_n(x) \not\equiv 0$. A formal generalized Weierstrass polynomial whose coefficients are convergent is called *generalized Weierstrass polynomial.* In [8] it was introduced, by first time, the definition of Weierstrass integrability although that this definition is given in a more general context. In reference [10] are studied the Liénard systems which have a generalized Weierstrass first integral or a generalized Weierstrass inverse integrating factor.

We recall that an analytic first integral $H: U \to \mathbb{C}$ of system (2) where U is an open subset of $\mathbb{C}^2$ is a non-locally constant analytic function such that it is constant on the solution of system (2) contained in U.

Let $V: W \to \mathbb{C}$ be a function satisfying

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}(a(x)y^4 + b(x)y^3 + c(x)y^2 + d(x)y + e(x)) = (4a(x)y^3 + 3b(x)y^2 + 2c(x)y + d(x))V.$$

Then V is an *inverse integrating factor*, and it is known that there exists a first integral H such that

$$\frac{1}{V} = \frac{\partial H}{\partial y}, \quad \frac{a(x)y^4 + b(x)y^3 + c(x)y^2 + d(x)y + e(x)}{V} = -\frac{\partial H}{\partial x}. \tag{4}$$

We say that a differential system (2) is *generalized Weierstrass integrable* if it admits a first integral or an inverse integrating factor which is a generalized Weierstrass polynomial.

When in (3) $a_n(x) = 1$ and $a_i(0) = 0$ for $i < n$ we have a *Weierstrass polynomial* instead of a generalized Weierstrass polynomial. In [13] the Weierstrass and the generalized Weierstrass integrability have been characterized for the Abel, the Riccati and the linear differential equations. Moreover, the Weierstrass integrability also has been characterized in [14] for the differential equations

$$\frac{dx}{dy} = b_n(x)y^n + \cdots + b_1(x)y + b_0(x), \tag{5}$$

with $b_n(0) \neq 0$ and $b_i(x)$ meromorphic functions for $i = 0, 1, \ldots, n$. But the characterization of the generalized Weierstrass integrability for the differential equations (5) remains open for $n > 3$. The goal of this paper is to characterize the generalized Weierstrass integrability for $n = 4$, i.e., for the differential equation (1). More precisely: *How to recognize functions* $a(x)$, $b(x)$, $c(x)$, $d(x)$ *aand* $e(x)$ *for which differential equation* (1) *is generalized Weierstrass integrable*?

Our first result is the following.

Proposition 1. *System* (2) *with* $a(x) \not\equiv 0$ *has no generalized Weierstrass first integrals.*

Now we shall look for inverse integrating factors of the form

$$V = V_s(x)y^s + V_{s-1}(x)y^{s-1} + \cdots + V_1(x)y + V_0(x) = \sum_{i=0}^{s} V_i(x)y^i \tag{6}$$

with $V_s(x) \not\equiv 0$.

Theorem 2. *System* (2) *with* $a(x) \not\equiv 0$ *admits a generalized Weierstrass inverse integrating factor of the form* (6) *if and only if* $s = 4$ *and one of the following three conditions holds.*

(a) $3b(x)^2 - 8a(x)c(x) \neq 0$,

$$d(x) = \frac{(3b(x)^2 - 8a(x)c(x))^{3/2}}{a(x)^2}\left(C_1 + \int \frac{f_2(x)}{(3b(x) - 8a(x)c(x))^{7/2}}\,dx\right),$$

$$e(x) = \frac{(3b(x)^2 - 8a(x)c(x))^2}{a(x)^3}\left(C_0 - \int \frac{a(x)f_1(x)}{(3b(x) - 8a(x)c(x))^4}\,dx\right),$$

where the functions $f_1(x)$ *and* $f_2(x)$ *are given in* (14) *and* (15), *respectively.*

(b) $c(x) = 3b(x)^2/(8a(x))$, $(3b(x)^4 - 64a(x)b(x)(a(x)d(x) + a'(x)) + 64a(x)^2(4a(x)e(x) + b'(x))^{3/4} \neq 0$ *and* $f_3(x) = 0$, *where the function* $f_3(x)$ *is given in* (16).

(c) $c(x) = 3b(x)^2/(8a(x))$ *and*

$$e(x) = \frac{-3b(x)^4 + 64a(x)^2d(x)b(x) + 64a(x)a'(x)b(x) - 64a(x)^2b'(x)}{256a(x)^3}.$$

The expressions for the generalized Weierstrass inverse integrating factors are given along the proof of Theorem 2.

The proofs of Proposition 1 and Theorem 2 are given in section 2, where we also provide two examples satisfying the conditions given in statements (b) and (c) of Theorem 2.

The analytic conditions for the existence of generalized Weierstrass inverse integrating factors have been computed with the help of the algebraic manipulator mathematica.

2. Proof of the results

Proof of Proposition 1. Imposing that system (2) has a first integral of the form

$$H = H_s(x)y^s + H_{s-1}(x)y^{s-1} + \cdots + H_1(x)y + H_0(x) = \sum_{i=0}^{s} H_i(x)y^i \tag{7}$$

(with $H_s(x) \neq 0$) we obtain the following polynomial in y

$$\sum_{i=0}^{s} H_i'(x)y^i + \sum_{i=0}^{s} iH_i(x)y^{i-1}(a(x)y^4 + b(x)y^3 + c(x)y^2 + d(x)y + e(x)) = 0, \tag{8}$$

whose coefficients must be zero. Since $a(x) \not\equiv 0$ we get a recursive differential system to determine the functions $H_i(x)$ for $i = 0, \ldots, s$. The highest power is y^{s+3} and its coefficient is $sH_s(x)a(x) = 0$. Since we are assuming that $a(x) \neq 0$ and $H_s(x) \neq 0$, we have that

$s = 0$. So $H = H_0(x)$. Then, from (8), it follows that $H_0'(x) = 0$, that is $H = H_0$ is a constant in contradiction with the fact that H is a first integral. $\square$

Proof of Theorem 2. Imposing that system (2) has an inverse integrating factor of the form (6) we obtain a polynomial in y whose coefficients must be zero. Hence we get that

$$\begin{aligned}&\sum_{i=0}^{s} V_i'(x)y^i + \sum_{i=0}^{s} iV_i(x)y^{i-1}(a(x)y^4 + b(x)y^3 + c(x)y^2 + d(x)y + e(x))\\ &= (4a(x)y^3 + 3b(x)y^2 + 2c(x)y + d(x))\Big(\sum_{i=0}^{s} V_i(x)y^i\Big).\end{aligned} \tag{9}$$

Now, computing the coefficients in (9) of y^{s+3} we get $sV_s(x)a(x) = 4V_s(x)a(x)$, i.e. $s = 4$, because $V_s(x)a(x) \neq 0$.

Computing (9) with $s = 4$ the coefficients of y^k for $k = 6, 5, 4, 3, 2, 1, 0$ are

$$\begin{aligned}
e_6 &= -a(x)V_3(x) + b(x)V_4(x),\\
e_5 &= -2a(x)V_2(x) + 2c(x)V_4(x),\\
e_4 &= -3a(x)V_1(x) - b(x)V_2(x) + c(x)V_3(x) + 3d(x)V_4(x) + V_4'(x),\\
e_3 &= -4a(x)V_0(x) - 2b(x)V_1(x) + 2d(x)V_3(x) + 4e(x)V_4(x) + V_3'(x),\\
e_2 &= -3b(x)V_0(x) - c(x)V_1(x) + d(x)V_2(x) + 3e(x)V_3(x) + V_2'(x),\\
e_1 &= -2c(x)V_0(x) + 2e(x)V_2(x) + V_1'(x),\\
e_0 &= -d(x)V_0(x) + e(x)V_1(x) + V_0'(x).
\end{aligned} \tag{10}$$

Solving $e_6 = 0$ we get that

$$V_3(x) = \frac{b(x)V_4(x)}{a(x)}. \tag{11}$$

Then, from $e_5 = 0$ we obtain

$$V_2(x) = \frac{c(x)V_4(x)}{a(x)}. \tag{12}$$

Now from $e_4 = 0$ we have that

$$V_1(x) = \frac{3d(x)V_4(x) + V_4'(x)}{3a(x)}. \tag{13}$$

Moreover, from $e_3 = 0$ we get

$$V_0(x) = \frac{12a(x)^2e(x)V_4(x) - 3b(x)V_4(x)a'(x) + 3a(x)V_4(x)b'(x) + a(x)b(x)V_4'(x)}{12a(x)^3}.$$

Case 1: $3b(x)^2 - 8a(x)c(x) \neq 0$. Substituting $V_k(x)$ for $k = 0, 1, 2, 3$ into $e_2 = 0$ we get that

$$V_4(x) = \frac{a(x)^3}{(3b(x)^2 - 8a(x)c(x))^{3/2}}.$$

Now we introduce $V_4(x)$ into the expressions $e_1 = 0$ and $e_0 = 0$. Solving $e_0 = 0$ with respect to $e(x)$ we get

$$e(x) = \frac{(3b(x)^2 - 8a(x)c(x))^2}{a(x)^3}\left(C_0 - \int \frac{a(x)f_1(x)}{(3b(x) - 8a(x)c(x))^4}\,dx\right) \tag{14}$$

being C_0 a constant and

$$f_1(x) = 3b(x)^3c(x)a'(x)^2 + 3a(x)b(x)(b(x)^2c(x)a''(x) + a'(x)(c(x)(4c(x)a'(x) - b(x)(8b'(x) + b(x)d(x))) + 3b(x)^2c'(x))) + a(x)^2(-16c(x)^2a'(x)b'(x) + 2b(x)c(x)(-4c(x)a''(x) + 4a'(x)(2c'(x) + c(x)d(x)) + 15b'(x)^2) + 6b(x)^2(b'(x)(c(x)d(x) - 3c'(x)) - c(x)b''(x)) + 3b(x)^3(c''(x) - d(x)c'(x))) + 4a(x)^3(2c(x)(c'(x)(b(x)d(x) - 4b'(x)) - b(x)c''(x)) + 4c(x)^2(b''(x) - d(x)b'(x)) + 5b(x)c'(x)^2).$$

Solving $e_1 = 0$ with respect to $d(x)$ we get

$$d(x) = \frac{(3b(x)^2 - 8a(x)c(x))^{3/2}}{a(x)^2}\left(C_1 + \int \frac{f_2(x)}{(3b(x) - 8a(x)c(x))^{7/2}}\,dx\right), \tag{15}$$

being C_1 a constant and

$$f_2(x) = -9b(x)^4a'(x)^2 + 3a(x)b(x)^2(-3b(x)^2a''(x) + b(x)a'(x)(15b'(x) + 2c(x)^2) - 4c(x)a'(x)^2) + a(x)^2(-16b(x)c(x)^3a'(x) + 16c(x)^2a'(x)^2 - 12b(x)^2(-3c(x)a''(x) + 7a'(x)c'(x) + b'(x)(3b'(x) + c(x)^2)) + b(x)^3(9b''(x) + 6c(x)c'(x))) - 4a(x)^3(4c(x)^2(2a''(x) + b(x)c'(x)) + 2c(x)(-8a'(x)c'(x) + 3b(x)b''(x) + 3b'(x)^2) + 3b(x)(b(x)c''(x) - 10b'(x)c'(x)) - 8c(x)^3b'(x)) + a(x)^4(32c(x)c''(x) - 80c'(x)^2).$$

Hence statement (a) is proved.

Case 2: $3b(x)^2 - 8a(x)c(x) = 0$, i.e.

$$c(x) = \frac{3b(x)^2}{8a(x)}.$$

We consider two subcases.

Subcase 2.1: $(3b(x)^4 - 64a(x)b(x)(a(x)d(x) + a'(x)) + 64a(x)^2(4a(x)e(x) + b'(x))^{3/4} \neq 0$. In this case, substituting $c(x)$ and $V_k(x)$ for $k = 0, 1, 2, 3$ into e_1 and e_0 we obtain two expressions where appear $V_4''(x)$. Isolating from both expressions $V_4''(x)$ and equating them we have a differential equation for $V_4'(x)$. Solving this differential equation we get

$$V_4(x) = \frac{a(x)^3}{(3b(x)^4 - 64a(x)b(x)(a(x)d(x) + a'(x)) + 64a(x)^2(4a(x)e(x) + b'(x))^{3/4}}.$$

Substituting $V_4(x)$ in $e_1=0$ and $e_0=0$ we obtain that the function $f_3(x)$ equal to

$$
\begin{aligned}
&-4a(x)\Big(1024\big(-16d'(x)e(x)^2+4(3d(x)e'(x)+e''(x))e(x)-7e'(x)^2\big)a(x)^7\\
&-512\big(6(e(x)b'(x)+b(x)e'(x))d(x)^2-\big(13b'(x)e'(x)+2e(x)(4e(x)a'(x)\\
&+5b(x)d'(x)+2b''(x))-2b(x)e''(x)\big)d(x)+8e(x)^2a''(x)+7e'(x)(b''(x)\\
&-b(x)d'(x))-2b'(x)e''(x)+2e(x)(10b'(x)d'(x)-2a'(x)e'(x)+b(x)d''(x)\\
&-b^{(3)}(x))\big)a(x)^6-64\big(12e(x)e'(x)b(x)^3+(4d'(x)d(x)^2-4d''(x)d(x)\\
&+7d'(x)^2-48e(x)^2b'(x))b(x)^2-2\big(6b'(x)d(x)^3+(8e(x)a'(x)-4b''(x))d(x)^2\\
&+\big(7b'(x)d'(x)-2(28a'(x)e'(x)+b^{(3)}(x))\big)d(x)+28e'(x)a''(x)+7d'(x)b''(x)\\
&-2b'(x)d''(x)-8a'(x)e''(x)+e(x)(76a'(x)d'(x)-8a^{(3)}(x))\big)b(x)\\
&-80e(x)^2a'(x)^2+19d(x)^2b'(x)^2+7b''(x)^2+24b'(x)^2d'(x)-22d(x)b'(x)b''(x)\\
&+40e(x)\big(b'(x)(d(x)a'(x)+2a''(x))+a'(x)b''(x)\big)-4b'(x)(30a'(x)e'(x)\\
&+b^{(3)}(x))\big)a(x)^5+16\big(3(-4e(x)d'(x)+7d(x)e'(x)+e''(x))b(x)^4\\
&-6\big(8e(x)(3e(x)a'(x)+d(x)b'(x))+9b'(x)e'(x)\big)b(x)^3-4\big(8a'(x)d(x)^3\\
&-4a''(x)d(x)^2+4(3a'(x)d'(x)-a^{(3)}(x))d(x)-33e(x)b'(x)^2+14d'(x)a''(x)\\
&-4a'(x)d''(x)\big)b(x)^2+8\big(24a'(x)b'(x)d(x)^2-(b'(x)a''(x)+6a'(x)b''(x))d(x)\\
&-60a'(x)^2e'(x)+7a''(x)b''(x)-2b'(x)a^{(3)}(x)+a'(x)(9b'(x)d'(x)+60e(x)a''(x)\\
&-2b^{(3)}(x))\big)b(x)-8b'(x)\big(-60e(x)a'(x)^2-9b''(x)a'(x)+b'(x)(21d(x)a'(x)\\
&+8a''(x))\big)\big)a(x)^4+4\big(3(d(x)d'(x)-d''(x))b(x)^5+3\big(-3b'(x)d(x)^2\\
&+2(26e(x)a'(x)+b''(x))d(x)+8b'(x)d'(x)+60a'(x)e'(x)-4e(x)a''(x)\\
&+b^{(3)}(x)\big)b(x)^4-42b'(x)(20e(x)a'(x)+d(x)b'(x)+b''(x))b(x)^3+4\big(21b'(x)^3\\
&-4(24d(x)^2a'(x)^2-6d'(x)a'(x)^2+8d(x)a''(x)a'(x)-4a^{(3)}(x)a'(x)\\
&+7a''(x)^2)\big)b(x)^2+32a'(x)\big(-60e(x)a'(x)^2-9b''(x)a'(x)+b'(x)(27d(x)a'(x)\\
&+7a''(x))\big)b(x)-48a'(x)^2b'(x)^2\big)a(x)^3+b(x)\big(-9e'(x)b(x)^6+36e(x)b'(x)b(x)^5\\
&+12\big(a'(x)(2d(x)^2-5d'(x))-d(x)a''(x)-a^{(3)}(x)\big)b(x)^4+24\big(58e(x)a'(x)^2\\
&+4b''(x)a'(x)+b'(x)(7d(x)a'(x)+5a''(x))\big)b(x)^3-504a'(x)b'(x)^2b(x)^2\\
&+128a'(x)^2(a''(x)-6d(x)a'(x))b(x)+384a'(x)^3b'(x)\big)a(x)^2-3b(x)^2a'(x)\\
&\cdot\big(9e(x)b(x)^5+4(3d(x)a'(x)+4a''(x))b(x)^3-60a'(x)b'(x)b(x)^2+64a'(x)^3\big)a(x)\\
&-12b(x)^5a'(x)^3\Big)
\end{aligned}
\tag{16}
$$

must be zero for having a generalized Weierstrass inverse integrating factor. This completes the proof of statement (b).

Subcase 2.2: $(3b(x)^4-64a(x)b(x)(a(x)d(x)+a'(x))+64a(x)^2(4a(x)e(x)+b'(x))^{3/4}=0$, that is,

$$
e(x)=\frac{-3b(x)^4+64a(x)^2d(x)b(x)+64a(x)a'(x)b(x)-64a(x)^2b'(x)}{256a(x)^3}.
$$

In this case, substituting $c(x)$, $e(x)$ and $V_k(x)$ for $k=0,1,2,3$ into e_1 and e_0 we obtain two expressions. In fact $e_0=b(x)e_1/(4a(x))$, and $e_1=0$ becomes

$$
\begin{aligned}
&(9V_4(x)a'(x)-3a(x)V_4'(x))b(x)^3-9a(x)V_4(x)b'(x)b(x)^2+16a(x)^2\Big(-a'(x)V_4'(x)\\
&+3d(x)(a(x)V_4'(x)-V_4(x)a'(x))+a(x)(3V_4(x)d'(x)+V_4''(x))\Big)=0.
\end{aligned}
$$

Taking $V_4(x)$ satisfying this differential equation, the proof of statement (c) is done. □

Example 1. We provide an example of the existence of a generalized Weierstrass inverse integrating factor of the differential system (2) satisfying the conditions of statement (b) of

Theorem 2. Consider the differential system (2) with $a(x) \not\equiv 0$, $b(x) = c(x) = 0$, $e(x) \not\equiv 0$, $D = 4a(x)d(x)e(x) + a'(x)e(x) - a(x)e'(x) \neq 0$ and

$$d(x) = \frac{1}{D}\Big(7e'(x)^2a(x)^2 + 16e(x)^2d'(x)a(x)^2 - 4e(x)e''(x)a(x)^2 - 2e(x)a'(x)e'(x)a(x) + 4e(x)^2a''(x)a(x) - 5e(x)^2a'(x)^2)\Big).$$

Then the generalized inverse integrating factor is of the form $V(x,y) = V_0(x) + y + V_4(x)y^4$ with

$$V_4(x) = \frac{4a(x)^2e(x)}{D}, \quad \text{and} \quad V_0(x) = \frac{4a(x)e(x)^2}{D}.$$

Example 2. Now we give an example of the existence of a generalized Weierstrass inverse integrating factor of the differential system (2) satisfying the conditions of statement (c) of Theorem 2. Consider the differential system (2) with $a(x) \not\equiv 0$ and $b(x) = c(x) = e(x) = 0$. Then the generalized inverse integrating factor is of the form $V(x,y) = y + V_4(x)y^4$ with

$$V_4(x) = e^{-3\int d(x)\,dx}K + 3e^{-3\int d(x)\,dx}\int e^{-\int_0^x -3d(s)\,ds}a(x)\,dx.$$

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