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DARBOUX INTEGRABILITY OF POLYNOMIAL DIFFERENTIAL SYSTEMS IN $\mathbb{R}^3$

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Abstract. In this article we study the Darboux integrability of the polynomial differential systems

$$\dot{x} = y - x^2, \quad \dot{y} = z - x, \quad \dot{z} = -d - ax - by - cz.$$

This system comes from the study of a Hopf bifurcation in slow-fast systems with two slow variables and one fast variable. The tools used here for studying the Darboux integrability can be applied to arbitrary polynomial differential systems in $\mathbb{R}^3$.

1. Introduction and statement of the main results

Recently the Hopf bifurcations have been studied intensively in two dimensional differential systems with one slow and one fast variable, see for instance [1, 5, 6, 10, 8]. Less analysis has been done of the Hopf bifurcations in slow-fast systems in $\mathbb{R}^3$ with two slow variables and one first variable, see [7, 9, 13, 14]. Guuckenheimer in [9] reduces the study of this Hopf bifurcation to study the zero Hopf bifurcation of the differential system

$$\begin{aligned} \dot{x} &= y - x^2, \\ \dot{y} &= z - x, \\ \dot{z} &= -d - ax - by - cz, \end{aligned} \tag{1}$$

where $(x, y, z) \in \mathbb{R}^3$, $a, b, c, d \in \mathbb{R}$ and the dot denotes derivative with respect to the independent variable t.

The vector field associated to (1) is

$$\mathcal{X} = (y - x^2)\frac{\partial}{\partial x} + (z - x)\frac{\partial}{\partial y} - (d + ax + by + cz)\frac{\partial}{\partial z}.$$

Let U be an open subset in $\mathbb{R}^3$ such that $\mathbb{R}^3 \setminus U$ has zero Lebesgue measure. We say that a real function $H = H(x, y, z)\colon U \subset \mathbb{R}^3 \to \mathbb{R}$ non-constant in any open subset of U is a *first integral* if $H(x(t), y(t), z(t))$ is constant on all solutions $(x(t), y(t), z(t))$ of $\mathcal{X}$ contained in U, i.e. $\mathcal{X} H_{|U} = 0$. The existence of a first integral for a differential system in $\mathbb{R}^3$ allows to reduce its study in one dimension. This is the main reason to look for first integrals.

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One of the more classical problems in the qualitative theory of differential equations depending on parameters is to characterize the existence or not of first integrals in function of these parameters. This is a very difficult problem and not many results can be found in the literature. One of the best tools to look for first integrals is the Darboux theory of integrability.

Our objective is to study the Darboux integrability of system (1). Probably the more interesting novelty of this paper is not the characterization of the Darboux integrability of system (1), but the method for reaching this result, which can be applied to other polynomial differential systems in $\mathbb{R}^3$.

Now we shall introduce the basic notions of the Darboux theory of integrability restricted to system (1). Let $\mathbb{C}[x, y, z]$ be the ring of polynomials in the variables x, y, z with coefficients in $\mathbb{C}$. We say that $f \in \mathbb{C}[x, y, z]$ is a *Darboux polynomial* of the vector field $\mathcal{X}$ if there exists a polynomial $K \in \mathbb{C}[x, y, z]$ such that $\mathcal{X}f = Kf$, i.e.

$$(y-x^2)\frac{\partial h}{\partial x} + (z-x)\frac{\partial h}{\partial y} - (d+ax+by+cz)\frac{\partial h}{\partial z} = Kh. \tag{2}$$

The polynomial $K = K(x, y, z)$ is called the *cofactor* of f. It is easy to show that the cofactor of a Darboux polynomial of the vector field $\mathcal{X}$ has degree at most one, i.e. $K = k_0 + k_1x + k_2y + k_3z$ with $k_i \in \mathbb{C}$ for $i = 0, \ldots, 3$. Note that we look for complex Darboux polynomials in real differential systems. The reason is that frequently the complex structure forces the existence of real first integrals, and sometimes if we only work with the real Darboux polynomials, we cannot detect all the real first integrals.

If $f \in \mathbb{C}[x, y, z]$ is a Darboux polynomial of the vector field $\mathcal{X}$ then $f(x, y, z) = 0$ is an *invariant algebraic surface* for the differential system (1), i.e. if an orbit has a point in this surface then the whole orbit is contained in it. Note that a Darboux polynomial with zero cofactor is a *polynomial first integral*.

The Darboux polynomials for system (1) with non–zero cofactor are characterized in the next result.

Theorem 1. *System* (1) *has an irreducible Darboux polynomial with non–zero cofactor if and only if* $a = b = 0$ *and* $c \neq 0$. *In this case the irreducible Darboux polynomial is* $cz + d$ *with cofactor* $-c$.

Theorem 1 is proved in section 3.

Let $I_{(x,y,z)}$ be the maximal interval of definition of the solution of system (1) such that at time zero pass through the point (x, y, z). We say that a real function $I = I(x, y, z, t)\colon U \times I_{(x,y,z)} \to \mathbb{R}$ non-constant in any open subset of U is an *invariant* if it is constant on every solution $(x(t), y(t), z(t))$ contained in U, i.e. if

$$\left.\frac{dI}{dt}\right|_U = \mathcal{X}I + \left.\frac{\partial I}{\partial t}\right|_U = 0.$$

Corollary 2. *System* (1) *with* $a = b = 0$ *and* $c \neq 0$ *has the invariant* $I = (cz + d)e^{ct}$.

The proof of Corollary 2 is immediate from Theorem 1 and the definition of invariant.

To know and invariant of a differential system is important because it allows to compute either the α-limits, or the ω-limits of the orbits of the system. More precisely, if $c > 0$ then when $t \to \overline{\alpha}$ the orbit of system (1) having maximal interval of definition $(\overline{\alpha}, \overline{\omega})$, under the assumption of Corollary 2, tends to the invariant plane $cz + d = 0$, and studying the dynamics on this invariant plane we can determine the α-limit sets. A similar study can be done if $c < 0$ for the ω-limit sets. For a definition of α- or ω-limit sets see [4].

An *exponential factor* $F(x, y, z)$ of the vector field $\mathcal{X}$ is an exponential function of the form $\exp(g/h)$ with g and h coprime polynomials in $\mathbb{C}[x, y, z]$ and satisfying $\mathcal{X}F = LF$ for some $L \in \mathbb{C}[x, y, z]$ with degree one. The exponential factors appear when some Darboux polynomial has multiplicity larger than one, or when the multiplicity of the plane at infinity is larger than one, for more details see [2, 12].

A first integral of system (1) is called of *Darboux type* if it is a first integral of the form

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q}$$

where $f_1, \ldots, f_p$ are Darboux polynomials and $F_1, \ldots, F_q$ are exponential factors.

The next theorem is the main result of the paper, and it characterizes the first integrals of Darboux type for system (1).

Theorem 3. *System* (1) *has Darboux first integrals if and only if $b = d = 0$ and $a + c = 0$. Moreover, when system* (1) *has Darboux first integrals these are functions of Darboux type in the variable $ay - z$.*

Theorem 3 is proved in section 4. In section 2 we introduce some auxiliary results that will be used all through the paper.

2. Auxiliary results

In the rest of this paper we will use the following well known result of the Darboux theory of integrability, see for instance Chapter 8 of [4].

Theorem 4 (Darboux theory of integrability)**.** *Suppose that a polynomial vector field $\mathcal{X}$ defined in $\mathbb{R}^n$ of degree m admits p Darboux polynomials f_i with cofactors K_i for $i = 1, \ldots, p$ and q exponential factors $F_j = \exp(g_j/h_j)$ with cofactors L_j for $j = 1, \ldots, q$. If there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0, \tag{3}$$

then the following real (multivalued) function of Darboux type

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q},$$

substituting $f_i^{\lambda_i}$ by $|f_i|^{\lambda_i}$ if $\lambda_i \in \mathbb{R}$, is a first integral of the vector field $\mathcal{X}$.

For a proof of the next result see [11, 12].

Proposition 5. *The following statements hold.*

(a) *If* $e^{g/h}$ *is an exponential factor for the polynomial differential system* (1) *and* h *is not a constant polynomial, then* $h = 0$ *is an invariant algebraic surface.*
(b) *Eventually* e^g *can be an exponential factor, coming from the multiplicity of the infinite invariant plane.*

The proof of the next result can be found in Chapter 8 of [4].

Lemma 6. *Let* f *be a polynomial and* $f = \prod_{j=1}^{s} f_j^{\alpha_j}$ *its decomposition into irreducible factors in* $\mathbb{C}[x, y, z]$*. Then* f *is a Darboux polynomial of system* (1) *if and only if all the* f_j *are Darboux polynomials of system* (1)*. Moreover if* K *and* K_j *are the cofactors of* f *and* f_j*, then* $K = \sum_{j=1}^{s} \alpha_j K_j$*.*

We note that in view of Lemma 6 to study the Darboux polynomials of system (1) it is enough to study the irreducible ones.

To prove Theorem 3 we also need one auxiliary result proved in [3]. We recall that a *generalized rational function* is a function which is the quotient of two analytic functions. In particular rational first integrals and analytic first integrals are particular cases of generalized rational first integrals. Clearly a Darboux type function is a generalized rational function.

Theorem 7. *Assume that the differential system* (1) *has* p *as a singular point and let* $\lambda_1, \lambda_2, \lambda_3$ *be the eigenvalues of the linear part of system* (1) *at* p*. Then the number of functionally independent generalized rational integrals of system* (1) *is at most the dimension of the minimal vector subspace of* $\mathbb{R}^3$ *containing the set*

$$\big\{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = 0,\ (k_1, k_2, k_3) \neq (0, 0, 0)\big\}.$$

3. Proof of Theorem 1

To prove Theorem 1 we state and prove some auxiliary results. We denote by $\mathbb{N}$ the set of positive integers.

Lemma 8. *If* h *is a Darboux polynomial of system* (1) *with non–zero cofactor* K *then* $K = k_0 - mx$ *for some* $k_0 \in \mathbb{C}$ *and* $m \in \mathbb{N} \cup \{0\}$*.*

Proof. Let h be a Darboux polynomial of system (1) with non–zero cofactor K then $K = k_0 + k_1x + k_2y + k_3z$ for some $k_0, k_1, k_2, k_3 \in \mathbb{C}$.

Let n be the degree of h. We write h in sum of its homogeneous parts as $h = \sum_{i=1}^{n} h_i$ where each h_i is a homogenous polynomial of degree i. Without loss of generality we can assume that $h_n \neq 0$ and $n \geq 1$.

Computing the terms of degree $n + 1$ in (2) we get that

$$-x^2\frac{\partial h_n}{\partial x} = (k_1x + k_2y + k_3z)h_n.$$

Solving it we get

$$h_n = C_n(y, z)x^{-k_1} \exp\Big(\frac{k_2y}{x} + \frac{k_3z}{x}\Big),$$

where C_n is any function in the variables y, z. Since h_n must be a homogeneous polynomial we must have $k_1 = -m$ with $m \in \mathbb{N} \cup \{0\}$, $k_2 = k_3 = 0$ and $C_n(y, z) \in \mathbb{C}[y, z] \setminus \{0\}$. This concludes the proof of the lemma. □

Proposition 9. *System* (1) *with either* $a^2+b^2 \neq 0$, *or* $a=b=c=0$ *has no Darboux polynomials with non–zero cofactor.*

Proof. For simplifying the computations we introduce the weight change of variables

$$x=\mu^{-1}X, \quad y=\mu^{-2}Y, \quad z=\mu^{-1}Z, \quad t=\mu T,$$

with $\mu \in \mathbb{R}\setminus\{0\}$. Then system (1) becomes

$$\begin{aligned} X' &= Y-X^2,\\ Y' &= \mu^2(Z-X),\\ Z' &= -bY-\mu(aX+cZ)-\mu^2 d, \end{aligned}$$

where the prime denotes derivative with respect to the variable T.

A polynomial $H(X,Y,Z)$ is said to be *weight-homogeneous* of degree $r \in \mathbb{N}$ with respect to the weight exponent $s=(s_1,s_2,s_3)$ for all $\mu \in \mathbb{R}\setminus\{0\}$ if we have $H(\mu^{s_1}X,\mu^{s_2}Y,\mu^{s_3}Z)=\mu^r H(X,Y,Z)$.

Let $h(x,y,z)$ be a Darboux polynomial of system (1) with cofactor $k(x,y,z)$. Without loss of generality, we can assume that $k(x,y,z)=k_0+k_1x+k_2y+k_3z$. In view of Lemma 8 we get $k_2=k_3=0$ and $k_1=-m$ with $m \in \mathbb{N}\cup\{0\}$. Set $H(X,Y,Z)=\mu^n h(\mu^{-1}X,\mu^{-2}Y,\mu^{-1}Z)=\sum_{i=0}^{n}\mu^i H_i(X,Y,Z)$ where H_i is the weight-homogeneous part with weight degree $n-i$ of H and n is the weight degree of H with weight exponent $s=(-1,-2,-1)$. We also set $K(X,Y,Z)=\mu k(\mu^{-1}X,\mu^{-2}Y,\mu^{-1}Z)=\mu(k_0-m\mu^{-1}X)=\mu k_0-mX$. From the definition of a Darboux polynomial, we have

$$\begin{aligned}&(y-x^2)\sum_{i=0}^{n}\mu^i\frac{\partial H_i}{\partial x}+\mu^2(z-x)\sum_{i=0}^{n}\mu^i\frac{\partial H_i}{\partial y}-(by+\mu(ax+cz)+\mu^2 d)\sum_{i=0}^{n}\mu^i\frac{\partial H_i}{\partial z}\\ &=(\mu k_0-mx)\sum_{i=0}^{n}\mu^i H_i,\end{aligned} \tag{4}$$

where we still use x,y,z instead of X,Y,Z.

Equating in (4) the terms with μ^i for $i=0,1,\ldots,n+2$ we get

$$\begin{aligned} L[H_0] &= -mxH_0,\\ L[H_1] &= -mxH_1+k_0H_0+(ax+cz)\frac{\partial H_0}{\partial z},\\ L[H_2] &= -mxH_2+k_0H_1+(ax+cz)\frac{\partial H_1}{\partial z}-(z-x)\frac{\partial H_0}{\partial y}+d\frac{\partial H_0}{\partial z},\\ L[H_j] &= -mxH_j+k_0H_{j-1}+(ax+cz)\frac{\partial H_{j-1}}{\partial z}-(z-x)\frac{\partial H_{j-2}}{\partial y}+d\frac{\partial H_{j-2}}{\partial z}, \end{aligned} \tag{5}$$

for $j=3,\ldots,n+2$, where $H_j=0$ for $j>n$, and L is the linear partial differential operator of the form

$$L=(y-x^2)\frac{\partial}{\partial x}-by\frac{\partial}{\partial z}.$$

Consider the first equation of (5), that is $L[H_0] = -mxH_0$, where H_0 is a weight-homogeneous polynomial of degree n.

By direct computation we get

$$H_0 = (x^2 - y)^{m/2} f_0(y, v),$$

where $f_0(y, v)$ is a differentiable function in y and $v = z + b\sqrt{y}\,\text{arctanh}(x/\sqrt{y})$. We consider different cases.

Case 1*:* $b \neq 0$. In this case since H_0 is a weight-homogeneous polynomial we must have $f_0 = f_0(y)$ being f_0 a homogeneous polynomial, i.e. $f_0(y) = C_0 y^l$ for some $C_0 \in \mathbb{C} \setminus \{0\}$. Thus,

$$H_0 = C_0(x^2 - y)^{m/2} y^l, \quad C_0 \in \mathbb{C} \setminus \{0\},$$

and the weight degree of H_0 is $n = m + 2l$.

From the second equation of (5) we get

$$L[H_1] = -mxH_1 + k_0 H_0.$$

Solving it we obtain

$$H_1 = C_0 k_0 (x^2 - y)^{m/2} y^{l-1/2} \text{arctanh}\Big(\frac{x}{\sqrt{y}}\Big) + (x^2 - y)^{m/2} f_1(y, v),$$

where $f_1(y, v)$ is a differentiable function. Again since H_1 is a weight-homogenous polynomial we must have $C_0 k_0 = 0$ (which yields $k_0 = 0$ because $C_0 \neq 0$), and f_1 must be a function of y but since the weight-degree of H_1 is $n - 1 = m + 2l - 1$ we get that $f_1 = 0$ and thus $H_1 = 0$. In short, $k_0 = 0$ and $H_1 = 0$.

The third equation in (5) is

$$L[H_2] = -mxH_2 - (z - x)\frac{\partial H_0}{\partial y}.$$

Solving it we get

$$\begin{aligned} H_2 = {} & C_0 m (x^2 - y)^{m/2-1} y^{l-1}((b - 2)y + 2xz) \\ & + C_0(x^2 - y)^{m/2} y^{l-3/2}\bigg(b(m + 4l)\sqrt{y}\,\text{arctanh}^2\Big(\frac{x}{\sqrt{y}}\Big) \\ & + ((m + 4l)z - 4l\sqrt{y}) \log(x - \sqrt{y}) + b(m + 4l)\sqrt{y}\,\text{arctanh}\Big(\frac{x}{\sqrt{y}}\Big) \log\frac{x - \sqrt{y}}{x + \sqrt{y}} \\ & - ((m + 4l)z + 4l\sqrt{y}) \log(x + \sqrt{y})\bigg) + (x^2 - y)^{m/2} f_2(y, v), \end{aligned}$$

where $f_2(y, v)$ is a differentiable function. Since H_2 is a weight-homogenous polynomial of degree $n - 2 = m + 2l - 2$, we must have $f_2 = f_2(y)$ and $4l = m + 4l = 0$. In short, $l = m = 0$ and $k_0 = 0$, which yields $K = 0$, in contradiction with the fact that h is a Darboux polynomial with non–zero cofactor. Hence if $b \neq 0$ there are no Darboux polynomials with non–zero cofactor.

Case 2*:* $b = 0$. In this case we consider two subcases.

Subcase 2.1*:* $a \neq 0$. The first equation in (5), i.e. $L[H_0] = -mxH_0$, together with the fact that H_0 is a homogeneous polynomial with weight-degree n yields

$$H_0 = (x^2 - y)^{m/2} f_0(y, z),$$

where $g_0 \in \mathbb{C}[y, z]$ and has weight-degree $n - m$. The second equation in (5) yields

$$L[H_1] = -mxH_1 + k_0 H_0 + (ax + cz)\frac{\partial H_0}{\partial z}.$$

Solving it we get

$$\begin{aligned} H_1 = {} & (x^2 - y)^{m/2}\Big(\frac{a}{2}\log(x^2 - y^2)\frac{\partial f_0}{\partial z} - y^{-1/2}\big(k_0 f_0 + cz\frac{\partial f_0}{\partial z}\big)\operatorname{arctanh}\big(x/\sqrt{y}\big)\Big) \\ & + (x^2 - y)^{m/2} f_1(y, z), \end{aligned}$$

where f_1 is a smooth function in the variables y, z. Taking into account that H_1 is a weight-homogeneous polynomial with weight-degree $n - 1$ and that $a \neq 0$ we get $\partial f_0/\partial z = 0$ and $k_0 = 0$. Hence $H_0 = C_0(x^2 - y)^{m/2} y^l$ with $C_0 \in \mathbb{C} \setminus \{0\}$, $n = m + 2l$ and $H_1 = (x^2 - y)^{m/2} f_1(y, z)$ with $f_1 \in \mathbb{C}[y, z]$ of weight-degree $n - m - 1$.

The third equation in (5) yields

$$L[H_2] = -mxH_2 + (ax + cz)\frac{\partial H_1}{\partial z} - (z - x)\frac{\partial H_0}{\partial y}.$$

Solving it we get

$$\begin{aligned} H_2 = {} & \frac{1}{4}(x^2 - y)^{m/2-1}\bigg(y^{l-1}m(xz - y) - (x^2 - y)\bigg(2\Big(ly^l + ay\frac{\partial f_1}{\partial z}\Big)\log(x^2 - y) \\ & - zy^{-1/2}\Big((m + 4l)y^l - 4cy\frac{\partial f_1}{\partial z}\Big)\operatorname{arctanh}\Big(\frac{x}{\sqrt{y}}\Big)\bigg)\bigg) + (x^2 - y)^{m/2} f_2(y, z), \end{aligned}$$

where f_2 is a smooth function in the variables y, z. Since H_2 must be a weight-homogeneous polynomial, we must have $ly^l + ay\partial f_1/\partial z = 0$ and $(m + 4l)y^l - 4cy\partial f_1/\partial z = 0$. Using that $a \neq 0$ we obtain $f_1 = -ly^{l-1}z/a + g_1(y)$ and $am + 4(a + c)l = 0$. Since f_1 has weight degree $n - m - 1 = 2l - 1$ and y has weight-degree 2, we must have $g_1 = 0$. Furthermore, if $a + c = 0$, then $m = 0$ which is not possible because then $K = 0$ in contradiction with the fact that the cofactor is non-zero. Hence, $a + c \neq 0$ and $l = -am/(4(a + c))$. In short, $f_1 = -ly^{l-1}z/a$, $a + c \neq 0$, $l = -am/(4(a + c))$ and $H_2 = m(x^2 - y)^{m/2-1} y^{l-1}(y - xz)/4 + f_2(y, z)$ where $f_2 \in \mathbb{C}[y, z]$ is a weight-homogeneous polynomial with weight-degree $n - 2$.

The fourth equation in (5) yields

$$L[H_3] = -mxH_3 + (ax + cz)\frac{\partial H_2}{\partial z} - (z - x)\frac{\partial H_1}{\partial y} + d\frac{\partial H_1}{\partial z}.$$

Solving it we get

$$\begin{aligned} H_3 = {} & \frac{1}{16(a+c)^2}(x^2-y)^{m/2-1}y^{l-2}\Big((a+c)m(2a(a+c)xy \\ & + (2c(a+c)-m)yz + mxz^2) + (x^2-y)\Big(\frac{1}{2}\Big(m(4c+a(4+m))z \\ & - 4a(a+c)^2y^{2-l}\frac{\partial f_2}{\partial z}\Big)\log(x^2-y) + y^{-1/2}\Big(m(2(a+c)(a(a+c)+2d)y \\ & + (4(a+c)-cm)z^2) + 4c(a+c)^2y^{2-l}z\frac{\partial f_2}{\partial z}\Big)\operatorname{arctanh}\Big(\frac{x}{\sqrt{y}}\Big)\Big) \\ & + (x^2-y)^{m/2}f_3(y,z), \end{aligned}$$

where f_3 is a function in the variables y, z. Since H_3 must be a weight-homogeneous polynomial, we have

$$\begin{aligned} & m(4c+a(4+m))z - 4a(a+c)^2y^{2-l}\frac{\partial f_2}{\partial z} = 0, \\ & m(2(a+c)(a(a+c)+2d)y + (4(a+c)-cm)z^2) + 4c(a+c)^2y^{2-l}z\frac{\partial f_2}{\partial z} = 0. \end{aligned} \tag{6}$$

From the first identity in (6) we get

$$f_2(y,z) = \frac{m(4c+a(m+4))}{8a(a+c)^2}y^{l-2}z^2 + g_2(y).$$

Substituting it in the second identity in (6) we obtain

$$\frac{2(a+c)m}{a}(a(a(a+c)+2d)y + 2(a+c)z^2) = 0.$$

Since $a(a+c) \neq 0$ we must have $m = 0$. In short, $k_0 = m = 0$, which yields $K = 0$, in contradiction with the fact that h is a Darboux polynomial with non–zero cofactor. Hence if $b = 0$ and $a \neq 0$ there are no Darboux polynomials with non–zero cofactor.

Subcase 2.2*:* $a = 0$. By hypothesis we also have $c = 0$. Proceeding as in Subcase 2.1 we get that $H_0 = (x^2-y)^{m/2}f_0(y,z)$ where $f_0 \in \mathbb{C}[y,z] \setminus \{0\}$ is a weight-homogeneous polynomial of weight degree $n-m$. The second equation in (5) becomes

$$L[H_1] = -mxH_1 + k_0H_0.$$

Solving it we get

$$H_1 = (x^2-y)^{m/2}y^{-1/2}k_0f_0(y,z)\operatorname{arctanh}\Big(\frac{x}{\sqrt{y}}\Big) + (x^2-y)^{m/2}f_1(y,z),$$

where f_1 is a function in y, z. Since H_1 is a weight-homogeneous polynomial of weight degree $n-1$, we must have $k_0f_0 = 0$ and f_1 be a weight-homogeneous polynomial of weight degree $n-m-1$. Since $f_0 \neq 0$ we must have $k_0 = 0$ and $H_1 = (x^2 - y)^{m/2}f_1(y,v)$.

From the the third equation in (5) we get

$$L[H_2] = -mxH_2 - (z-x)\frac{\partial H_0}{\partial y} + d\frac{\partial H_0}{\partial z}.$$

Solving it we obtain

$$
\begin{aligned}
H_2 = {} & \frac{1}{4}(x^2-y)^{m/2-1}\Big(my^{-1}(-y+xz)f_0(y,z) - 2(x^2-y)\log(x^2-y)\frac{\partial f_0}{\partial y} \\
& - \frac{(x^2-y)}{y^{3/2}}\Big(mzf_0 - 4dy\frac{\partial f_0}{\partial z} + 4yz\frac{\partial f_0}{\partial y}\Big)\operatorname{arctanh}\Big(\frac{x}{\sqrt{y}}\Big)\Big) \\
& + (x^2-y)^{m/2} f_2(y,z),
\end{aligned}
$$

where f_2 is a function in y, z. Since H_2 is a weight-homogeneous polynomial of weight degree $n-2$, we must have $\partial f_0/\partial y = 0$ and $mzf_0 - 4dy\partial f_0/\partial z = 0$. Since $f_0 \neq 0$ we must have $m = 0$. In short, $k_0 = m = 0$ which yields $K = 0$, in contradiction with the fact that h is a Darboux polynomial with non–zero cofactor. Hence if $b = a = 0$ and $c \neq 0$ there are no Darboux polynomials with non–zero cofactor. □

In view of Proposition 9, to complete the proof of Theorem 1 we should study the case $a = b = 0$ and $c \neq 0$ and show that it has the unique irreducible Darboux polynomial $cz + d$. We introduce the change of variables $u = cz + d$. Then system (1) becomes

$$
\begin{aligned}
\dot{x} &= y - x^2, \\
\dot{y} &= -\frac{d}{c} + \frac{u}{c} - x, \\
\dot{u} &= -cu.
\end{aligned} \tag{7}
$$

We introduce the change of variables For simplifying the computations we introduce the weight change of variables

$$
x = \mu^{-1}X, \quad y = \mu^{-2}Y, \quad u = \mu^{-1}U, \quad t = \mu T,
$$

with $\mu \in \mathbb{R} \setminus \{0\}$. Then system (7) becomes

$$
\begin{aligned}
X' &= Y - X^2, \\
Y' &= -\mu^2 X + \mu^2\frac{U}{c} - \frac{d}{c}\mu^3, \\
U' &= -c\mu U,
\end{aligned}
$$

where the prime denotes derivative with respect to the variable T. From the definition of a Darboux polynomial, we have

$$
\begin{aligned}
& (y-x^2)\sum_{i=0}^{n}\mu^i\frac{\partial H_i}{\partial x} - \mu^2\Big(x - \frac{u}{c} + \mu\frac{d}{c}\Big)\sum_{i=0}^{n}\mu^i\frac{\partial H_i}{\partial y} - c\mu u\sum_{i=0}^{n}\mu^i\frac{\partial H_i}{\partial u} \\
& = (\mu k_0 - mx)\sum_{i=0}^{n}\mu^i H_i,
\end{aligned} \tag{8}
$$

where we still use x, y, u instead of X, Y, U.

Equating in (8) the terms with μ^i for $i = 0, 1, \dots, n+2$ we get

$$
\begin{aligned}
L[H_0] &= -mxH_0,\\
L[H_1] &= -mxH_1 + k_0H_0 + cu\frac{\partial H_0}{\partial u},\\
L[H_2] &= -mxH_2 + k_0H_1 + cu\frac{\partial H_1}{\partial u} - \Big(\frac{u}{c} - x\Big)\frac{\partial H_0}{\partial y},\\
L[H_3] &= -mxH_3 + k_0H_2 + cu\frac{\partial H_2}{\partial u} - \Big(\frac{u}{c} - x\Big)\frac{\partial H_1}{\partial y} + \frac{d}{c}\frac{\partial H_0}{\partial y},\\
L[H_j] &= -mxH_j + k_0H_{j-1} + cu\frac{\partial H_{j-1}}{\partial u} - \Big(\frac{u}{c} - x\Big)\frac{\partial H_{j-2}}{\partial y} + \frac{d}{c}\frac{\partial H_{j-3}}{\partial y},
\end{aligned}
\tag{9}
$$

for $j = 4, \dots, n+3$, where $H_j = 0$ for $j > n$ and L is the linear partial differential operator of the form

$$L = (y - x^2)\frac{\partial}{\partial x}.$$

Consider the first equation of (9). By direct computation we get

$$H_0 = (y - x^2)^{m/2} f_0(y, u),$$

where $f_0 \in \mathbb{C}[y, u]$ has weight-degree $n - m$. The second equation in (5) gives

$$L[H_1] = -mxH_1 + k_0H_0 + cu\frac{\partial H_0}{\partial u}.$$

Solving it and using that it has weight-degree $n - 1$ we get

$$H_1 = (x^2 - y)^{m/2} y^{-1/2}\Big(k_0 f_0(y, u) + cu\frac{\partial f_0}{\partial u}\Big)\text{arctanh}\Big(\frac{x}{\sqrt{y}}\Big) + (x^2 - y)^{m/2} f_1(y, u),$$

which yields $f_0(y, u) = u^{-k_0/c} g_0(y)$. Moreover, $k_0 = -lc$, g_0 has weight-degree $n-m-l$ and $H_1 = (x^2 - y)^{m/2} f_1(y, u)$ with $f_1 \in \mathbb{C}[y, u]$ and has weight-degree $n - 1$.

The third equation in (5) gives

$$L[H_2] = -mxH_2 + k_0H_1 + cu\frac{\partial H_1}{\partial u} - \Big(\frac{u}{c} - x\Big)\frac{\partial H_0}{\partial y}.$$

Solving it we obtain

$$
\begin{aligned}
H_2 = \frac{(x^2 - y)^{m/2-1} u^l}{4c}&\Big(my^{-1}(ux - cy)g_0(y) - 2c(x^2 - y)\log(x^2 - y)g_0'(y)\\
&+ \frac{x^2 - y}{y^{3/2}}\Big(-u(mg_0 + 4yg_0'(y)) + 4c^2yu^{-l}\Big(-lf_1 + u\frac{\partial f_1}{\partial u}\Big)\Big)\text{arctanh}\Big(\frac{x}{\sqrt{y}}\Big)\\
&+ (x^2 - y)^{m/2} f_2(y, u),
\end{aligned}
$$

where $f_2 \in \mathbb{C}[y, u]$ and has weight-degree $n - 2$. Since H_2 is a polynomial and $g_0 \neq 0$ we must have $g_0'(y) = 0$, i.e, $g_0 = C_0 \in \mathbb{C} \setminus \{0\}$, $m = 0$ and $f_1 = C_1 u^l$. This yields $n = l$. Moreover, since H_1 has weight-degree $n - 1 = l - 1$ we must have $C_1 = 0$. In short, $H_0 = C_0 u^n$, $n = l$, $H_1 = 0$ and $H_2 = f_2(y, u)$ with weight-degree $n - 2 = l - 2$. Proceeding inductively we get that $H_i = 0$ for $i = 2, \dots, n$. Hence the

Darboux polynomial is $h = C_0(d+cz)^n$ with cofactor $-cn$. The irreducible Darboux polynomial is $d+cz$ with cofactor $-c$. This concludes the proof of Theorem 1.

4. Proof of Theorem 3

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We will divide the proof of Theorem 3 into different results. The first one characterizes the polynomial first integrals of system (1).

Lemma 10. *System* (1) *has a polynomial first integral if and only if* $b = d = 0$ *and* $a + c = 0$. *In this case a polynomial first integral is* $ay - z$.

Proof. Let $h \in \mathbb{C}[x, y, z] \setminus \mathbb{C}$ be a polynomial first integral of system (1). We write h as a polynomial in the variable x as

$$h = \sum_{i=0}^{n} h_i(y, z)x^i,$$

where each $h_i \in \mathbb{C}[y, z]$. The coefficient of x^{n+1} from (2) with $K = 0$ is

$$-nh_n - \frac{\partial h_n}{\partial y} - a\frac{\partial h_n}{\partial z} = 0.$$

Solving it we get

$$h_n = F(ay - z)e^{-ny},$$

for some function F. Thus, since h_n must be a polynomial we get that $n = 0$ and F is a polynomial in the variable $ay - z$. In short $h = h_0 = F(ay - z)$ being F a polynomial. Then from (2) with $K = 0$ we get

$$\frac{dF}{d(ay - z)}\big(a(z - x) + (d + ax + by + cz)\big) = \frac{dF}{d(ay - z)}\big(d + by + (c + a)z\big) = 0.$$

Since F is a polynomial first integral it is not constant and thus $dF/d(ay - z) \neq 0$. This implies that $d = b = 0$ and $c = -a$. $\square$

Theorem 3 follows directly from the following two propositions.

Proposition 11. *The unique first integrals of Darboux type for system* (1) *with* $b = d = 0$ *and* $a + c = 0$ *are functions of Darboux type in the variable* $ay - z$.

Proof. We consider system (1) with $b = d = 0$ and $a + c = 0$, that is,

$$\dot{x} = y - x^2, \quad \dot{y} = z - x, \quad \dot{z} = a(z - x). \tag{10}$$

It follows from Lemma 10 that this system has the polynomial first integral $H_1 = ay - z$, which obviously is a generalized rational first integral. To conclude the proof of the proposition we shall show that system (10) has no other first integrals of Darboux type independent with H_1. To prove this we will use Theorem 7. First we note that the singular points of system (10) are of the form (x, x^2, x) with $x \in \mathbb{R}$. We compute the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the Jacobian matrix of this system on these singular points and we get

$$\lambda_1 = 0, \quad \lambda_{2,3} = \frac{1}{2}(a - 2x \pm \sqrt{(a + 2x)^2 - 4}).$$

Therefore $k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = 0$ is equivalent to

$$k_2(a - 2x + \sqrt{(a+2x)^2 - 4}) + k_3(a - 2x + \sqrt{(a+2x)^2 - 4}) = 0,$$

or in other words,

$$\frac{k_2}{k_3} = -\frac{a - 2x + \sqrt{(a+2x)^2 - 4}}{a - 2x - \sqrt{(a+2x)^2 - 4}}. \tag{11}$$

It is clear that the left-hand side of (11) is a rational number (because $k_2, k_3 \in \mathbb{Z}$), and that choosing x in a convenient way the right-hand side of (11) is irrational. Therefore (11) cannot hold for this convenient choice of x. Hence for this special singular point (x, x^2, x), the dimension of the minimal vector space of $\mathbb{R}^3$ containing the set

$$\big\{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = 0,\ (k_1, k_2, k_3) \neq (0,0,0)\big\}$$

is clearly one generated by $(1, 0, 0)$. Thus it follows from Theorem 7 that system (10) can only have one generalized rational first integral, which is a function of H_1. This completes the proof of the proposition. $\square$

In the next result we characterize the exponential factors of system (1) with either $b \neq 0$, or $d \neq 0$, or $c + a \neq 0$.

Lemma 12. *The unique exponential factors of system* (1) *with either $b \neq 0$, or $d \neq 0$, or $c + a \neq 0$ are:*

(a) *e^y and e^{ay-z} with cofactors $z - x$ and $d + by + (c+a)z$ if $a + c \neq 0$, or $a + c = 0$ and $b \neq 0$.*

(b) *e^y, e^{ay-z} and $e^{(ay-z)^2}$ with cofactors $z - x$, d and $2d(ay - z)$ if $a + c = 0$ and $b = 0$.*

Proof. Let $F = \exp(g/h)$ be an exponential factor of system (1) with cofactor L, where $g, h \in \mathbb{C}[x, y, z]$ with $(g, h) = 1$. Then from the definition of exponential factor and in view of Proposition 5 we have that either h is a constant the we can take $h = 1$, or h is a Darboux polynomial of system (1).

Case 1. We first assume that F is of the form $F = \exp(g)$ where $g = g(x, y, z) \in \mathbb{C}[x, y, z] \setminus \mathbb{C}$, with cofactor $L = L(x, y, z)$ of degree at most one. Without loss of generality we can assume that g has no constant term and that we can write $L = l_0 + l_1x + l_2y + l_3z$. So we have

$$(y - x^2)\frac{\partial g}{\partial x} + (z - x)\frac{\partial g}{\partial y} - (d + ax + by + cz)\frac{\partial g}{\partial z} = l_0 + l_1x + l_2y + l_3z. \tag{12}$$

We write g as a polynomial in the variable x as

$$g = \sum_{i=0}^{n} g_i(y, z)x^i,$$

where each $g_i \in \mathbb{C}[y, z]$. The coefficient of x^{n+1} from (12) if $n \geq 1$ is

$$-ng_n - \frac{\partial g_n}{\partial y} - a\frac{\partial g_n}{\partial z} = 0.$$

Solving it we get
$$g_n = F(ay-z)e^{-ny},$$
for some function F. Thus $g_n = 0$ if $n \geq 1$. For $n = 0$ we have $g = g_0(y,z)$ and the coefficients of x from (12) are
$$-\frac{\partial g}{\partial y} - a\frac{\partial g}{\partial z} = l_1,$$
$$z\frac{\partial g}{\partial y} - (d+by+cz)\frac{\partial g}{\partial z} = l_0 + l_2 y + l_3 z.$$
Now solving it we get from the first equation
$$g = G(ay-z) - l_1 y;$$
and from the second, introducing the variable $u = ay - z$, we obtain
$$(d+by+(a+c)(ay+u))G'(u) = -l_0 - l_2 y - (l_1+l_3)(ay+u). \tag{13}$$
We consider different subcases.

Subcase 1.1*: $a+c \neq 0$.* In this case solving (13) we get
$$G(u) = -\frac{1}{(a+c)^2}\big((a+c)(l_1+l_3)u + (-(l_1+l_3)(d+by) + (a+c)(l_0+l_2y))\log(d+(a+c)u+(b+a(a+c))y)\big).$$
Since $G(u)$ must be a polynomial we get
$$l_0 = \frac{d(l_1+l_3)}{a+c}, \quad l_2 = \frac{b(l_1+l_3)}{a+c} \quad \text{and} \quad G(u) = -\frac{l_1+l_3}{a+c}u.$$
Then $g = -l_1 y - (l_1+l_3)(ay-z)/(a+c)$.

Subcase 1.2*: $a+c = 0$ and $b = 0$.* In this case, $d \neq 0$ and solving (13) we get
$$G(u) = -\frac{u}{2d}\big(2l_0 + (l_1+l_3)u + 2(l_2 + a(l_1+l_3))y\big).$$
Since $G(u)$ must be a polynomial independent of y we get
$$l_2 = -a(l_1+l_3) \quad \text{and} \quad G(u) = -\frac{u}{2d}\big(2l_0 + (l_1+l_3)u\big).$$
Then $g = -l_1 y - (ay-z)\big(2l_0 + (l_1+l_3)(ay-z)\big)/(2d)$.

Subcase 1.3*: $a+c = 0$, $b \neq 0$ and $d = 0$.* In this case solving (13) we get
$$G(u) = -\frac{u}{2by}\big(2l_0 + (l_1+l_3)u + 2(l_2 + a(l_1+l_3))y\big).$$
Since $G(u)$ must be a polynomial independent of y we get
$$l_2 = 0, \quad l_3 = -l_1 \quad \text{and} \quad G(u) = -\frac{l_2 u}{b}.$$
Then $g = -l_1 y - l_2(ay-z)/b$.

Subcase 1.4*: $a+c = 0$, $bd \neq 0$.* In this case solving (13) we get
$$G(u) = -\frac{u}{2(d+by)}\big(2l_0 + (l_1+l_3)u + 2(l_2 + a(l_1+l_3))y\big).$$

Since $G(u)$ must be a polynomial we get

$$2l_0 + (l_1 + l_3)u + 2(l_2 + a(l_1 + l_3))y = \kappa 2(d + by), \quad \kappa \in \mathbb{C} \setminus \{0\}.$$

Solving it, we obtain

$$\kappa = \frac{l_0}{d}, \quad l_3 = -l_1, \quad l_2 = \frac{bl_0}{d} \quad \text{and} \quad G(u) = -\frac{l_0}{d}u.$$

Then $g = -l_1 y - l_0(ay - z)/d$.

Case 2. We study the exponential factors of the form $\exp(g/h)$ with cofactor $L = L(x, y, z)$ of degree at most one, $(g, h) = 1$ and h is a Darboux polynomial. Therefore g and h satisfy

$$(y - x^2)\frac{\partial g}{\partial x} + (z - x)\frac{\partial g}{\partial y} - (d + ax + by + cz)\frac{\partial g}{\partial z} = Kg + Lh, \tag{14}$$

where we have simplified the common factor $\exp(g/h)$ and we have used the fact that h is a Darboux polynomial of system (1) with cofactor K. By Theorem 1 and Lemma 6, $a = b = 0$, $c \neq 0$, $h = (cz + d)^n$ with $n \geq 1$ and $K = -nc$. Then (14) becomes

$$(y - x^2)\frac{\partial g}{\partial x} + (z - x)\frac{\partial g}{\partial y} - (d + cz)\frac{\partial g}{\partial z} = -ncg + (l_0 + l_1 x + l_2 y + l_3 z)(d + cz)^n, \tag{15}$$

with $n \geq 1$. We denote by $\bar{g}$ the restriction of g to $z = -d/c$. Then $\bar{g} \neq 0$ (since otherwise $(h, g) \neq 1$). In this case, $\bar{g}$ satisfies

$$(y - x^2)\frac{\partial \bar{g}}{\partial x} - \Big(\frac{d}{c} + x\Big)\frac{\partial \bar{g}}{\partial y} = -nc\bar{g}. \tag{16}$$

We write $\bar{g}$ as a polynomial in the variable x that is $\bar{g} = \sum_{i=0}^m \bar{g}_i(y)x^i$. Then the coefficient of x^{n+1} in (16) satisfy

$$-m\bar{g}_m - \frac{d\bar{g}_m}{dy} = 0,$$

which yields $\bar{g}_m = C_m e^{-my}$. Since $\bar{g}_m$ must be a polynomial we have $m = 0$ and $\bar{g} = \bar{g}_0 = \bar{g}_0(y)$. Therefore, from (16) we get

$$-\Big(\frac{d}{c} + x\Big)\frac{d\bar{g}_0}{dy} = -nc\bar{g}_0,$$

which yields $\bar{g}_0 = 0$, in contradiction with the fact that $\bar{g} \neq 0$. □

Proposition 13. *System* (1) *with either* $b \neq 0$, *or* $d \neq 0$, *or* $c + a \neq 0$ *has no first integrals of Darboux type.*

Proof. It follows from Theorem 4 that system (1) has a first integral of Darboux type if and only if there exists $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that equation (3) is satisfied where p, q are the numbers of Darboux polynomials and exponential factors, respectively. Furthermore, K_j and L_j are the cofactors of Darboux polynomials and exponential factors, respectively. It follows from Theorem 1 that the cofactor of the Darboux polynomials of system (1) when $a = b = 0$ and $c \neq 0$ is $-c$ and otherwise there are no Darboux polynomials of system (1). We consider three different cases.

Case 1*: $a = b = 0$ and $c \neq 0$.* In this case $a + c \neq 0$ and it follows from Theorem 1 and Lemma 12 that equation (3) is equivalent to

$$-c\lambda_1 + \mu_1(z - x) + \mu_2(d + cz) = 0.$$

Solving it we get $\lambda_1 = \mu_1 = \mu_2 = 0$. In short there are no first integrals of Darboux type in this case.

Case 2*: $a + c = 0$ and $b = 0$.* In this case $d \neq 0$ and it follows from Theorem 1 and Lemma 12 that equation (3) is equivalent to

$$\mu_1(z - x) + \mu_2 d + 2\mu_3 d(ay - z) = 0.$$

Solving it we get $\mu_1 = \mu_2 = \mu_3 = 0$ and there are no first integrals of Darboux type in this case.

Case 3*: Remaining cases.* It follows from Theorem 1 and Lemma 12 that equation (3) is equivalent to

$$\mu_1(z - x) + \mu_2(d + by + (c + a)z) = 0.$$

Solving it we get $\mu_1 = \mu_2 = 0$ which concludes the proof of Proposition 13. □

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