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LIOUVILLIAN FIRST INTEGRALS FOR GENERALIZED LIÉNARD POLYNOMIAL DIFFERENTIAL SYSTEMS

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ABSTRACT. We study the Liouvillian first integrals for the generalized Liénard polynomial differential systems of the form $x' = y$, $y' = -g(x) - f(x)y$, where $g(x)$ and $f(x)$ are arbitrary polynomials such that $2 \leq \deg g \leq \deg f$.

1. Introduction and statement of the main result

One of the more classical problems in the qualitative theory of planar differential systems depending on parameters is to characterize the existence or not of first integrals. This is a difficult problem because in general there are no tools for solving it.

We consider the system

$$x' = y, \quad y' = -g(x) - f(x)y, \tag{1}$$

called the *generalized Liénard differential system*, where x and y are complex variables and the prime denotes derivative with respect to the time t, which can be either real or complex. Such differential systems appear in several branches of the sciences, such as biology, chemistry, mechanics, electronics, etc. For $g(x) = x$ the Liénard differential systems (1) are called the *classical Liénard systems*.

Let $U \subset \mathbb{C}^2$ be an open set. We say that the non–constant function $H: \mathbb{C}^2 \to \mathbb{C}$ is a first integral of the polynomial vector field X on U, if $H(x(t), y(t)) =$ constant for all values of t for which the solution $(x(t), y(t))$ of X is defined on U. Clearly H is a first integral of X on U if and only if $XH = 0$ on U.

A *Liouvillian first integral* is a first integral H which is a Liouvillian function, that is, roughly speaking which can be obtained "by quadratures" of elementary functions. For a precise definition see [13]. The study of the Liouvillian first integrals is a classical problem of the integrability theory of the differential equations which goes back to Liouville, see for details again [13].

As far as we know there are few results providing all the Liouvillian first integral of some multi–parameter family of planar polynomial differential systems. One of the few families where the Liouvillian first integrals have been completely classified is the planar Lotka–Volterra quadratic polynomial differential system, see [1, 5, 9, 10, 11, 12].

The main objective of this paper is to study the *Liouvillian first integrals* of system (1) depending on the polynomial functions $f(x)$ and $g(x)$. We denote by m and n the degrees of g and f, respectively. We assume that $2 \leq m \leq n$. Note that when $m = 1$ the Liouvillian first integrals of system (1) were studied in [7]. More precisely, after a change of variables of the form $(x, y) \to (ax + b, y)$ system (1) with $m = 1$ can be written as

$$x' = y, \quad y' = -cx - f(x)y. \tag{2}$$

Their Liouvillian first integrals are characterized in [7] as follows.

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Theorem 1. *The unique Liouvillian first integrals $H = H(x, y)$ of the Liénard polynomial differential system* (2) *are:*

(a) $H = cx^2 + y^2$ *if* $f(x) = 0$*;*
(b) $H = y + \int f(x)\,dx$ *if* $c = 0$*;*
(c) $H = H = \Big(\frac{1}{2}(1-\sqrt{1+4\alpha})x + y\Big)^{-1+\sqrt{1+4\alpha}} \Big(\frac{1}{2}(1+\sqrt{1+4\alpha})x + y\Big)^{1+\sqrt{1+4\alpha}}$ *with* $\alpha = -c/f(0)^2$ *if* $f(x) = f(0) \neq 0$*; and*
(d) $H = e^{f'(0)^2x^2+2f'(0)y}(c + f'(0)y)^{-2c}$ *if* $f(x) = f'(0)x \neq 0$*.*

Our main result is the following one.

Theorem 2. *The unique Liouvillian first integrals $H = H(x, y)$ of the generalized Liénard polynomial differential system* (1) *with* $\deg g = m$*,* $\deg f = n$ *and* $2 \le m \le n$ *are* $H = (a + y)e^{-(y+F(x))/a}$ *if* $g(x) = af(x)$ *where* $a \in \mathbb{C} \setminus \{0\}$ *and* $dF(x)/dx = f(x)$*.*

The proof of Theorem 2 is given in section 2.

2. Proof of Theorem 2

If $g(x) = af(x)$ with $a \in \mathbb{C} \setminus \{0\}$ it is easy to check that

$$H = H(x, y) = (a + y)e^{-(y+F(x))/a}$$

is a Liouvillian first integral of system (1). From now on we will assume that g/f is not constant.

Let $h(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}$. As usual $\mathbb{C}[x, y]$ denotes the ring of all complex polynomials in the variables x and y. We say that $v = 0$ is an *invariant algebraic curve* of the vector field X associated to system (1) if it satisfies

$$y\frac{\partial h}{\partial x} - (g(x) + f(x)y)\frac{\partial h}{\partial y} = Kh,$$

the polynomial $K = K(x, y) \in \mathbb{C}[x, y]$ is called the *cofactor* of $h = 0$ and has degree at most n. We also say that h is a *Darboux polynomial* of system (1). Note that a polynomial first integral is a Darboux polynomial with zero cofactor.

The invariant algebraic curves are important because a sufficient number of them forces the existence of a first integral. This result is the basis of the Darboux theory of integrability, see for instance [4, 6].

An *exponential factor* E of system (1) is a function of the form $E = e^{u/v} \notin \mathbb{C}$ with $u, v \in \mathbb{C}[x, y]$ satisfying that

$$y\frac{\partial E}{\partial x} - (g(x) + f(x)y)\frac{\partial E}{\partial y} = LE,$$

for some polynomial $L = L(x, y)$ of degree at most n, called the *cofactor* of E.

The existence of exponential factors $e^{u/v}$ with $v \notin \mathbb{C}$ is due to the fact that the multiplicity of the invariant algebraic curve $v = 0$ is larger than 1, and the existence of exponential factors of the form e^u is due to the multiplicity of the invariant straight line at infinity, for more details see [3].

In 1992 Singer [13] proved that a polynomial differential system has a Liouvillian first integral, if and only if it has an inverse integrating factor of the form

$$e^{\left(\int U_1(x,y)dx + \int U_2(x,y)dy\right)} \tag{3}$$

where U_1 and U_2 are rational functions which verify $\partial U_1/\partial y = \partial U_2/\partial x$. In 1999 Christopher [2] improved the results of Singer showing that the inverse integrating factor (3) can be written in the form

$$e^{u/v}\prod_{i=1}^{k} h_i^{\lambda_i} \tag{4}$$

where u, v and h_i are polynomials and $\lambda_i \in \mathbb{C}$.

From (4) and the Darboux theory of integrability (see [6]) we have the following result.

Theorem 3. *The polynomial differential system* (1) *has a Liouvillian first integral if and only if system* (1) *has an integrating factor of the form* (4)*, or equivalently there exist* p *invariant algebraic curves* $h_i = 0$ *with cofactors* K_i *for* $i = 1,\dots,p$, q *exponential factors* $E_j = e^{u_j/v_j}$ *with cofactors* L_j *for* $j = 1,\dots,q$ *and* $\lambda_j, \mu_j \in \mathbb{C}$ *not all zero such that*

$$\sum_{i=1}^{p}\lambda_i K_i + \sum_{j=1}^{q}\mu_j L_j = -\textit{divergence of } (1) = f(x).$$

In short, for proving Theorem 2 we need to characterize the Darboux polynomials and the exponential factors of system (1). We start with some preliminaries. Let $n = \deg f$ and $m = \deg g$. We write

$$f(x) = \sum_{j=0}^{n} a_j x^j \quad \text{and hence} \quad F(x) = \sum_{j=0}^{n}\frac{a_j}{j+1}x^{j+1}, \quad n \ge 2 \tag{5}$$

and

$$g(x) = \sum_{j=0}^{m} b_j x^j, \quad 2 \le m \le n.$$

We will also assume that g/f is not constant.

Proposition 4. *System* (1) *under the assumptions of Theorem 2 and with* g/f *not constant has no Darboux polynomials.*

Proof. See Odani in [8]. □

Since there are no Darboux polynomials of system (1) under the assumptions of Theorem 2, all the exponential factors are of the form e^u with $u \in \mathbb{C}[x, y]$.

The first result that we will prove is the following.

Proposition 5. *System* (1) *under the assumptions of Theorem 2 and with* g/f *not constant has the following exponential factors.*

(a) e^{x^k} *for* $k = 1,\dots,n$,
(b) $e^{y+F(x)}$,
(c) $e^{(yx^{k-1}+\int f(x)x^{k-1}dx)}$ *for* $k = 2,\dots,n-m+1$.

Moreover, the exponential of any linear combination of the exponents of the exponentials given in statemens (a), (b) and (c) is also an exponential factor.

Proof. Let $E = e^u$ be an exponential factor of system (1) and

$$L = L(x, y) = \sum_{k=0}^{n}\sum_{l=0}^{k}\beta_{k-l,l}x^{k-l}y^l, \quad \beta_{k-l,l} \in \mathbb{C}$$

be its cofactor. We have

$$y\frac{\partial u}{\partial x} - (g(x) + f(x)y)\frac{\partial u}{\partial y} = \sum_{k=0}^{n}\sum_{l=0}^{k}\beta_{k-l,l}x^{k-l}y^l. \tag{6}$$

We write u as a polynomial in the variable y as $u = \sum_{j=0}^{r} u_j(x)y^j$. Without loss of generality we can assume that $u_r(x) \neq 0$. We rewrite (6) as

$$\begin{aligned}
&\sum_{j=0}^{r} u_j'(x)y^{j+1} - g(x)\sum_{j=1}^{r} ju_j(x)y^{j-1} - f(x)\sum_{j=0}^{r} ju_j(x)y^j = \\
&\sum_{k=0}^{n}\sum_{l=0}^{k} \beta_{k-l,l}x^{k-l}y^l = \sum_{l=0}^{n}\sum_{k=l}^{n} \beta_{k-l,l}x^{k-l}y^l = \\
&\sum_{k=0}^{n} \beta_{k,0}x^k + y\sum_{k=1}^{n} \beta_{k-1,1}x^{k-1} + \sum_{l=2}^{n} y^l \sum_{k=l}^{n} \beta_{k-l,l}x^{k-l}.
\end{aligned} \tag{7}$$

Assume that $r \geq n$. Then computing in (7) the coefficient of y^{r+1} we get that $u_r'(x) = 0$, that is, $u_r(x) = \gamma_r \in \mathbb{C} \setminus \{0\}$. Now we will show that if we write

$$u = \gamma_r y^r + \sum_{j=1}^{r} u_{r-j}(x)y^{r-j},$$

then

$$u_{r-j}(x) = \gamma_r \frac{a_n^j}{j!(n+1)^j} x^{j(n+1)} \prod_{i=0}^{j-1}(r-i) + l.o.t., \quad \text{for } j = 1, \ldots, r, \tag{8}$$

where $l.o.t$ means lower order terms in x.

For $j = 1$ computing the coefficient of y^r in (7), we get that

$$u_{r-1}'(x) - f(x)ru_r(x) = 0, \quad \text{if } r > n,$$

and

$$u_{r-1}'(x) - f(x)ru_r(x) = \beta_{0,r}, \quad \text{if } r = n.$$

Integrating it we obtain

$$u_{r-1}(x) = \gamma_r rF(x) + \text{constant} = \gamma_r \frac{a_n}{n+1} rx^{n+1} + l.o.t., \quad \text{if } r > n,$$

and

$$u_{r-1}(x) = \gamma_r rF(x) + \beta_{0,r}x + \text{constant} = \gamma_r \frac{a_n}{n+1} rx^{n+1} + l.o.t., \quad \text{if } r = n.$$

These last two expressions coincide with (8) for $j = 1$.

Now we assume that (8) is true for $j = 0, \ldots, J$ with $J < r$ and we will prove it for $j = J + 1$. Computing the coefficient of y^{r-J} in (7) we get

$$u_{r-J-1}'(x) = g(x)(r - J + 1)u_{r-J+1}(x) + f(x)(r - J)u_{r-J}(x) + l.o.t.$$

Now using the induction hypothesis and since $x^{(J-1)(n+1)+m}$ belongs to the lower terms in comparison with $x^{J(n+1)+n}$, we obtain that

$$\begin{aligned}
u_{r-J-1}'(x) &= f(x)(r-J)\frac{\gamma_r a_n^J}{J!(n+1)^J} x^{J(n+1)} \prod_{i=0}^{J-1}(r-i) + l.o.t. \\
&= \frac{\gamma_r a_n^{J+1}}{J!(n+1)^J} x^{J(n+1)+n} \prod_{i=0}^{J}(r-i) + l.o.t.
\end{aligned}$$

Now integrating the previous equation we obtain

$$\begin{aligned}
u_{r-J-1}(x) &= \frac{\gamma_r a_n^{J+1}}{J!(n+1)^J(J+1)(n+1)} x^{(J+1)(n+1)} \prod_{i=0}^{J}(r-i) + l.o.t. \\
&= \frac{\gamma_r a_n^{J+1}}{(J+1)!(n+1)^{J+1}} x^{(J+1)(n+1)} \prod_{i=0}^{J}(r-i) + l.o.t.,
\end{aligned}$$

which is equation (8) with $j = J+1$. This completes the proof of (8).

From (8) with $j = r-1$ we obtain

$$u_1(x) = \frac{\gamma_r a_n^{r-1}}{(r-1)!(n+1)^{r-1}} x^{(r-1)(n+1)} \prod_{i=0}^{r-2} (r-i) + l.o.t.$$

Then computing the coefficient of y^0 in (7) we get

$$-g(x)u_1(x) = \sum_{k=0}^{n} \beta_{k,0} x^k,$$

or equivalently

$$-g(x)u_1(x) = -\frac{b_m \gamma_r a_n^{r-1}}{(r-1)!(n+1)^{r-1}} \prod_{i=0}^{r-2} (r-i) x^{(r-1)(n+1)+m} + l.o.t. = \sum_{k=0}^{n} \beta_{k,0} x^k. \tag{9}$$

Since $n \geq 2$ and $r > n-1 \geq 1$ we have that $(r-1)(n+1) > n$, from (9) we have a contradiction. Hence $r \leq n-1$.

We first assume that $r \geq 2$ and again we will reach a contradiction. We claim that (7) becomes

$$\begin{aligned} &\sum_{j=0}^{r} u_j'(x) y^{j+1} - g(x) \sum_{j=1}^{r} j u_j(x) y^{j-1} - f(x) \sum_{j=0}^{r} j u_j(x) y^j = \\ &\sum_{k=0}^{n} \beta_{k,0} x^k + y \sum_{k=1}^{n} \beta_{k-1,1} x^{k-1} + \sum_{l=2}^{r+1} y^l \sum_{k=l}^{n} \beta_{k-l,l} x^{k-l}. \end{aligned} \tag{10}$$

Indeed, since all the coefficients with y^l for $l = r+2, \ldots, n$ in (7) only appears in the right-hand we have that

$$\sum_{l=r+2}^{n} y^l \sum_{k=l}^{n} \beta_{k-l,l} x^{k-l} = 0.$$

This implies that (10) holds, and consequently the claim is proved.

Computing the coefficient of y^{r+1} in (10) we get that

$$u_r'(x) = \sum_{k=r+1}^{n} \beta_{k-r-1,r+1} x^{k-r-1}, \quad \text{that is} \quad u_r(x) = c_r + \sum_{k=r+1}^{n} \frac{\beta_{k-r-1,r+1}}{k-r} x^{k-r} = \sum_{k=r}^{n} \tilde{\beta}_k x^{k-r},$$

where $\tilde{\beta}_r = c_r \in \mathbb{C}$ and $\tilde{\beta}_k = \beta_{k-r-1,r+1}/(k-r)$ for $k = r+1, \ldots, n$. Without loss of generality and since $u_r(x) \neq 0$ we denote by k^* the greatest integer of $\{r, \ldots, n\}$ such that $\tilde{\beta}_{k^*} \neq 0$. Then it is clear that

$$u_r(x) = \tilde{\beta}_{k^*} x^{k^*-r} + l.o.t.$$

We claim that

$$u_{r-j}(x) = \frac{\tilde{\beta}_{k^*} a_n^j}{\prod_{i=1}^{j} (i(n+1) + k^* - r)} x^{j(n+1)+k^*-r} \prod_{i=0}^{j-1} (r-i) + l.o.t., \tag{11}$$

for $j = 1, \ldots, r-1$.

Computing the coefficient of y^r in (10) we get

$$u_{r-1}'(x) - f(x) r u_r(x) = \sum_{k=r}^{n} \beta_{k-r,r} x^{k-r}.$$

Since $k^* \geq r \geq 2$, the terms x^{n-r} belongs to the lower terms in comparison with x^{n-r+k^*}. Then we obtain that

$$u_{r-1}'(x) = a_n r \tilde{\beta}_{k^*} x^{n+k^*-r} + l.o.t.$$

Integrating this last expression we get

$$u_{r-1}(x)=\frac{\tilde\beta_{k^*}a_n}{n+1+k^*-r}x^{n+1+k^*-r}r+l.o.t.,$$

which coincides with (11) with $j=1$.

Now we assume that (11) is true for $j=1,\dots,J$ with $1\le J<r-1$ and we will prove it for $j=J+1$. Computing the coefficient of y^{r-J} in (10) we get

$$u'_{r-J-1}(x)-g(x)(r-J+1)u_{r-J+1}(x)-f(x)(r-J)u_{r-J}(x)=\sum_{k=r-J}^{n}\beta_{k-r+J,r-J}x^{k-r+J}.$$

Now using the induction hypothesis and since $x^{(J-1)(n+1)+m+k^*-r}$ and x^{n-r+J} belong to the lower terms in comparison with $x^{J(n+1)+k^*-r+n}$ (note that $J\ge 1$), from the last equation we obtain that

$$\begin{aligned}u'_{r-J-1}(x)&=a_nx^n(r-J)\frac{\tilde\beta_{k^*}a_n^J}{\prod_{i=1}^J(i(n+1)+k^*-r)}x^{J(n+1)+k^*-r}\prod_{i=0}^{J-1}(r-i)+l.o.t.\\&=\frac{\tilde\beta_{k^*}a_n^{J+1}}{\prod_{i=1}^J((i(n+1)+k^*-r)}x^{J(n+1)+k^*-r+n}\prod_{i=0}^{J}(r-i)+l.o.t.\end{aligned}$$

Now integrating the previous equation we obtain

$$\begin{aligned}u_{r-J-1}(x)&=\frac{\tilde\beta_{k^*}a_n^{J+1}x^{(J+1)(n+1)+k^*-r}}{((J+1)(n+1)+k^*-r)\prod_{i=1}^J(i(n+1)+k^*-r)}\prod_{i=0}^{J}(r-i)+l.o.t.\\&=\frac{\tilde\beta_{k^*}a_n^{J+1}}{\prod_{i=1}^{J+1}(i(n+1)+k^*-r)}x^{(J+1)(n+1)+k^*-r}\prod_{i=0}^{J}(r-i)+l.o.t.,\end{aligned}$$

which is equation (11) with $j=J+1$. This proves the claim done in (11).

From (11) with $j=r-1$ we obtain

$$u_1(x)=\frac{\tilde\beta_{k^*}a_n^{r-1}}{\prod_{i=1}^{r-1}(i(n+1)+k^*-r)}x^{(r-1)(n+1)+k^*-r}\prod_{i=0}^{r-2}(r-i)+l.o.t.,$$

Then we have that the coefficient of y^0 in (10) satisfies

$$-g(x)u_1(x)=\sum_{k=0}^{n}\beta_{k,0}x^k,$$

or equivalently

$$-g(x)u_1(x)=-\frac{b_m\tilde\beta_{k^*}a_n^{r-1}}{\prod_{i=1}^{r-1}(i(n+1)+k^*-r)}\prod_{i=0}^{r-2}(r-i)x^{(r-1)(n+1)+k^*-r+m}+l.o.t.=\sum_{k=0}^{m}\beta_{k,0}x^k.$$

Since $r\ge 2$ we have that $(r-1)(n+1)+k^*-r=(r-1)n+k^*-1\ge n+k^*-1\ge n+r-1\ge n+1$, we have a contradiction. Hence $r<2$, that is, $r\le 1$.

We write $u(x,y)=u_0(x)+yu_1(x)$. From (10) we have

$$yu_0'(x)+y^2u_1'(x)-(g(x)+f(x)y)u_1(x)=\sum_{k=0}^{n}\beta_{k,0}x^k+y\sum_{k=1}^{n}\beta_{k-1,1}x^{k-1}+y^2\sum_{k=2}^{n}\beta_{k-2,2}x^{k-2}. \tag{12}$$

Computing the coefficient of y^2 in (12) we get

$$u_1'(x)=\sum_{k=2}^{n}\beta_{k-2,2}x^{k-2},\quad \text{i.e.}\quad u_1(x)=\beta^*+\sum_{k=2}^{n}\frac{\beta_{k-2,2}}{k-1}x^{k-1}\quad\text{with}\quad \beta^*\in\mathbb{C}. \tag{13}$$

Furthermore the coefficient of y in (12) gives

$$u_0'(x) - f(x)u_1(x) = \sum_{k=1}^{n} \beta_{k-1,1} x^{k-1}.$$

Integrating we have

$$\begin{aligned} u_0(x) &= \int f(x)\left(\beta^* + \sum_{k=2}^{n} \frac{\beta_{k-2,2}}{k-1} x^{k-1}\right) dx + \sum_{k=1}^{n} \frac{\beta_{k-1,1}}{k} x^k \\ &= \beta^* F(x) + \sum_{k=2}^{n} \frac{\beta_{k-2,2}}{k-1} \int f(x) x^{k-1}\, dx + \sum_{k=1}^{n} \frac{\beta_{k-1,1}}{k} x^k. \end{aligned} \tag{14}$$

Finally the coefficient of y^0 in (12) gives

$$-g(x)\left(\beta^* + \sum_{k=2}^{n} \frac{\beta_{k-2,2}}{k-1} x^{k-1}\right) = \sum_{k=0}^{n} \beta_{k,0} x^k. \tag{15}$$

Since $g(x) = \sum_{j=0}^{m} b_j x^j$ and $b_m \neq 0$ with $m \geq 2$ we have that

$$\beta_{k-2,2} = 0 \quad \text{for } k \geq n - m + 2. \tag{16}$$

Thus we can rewrite (15) as

$$\begin{aligned} &-\beta^* \sum_{j=0}^{m} b_j x^j - \sum_{j=0}^{m} b_j x^j \sum_{k=2}^{n-m+1} \frac{\beta_{k-2,2}}{k-1} x^{k-1} \\ &\quad = -\beta^* \sum_{j=0}^{m} b_j x^j - \sum_{j=0}^{m} \sum_{k=2}^{n-m+1} \frac{b_j \beta_{k-2,2}}{k-1} x^{j+k-1} \\ &\quad = -\beta^* \sum_{j=0}^{m} b_j x^j - \sum_{l=0}^{n-m-1} \sum_{j=0}^{m} \frac{b_j \beta_{l,2}}{l+1} x^{j+l+1} \\ &\quad = -\beta^* \sum_{j=0}^{m} b_j x^j - \sum_{k=1}^{n} x^k \sum_{\substack{j+l=k-1 \\ j \in \{0,\ldots,m\} \\ l \in \{0,\ldots,n-m-1\}}} \frac{b_j \beta_{l,2}}{l+1} = \sum_{k=0}^{n} \beta_{k,0} x^k, \end{aligned}$$

which yields

$$\beta_{0,0} = -\beta^* b_0, \quad \beta_{k,0} = -\beta^* b_k \lambda_k - \sum_{\substack{j+l=k-1 \\ j \in \{0,\ldots,m\} \\ l \in \{0,\ldots,n-m-1\}}} \frac{b_j \beta_{l,2}}{l+1}, \tag{17}$$

for $k = 1, \ldots, n$ with $\lambda_k = 0$ for $k \geq m+1$.

From (13), (14) and (16) we have that

$$u(x,y) = \sum_{k=2}^{n-m+1} \frac{\beta_{k-2,2}}{k-1}\left(\int f(x) x^{k-1}\, dx + y x^{k-1}\right) + \beta^*(y + F(x)) + \sum_{k=1}^{n} \frac{\beta_{k-1,1}}{k} x^k. \tag{18}$$

In short, taking $u(x,y)$ from (18) we have that $e^{u(x,y)}$ is an exponential factor. Note that it depends on the constants $\beta_{k-2,2}$, $\beta_{k-1,1}$ and β^*. Now the proof of the proposition follows taking each independent solution. For example, setting $\beta_{k-2,2} = 0$ for $k = 2, \ldots, n-m+1$, $\beta^* = 0$ and all the $\beta_{k-1,1} = 0$ except one we get all the solutions of statement (a). Moreover, setting $\beta_{k-1,1} = 0$ for $k = 1, \ldots, n$ and $\beta_{k-2,2} = 0$ for $k = 2, \ldots, n-m+1$ we get the solution of statement (b). Finally, setting $\beta_{k-1,1} = 0$ for $k = 1, \ldots, n$, $\beta^* = 0$ and all the

$\beta_{k-2,2} = 0$ except one we get all the solutions of statement (c). This completes the proof of the proposition. $\square$

Note that from (12) we have that

$$L = L(x,y) = \sum_{k=0}^{n} \beta_{k,0} x^k + y \sum_{k=1}^{n} \beta_{k-1,1} x^{k-1} + y^2 \sum_{k=2}^{n} \beta_{k-2,2} x^{k-2},$$

and using (16) and (17) we conclude that

$$\begin{aligned} L &= -\beta^* g(x) - \sum_{k=1}^{n} x^k \sum_{\substack{j+l=k-1 \\ j\in\{0,\ldots,m\} \\ l\in\{0,\ldots,n-m-1\}}} \frac{b_j \beta_{l,2}}{l+1} + y \sum_{k=1}^{n} \beta_{k-1,1} x^{k-1} + y^2 \sum_{k=2}^{n} \beta_{k-2,2} x^{k-2} \\ &= -\beta^* g(x) - \sum_{k=1}^{n} x^k \sum_{\substack{j+l=k-1 \\ j\in\{0,\ldots,m\} \\ l\in\{0,\ldots,n-m-1\}}} \frac{b_j \beta_{l,2}}{l+1} + y \sum_{l=0}^{n-1} \beta_{l,1} x^l + y^2 \sum_{l=0}^{n-m-1} \beta_{l,2} x^l. \end{aligned} \tag{19}$$

Proof of Theorem 2. The general form of the exponential factors is $e^{u(x,y)}$ with $u(x,y)$ given in (18) and the general form of the cofactor $L(x,y)$ is given in (19).

In order that system (1) has a Liouvillian first integral, by Theorem 3, we must have

$$L(x,y) = f(x).$$

Hence, using (2) we get

$$-\beta^* g(x) - \sum_{k=1}^{n} x^k \sum_{\substack{j+l=k-1 \\ j\in\{0,\ldots,m\} \\ l\in\{0,\ldots,n-m-1\}}} \frac{b_j \beta_{l,2}}{l+1} + y \sum_{l=0}^{n-1} \beta_{l,1} x^l + y^2 \sum_{l=0}^{n-m-1} \beta_{l,2} x^l = f(x). \tag{20}$$

Since the right-hand side of (20) is independent of y, we get that $\beta_{l,1} = 0$ for $j = 0, \ldots, n-1$ and $\beta_{l,2} = 0$ for $l = 0, \ldots, n-m-1$. Then (20) becomes $-\beta^* g(x) = f(x)$, which is again impossible since $g(x)/f(x)$ is not constant. This ends the proof of Theorem 2. $\square$

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