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Study of the period function of a biparametric family of centers

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Introduction

A critical point p of a planar differential system is called *center* if it has a punctured neighbourhood that consist entirely of periodic orbits surrounding p. The largest neighbourhood with this property is called *period annulus* and henceforth it will be denoted by $\mathscr{P}$. The *period function* is the function that assigns to each periodic orbit of $\mathscr{P}$ its period. The present work is an study of the period function of an specific biparametric family of planar vector fields which has a center at the origin for all value of the parameter and has two principal motivations.

The first one is to give an introduction in the study of period functions for planar vector fields. More concretely, we will introduce general results about analytic systems in the plane and we will emphasise them for potential systems. The second motivation is that the family chosen for this study comes from a work of Y. Miyamoto and K. Yagasaki [13]. The authors in [13] proved that period function associated to the center at $x = 1$ of the differential equation

$$\ddot{x} = x - x^p$$

with $p \in \mathbb{N}$ is monotone increasing in all the parameter space. Lately, Yagasaki [18] extended this result for $p \in \mathbb{R}$ with $p > 1$ by using a monotonicity criterion of Chicone [2]. These two works motived us to study that family in a more extended way. More concretely, we will study the family of differential equations

$$\ddot{x} = x^q - x^p$$

with $p > q$ and restricting them to not take the value -1 in order to have the same expression for the potential function of the system for all the treated parameters. Moreover, for convenience we translate the center at the origin to make the computations easier so finally the potential system under consideration will be

$$X_\mu \begin{cases} \dot{x} = -y \\ \dot{y} = (1+x)^p - (1+x)^q \end{cases} , \quad \mu = (q, p), \ p > q$$

with $p, q \neq -1$. Notice that the family X_μ is a generalization of the one studied by Miyamoto and Yagasaki, which corresponds to $q = 1$.

In the first chapter, we will introduce all the general tools that we will use in following chapters for study the period function. We will present the main background needed to follow this work and also we will introduce in a more concrete way the family of systems under consideration. Particularly, we will show an original result on the computation of the commutator vector field in a neighbourhood of the center for potential systems that have an isochronous center at the origin (Proposition 1.2).

The second chapter is dedicated to study the monotonicity of the period function. We will use the well known Schaaf's criterion [16] in order to prove that the period function of the family X_μ is monotone increasing in some regions and monotone decreasing in others (Theorem 2.2). Particularly, the result of Yagasaki in [18] is included in that proof.

In the third chapter, we will present the notion of period constants and the relation with the isochronous centers, the centers such that all the periodic orbits have the same period. We will also introduce the bifurcation theory to study the appearance of critical periods for perturbations of the system in the family. There will be two different kind of bifurcation that we will study: bifurcations from the center and bifurcations from the period annulus. In the case of bifurcations from the origin itself, we will prove a result that ensures that at most one critical period bifurcates from the center (Propositions 3.2 and 3.3). On the other hand, the result for bifurcations from the period annulus also will give us an upper bound of one critical period that bifurcates from the isochrones. Moreover, the upper bound will be taken if we perturb the system in a specific way (Theorem 3.7).

The fourth chapter is dedicated to study the period function in the outer boundary of the period annulus. The study near the outer boundary is generally much more difficult than the one in the center. In this chapter we will present a general result for the value of the extension at the outer boundary of the period function in potential systems (Theorems 4.1 and 4.2) and we will apply it in the family under consideration (Theorem 4.3).

In the final chapter, we will sum up the main dynamical results about the period function of the family and we will give a conjecture for the whole dynamic behaviour of the period function.

Chapter 1

General results

1.1 The period function

In this section we will introduce the general notions needed to follow the thinking line of this work in the study of the period function of a family of vector fields. Let $\{X_\mu\}$ be an analytic family of planar vector fields which depends on a k-parameter $\mu \in \Omega \subset \mathbb{R}^k$. Let us suppose that, for each $\mu \in \Omega$, $\{X_\mu\}$ has a center at the point p. The study of the period of the periodic orbits that foliates the center is the main aim of this work. More concretely, the study of the so-called *period function* of the center.

Through this section we introduce the main objects and general results related with the period function and more concretely with period functions of potential vector fields. At the end of the section, we will present the particular family of vector fields that we are going to study and some immediate results about the period function and the period annulus. Let us start with some definitions.

Definition 1. Let X be a planar differential system with a center at the point p. The *period annulus* associated to this center is the largest punctured neighbourhood entirely foliated by periodic orbits surrounding p. Henceforth will be denoted by $\mathscr{P}$. Moreover, we will denote by $\mathcal{I} = (x_L, x_R)$ the projection of $\mathscr{P}$ over the x-axis. Then, if $p = (p_1, p_2) \in \mathscr{P}$ then $p_1 \in \mathcal{I}$. $\triangle$

Definition 2. Let X be a planar differential system with a center at the point p. The *period function* associated to this center is the function that assigns to each periodic orbit of $\mathscr{P}$ its period. $\triangle$

Notice that this definition does not give us an explicit expression for the period function. In fact there are many ways to parametrize the orbits, so there are many different expressions for the period function. For instance, one can think in the function

$$x \in \Sigma \longmapsto T(x)$$

where Σ is a section of the period annulus $\mathscr{P}$ that cross each orbit ones and $T(x)$ the period of the orbit that cross Σ at point x. Therefore we need to fix what is a parametrization of the period function. In other words, different parametrizations do not change the critical points of the functions neither its character. In order to fix ideas, let us consider an analytic transversal curve $\varphi : [0, 1) \mapsto \Sigma \subset \mathscr{P}$ with $\varphi(0) = p$ such that each periodic orbit γ in $\mathscr{P}$ intersects exactly once with. Let $\phi(x,t)$ the flow of the system such that $\phi(x,0) = x \in \mathscr{P} \cup \{p\}$. The map

$$s \in (0,1) \longmapsto \phi(\varphi(s), t)$$

is a parametrization of the periodic orbit $\phi(\varphi(s), t)$. This definition is good because we can pass from one transversal curve to another by a diffeomorphism. Indeed, let $\psi(r)$ be another transversal curve parametrized by $r \in (0,1)$ and $\psi(0) = p$. Since $\psi(r)$ is a transversal curve in $\mathscr{P}$, there exists a time $t(s)$ such that $\phi(\varphi(s), t(s)) = \psi(r)$. Therefore, the map

$$s \in (0,1) \longmapsto \varphi(s) \longmapsto \phi(\varphi(s), t(s)) = \psi(r) \longmapsto r \in (0,1)$$

is a diffeomorphism because of the Theorem of differentiability with respect to parameters. Moreover, any other map which results from the composition of one of this parametrizations with a diffeomorphism is also a parametrization of the period function.

Definition 3. Let $\mathscr{P} \subset \mathcal{U} \subset \mathbb{R}^2$ the period annulus of a system defined in an open subset $\mathcal{U}$ with a center at $p \in \mathcal{U}$. The boundary of $\mathscr{P}$ in $\mathbb{R}^2$ is formed by the critical point p and maybe some others connected components. We call *inner boundary of the period annulus* the center p of the system. We call *outer boundary of the period annulus* the union of the others connected components, $\partial\mathscr{P} \setminus \{p\}$. △

Remark 1.1. Notice that the inner boundary is always well-defined, whereas the outer boundary is not topologically defined a priori. In fact, in the case where the outer boundary of $\mathscr{P}$ is strictly contained in $\mathcal{U}$ the system is well-defined in the outer boundary and then it is a polycycle formed by a set of critical points with orbits that interconnect them forming a topologic $\mathcal{S}^1$. However, the boundary may intersect $\partial\mathcal{U}$ and therefore the vector field would be not well defined in the outer boundary. In this case, the outer boundary will have some connected components strictly contained in $\mathcal{U}$. Another case is when the center is globally defined in all $\mathbb{R}^2$. Then, the outer boundary with the above definition is empty. In this case, we will think that the outer boundary is the disk at infinity in the sense of Poincaré's compactification. □

Definition 4. Let $T(s)$ be the period function of a planar vector field X. A *critical period* of $T(s)$ is a value $s = s_0$ of the parameter such that $T'(s_0) = 0$. △

Remark 1.2. The number of critical periods and its character are independent of the parametrization. Let s and $\xi(s)$ be two parametrizations. Since ξ is a diffeomorphism, then $T'(s) = \xi'(s)T'(\xi(s))$ with $\xi'(s) \neq 0$. Therefore, $T'(s_0) = 0$ if and only if $T'(\xi(s_0)) = 0$. That allow us to use freely any parametrization in the study of the critical periods. Moreover, if the diffeomorphism $\xi(s)$ keeps the direction, that is $\xi'(s) > 0$ for all s, then the character of the critical points is also kept. □

Besides critical periods, another object to study are the so-called isochronous centers.

Definition 5. Let X be a planar vector field with a center in p. The center is called *isochronous* if the period function associated to the center is constant though all the periodic orbits. That is, all the periodic orbits of $\mathscr{P}$ have the same period. △

Isochronous centers are not only interesting by itself, but also because, at least intuitively, little perturbations can bring up critical periods. Isochronicity is strongly related with two elements that we will present. One of them comes from the following definition.

Definition 6. Let X and Y be two smooth vector fields defined on a manifold of $\mathbb{R}^n$. The *Lie bracket* of X and Y is another smooth vector field $[X,Y]$ such that

$$[X,Y](f) = X(Y(f)) - Y(X(f)) \tag{1.1}$$

for all $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. In coordinates, if $X = \sum_{i=1}^{n} X^i \partial_i$ and $Y = \sum_{i=1}^{n} Y^i \partial_i$,

$$[X,Y] = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(X^j \partial_j Y^i - Y^j \partial_j X^i \right) \partial_i. \tag{1.2}$$

$\triangle$

Definition 7. Let X be a planar vector field in $\mathcal{V} \subset \mathbb{R}^2$ with $X(p) = 0$. We say that U is a *commutator* of X in a neighbourhood $\Omega \subset \mathcal{V}$ of p if U is transversal to X in $\Omega \setminus \{p\}$ and commutes in time. That is, if $\phi_X(x,t)$ denotes the flow of the system X such that $\phi_X(x,0) = x$ and $\phi_U(x,t)$ denotes the flow of the system U such that $\phi_U(x,0) = x$ then

$$\phi_X(\phi_U(x,t_1),t_2) = \phi_U(\phi_X(x,t_2),t_1), \text{ for all } t_1, t_2,\ x \in \Omega \setminus \{p\}. \tag{1.3}$$

This definition is equivalent to say that $[X,U] \equiv 0$ at Ω where $[X,U]$ denotes the Lie bracket of X and U. $\triangle$

Notice that the definition does not take into account about uniqueness of the commutator. Indeed, let λU with $\lambda \in \mathbb{R} \setminus \{0\}$ be a planar vector field. If U is a commutator of a vector field X, so it is λU by the bilineality of the Lie bracket. The notion of a commutator is local in a neighbourhood of the critical point p. If a commutator U can be extended to $\mathcal{V}$ we say that U is a *global commutator*.

Definition 8. Let X be a vector field with $X(p) = 0$. We say that X is *linearizable* at p if there exists a neighbourhood $\mathcal{U}$ of p such that $X|_{\mathcal{U}}$ is $\mathcal{C}^1$-conjugated to a linear system. $\triangle$

The following result is the one that gives the equivalence relation between these three elements. The equivalence between isochronous centers and the existence of a local commutator was given by Sabatini in [15], whereas the result that an isochronous center is linearizable near the origin is from Montgomery and Zippin in [14].

Theorem 1.1. *Let X be an analytic system with a center at the origin. The following statements are equivalent:*

(i) The origin is an isochronous center.

(ii) The system is linearizable in a neighbourhood of the origin.

(iii) The system has a commutator in a neighbourhood of the origin.

Proof. Let us prove that (i) implies (ii). Since the center is isochronous, it is, particularly, non-degenerate. That is the system X has a linear part of the form $\dot{x} = Ax$ at the origin with $\det(A) \neq 0$. Let us see that, in fact, X is locally $\mathcal{C}^1$-conjugated to its linear part at the origin. Indeed, let us denote by $\varphi(t,x)$ the flow of the system X such that $\varphi(0,x) = x$ and let $\phi(t,x) = e^{At}x$ the flow of the linear system. Let $T > 0$ be the period of the center and consider the map

$$\psi(x) = \frac{1}{T}\int_0^T e^{-As}\varphi(s,x)ds \tag{1.4}$$

which corresponds to let pass a time s through the flow of X, come back the same time through the linear system and integrate in s. We will prove that $\psi(x)$ is a conjugation between X and the linear system. We need to check the next two facts:

(a) $\psi(x)$ is a diffeomorphism in a neighbourhood of the origin.

(b) $\psi(x)$ satisfies $\psi(\varphi(t,x)) = e^{At}\psi(x)$.

Indeed, (a) comes from differentiate the map,

$$\psi'(0) = \frac{1}{T}\int_0^T e^{-As}\frac{\partial}{\partial x}\left[\varphi(s,x)\right]_{x=0} ds. \tag{1.5}$$

Moreover, $\frac{\partial}{\partial x}\varphi(s,x)$ satisfies the variational equation

$$\begin{cases} \left(\frac{\partial}{\partial x}\varphi(s,x)\right)' = A\frac{\partial}{\partial x}\varphi(s,x) \\ \frac{\partial}{\partial x}\varphi(s,0) = \mathrm{Id}, \end{cases}$$

where Id denotes the identity matrix, which has as solution $\frac{\partial}{\partial x}\varphi(s,x) = e^{As}$. Therefore, by (1.5),

$$\psi'(0) = \frac{1}{T}\int_0^T \mathrm{Id}ds = \mathrm{Id}$$

that have $\det(\psi'(0)) = 1$, so the map $\psi(x)$ is a local diffeomorphism of the origin. Let us prove (b). By the definition of $\psi(x)$,

$$\begin{aligned} \psi(\varphi(t,x)) &= \frac{1}{T}\int_0^T e^{-As}\varphi(s,\varphi(t,x))ds = \frac{1}{T}\int_0^T e^{-As}\varphi(s+t,x)ds \\ &= \frac{1}{T}\int_t^{t+T} e^{-Au}e^{At}\varphi(u,x)du = e^{At}\frac{1}{T}\int_t^{t+T} e^{-Au}\varphi(u,x)du. \end{aligned}$$

Since T is the period of system X,

$$\psi(\varphi(t,x)) = e^{At}\frac{1}{T}\int_0^T e^{-Au}\varphi(u,x)du = e^{At}\psi(x)$$

as we desired prove.

The proof of (ii) implies (iii) follows easily as well. Since the system X is linearizable, there exists a diffeomorphism ψ such that the system becomes linear through ψ. It is immediate that the orthogonal system is a commutator of the linearization of X, so ψ^{-1} give us a commutator of X at the origin.

Finally, let us prove that (iii) implies (i). Let $\varphi(t,x)$ the flow of the system X and $\psi(t,x)$ the flow of the commutator U. Then, since U is a local commutator in a neighbourhood $\mathcal{U}$ of the origin, $\varphi(t_2,\psi(t_1,x)) = \psi(t_1,\varphi(t_2,x))$ for $x \in \mathcal{U}$. Let $\gamma_1, \gamma_2 \subset \mathcal{U}$ two different periodic orbits of the center with periods T_1 and T_2 respectively. Let $x \in \gamma_1$ and let s such that $\psi(s,x) \in \gamma_2$. Then,

$$\psi(s,\varphi(T_1,x)) = \psi(s,x) = \varphi(T_2,\psi(s,x)) = \psi(s,\varphi(T_2,x))$$

where in the first equality we have used that $\varphi(t,x)$ has period T_1, in the second equality we have used that $\psi(t,x)$ has period T_2 and in the last equality we have used the commutator property. With this, we can ensure that $T_1 = nT_2$ for $n \in \mathbb{N}$ but with the same argument interchanging the roles of γ_1 and γ_2 we get the equality $T_1 = T_2$. Therefore, the center is isochronous. ■

Remark 1.3. The same proof of the result that a center of a vector field is isochronous if, and only if, it is linearizable can be extended for families of vector fields $\{X_\mu, \mu \in \Lambda\}$ where $\Lambda \subset \mathbb{R}^m$ thought using the family of diffeomorphisms $\psi_\mu(s) = \frac{1}{T_\mu}\int_0^{T_\mu} e^{-A_\mu s}\varphi(s;x,\mu)dx$, where $A_\mu = \left(\frac{\partial X_\mu}{\partial x}\right)(0)$ and $\varphi(t;x,\mu)$ is the solution of $x' = X_\mu$ with $\varphi(0;x,\mu) = x$. □

Henceforth we will restrict a little bit more the notion of the planar vector field X. We will suppose that X is a potential system with a center at the origin $p = (0,0)$ and with an analytic potential function $V(x)$ chosen such that the Hamiltonian $H(x,y) = \frac{y^2}{2} + V(x)$ vanish at the origin.

Definition 9. Let $V(x)$ be the potential function associated to a potential system X such that $V(0) = 0$. Let $g(x)$ be the function defined as

$$g(x) = \operatorname{sgn}(x)\sqrt{V(x)} = x\sqrt{\frac{V(x)}{|x^2|}}. \tag{1.6}$$

△

Notice that $g(x)$ is analytic at $x = 0$. The function $g(x)$ will be very important in all the development of the work and we will study its properties in each moment when it will be needed.

Is well known that the a center in potential system is a local minimum of the potential function $V(x)$. Then, if the center is at the origin and choosing $V(x)$ in the way before, $V(x)$ is a quadratic function in a neighbourhood of the origin. This fact justify the following definition.

Definition 10. Let X be a potential system with a center at the origin. The *involution* associated to the potential $V(x)$ is the function $\sigma \neq \mathrm{Id}$ such that $V(x) = V(\sigma(x))$ for all x of the projection of the period annulus (see Figure 1.1). Moreover, $\sigma^2(x) = x$. △

Notice that the involution is also an analytic function at $x = 0$ since it can be written in the following way

$$\sigma(x) = g^{-1}(-g(x)).$$

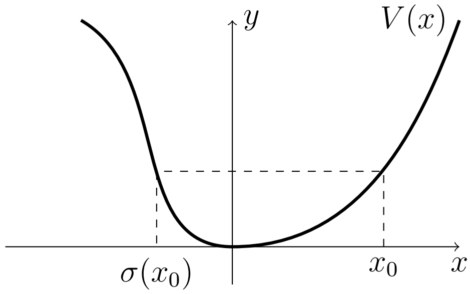


Figure 1.1: Involution associated to the potential $V(x)$.

Then, by composition we have the analyticity. Directly related with the involution $\sigma(x)$ is the function

$$h(x) = \frac{x - \sigma(x)}{2}, \tag{1.7}$$

which is essentially the length between the two points with same image of $V(x)$. With this definition, one can prove that $V(x)$ can be written as an even function of $h(x)$. That is, $V(x) = \widehat{V}(h(x)^2)$. Indeed, we can write $V(x) = \bar{V}(h(x))$ with $\bar{V} = V \circ h^{-1}$ which is analytic. Furthermore, since $h(\sigma(x)) = -h(x)$,

$$\bar{V}(-h(x)) = V(h^{-1}(-h(x))) = V(\sigma(x)) = V(x) = \bar{V}(h(x)),$$

so $\bar{V}$ is an even function of $h(x)$. In the case of isochronous centers, $h(x)$ determines the potential in a stronger way. Next proposition is proved by Cima, Mañosas and Villadelprat in [5].

Proposition 1.1. *Let X be a potential vector field with Hamiltonian $H(x,y) = \frac{y^2}{2} + V(x)$ and with a center at the origin. The following statements are equivalent:*

(i) The origin is an isochronous center of period ω.

(ii) There exists an involution σ such that $V(x) = \frac{\pi^2}{2\omega^2}(x - \sigma(x))^2$ for all $x \in \mathcal{I}$.

Next result provide us a general way to obtain the commutator vector field of a potential system which has an isochronous center at the origin.

Proposition 1.2. *Let X be a potential vector field with an isochronous center at the origin. Let $h(x)$ be the function in (1.7) related to the involution defined by $V(x)$. Then*

$$\begin{cases} \dot{u} &= u + v\dfrac{r\int_0^\theta (h^{-1})''(r\cos\alpha)\cos\alpha d\alpha}{(h^{-1})'(u)}, \\ \dot{v} &= v - u\dfrac{r\int_0^\theta (h^{-1})''(r\cos\alpha)\cos\alpha d\alpha}{(h^{-1})'(u)}, \end{cases}$$

is a commutator of X, where $u = h(x)$, $v = \frac{y}{\sqrt{k}}$, $k = \frac{4\pi^2}{\omega^2}$ and (r,θ) are the polar coordinates associated to (u,v).

Proof. By Proposition 1.1, $V(x)=\frac{2\pi^2}{\omega^2}h(x)^2$ so the system X is written as

$$\begin{cases} \dot{x} &= -y, \\ \dot{y} &= kh(x)h'(x). \end{cases}$$

Let take the following change of variable $u=h(x)$, $v=\frac{y}{\sqrt{k}}$. Then, the vector field can be written in this variables as

$$\begin{cases} \dot{u} &= -yh'(x) = -\frac{\sqrt{k}}{(h^{-1})'(u)}v, \\ \dot{v} &= \frac{\sqrt{k}}{(h^{-1})'(u)}u. \end{cases}$$

We will find a commutator for the system

$$\begin{cases} \dot{u} &= -\frac{1}{(h^{-1})'(u)}v, \\ \dot{v} &= \frac{1}{(h^{-1})'(u)}u. \end{cases}$$

because the time scale does not have any important difference. If we pass the system to polar coordinates,

$$\begin{cases} \dot{r} &= 0 \\ \dot{\theta} &= \frac{1}{(h^{-1})'(r\cos\theta)}. \end{cases} \tag{1.8}$$

Let $F_r(\theta)$ be the function defined as

$$F_r(\theta) = \int_0^\theta \left(h^{-1}\right)'(r\cos\alpha)d\alpha.$$

Let us see that the function $F_r(\theta)$ is a degree-one circle diffeomorphism. That is $F_r(\theta+2\pi)=F_r(\theta)+2\pi$ and $F_r'(\theta)\neq 0$ for all θ. Indeed, let σ be the associated involution of the potential. Then,

$$h\left(\sigma(x)\right) = -h(x) \Rightarrow \sigma(x) = h^{-1}(-h(x)).$$

Taking $x=h^{-1}(z)$,

$$\sigma(h^{-1}(z)) = h^{-1}(-z).$$

Therefore,

$$z = h(h^{-1}(z)) = \frac{h^{-1}(z)-\sigma(h^{-1}(z))}{2} = \frac{h^{-1}(z)-h^{-1}(-z)}{2}$$

so the odd part of h^{-1} is the identity. Then,

$$h^{-1}(z) = z + G(z),$$

where $G(z)$ is the even part of h^{-1}. Deriving,

$$(h^{-1})'(z) = 1 + G'(z)$$

with $G'(y)$ an odd function, so

$$\begin{aligned} F_r(\theta+2\pi) &= \int_0^{\theta+2\pi} \left(h^{-1}\right)'(r\cos\alpha)d\alpha \\ &= F_r(\theta) + \int_0^{2\pi} 1 + G'(r\cos\alpha)d\alpha \\ &= F_r(\theta) + 2\pi + \int_0^{2\pi} G'(r\cos\alpha)d\alpha \\ &= F_r(\theta) + 2\pi \end{aligned}$$

because $\int_0^{2\pi} G'(r\cos\alpha)d\alpha = 0$ since G' is odd and $\cos\alpha$ is 2π-periodic. In addition, $F_r'(\theta) = (h^{-1})'(r\cos\theta)$ thus $F_r'(\theta) \neq 0$ for all θ.

Therefore $F_r(\theta)$ is a degree-one circle diffeomorphism and the change $r = \rho$, $\phi = F_r(\theta)$ is a diffeomorphism that linearizes system (1.8). That is, the system becomes $\dot{\rho} = 0$, $\dot{\phi} = 1$ in the new coordinates. To obtain the commutator, we want a system $\dot{r} = P(r,\theta)$, $\dot{\theta} = Q(r,\theta)$ such that the same change of variables transform it in $\dot{\rho} = \rho$, $\dot{\phi} = 0$. Then, we will have

$$\begin{cases} \dot{\rho} &= P(r,\theta) = r = \rho, \\ \dot{\phi} &= \frac{\partial}{\partial r}F_r(\theta)P + \frac{\partial}{\partial \theta}F_r(\theta)Q = 0, \end{cases}$$

so

$$Q(r,\theta) = -\frac{r\frac{\partial}{\partial r}F_r(\theta)}{\frac{\partial}{\partial\theta}F_r(\theta)} = \frac{-r\displaystyle\int_0^{\theta}\left(h^{-1}\right)''(r\cos\alpha)\cos\alpha d\alpha}{\left(h^{-1}\right)'(r\cos\theta)}.$$

Then, the commutator can be written as

$$\begin{cases} \dot{r} &= r, \\ \dot{\theta} &= \dfrac{-r\displaystyle\int_0^{\theta}\left(h^{-1}\right)''(r\cos\alpha)\cos\alpha d\alpha}{(h^{-1})'(r\cos\theta)} \end{cases}$$

and

$$\begin{cases} \dot{u} &= \dot{r}\cos\theta - r\sin\theta\dot{\theta} = u + v\dfrac{r\int_0^{\theta}(h^{-1})''(r\cos\alpha)\cos\alpha d\alpha}{(h^{-1})'(u)}, \\ \dot{v} &= \dot{r}\sin\theta + r\cos\theta\dot{\theta} = v - u\dfrac{r\int_0^{\theta}(h^{-1})''(r\cos\alpha)\cos\alpha d\alpha}{(h^{-1})'(u)}. \end{cases}$$

■

The previous results provide an alternative way for studying isochronicity and also to obtain a commutator of the vector field. All these things will be useful in following sections for the study of the behaviour of the period function in the system X_μ commented in the introduction and that we will introduce more in detail in the following section. After all those general results, now we will go into detail on the system that we study.

The parametrization that we will use is the classical one when dealing with a potential system. The vector field X_μ has a first integral defined by

$$H(x,y) = \frac{y^2}{2} + V(x) \tag{1.9}$$

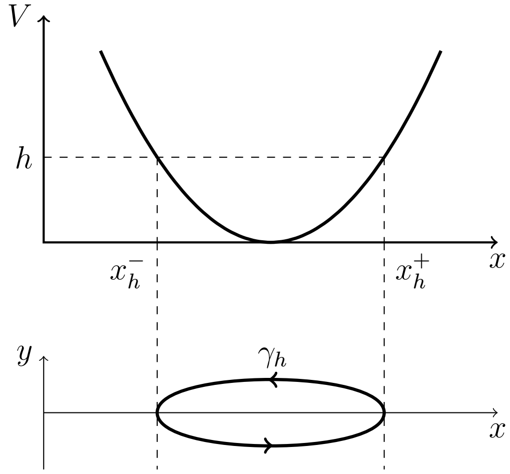


Figure 1.2: Interpretation of an orbit.

that vanishes at the origin since we choose the potential such that $V(0) = 0$. Let us denote γ_h the period orbit of $\mathscr{P}$ contained in the energy level h of the Hamiltonian function, $\gamma_h \subset \{H(x,y) = h\}$. With this choice of the potential, the energy level $h = 0$ corresponds to the center itself.

The parametrization that we will use henceforth is the one given by the energy level h of the Hamiltonian,

$$h \in (0, h^*) \longmapsto T(h)$$

where h^* can be finite or infinite and $T(h)$ is the period of the periodic orbit γ_h. Notice that this parametrization satisfies the characterization introduced at the beginning of the chapter. Let $\mathcal{I} = (x_L, x_R)$ be the projection of the period annulus $\mathscr{P}$ over the x-axis and let us consider the parametrized curve $\varphi(s) = (s, 0)$ for $s \in (0, x_L)$ with corresponds to the left part of $\mathcal{I}$ with respect to the center. Then, since $V(x)$ is a diffeomorphism in $(0, x_L)$, the mapping $h \in (0, h^*) \mapsto (V^{-1}(h), 0)$ proves that h is also a parametrization because $H(s, 0) = V(s) = h$. Since the Hamiltonian is a diffeomorphism then $H(\varphi(x), 0) = h$ determines the parametrization which we were talking about.

With this parametrization, one can define the period function by using a first integral. Let us denote by (x_h^-, x_h^+) the projection over $\{y = 0\}$ of the periodic orbit γ_h (See Figure 1.2). The following lemma tell us about the expression of the period function.

Lemma 1.1. *The period function can be written as*

$$T(h) = \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g^{-1})'(\sqrt{h} \sin\theta) d\theta \tag{1.10}$$

or

$$T(h) = 2\sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{h}\sin\theta}{V'(g^{-1}(\sqrt{h}\sin\theta))} d\theta \tag{1.11}$$

with $h \in (0, h^*)$.

Proof. Since $\frac{dx}{dt} = -y$,

$$T(h) = \int_0^{T(h)} dt = \int_{\gamma_h} \frac{dx}{y} = 2 \int_{x_h^-}^{x_h^+} -\frac{dx}{y}$$

where $x_h^{\pm}$ are the solutions of $H(x,0) = V(x) = h$ which correspond to the intersections of the periodic orbit γ_h with the x-axis. So the period function is

$$T(h) = \sqrt{2} \int_{x_h^-}^{x_h^+} \frac{dx}{\sqrt{h - V(x)}}. \tag{1.12}$$

Notice that the expression above has always poles at the both endpoints of the integral $x_h^{\pm}$ because by definition $V(x_h^{\pm}) = h$. However, through a change of variable we can show that these poles disappear. Let us take $x = g^{-1}(\sqrt{h}\sin\theta)$. Then, the expression above can be written, using the Change of Variables Theorem, as

$$T(h) = \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g^{-1})'(\sqrt{h}\sin\theta) d\theta = \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{g'(g^{-1}(\sqrt{h}\sin\theta))} \tag{1.13}$$

or

$$T(h) = 2\sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{h}\sin\theta}{V'(g^{-1}(\sqrt{h}\sin\theta))} d\theta \tag{1.14}$$

since $V' = 2gg'$.

■

1.2 The system under consideration

Now let us introduce the bi-parametric family of potential systems that we will study in this work. Let $\{X_\mu\}$ be the analytic family of planar vector fields determined by

$$X_\mu \begin{cases} \dot{x} = -y \\ \dot{y} = (x+1)^p - (x+1)^q \end{cases} , \ \mu := (q,p) \in \mathscr{R}, \ x > -1 \tag{1.15}$$

where $\mathscr{R} := \{(q,p) \in \mathbb{R}^2 \ : \ p > q \text{ and } p, q \neq -1\}$. The vector field X_μ has an associated potential function defined by

$$V_\mu(x) := \frac{(x+1)^{p+1}}{p+1} - \frac{(x+1)^{q+1}}{q+1} + \frac{p-q}{(p+1)(q+1)} \tag{1.16}$$

in order to have $V(0) = 0$ for all $\mu \in \mathscr{R}$. Clearly the centers are determined by the local minimum points of $V_\mu(x)$. In this case, for all $\mu \in \mathscr{R}$, the point $(x,y) = (0,0)$ is the only center of the system X_μ.

Lemma 1.2. *The point $(x,y) = (0,0)$ is the unique critical point of X_μ and it is a center for all $\mu \in \mathscr{R}$.*

Proof. It is only necessary to see that the point $(x,y) = (0,0)$ is the only local minimum of the potential $V(x)$ for $x > -1$. One can check easily that $V'(0) = 0$ and $V''(0) = p - q > 0$ since $p > q$. The uniqueness comes from the computation of the derivative of V searching for other zeros. That is

$$V'(x) = (x+1)^p - (x+1)^q = (x+1)^q \left[(x+1)^{p-q} - 1\right] = 0,$$

that only vanishes if $(x+1)^{p-q} - 1 = 0$ since $x > -1$, so we only obtain $x = 0$ which is the center that we have already found.

■

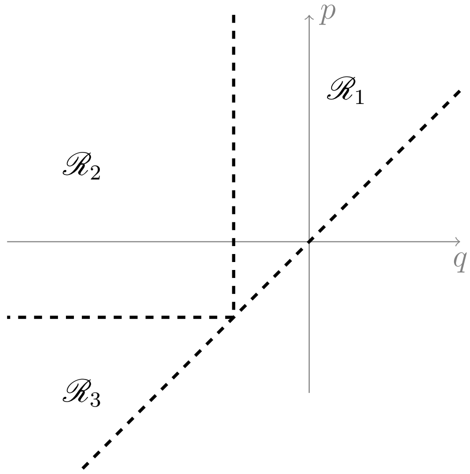


Figure 1.3: Space of parameters.

The family of vector fields $\{X_\mu\}_{\mu\in\mathscr{R}}$ has a center at the origin for all μ, then we can study the period annulus and the period function associated to the center for each choice of the parameter. Let us define the following three subsets of $\mathscr{R}$,

$$\begin{aligned}\mathscr{R}_1 &:= \mathscr{R}\cap\{(q,p)\in\mathbb{R}^2 \ : \ -1<q<p\},\\ \mathscr{R}_2 &:= \mathscr{R}\cap\{(q,p)\in\mathbb{R}^2 \ : \ q<-1<p\},\\ \mathscr{R}_3 &:= \mathscr{R}\cap\{(q,p)\in\mathbb{R}^2 \ : \ q<p<-1\},\end{aligned} \tag{1.17}$$

which is a partition of the region $\mathscr{R}=\mathscr{R}_1\cup\mathscr{R}_2\cup\mathscr{R}_3$ and $\mathscr{R}_i\cap\mathscr{R}_j=\emptyset$ if $i\neq j$. Figure 1.3 shows these regions. The following lemma tell us about how the period annulus is in each of those regions.

Lemma 1.3. *Let X_μ be defined by* (1.15). *The projection of the period annulus on the x-axis, $\mathcal{I}$, is*

(a) $\mathcal{I}=(-1,\rho)$ if $\mu\in\mathscr{R}_1$.

(b) $\mathcal{I}=(-1,+\infty)$ if $\mu\in\mathscr{R}_2$.

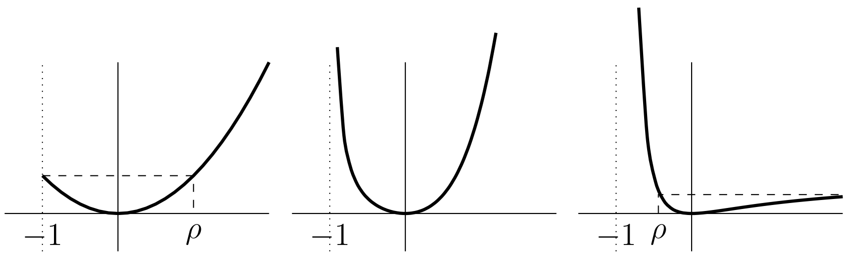
(c) $\mathcal{I}=(\rho,+\infty)$ if $\mu\in\mathscr{R}_3$.

where $\rho:=\left(\frac{p+1}{q+1}\right)^{\frac{1}{p-q}}-1$.

Proof. $V_\mu(x)$ can be expressed as

$$V_\mu(x)=(1+x)^{q+1}\left[\frac{(1+x)^{p-q}}{p+1}-\frac{1}{q+1}\right]+\frac{p-q}{(p+1)(q+1)}.$$

Since $p-q>0$, taking the limit $x\to-1$ we have two possibilities: if $q+1>0$, $V_\mu(-1)=\frac{p-q}{(p+1)(q+1)}$ and if $q+1<0$, $V_\mu(-1)\to+\infty$. Recall that we are not considering the cases $q=-1$ or $p=-1$ in $\mathscr{R}$. In the case of x tending to infinity, we have that



(a) $V_\mu(x)$ with $\mu \in \mathscr{R}_1$ (b) $V_\mu(x)$ with $\mu \in \mathscr{R}_2$ (c) $V_\mu(x)$ with $\mu \in \mathscr{R}_3$

Figure 1.4: Shape of the potential $V_\mu(x)$ depending on the parameter region.

if $q+1>0$ then $\lim_{x\to -1} V_\mu(x) = +\infty$, and if $q+1<0$ two different cases can happen depending on p. Since $q+1<0$ we can write it as

$$V_\mu(x) = \frac{\frac{(1+x)^{p-q}}{p+1} - \frac{1}{q+1}}{(1+x)^{-(q+1)}} + \frac{p-q}{(p+1)(q+1)}. \tag{1.18}$$

and then, if $p+1>0$, $p-q>-(q+1)$ and the quotient tends to infinity so $V_\mu(x) \to +\infty$. Otherwise the quotient tends to zero and $\lim_{x\to+\infty} V_\mu(x) = \frac{p-q}{(p+1)(q+1)}$.

We have three different situations:

- The case $\mu \in \mathscr{R}_1$ ($p>q>-1$) where $V_\mu(x)$ tends to infinity at infinity and the image in $x=-1$ is finite. Then, the period annulus is determined by the pre-images of $V(-1)$ which are $x=-1$ and $x=\left(\frac{p+1}{q+1}\right)^{\frac{1}{p-q}}-1$ (see Figure 1.4a).
- The case $\mu \in \mathscr{R}_2$ ($p>-1>q$) where $V_\mu(x)$ tends to infinity in both sides. In this case, the period annulus has no restrictions and it is given by $(-1,+\infty)$ (see Figure 1.4b).
- The case $\mu \in \mathscr{R}_3$ ($-1>p>q$) where $V_\mu(x)$ tends to infinity in $x=-1$ and tends to $\frac{p-q}{(p+1)(q+1)}$ at infinity. The period annulus is given in this case by the pre-image of the horizontal asymptote where x tends to infinity. The limit at infinity is $\frac{p-q}{(p+1)(q+1)}$ so the pre-image is $\left(\frac{p+1}{q+1}\right)^{\frac{1}{p-q}}-1$ (see Figure 1.4c).

$\blacksquare$

Notice that ρ is positive in the first case and negative in the third one, so the x-coordinate of the center is always inside the projection of the period annulus. By this lemma, the space of parameters is divided in three different areas that will be studied separately such as the Figure 1.3 shows.

The last fact we will study in this section is the image of the energy level $H(x,y)=h$ for each region $\mathscr{R}_i$, $i=1,2,3$. The next lemma says how is the domain of the parameter h for the period function.

Lemma 1.4. *The energy level of the potential vector field X_μ ranges between $[0,h^*)$ where* *(a)* $h^* = \dfrac{p-q}{(p+1)(q+1)}$ *if* $\mu \in \mathscr{R}_1 \cup \mathscr{R}_3$.

(b) $h^* = \infty$ *if* $\mu \in \mathscr{R}_2$.

Proof. Let $V_\mu(x)$ be the potential function. Then, the energy level h is determined by $H(x,y) = \frac{y^2}{2} + V_\mu(x) = h$. As we said at the beginning of the chapter, $h = 0$ corresponds to the center itself, so the only we need is the associated energy level of the outer boundary.

Particularly, $H(x,0) = V_\mu(x) = h$. Then, the energy level of the outer boundary is $\lim\limits_{x \to x_R} V_\mu(x)$. If $\mu \in \mathscr{R}_1$, $x_R = \rho$ so $\lim\limits_{x\to\rho} V_\mu(x) = V_\mu(\rho) = \frac{p-q}{(p+1)(q+1)}$. If $\mu \in \mathscr{R}_2$, $\lim\limits_{x\to\rho} V_\mu(x) = +\infty$, then $h \in (0, +\infty)$. Finally if $\mu \in \mathscr{R}_3$, $\lim\limits_{x\to\rho} V_\mu(x) = \frac{p-q}{(p+1)(q+1)}$.

■

Chapter 2

Monotonicity of the period function

In this section we will study the monotonicity of the period function in some region of the parameter space. To this end we shall apply the following result of R. Schaaf [16].

Theorem 2.1. *Let X be a potential system of the form*

$$\begin{cases} \dot{x} &= -y, \\ \dot{y} &= V'(x), \end{cases}$$

with a center at the origin. Let $\mathcal{I} = (x_L, x_R)$ be the projection of the period annulus $\mathscr{P}$. The center has a monotonous increasing period function in the case that

$$5V'''(x)^2 - 3V''(x)V^{(4)}(x) > 0 \text{ for any } x \in \mathcal{I} \text{ with } V''(x) > 0$$

and

$$V'(x)V'''(x) < 0 \text{ for any } x \in \mathcal{I} \text{ with } V''(x) = 0.$$

On the other hand, in the case that

$$5V'''(x)^2 - 3V''(x)V^{(4)}(x) < 0 \text{ for any } x \in \mathcal{I} \text{ with } V''(x) \geq 0$$

then the period function is monotonous decreasing.

The monotonicity study will be treated separately depending on the behaviour of $V''(x)$ and where the assumptions of the theorem are satisfied. First of all notice that the assumptions have to be satisfied only for $x \in \mathcal{I}$ such that $V''(x) \geq 0$.

Let $V_\mu(x)$ be the potential function of the system X_μ. Then,

$$V_\mu''(x) = p(x+1)^{p-1} - q(x+1)^{q-1} = (x+1)^{q-1}\left(p(x+1)^{p-q} - q\right) = 0$$

if and only if $x = x_\mu^*$ where $x_\mu^* := (q/p)^{\frac{1}{p-q}} - 1$. Therefore, for each case of the period annulus projection showed in Lemma 1.3, x_μ^* determine a different interval where the Schaaf's conditions need to be verified depending on the parameter μ. In order to make the computations easier, let us consider the map $x \longmapsto z(x) = (1+x)^{p-q}$. Then, $z_\mu^* = (1+x_\mu^*)^{p-q} = q/p$.

Let us divide regions $\mathscr{R}_1$ and $\mathscr{R}_2$ such that

$$\mathscr{R}_1 = \mathscr{R}_{11} \cup \mathscr{R}_{12} \cup \mathscr{R}_{13} \cup \mathscr{R}_{14}$$
$$\mathscr{R}_2 = \mathscr{R}_{21} \cup \mathscr{R}_{22}$$

where

$$\begin{aligned}
\mathscr{R}_{11} &= \mathscr{R}_1 \cap \{(q,p) \ : \ p > q > 0\},\\
\mathscr{R}_{12} &= \mathscr{R}_1 \cap \{(q,p) \ : \ p > 0 \geq q\},\\
\mathscr{R}_{13} &= \mathscr{R}_1 \cap \{(q,p) \ : \ 0 > p > q \text{ and } p+q > -1\},\\
\mathscr{R}_{14} &= \mathscr{R}_1 \cap \{(q,p) \ : \ 0 > p > q \text{ and } p+q \leq -1\},\\
\mathscr{R}_{21} &= \mathscr{R}_2 \cap \{(q,p) \ : \ p \geq 0 \text{ and } q < -1\},\\
\mathscr{R}_{22} &= \mathscr{R}_2 \cap \{(q,p) \ : \ q < -1 < p < 0\}.
\end{aligned} \tag{2.1}$$

The next lemma shows us which are the intervals where the conditions of Schaaf's criterion presented in Theorem 2.1 have to be satisfied.

Lemma 2.1. *The interval where $V''(x)$ is positive is mapped by $x \mapsto (1+x)^{p-q}$ to*

(a) $\left(\frac{q}{p}, \frac{p+1}{q+1}\right)$ *if* $\mu \in \mathscr{R}_{11}$.

(b) $\left(0, \frac{p+1}{q+1}\right)$ *if* $\mu \in \mathscr{R}_{12}$.

(c) $\left(0, \frac{p+1}{q+1}\right)$ *if* $\mu \in \mathscr{R}_{13}$.

(d) $\left(0, \frac{q}{p}\right)$ *if* $\mu \in \mathscr{R}_{14}$.

(e) $(0, +\infty)$ *if* $\mu \in \mathscr{R}_{21}$.

(f) $\left(0, \frac{q}{p}\right)$ *if* $\mu \in \mathscr{R}_{22}$.

(g) $\left(0, \frac{q}{p}\right)$ *if* $\mu \in \mathscr{R}_{3}$.

Proof. First of all notice that the mapping $x \mapsto z = (1+x)^{p-q}$ preserves orientation. Then, we can map first before describe the interval. The projections of the period annulus $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ in each case are mapped to $\widehat{\mathcal{I}}_1 = (0, \frac{p+1}{q+1})$, $\widehat{\mathcal{I}}_2 = (0, \infty)$ and $\widehat{\mathcal{I}}_3 = (\frac{p+1}{q+1}, \infty)$ respectively. Moreover, the center $x = 0$ is mapped to $z = 1$ and x^*_μ is mapped to $z^*_\mu = q/p$.

Let us denote by $(z_L, z_R) = \widehat{\mathcal{I}}_i$ for $i = 1, 2, 3$. Notice that, since V'' is always positive at the center, $z = 1$ is always inside the interval. Furthermore, since $p > q$, if $p > 0$ then $q/p < 1$ and if $p > 0$ then $q/p > 1$. Then, if $p > 0$ the interval will be $S = (\widehat{z}_1, z_R)$ and if $p < 0$ it will be $S = (z_L, \widehat{x}_2)$ where $\widehat{z}_1 = \max\{z^*_\mu, z_L\}$ and $\widehat{z}_2 = \min\{z^*_\mu, z_R\}$. Then, for each case the statement holds. ■

The expression to be studied is the one given by the first condition of Schaaf's criterion,

$$5V'''(x)^2 - 3V''(x)V^{(4)}(x) = \frac{P_\mu\left((1+x)^{p-q}\right)}{(1+x)^4}$$

with

$$P_\mu(z) = (q-1)q^2(1+2q) + pq(3p^2+3q^2-10pq+p+q+2)z + (p-1)p^2(1+2p)z^2.$$

Since $(1+x)^4 > 0$ for all $x \in \mathbb{R}$, and by the map $x \mapsto (1+x)^{p-q}$, we have to study if the quadratic polynomial has a constant sign in each of the intervals given in Lemma 2.1. To this end, we shall apply the following approach.

Let us suppose that we are interested in the number Z_λ of roots (counting multiplicity) of a family of polynomials $\{P_\lambda(x)\}_{\lambda\in\mathbb{R}^k}$ in an interval $I_\lambda = [a_\lambda, b_\lambda]$, where $P_\lambda(x)$, a_λ and b_λ depend continuously on the parameter λ.

Let us denote by $\Delta(Q(x), x)$ the discriminant of a polynomial $Q \in \mathbb{R}[x]$ with respect to x. The functions $P(a_\lambda)$, $P(b_\lambda)$ and $\Delta(P_\lambda, x)$ are continuous in λ by composition of continuous functions. There are three different ways to change Z_λ inside I_λ. Those are that a root is coming inside or leaving through a_λ, through b_λ or that two complex conjugated roots collapses into a double real root. The vanishing of $P(a_\lambda)$, $P(b_\lambda)$ and $\Delta(P_\lambda, x)$ determine that a root is coming or leaving by the respective way.

Proposition 2.1. *Let $P_\lambda(x)$ be a family of polynomials with the coefficients continuous functions of $\lambda \in \mathcal{U} \subset \mathbb{R}^k$. Let Z_λ be the number of roots, counting multiplicities, of $P_\lambda(x)$ in an interval $I_\lambda = (a_\lambda, b_\lambda)$ where a_λ and b_λ continuous functions of λ. Let us consider*

$$\mathcal{B} = \{\lambda : \ P_\lambda(a_\lambda)P_\lambda(b_\lambda)\Delta(P_\lambda(x), x) = 0\}.$$

Then Z_λ is constant in the connected components of $\mathcal{U} \setminus \mathcal{B}$.

Remark 2.1. The previous result is also true if one or both endpoints of the interval I_λ is infinity. In this case, $P_\lambda(a_\lambda)$ with $a_\lambda = \infty$ denotes the coefficient of the highest degree term. □

Proof of Proposition 2.1. Let $\Omega \subset \mathcal{U}\setminus\mathcal{B}$ a connected component of $\mathcal{U}\setminus\mathcal{B}$ and let $\lambda_0 \in \Omega$. Since Ω is a connected component of $\mathcal{U} \setminus \mathcal{B}$, particularly $P_{\lambda_0}(a_{\lambda_0}) \neq 0$, $P_{\lambda_0}(b_{\lambda_0}) \neq 0$ and $\Delta(P_{\lambda_0}(x), x) \neq 0$. Since the discriminant does not vanish at λ_0, all the roots of $P_{\lambda_0}(x)$ in I_{λ_0} are simple. Let us denote $\mathcal{U}_\delta := \{\lambda \ : \ |\lambda - \lambda_0| < \delta\}$. By continuity with respect to the parameter, there exists $\delta_1 > 0$ small enough such that for all $\lambda \in \mathcal{U}_{\delta_1}$ the roots of $P_\lambda(x)$ are also simple in I_λ. Moreover, since $P_{\lambda_0}(a_{\lambda_0}) \neq 0$ and $P_{\lambda_0}(b_{\lambda_0}) \neq 0$, for a given $\epsilon > 0$ there exists δ_2 such that for all $\mathcal{U}_{\delta_2}$ there are no roots in the intervals $(a_\lambda, a_\lambda + \epsilon]$ and $[b_\lambda - \epsilon, b_\lambda)$. Then, taking $\delta = \min\{\delta_1, \delta_2\}$ the number of zeros inside I_λ is constant in $\mathcal{U}_\delta$. ■

For this result is essentially that functions $P(a_\lambda)$,$P(b_\lambda)$ and $\Delta(P_\lambda, x)$ are continuous on the parameter λ. Therefore,

$$P(a_\lambda)P(b_\lambda)\Delta(P_\lambda, x) = 0$$

collect these three ways and separates the parametric space in arc connected regions such that for all λ inside this region the number Z_λ does not change.

In the case of $P_\mu(z)$, the curves that determine the different regions to be studied are given by

$$\begin{aligned}
\Delta(P_\mu(z)) &= 3p^2(p-q)^2q^2(7+3p^2+p(2-14q)+q(2+3q))=0,\\
P_\mu(0) &= (q-1)q^2(1+2q)=0,\\
P_\mu\left(\frac{q}{p}\right) &= 5(p-q)^2q^2=0,\\
P_\mu(\infty) &= (p-1)p^2(1+2p)=0,\\
P_\mu\left(\frac{p+1}{q+1}\right) &= \frac{(p-q)^2}{(1+q)^2}\big(2p^4+p^3(3+4q)+(1+q)^2(-1-q+2q^2)+\\
&\quad +p^2(-1+9q+9q^2)+p(-3+2q+9q^2+4q^3)\big)=0,
\end{aligned}$$

where $P(\infty)$ denotes the zeros that comes from infinity. That is, when the coefficient of z^2 vanish. In general the product of all these curves separates the parameter space $\mathscr{R}$ in regions where the number Z_μ does not change. However, depending on the interval given by Lemma 2.1, we will use one or another curves to divide the regions optimizing the number of curves needed to ensure the same number of roots inside the interval in each region. Let us denote by

$$\Delta := 7+3p^2+p(2-14q)+q(2+3q)=0,$$

$$\Theta := 2p^4+p^3(3+4q)+(1+q)^2(-1-q+2q^2)+p^2(-1+9q+9q^2)+p(-3+2q+9q^2+4q^3)=0,$$

the curves that comes from $\Delta(P_\mu(z))=0$ and $P_\mu\left(\frac{p+1}{q+1}\right)=0$ respectively. The following theorem is the main result of this section. Before it, let us notice that indeed the endpoints of the interval $I_\mu=[a_\mu,b_\mu]$ given in Lemma 2.1 vary continuously with respect to the parameter μ. For each region, the endpoints are continuous functions of p and q, so it is only necessary to check it at the boundaries of the regions. However, the boundaries where the interval vary are given by $\partial\mathrm{Cl}(\mathscr{R}_{11})\cap\partial\mathrm{Cl}(\mathscr{R}_{12})=\{q=0\}$, $\partial\mathrm{Cl}(\mathscr{R}_{13})\cap\partial\mathrm{Cl}(\mathscr{R}_{14})=\{p+q=-1\}$, $\partial\mathrm{Cl}(\mathscr{R}_{12})\cap\partial\mathrm{Cl}(\mathscr{R}_{21})=\{q=-1\}$ and $\partial\mathrm{Cl}(\mathscr{R}_{21})\cap\partial\mathrm{Cl}(\mathscr{R}_{22})=\{p=0\}$, where the interval vary also continuously looking at the expression of Lemma 2.1.

Theorem 2.2. *The period function of the center at the origin of system* (1.15) *is monotonous increasing in regions* $1,2,9,11-13,16-19$ *and is monotonous decreasing in regions* 6 *and* 8 *(see Figure 2.1 and Figure 2.2).*

Proof. For each region Ω, we will take only one point $(q_0,p_0)\in\Omega$. Then, $P(y):=P(y,q_0,p_0)$ will be a polynomial of degree 2 that will have two different roots because Ω has no points with discriminant zero. By the choice of the curves that divides the space of parameters, if these two roots does not lay in the interval associated to the region Ω, then any point $(q,p)\in\Omega$ will have any root inside this interval. In this situation, we will be able to apply Theorem 2.1 checking if $P(y,q_0,p_0)$ is positive or negative on the interval.

We will only prove the theorem for $\mathscr{R}_{11}$ and the others are analogue. By Lemma 2.1, if $\mu\in\mathscr{R}_{11}$ the interval where V'' is positive after the mapping is $\left(\frac{q}{p},\frac{p+1}{q+1}\right)$. For this region, we will have the following curves

$$\Delta(q,p)\Theta(q,p)pq=0$$

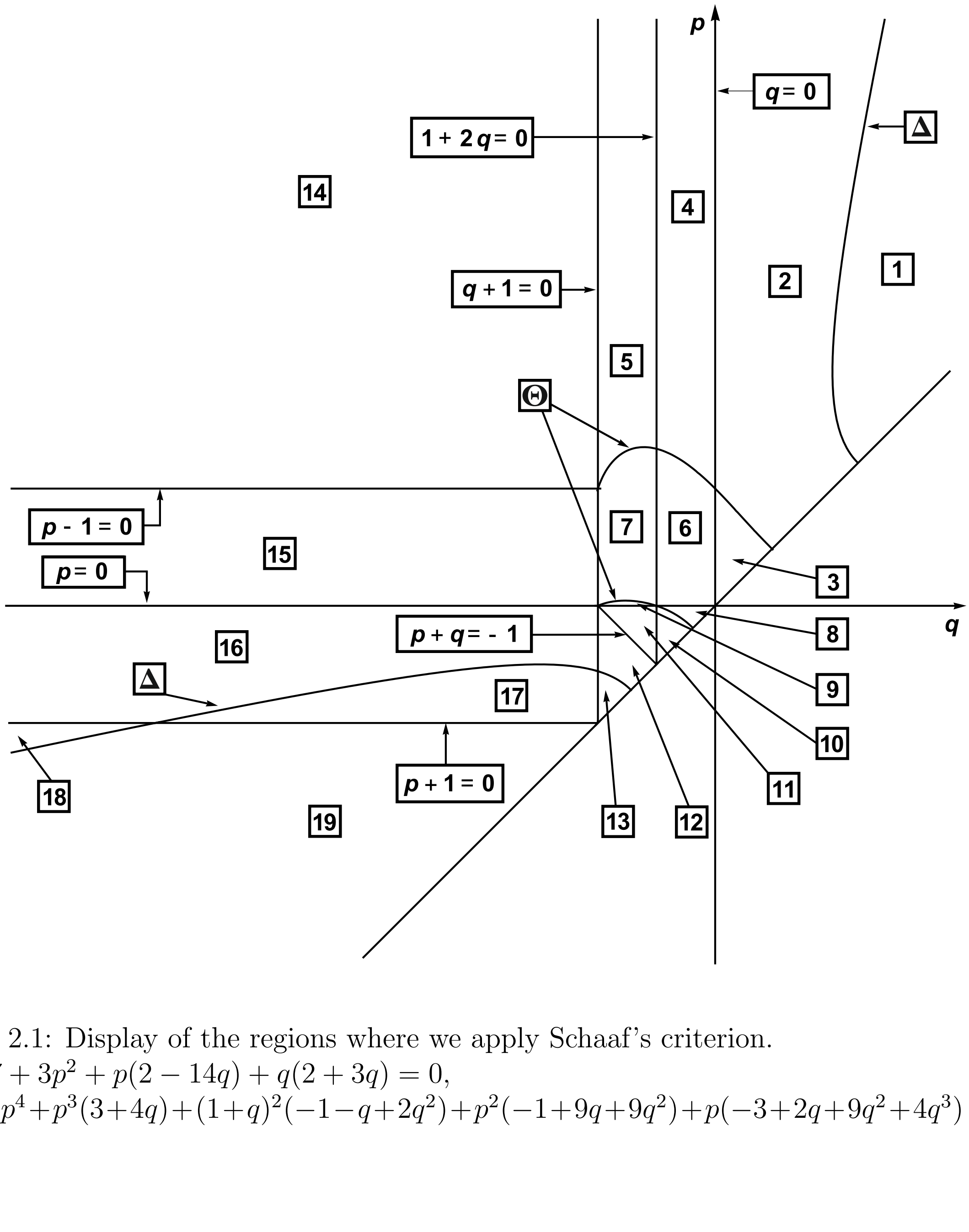


Figure 2.1: Display of the regions where we apply Schaaf's criterion.
$\Delta := 7 + 3p^2 + p(2 - 14q) + q(2 + 3q) = 0$,
$\Theta := 2p^4 + p^3(3+4q) + (1+q)^2(-1-q+2q^2) + p^2(-1+9q+9q^2) + p(-3+2q+9q^2+4q^3) = 0$.

that comes from $\Delta(P_\mu(z),z)P(q/p)P((p+1)/(q+1)) = 0$ restricted to $\mathscr{R}$. Moreover, if we restrict now to $\mathscr{R}_{11}$, then the curves are given by $\Delta(q,p)\Theta(q,p)pq = 0$ that divides the region in three different zones called **Region 1**, **2** and **3** in Figure 2.1.

It is easy to check that the following respective points p_j, $j = 1,2,3$ lies in each of these regions

$$p_1 = (\frac{3}{2},3), \quad p_2 = (\frac{1}{2},3), \quad p_3 = (\frac{1}{5},\frac{2}{5}).$$

First notice that the function $V'(x)V'''(x)$ only vanishes at $x=-1$, $x=0$ (for $V'(x)$) and $(x+1)^{p-q} = \frac{(q-1)q}{(p-1)p}$ because $V'''(x) = (p-1)p(x+1)^{p-2} - (q-1)q(x+1)^{q-2}$. Since the only root of $V''(x) = 0$ is x_μ^*, inside the region we are studying $V'(x)V'''(x) \neq 0$ and then does not change of sign. Hence, taking for instance $p_1 = (\frac{3}{2},3)$ then we can see that $V'(x^*(q,p))V'''(x^*(q,p)) < 0$ so the second condition of the Schaaf's criterion holds. Now, the roots of the polynomial $P(y)$ in each point are

$$\begin{gathered}
y_1^- = \frac{1}{224}(19 - 3\sqrt{159}i),\ y_1^+ = \frac{1}{224}(19 + 3\sqrt{159}i).\\
y_2^- = \frac{1}{160}(-29 - 3\sqrt{129}),\ y_2^+ = \frac{1}{160}(-29 + 3\sqrt{129}).\\
y_3^- = \frac{1}{3},\ y_3^+ = \frac{7}{9}.
\end{gathered}$$

Since $y_1^\pm$ are complex points, then $P(y,p_1)$ has no roots inside the interval. Moreover, $P(1,p_1) > 0$ so the first condition of Schaaf's criterion holds and the period function is monotonous increasing at **Region 1**. In the case of **Region 2**, $y_2^\pm \notin (\frac{1}{4},2)$ which is the interval in the case of p_2. Since $P(1,p_2) > 0$ also in this point, then the period function is monotonous increasing in **Region 2**. For **Region 3** we can not apply the Schaaf's criterion because $y_3^+ \in (\frac{1}{2},\frac{7}{6})$. ■

Figure 2.2 shows us the regions where we proved that the period function is monotone.

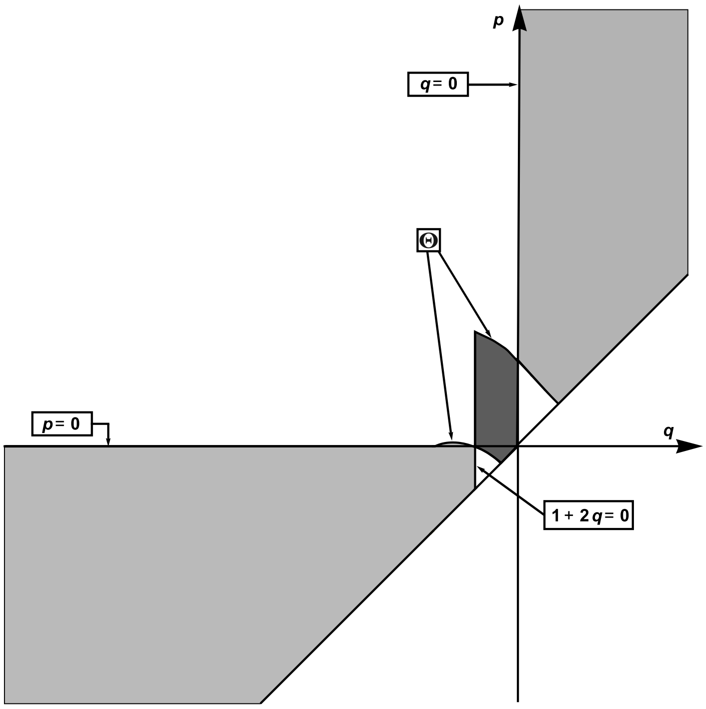


Figure 2.2: Display of the regions where the period function is monotone. Dark region decreasing, light region increasing.

Chapter 3

Isochronicity and bifurcation

3.1 Period constants

As we introduced at Chapter 1, isochronous centers are the centers which all the periodic orbits of the period annulus $\mathscr{P}$ have the same period. That notion does not depend on the parametrization of the orbits, so we will choose one that will allow us to make explanations more comfortable. The definition given in (1.13) allows us to write the period function parametrized by the energy level h as a Taylor development at $h = 0$ because $g(x)$ is an analytic diffeomorphism at $x = 0$ (see (1.6)). However, the coefficients in the series development are not polynomials but general functions in p and q.

Instead of study the period function of X_μ, in this section and in the following one, we will consider another family of vector fields which is essentially given by a conjugated vector field of X_μ with a rescaling of time. The following Lemma shows us that the period function of this other system parametrized by its correspondent energy level is such that the coefficients in the series development are polynomials in p and q.

Proposition 3.1. *Let Y_μ the family of vector fields that results from conjugate the vector field X_μ by the map $(x, y) \mapsto (u, v) = (x, \frac{1}{\sqrt{p-q}}y)$ and rescaling the time by $1/\sqrt{p-q}$. Let $T(h)$ be the period function of the center Y_μ parametrized by the energy level h. Then,*

$$T(h) = 2\pi + \sum_{i=1}^{\infty} \Delta_i(q, p)h^i, \tag{3.1}$$

where $T(h)$ is the period function parametrized by the energy level h associated to Y_μ and $\Delta_i(q, p) \in \mathbb{R}[q, p]$.

Proof. Notice that

$$V'(x) = (1 + x)^p - (1 + x)^q = \sum_{k=1}^{\infty} \left[\binom{p}{k} - \binom{q}{k} \right] x^k = \sum_{k=1}^{\infty} \alpha_k(q, p)x^k$$

where

$$\alpha_k(q, p) = \frac{p(p-1)\dots(p-k+1) - q(q-1)\dots(q-k+1)}{k!}.$$

Since $\alpha_k(p, p) = 0$, then $\alpha_k(q, p) = (p - q)\widehat{\alpha}_k(q, p)$ with $\widehat{\alpha}_k(q, p) \in \mathbb{R}[q, p]$ for all $k \geq 1$.

Then, $V(x)=(p-q)\sum_{k=2}^{\infty}\widehat{\alpha}_k(q,p)x^k$ where $\widehat{\alpha}_k(q,p)\in\mathbb{R}[q,p]$ and $\widehat{\alpha}_2(q,p)=1$ because $V(x)=\dfrac{p-q}{2}x^2+o(x^2)$. This shows that

$$V(x)=(p-q)\widehat{V}(x)$$

with

$$\widehat{V}(x)=\frac{x^2}{2}+\sum_{k\geq 3}\widehat{\alpha}_k(q,p)x^k.$$

Let us consider the system X_μ,

$$\begin{cases}\dot{x}=-y,\\ \dot{y}=V'(x).\end{cases}$$

Through the change of variables $u=x$, $v=\frac{1}{\sqrt{p-q}}y$, the system becomes

$$\begin{cases}\dot{u}=-\sqrt{p-q}v,\\ \dot{v}=\frac{1}{\sqrt{p-q}}V'(u).\end{cases}$$

Therefore, changing the time multiplying by $1/\sqrt{p-q}$,

$$\begin{cases}u'=-v,\\ v'=\frac{1}{p-q}V'(u).\end{cases}\tag{3.2}$$

The system (3.2) is

$$Y_\mu\begin{cases}u'=-v,\\ v'=\widehat{V}'(u).\end{cases}\tag{3.3}$$

By Proposition 12 of [4], since the coefficients of $\widehat{V}'(u)$ are polynomials on the parameters and $\widehat{V}'(u)=u^2/2+o(u^2)$, the coefficients of the Taylor's development of $\widehat{T}(u)$ parametrized by u associated to the system (3.3) are also polynomials on the parameters.

Let $\widehat{g}$ defined by (1.6) associated to $\widehat{V}$. Let us see that $\widehat{g}^{-1}(u)$ has polynomial coefficients in q,p in the Taylor development at $u=0$. Indeed, $\widehat{g}$ is analytic at 0 and $\widehat{g}'(0)\neq 0$, so $\widehat{g}^{-1}(u)$ is analytic at $u=0$ and since $\widehat{g}^{-1}(0)=0$ then $\widehat{g}^{-1}(u)=\sum_{i=1}^{\infty}a_i(q,p)u^i$. Let us prove that $a_i(q,p)\in\mathbb{R}[q,p]$ by induction on i. By definition of $\widehat{g}$, $\widehat{V}(\widehat{g}^{-1}(u))=u^2$. Then,

$$\widehat{V}(\widehat{g}^{-1}(u))=\frac{1}{2}\left(\sum_{i=1}^{\infty}a_i(q,p)u^i\right)^2+\sum_{k\geq 3}\widehat{\alpha}_k(q,p)\left(\sum_{i=1}^{\infty}a_i(q,p)u^i\right)^k=u^2.$$

For $i=1$, let us notice that the only way to obtain u^2 on the left side of the above equality is $\frac{1}{2}a_1(q,p)^2u^2$. Then, $\frac{1}{2}a_1(q,p)^2u^2=u^2$ so $a_1(q,p)=\sqrt{2}$. Let us assume that for $i=1,\ldots,j-1$, $a_i(q,p)\in\mathbb{R}[q,p]$. Then, notice again in the equality above that the coefficient of u^{j+1} is

$$a_1(q,p)a_j(q,p)u^{j+1}+\{\text{combination of } a_i(q,p) \text{ with } i=1,\ldots,j-1\}u^{j+1}.$$

Then, $\sqrt{2}a_j(q,p)u^{j+1} + \{\text{combination of } a_i(q,p) \text{ with } i = 1, \dots, j-1\}u^{j+1} = 0$ so

$$a_j(q,p) = \frac{\rho(q,p)}{\sqrt{2}}$$

for $\rho(q,p) \in \mathbb{R}[q,p]$. Then, $a_j(q,p) \in \mathbb{R}[q,p]$.

Let $T(h)$ be the period function parametrized by the energy level h of (3.3). Since $V(u) = h$, $T(V(u)) = \widehat{T}(u)$ so $T(u^2) = T(V(g^{-1}(u))) = \widehat{T}(g^{-1}(u))$. Therefore by composition, the coefficients of the Taylor development of $T(u^2)$ are polynomials in p, q. Finally, since $\widehat{T}(g^{-1}(-u))$ is an even function, $T(h) = \widehat{T}(g^{-1}(\sqrt{h}))$ with the coefficients polynomials in p, q. ■

Remark 3.1. The vector field X_μ is conjugated to (3.3) with a rescaling of time. Therefore, all the results for (3.3) transfers directly to X_μ. For this reason, and abusing of notation, we will continue referencing X_μ and assuming that the coefficients of the Taylor development at $h = 0$ of the period function $T(h)$ of X_μ are polynomials in p, q. □

Recall that the period function $T(h)$ was not defined for $h = 0$, which corresponds to the period at the center. However, expression (3.1) shows that $T(h)$ can be extended analytically to $h = 0$.

The coefficient $\Delta_n(q,p)$ is known as the *nth period constant*. A necessary and sufficient condition for a point $\mu = (q,p)$ to be an isochronous center is that all the coefficients $\Delta_n(q,p)$, $n \geq 1$ vanish. Let us consider the ideal generated by those polynomials

$$I = (\Delta_1(q,p), \Delta_2(q,p), \dots, \Delta_n(q,p), \dots)$$

over the ring $\mathbb{R}[q,p]$. This ideal is interesting since its complex variety, $V_{\mathbb{C}}(I) = \{(q,p) \in \mathbb{C}^2 \ : \ \Delta_n(q,p) = 0, \ \forall n \geq 1\}$ is in fact the complex set of isochronous centers. In the way to obtain the real ones, $V_{\mathbb{R}}(I) = V_{\mathbb{C}}(I) \cap \mathbb{R}^2$. The ideal I can be reduced in a first approach using the Hilbert's basis Theorem, saying that there exists $N \geq 1$ such that

$$I = (\Delta_1(q,p), \Delta_2(q,p), \dots, \Delta_N(q,p)) \,. \tag{3.4}$$

The main problem is that N is not generally known a priori. However, a necessary but not sufficient condition for X_μ to have an isochronous center at the origin is that $\mu = (q,p)$ must vanish at least the first period constants.

Lemma 3.1. *The first three constant periods of $T(h)$ are:*

$$\begin{aligned}
\Delta_1(q,p) &= \frac{1}{3\sqrt{2}}(2p^2 + 2q^2 + 7pq - p - q - 1), \\
\Delta_2(q,p) &= \frac{1}{18\sqrt{2}}(-23 + 4p^4 - 46q + 21q^2 + 44q^3 + 4q^4 + 4p^3(11 + 43q) + \\
&\quad + 3p^2(7 + 122q + 139q^2) + 2p(-23 + 42q + 183q^2 + 86q^3)), \\
\Delta_3(q,p) &= \frac{1}{2721600\sqrt{2}}(-11237 - 1112p^6 - 33711q - 10641q^2 + 34903q^3 + 22434q^4 \\
&\quad - 636q^5 - 1112q^6 + 12p^5(-53 + 803q) + 6p^4(3739 + 25888q + 27289q^2) \\
&\quad + p^3(34903 + 390273q + 734277q^2 + 336347q^3) \\
&\quad + 3p^2(-3547 + 88309q + 284637q^2 + 244759q^3 + 54578q^4) \\
&\quad + 3p(-11237 - 2951q + 88309q^2 + 130091q^3 + 51776q^4 + 3212q^5)).
\end{aligned} \tag{3.5}$$

Proof. Let $V(x) = (p-q)\widehat{V}(x)$ defined as in the proof of Proposition 3.1, where $V(x)$ is the potential function of the vector field X_μ and $\widehat{V}(x)$ is the one of Y_μ. Then, by expression (1.6),

$$\widehat{g}(x) = \frac{g(x)}{\sqrt{p-q}}$$

where $g(x)$ is associated to X_μ and $\widehat{g}(x)$ to Y_μ. The relation between the derivative of the inverse functions follows immediately as

$$\left(\widehat{g}^{-1}\right)'(x) = \sqrt{p-q}\left(g^{-1}\right)'(x\sqrt{p-q}). \tag{3.6}$$

By (1.13), we have an expression for the period function associated to Y_μ parametrized by the energy level h,

$$T(h) = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\widehat{g}^{-1})'(\sqrt{h}\sin\theta)d\theta.$$

The Taylor development for $h \approx 0$ of $(\widehat{g}^{-1})'(\sqrt{h}\sin\theta)$ can be written in terms of g by using (3.6),

$$(\widehat{g}^{-1})'(\sqrt{h}\sin\theta) = \sum_{i=0}^{\infty}(\widehat{g}^{-1})^{(i+1)}(0)\sin^i\theta\sqrt{h}^i = \sqrt{p-q}\sum_{i=0}^{\infty}(g^{-1})^{(i+1)}(0)\sin^i\theta\sqrt{h}^i\sqrt{p-q}^i.$$

Then, for $h \approx 0$,

$$T(h) = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{p-q}\sum_{i=0}^{\infty}(g^{-1})^{(i+1)}(0)\sin^i\theta\sqrt{h}^i\sqrt{p-q}^i d\theta$$

that converges uniformly in p, q, and from it follows that

$$T(h) = \sqrt{2}\sqrt{p-q}\sum_{i=0}^{\infty}(g^{-1})^{(i+1)}(0)\sqrt{h}^i\sqrt{p-q}^i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^i\theta d\theta.$$

Notice that for i odd, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^i\theta d\theta = 0$ so finally we get the expression

$$T(h) = \sum_{i=0}^{\infty}\Delta_i(q,p)h^i$$

where $\Delta_i(q,p) = \sqrt{2}(p-q)^{\frac{2i+1}{2}}(g^{-1})^{(2i+1)}(0)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^{2i}\theta d\theta$. Then, by using this formula for $i = 1, 2, 3$ the statement holds. ■

The following result characterizes the isochronous centers of the family that we are studying.

Theorem 3.1. *Let X_μ be the family described in* (1.15). *Then, the center is isochronous if and only if $\mu \in \{(-3,1), (-1/2,0), (0,1)\}$.*

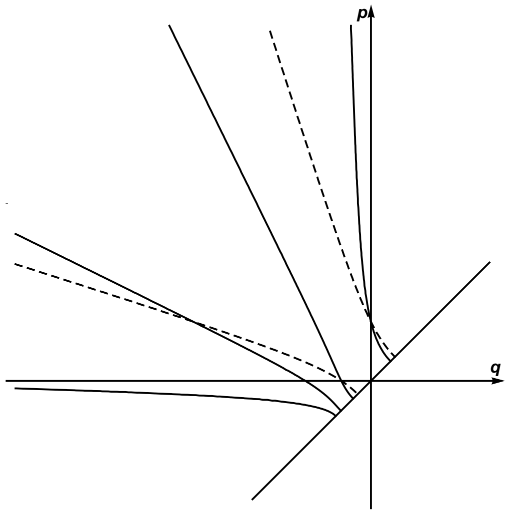


Figure 3.1: The two first period constants: $\Delta_1(q,p) = 0$ (dashed) and $\Delta_2(q,p) = 0$ (not dashed).

Proof. Since the isochronous must verify $\Delta_k(q,p) = 0$ for all $k \geq 1$, a necessary condition for X_μ to have an isochronous center at the origin is that $\mu = (q,p)$ satisfies $\Delta_1(q,p) = 0$ and $\Delta_2(q,p) = 0$. Using Lemma 3.1, the intersection of the curves $\Delta_1(q,p) = 0$ and $\Delta_2(q,p) = 0$ on the parameter space gives us the collection of points

$$\mu_1 = (-3,1),\ \mu_2 = (-1/2,0),\ \mu_3 = (0,1), \mu_4 = (1,-3),\ \mu_5 = (0,-1/2),\ \mu_6 = (1,0),$$
$$\mu_7 = \left(\frac{i}{\sqrt{3}}, -\frac{i}{\sqrt{3}}\right),\ \mu_8 = \left(-\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}\right), \tag{3.7}$$

and those of them which are $\mu = (q,p) \in \mathscr{R}$ are

$$\mu_1 = (-3,1),\ \mu_2 = (-1/2,0),\ \mu_3 = (0,1), \tag{3.8}$$

(see Figure 3.1). As we said, the vanishing of the first two terms is a necessary condition but not sufficient one for the isochronous centers. Fortunately, since there are only three available cases that can be isochronous, we can check it by hand.

For proving that μ_j are isochronous centers for $j = 1,2,3$ we will use Proposition 1.1. We will see that for each vector field X_{μ_j} we can compute the involution associated to its potential function and moreover this involution satisfies condition (ii) of Proposition 1.1.

As we mentioned, there are only three available candidates to be isochronous centers for $\mu \in \mathscr{R}$: μ_1, μ_2 and μ_3 (see (3.8)). Therefore, we only need to prove that they are really isochronous. Computing for each case $V(x) = V(\sigma(x))$ such that $\sigma(x) \neq x$ and $\sigma^2(x) = x$ we obtain

$$\sigma_1(x) = -\frac{x}{x+1},\ \sigma_2(x) = 4 + x - 4\sqrt{x+1},\ \sigma_3(x) = -x. \tag{3.9}$$

For Proposition 1.1 (ii), we only need to check the equality

$$V(x) = K\,(x - \sigma(x))^2 \text{ for all } x \in \mathcal{I}$$

for $K = \frac{\pi^2}{2\omega^2}$.

- In the case $\mu_1 = (-3, 1)$, $K = 2$:

$$\begin{aligned}(x - \sigma_1(x))^2 - 2V(x) &= \left(x + \frac{x}{1+x}\right)^2 - 2\left(\frac{(x+1)^2}{2} + \frac{1}{2(x+1)^2} - 1\right) \\ &= x^2 + \frac{2x^2}{(1+x)} + \frac{x^2}{(1+x)^2} - (x+1)^2 - \frac{1}{(x+1)^2} + 2 = 0.\end{aligned}$$

- In the case $\mu_2 = (-1/2, 0)$, $K = 16$:

$$\begin{aligned}(x - \sigma_2(x))^2 - 16V(x) &= \left(-4 + 4\sqrt{x+1}\right)^2 - 16\left(2 + x - 2\sqrt{x+1}\right) \\ &= 16(x+1) - 32\sqrt{x+1} + 16 - 16\left(2 + x - 2\sqrt{x+1}\right) = 0.\end{aligned}$$

- In the case $\mu_3 = (0, 1)$, $K = 8$:

$$\begin{aligned}(x - \sigma_3(x))^2 - 8V(x) &= (x + x)^2 - 8\frac{x^2}{2} \\ &= 4x^2 - 4x^2 = 0.\end{aligned}$$

Then, the three vector fields X_{μ_1}, X_{μ_2}, X_{μ_3} have an isochronous center at the origin. ■

Although the case $\mu_3 = (0, 1)$ was obvious since X_{μ_3} is the linear center, we wanted to follow the same argument of the three cases to show a general searching technique. For this particular case, it is clear that the period function is constant with value 2π.

In order to increase the knowledge about the period function, we will use in the following sections some techniques of local perturbation that will give us local information about the behaviour of the period function close to the isochronous centers and near the curve $\Delta_1(q, p) = 0$.

More concretely, we will study the bifurcation of critical periods, roughly speaking, the disappearance or emergence of critical periods as we perturb the system. There are three different situations to study:

(a) Bifurcation of the period function from the inner boundary of $\mathscr{P}$ (that is, the center itself).

(b) Bifurcation of the period function from $\mathscr{P}$.

(c) Bifurcation of the period function from the outer boundary of $\mathscr{P}$.

In this work we are only concerned with the first two kind of bifurcation. In the next sections we will study local bifurcations in the interior of the period annulus (Section 3.1) and local bifurcation from the inner boundary (Section 3.2).

3.2 Bifurcations from the center

The aim of this section is to study the bifurcation of critical periods from the center itself. As we said in sections before, the function $T'(h)$ can be written as a Taylor series in $h = 0$ with the called *period constants*, that are essentially odd derivatives of g^{-1} at 0.

$$T'(h) = \sum_{i=0}^{\infty} \Delta_{i+1}(q,p)h^i. \tag{3.10}$$

The study of critical period close to the center is essentially study if the function $T'(h)$ vanishes for the energy level h close to 0, which is the parameter corresponding to the center.

Definition 11. Let $T(h;\mu)$ the period function parametrized by h of the center X_μ. We say that l local critical periods bifurcate from the center corresponding to the parameter value μ_* if for every $\epsilon > 0$ and every neighbourhood $\mathcal{U}$ of μ_* there exists $\mu_0 \in \mathcal{U}$ such that $T'(h;\mu_0) = 0$ has l solutions on $(0,\epsilon)$. △

Definition 12. If $\Delta_1(\mu_*) = \Delta_2(\mu_*) = \cdots = \Delta_k(\mu_*) = 0$ and $\Delta_{k+1}(\mu_*) \neq 0$ with $k \geq 1$, then the center at the origin corresponding to the parameter vale μ_* is called a *weak center of order k*. The center is isochronous if $\Delta_k(\mu_*) = 0$ for all $k \geq 1$. In this case the center is also called a *weak center of infinite order.* △

The aim is study the bifurcation of critical periods from weak centers both order finite and infinite. In case of weak centers of finite order, the following result tells us that there are only weak center of order 1 and also refers about the bifurcation of critical periods from the center in these cases.

Proposition 3.2. *The highest order of a weak center of finite order of the system X_μ is one. From these centers, bifurcate at most one critical period. Moreover, this upper bound is achieve.*

Proof. We have seen in the proof of Theorem 3.1 that the condition $\Delta_1(\mu) = 0$ and $\Delta_2(\mu) = 0$ implies that $\mu \in \mathscr{R}$ is an isochronous for the system X_μ. Then, the highest order of weak centers of finite order is one.

Let us consider the Taylor development of $T'(h)$,

$$T'(h) = \sum_{i=0}^{\infty} \Delta_{i+1}(\mu)h^i = \Delta_1(\mu) + \Delta_2(\mu)h + o(h),$$

and let $\mu_0 \in \{\mu \in \mathscr{R} \ : \ \Delta_1(\mu) = 0\} \cap \{\mu \in \mathscr{R} \ : \ \Delta_2(\mu) \neq 0\}$. Since $\Delta_2(\mu_0) \neq 0$ and $\Delta_1(\mu_0) = 0$, for a given $\xi > 0$ small enough there exists $\delta > 0$ such that for any $\mu \in \mathcal{U}_\delta \subset \mathscr{R}$ neighbourhood of μ with distance lest than δ of μ_0, μ satisfies $\Delta_2(\mu) \neq 0$ and $\Delta_1(\mu) < \xi$. Then, by the Weierstrass Preparation Theorem, there are at most one root $h_0 \in (0,\epsilon)$ of $T'(h)$ for $\epsilon > 0$ small enough. Then, at most one critical period bifurcates from the origin at μ_0. The upper bound is achieve because, by the Implicit Function Theorem, for h_0 small enough,

$$h_0 = -\frac{\Delta_1(\mu)}{\Delta_2(\mu)}.$$

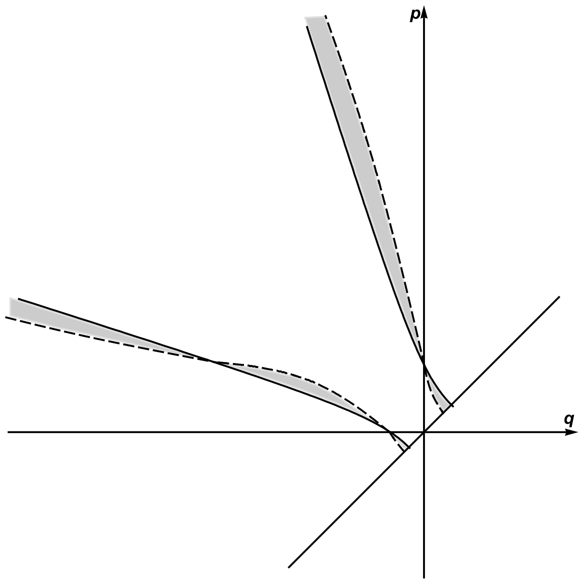


Figure 3.2: Bands with critical periods bifurcating from the center.

Then, $T'(h)$ has a root $h_0 \in (0, \epsilon)$ if $\Delta_1(\mu)\Delta_2(\mu) < 0$. Since $\Delta_2(\mu_0) \neq 0$, $\Delta_2(\mu)$ has always a determined sign. Therefore, by $\Delta_1(\mu_0) \neq 0$ we can choose μ such that $\Delta_1(\mu)$ has the appropriate sign. ■

Remark 3.2. The neighbourhood of the weak centers of order one that have a critical period which bifurcates from these centers forms an infinitesimal band in one side of the curve $\Delta_1(\mu) = 0$. Since for weak centers of order one $\Delta_2(\mu)$ has a fixed sign, the necessary condition to bifurcate a critical period is that $\Delta_1(\mu)\Delta_2(\mu) < 0$ as we saw in the proof. This implies that the neighbourhood of each weak center of order one μ_0 has to lie in one side of the curve $\Delta_1(\mu) = 0$. Moreover, the side changes when the curve $\Delta_1(\mu) = 0$ crosses an isochronous center since then $\Delta_2(\mu) = 0$ also and changes of sign beyond the isochronous (see Figure 3.2 and 3.3). Furthermore, we are able to know if the critical period is a maximum period or a minimum one depending on the sign of Δ_2, since $T''(h) = 2\Delta_2 + o(h)$ for h close to 0.

With both information, we can say that in the right part of $\Delta_1 = 0$ there is a close band that starts in $p - q = 0$ in the left side of the curve where there is, for each point on the band, a minimum period that have bifurcated from the center. The band changes its side with respect to $\Delta_1 = 0$ when it crosses the isochronous and the minimum periods becomes maximum periods. The same occurs with the other part of $\Delta_1 = 0$, starting from $p - q = 0$ in the left side of the curve with maximum periods and changing the side and the character (maximum or minimum) when the band crosses an isochronous (See Figure 3.2). □

This study is only valid for points that are not isochronous. In the case of bifurcations from the center in isochronous centers the study is in general more complicated because all period constants vanish. However, in this case we have the following result about it.

Proposition 3.3. *Let X_μ be the system described in* (1.15)*. There are at most one critical period that bifurcates from the center at the isochronous centers. Moreover, the upper-bound is taken.*

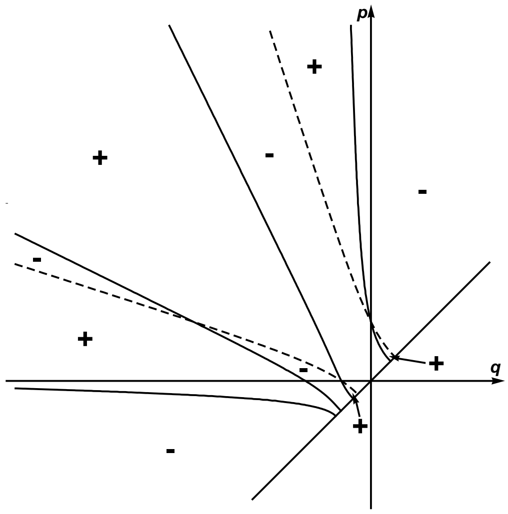


Figure 3.3: Curves $\Delta_1 = 0$ (dashed) and $\Delta_2 = 0$ with the sign of h_0 in each region.

This result uses the following Theorem of Chicone-Jacobs in [3] that give us an upper-bound of the number of critical periods that can bifurcate from the center in case of isochronous centers.

Theorem 3.2. *Let X_μ a family of analytic planar systems with a non-degenerate center at the origin for all $\mu \in \mathscr{R}$. If X_μ has an isochronous center at the origin corresponding to the parameter value μ_* and if all Taylor coefficients of the function $T'(s)$ are in the ideal $(\Delta_1, \Delta_2, \ldots, \Delta_{k+1})$ over $\mathbb{R}\{q,p\}_{\mu_*}$, the ring of convergent power series at μ_*, then there are at most k local critical periods which bifurcate from the isochronous center at μ_*. Moreover, if $\Delta_1, \Delta_2, \ldots, \Delta_k$ are independent with respect to Δ_{k+1} at μ_*, then exactly n local critical periods bifurcate from the center at the parameter value μ_* for each $n \leq k$.*

We will also need the following result about ideals which is deduced from Theorem A.1 of [3].

Definition 13. The *algebraic variety* defined by an ideal I over the ring $\mathbb{R}[x_1, \ldots, x_n]$ is

$$V(I) = \{z \in \mathbb{C}^n \ : \ f(z) = 0 \text{ for all } f \in I\}.$$

$\triangle$

Theorem 3.3. *Let the ideal $I = (f_1, \ldots, f_r)$ over $\mathbb{C}[x]$ be zero-dimensional. Suppose that all the points $a \in V(I)$ are such that rank$(\nabla f_1(a), \nabla f_2(a), \ldots, \nabla f_r(a))$ is maximal. Then in order that $f \in I$, it is necessary and sufficient that the following two conditions be satisfied:*

(i) $f(V(I)) = 0$,

(ii) $\nabla f(a) \in span\{\nabla f_1(a), \nabla f_2(a), \ldots, \nabla f_r(a)\},\ a \in V(I)$.

Proof of Proposition 3.3. As we saw, conditions $\Delta_1(q,p) = 0$ and $\Delta_2(q,p) = 0$ determine isochronous center in $\mathscr{R}$. Let us consider the ideal generated by the first three period constants

$$I = (\Delta_1(q,p), \Delta_2(q,p), \Delta_3(q,p)). \tag{3.11}$$

The variety $V(I) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$ is given by the points described in (3.7), which is a zero-dimensional variety, so I is zero-dimensional over $\mathbb{R}[q,p] \subset \mathbb{C}[q,p]$. Moreover, the Jacobian matrix $(\nabla\Delta_1(\mu_i), \nabla\Delta_2(\mu_i))$ in each point $\mu_i \in V(I)$ has maximal rank because

$$
\begin{aligned}
\det(\mathrm{J}(\mu_1)) &= \det\begin{pmatrix} -6 & -18 \\ 1152 & 2304 \end{pmatrix} = 6912 \neq 0,\\
\det(\mathrm{J}(\mu_2)) &= \det\begin{pmatrix} -3 & -9/2 \\ -36 & -18 \end{pmatrix} = -108 \neq 0,\\
\det(\mathrm{J}(\mu_3)) &= \det\begin{pmatrix} 6 & 3 \\ 576 & 144 \end{pmatrix} = -864 \neq 0,
\end{aligned}
$$

and the matrices $\mathrm{J}(\mu_i)$ with $i = 4,5,6$ are one of the previous ones permuting columns, then $\{\nabla\Delta_1(\mu_i), \nabla\Delta_2(\mu_i), \nabla\Delta_3(\mu_i)\}$ generates all the vector space $\mathbb{R}^2[q,p]$ for each μ_i. In fact, it is enough with Δ_1 and Δ_2. Therefore, condition (ii) of Theorem 3.3 is satisfied for any f. Therefore, the only condition necessary and sufficient for f to be in the ideal I is that $f(V(I)) = \{0\}$.

Let Δ_k the kth period constant for $k \geq 4$. Since $V(I)$ are the isochronous centers, Δ_k vanish in $V(I)$ so by Theorem 3.3, $\Delta_k \in I$ for all $k \geq 4$.

Moreover, since $\Delta_3(\mu_7) \neq 0$, $\Delta_3(q,p) \notin (\Delta_1(q,p), \Delta_2(q,p))$ over $\mathbb{R}[q,p]$. However, $(3p^2 + 3q^2 + 2)\Delta_3 \in I$ over $\mathbb{R}[q,p]$, so we can localize the ring in each isochronous μ_i, $i = 1,2,3$ such that

$$
\Delta_3 = \frac{m_1}{3p^2 + 3q^2 + 2}\Delta_1 + \frac{m_2}{3p^2 + 3q^2 + 2}\Delta_2
$$

with $m_1, m_2 \in \mathbb{R}[q,p]$. The functions $\dfrac{m_1}{3p^2 + 3q^2 + 2}$ and $\dfrac{m_2}{3p^2 + 3q^2 + 2}$ are particularly convergent power series at μ_i for $i = 1,2,3$ since the polynomial $3p^2 + 3q^2 + 2$ does not vanish at the isochronous. Therefore, $\Delta_3 \in (\Delta_1(q,p), \Delta_2(q,p))$ over $\mathbb{R}\{q,p\}_{\mu_i}$. Then, by Theorem 3.2 there are at most one critical period that bifurcates from each isochronous center.

Moreover, as $\Delta_1(q,p)$ and $\Delta_2(q,p)$ are independent in each μ_i, then exactly n local critical periods bifurcate from the center at the parameter value μ_i for each $n \leq k$. ■

3.3 Bifurcations from the period annulus

The aim of this section is to study the appearance of critical periods when we move μ close to the isochronous centers in the space of parameters. We will fix $\mu = \mu_i$ for $i = 1,2,3$ on of the isochronous centers of the system X_μ (see Theorem 3.1) and we will consider an analytic perturbation

$$
Z^i_\epsilon := X_{\mu_i + \psi(\epsilon)} \tag{3.12}
$$

with $\psi(\epsilon) = (\psi_1(\epsilon), \psi_2(\epsilon)) = \left(\sum_{i=1}^{\infty} a_i\epsilon^i, \sum_{i=1}^{\infty} b_i\epsilon^i\right)$ for ϵ small enough in order to ensure that X_ϵ^i has a center at the origin. The perturbed vector field is given by

$$Z_\epsilon^i = \begin{cases} \dot{x} = -y \\ \dot{y} = (x+1)^{p_i+\psi_1(\epsilon)} - (x+1)^{q_i+\psi_2(\epsilon)} \end{cases}. \tag{3.13}$$

Here $\{Z_\epsilon^i\}$ is an analytic family of planar vector fields with a center at the origin for all $\epsilon \approx 0$ and for $\epsilon = 0$ the center is an isochronous for each $i = 1, 2, 3$. Let U^i be a transversal vector field of $Z_0^i = X_{\mu_i}$. Let $q \in \mathscr{P}$ be fixed and consider $\varphi(s, q)$ the solution of U^i such that $\varphi(0, q) = q$. Then, we parametrize the period function by the transversal section defined by this solution. That is, we define $T(s;\epsilon)$ as the period of the periodic orbit of Z_ϵ^i passing through $\xi(s) := \varphi(s;q)$, where $s \in I$ and I is the maximal interval of definition of the solution $\varphi(s;q)$.

By Theorem 1.1, since Z_0^i have isochronous centers at the origin for $i = 1, 2, 3$, there exists a commutator U^i such that $[Z_0^i, U^i] = 0$ in a neighbourhood of the origin. However, in these cases the commutator will be global defined in all the period annulus $\mathscr{P}$, so we will parametrize the periodic orbit assuming U^i be the commutator of X_0^i.

Definition 14. We say that k *critical periods bifucate from* $\mathscr{P}$ in the family $\{Z_\epsilon\}$ as $\epsilon \to 0$ if there exist k functions $s_i(\epsilon)$, continuous in a neighbourhood of $\epsilon = 0$, and such that $T'(s_i(\epsilon);\epsilon) \equiv 0$ and $s_i(0) = s_i^* \in I$ for each $i = 1, 2, \ldots, k$. △

Remark 3.3. Since $T(s;\epsilon)$ is analytic for $\epsilon \approx 0$, we can consider its Taylor development at $\epsilon = 0$,

$$T(s;\epsilon) = T_0 + \sum_{i=1}^{+\infty} T_i(s)\epsilon^i. \tag{3.14}$$

Notice that T_0 is constant because $X_{\mu_i,0}$ is an isochronous. Then, if the center is not isochronous for $\epsilon \neq 0$, there exists $l \geq 1$ such that

$$T'(s;\epsilon) = T_l'(s)\epsilon^l + o(\epsilon^l),$$

where T_l' is not identically zero and the remainder is uniform in s on each compact subinterval of I. Then, applying the Weierstrass Preparation Theorem, the number of zeros of $T_l'(s)$ for $s \in I$, counted by multiplicities, provides an upper bound for k. A lower bound for k is given by the number of simple zeros of $T_l'(s)$ in I by using the Implicit Function Theorem. □

In the case of linear perturbations, Gasull and Yu [7] give an explicit formula for the first order term in ϵ, $T_1'(s)$, of the derivative of the period function $T(s;\epsilon)$. A more general result from Grau-Villadelprat [9] based on the one given by Gasull-Yu gives a formula to compute $T_l'(s)$ assuming that all the previous ones vanish. We will use that general result to prove that at most one critical period bifurcates from the periodic orbits of the above isochronous systems when we perturb them through $\psi(\epsilon)$ inside the parameter space.

Remark 3.4. The number of critical periods does not depend on the parametrization used to describe the period function. If $T(s;\epsilon)$ and $\widehat{T}(s;\epsilon)$ are two parametrization, then there exists a diffeomorphism $\eta : \widehat{I} \mapsto I$ such that $T(\eta(s);\epsilon) = \widehat{T}(s;\epsilon)$. Therefore, $\widehat{T}_l'(s) = \eta'(s)T_l'(\eta(s))$. Moreover, if the diffeomorphism preserves orientation, the character of the critical period also does not changes. □

The result about the formula for the first order term in ϵ for the period function in case of linear perturbations of Gasull and Yu is **Theorem 2** in [7]. Here we will proof a particular case when the transversal field is a commutator defined in all the period annulus.

Theorem 3.4. *Let X be a smooth planar vector field having an isochronous center of period T_0 at $p \in \mathcal{D}$. Let U be a commutator of X in $\mathcal{D} \setminus \{p\}$ such that $[X, U] = 0$.*

Consider the family of vector fields $X + \epsilon Y$ and assume that for $|\epsilon|$ small enough all of them have a center at p. Write $Y = aX + bU$ for some scalar functions a and b. Fix a point $q \in \mathcal{D} \setminus \{p\}$, and let $\boldsymbol{y}(s; q)$ denote the flow of U satisfying $\boldsymbol{y}(0; q) = q$ and let $\boldsymbol{x}(t, s) = \boldsymbol{x}(t; \boldsymbol{y}(s; q))$ be a parametrization of the periodic orbits of X. Then:

(i) The period function $T(s, \epsilon)$ of the periodic orbit $\boldsymbol{x}_\epsilon(t; \boldsymbol{y}(s, q))$ of $X + \epsilon Y$ is

$$T(s, \epsilon) = T_0 + \epsilon T_1(s) + O(\epsilon^2)$$

where

$$T_1'(s) = -\int_0^{T_0} \nabla a(\boldsymbol{x}(t, s)) \cdot U(\boldsymbol{x}(t, s))dt. \tag{3.15}$$

(ii) If s^* is a simple zero if $T_1'(s)$ then for $|\epsilon|$ small enough there is exactly one critical period of $X + \epsilon Y$ corresponding to a value of s that tends to s^* as ϵ tends to zero.

In order to give the proof of the theorem above, firstly we need to prove another result. This result is an extension of **Theorem 1** of [6] and it is **Theorem 4** of [7].

Theorem 3.5. *Assume that a vector field X has a center at p with period annulus $\mathscr{P}$. Take any vector field $U \in \mathcal{C}^1(\mathscr{P} \cup \{p\})$ transversal to X in $\mathscr{P}$. Let $\alpha, \beta \in \mathcal{C}^1(\mathscr{P})$ such that $[X, U] = \alpha X + \beta U$. Fix a point $q \in \mathscr{P}$, and let $\boldsymbol{y}(s; q)$ denote the flow of U satisfying $\boldsymbol{y}(0; q) = q$ and let $\boldsymbol{x}(t, s) := \boldsymbol{x}(t; \boldsymbol{y}(s, q))$ be a parametrization of the periodic orbits of X. Then*

$$T'(s) = \int_0^{T(s)} \alpha(\boldsymbol{x}(t, s))e^{-\int_0^t \beta(\boldsymbol{x}(\tau,s))d\tau}dt, \tag{3.16}$$

where $T(s)$ is the period of $\boldsymbol{x}(t, s)$. It also holds that $\int_0^{T(s)} \beta(\boldsymbol{x}(t, s))dt \equiv 0$.

Proof. Let $\gamma = \{\mathbf{x}(t) := \mathbf{x}(t; q) : \mathbf{x}(0) = \mathbf{x}(T) = q\}$ be a periodic orbit of X with period T. Take an arc Σ of the orbit $\mathbf{y}(s) := \mathbf{y}(s; q)$ of the vector field U transversal to X in $\mathscr{P}$ such that $\mathbf{y}(0) = q$. Since U is transversal to X and $\mathbf{y}(0) = \mathbf{x}(0) = q$, Σ is a transversal section of the center of X at $\mathbf{x}(0) = q$. Hence we define the return map on Σ as

$$\pi : \Sigma_0 \subset \Sigma \to \Sigma,$$

and we have

$$\pi(\mathbf{y}(s)) = \mathbf{x}(T + \tau(s), \mathbf{y}(s)),$$

where $T(s) := T + \tau(s)$ is the period of the closed orbit of X passing through $y(s)$.

Consider the variational equation of X along the periodic orbit $x(t)$,

$$\frac{d\eta}{dt} = DX(\mathbf{x}(t))\eta. \tag{3.17}$$

Let us see that the following function

$$\eta(t) = U(\mathbf{x}(t))e^{-\int_0^t \beta(\mathbf{x}(\tau))d\tau} - X(\mathbf{x}(t))\int_0^t \alpha(\mathbf{x}(u))e^{-\int_0^u \beta(\mathbf{x}(\tau))d\tau}du \tag{3.18}$$

is one of the solutions of (3.17). Since $[X, U] = DU \cdot X - DX \cdot U = \alpha X + \beta U$,

$$\begin{aligned}
\frac{d}{dt}\eta(t) = & \left[\left(DU \cdot \frac{d\mathbf{x}(t)}{dt}\right)(\mathbf{x}(t)) - (U \cdot \beta)(\mathbf{x}(t))\right] e^{-\int_0^t \beta(\mathbf{x}(\tau))d\tau} \\
& - \left[\left(DX \cdot \frac{d\mathbf{x}(t)}{dt}\right)(\mathbf{x}(t)) \int_0^t \alpha(\mathbf{x}(u))e^{-\int_0^u \beta(\mathbf{x}(\tau))d\tau}du \right. \\
& \left. + (X \cdot \alpha)(\mathbf{x}(t))e^{-\int_0^t \beta(\mathbf{x}(\tau))d\tau}\right] e^{-\int_0^t \beta(\mathbf{x}(\tau))d\tau} \\
& = [(DU \cdot X) - (U \cdot \beta) - (X \cdot \alpha)](\mathbf{x}(t))e^{-\int_0^t \beta(\mathbf{x}(\tau))d\tau} \\
& - (DX \cdot X)(\mathbf{x}(t)) \int_0^t \alpha(\mathbf{x}(u))e^{-\int_0^u \beta(\mathbf{x}(\tau))d\tau}du \\
& = (DX \cdot U)(\mathbf{x}(t))e^{-\int_0^t \beta(\mathbf{x}(\tau))d\tau} - (DX \cdot X)(\mathbf{x}(t)) \int_0^t \alpha(\mathbf{x}(u))e^{-\int_0^u \beta(\mathbf{x}(\tau))d\tau}du \\
& = DX(\mathbf{x}(t))\eta(t).
\end{aligned}$$

From (3.18) it follows that

$$\begin{aligned}
\eta(0) &= U(q), \\
\eta(T) &= U(q)e^{-\int_0^T \beta(\mathbf{x}(\tau))d\tau} - X(q)\int_0^T \alpha(\mathbf{x}(u))e^{-\int_0^u \beta(\mathbf{x}(\tau))d\tau}du.
\end{aligned}$$

By the theorem of derivation with respect to initial condition, $D_2\mathbf{x}(t, q)$ is the fundamental matrix $\Phi(t)$ of the equation (3.17), therefore the monodromy matrix is $\Phi(T) = D_2\mathbf{x}(T, q)$. Since $X(q)$ and $U(q)$ are orthogonal, we can write the monodromy matrix in the basis $\{X(q), U(q)\}$. The first component is

$$D_2\mathbf{x}(T, q) \cdot X(q) = \frac{\partial}{\partial t}\mathbf{x}(T, \mathbf{x}(t, q))\bigg|_{t=0} = \frac{\partial}{\partial t}\mathbf{x}(T + t, q)\bigg|_{t=0} = \frac{\partial}{\partial t}\mathbf{x}(t, q)\bigg|_{t=0} = X(q).$$

On the other hand, since $\pi(\mathbf{y}(s)) = \mathbf{x}(T(s), \mathbf{y}(s))$,

$$\pi'(\mathbf{y}(0))\mathbf{y}'(0) = \frac{\partial}{\partial t}(\pi \circ \mathbf{y}(s))\bigg|_{s=0} = D_1\mathbf{x}(T, \mathbf{y}(0))T'(0) + D_2\mathbf{x}(T, \mathbf{y}(0))\mathbf{y}'(0).$$

By definition, $\mathbf{y}(0) = q$ and $\mathbf{y}'(0) = U(q)$, so the second component of the monodromy matrix is

$$D_2\mathbf{x}(T, \mathbf{y}(0))U(q) = \pi'(q)U(q) - T'(0)X(q).$$

Moreover, since the system X has a center, $\det\Phi(T) = 1$ by the Liouville's Theorem, and therefore $\pi'(q) = 1$ giving

$$\Phi(T) = \begin{pmatrix} 1 & -T'(0) \\ 0 & \pi'(q) \end{pmatrix} = \begin{pmatrix} 1 & -T'(0) \\ 0 & 1 \end{pmatrix}.$$

Hence in this basis we have

$$\begin{pmatrix} -\int_0^T \alpha(\mathbf{x}(u))e^{-\int_0^u \beta(\mathbf{x}(\tau))d\tau}du \\ e^{-\int_0^T \beta(\mathbf{x}(\tau))d\tau} \end{pmatrix} = \begin{pmatrix} 1 & -T'(0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, $T'(0) = \int_0^T \alpha(\mathbf{x}(u))e^{-\int_0^u \beta(\mathbf{x}(\tau))d\tau}du$ and $\int_0^T \beta(\mathbf{x}(\tau))d\tau = 0$. Since this is true for a general q, then $T'(s) = \int_0^{T(s)} \alpha(\mathbf{x}(u))e^{-\int_0^u \beta(\mathbf{x}(\tau))d\tau}du$ and $\int_0^{T(s)} \beta(\mathbf{x}(\tau))d\tau = 0$ where $T(s)$ is the period of $\mathbf{x}(t,s)$ such that $\mathbf{x}(0,s) = q$. ∎

Proof of Theorem 3.4. From the equality $Y = aX + bU$ we have

$$X = \frac{1}{1+\epsilon a}(X + \epsilon Y - \epsilon bU).$$

Hence

$$[X + \epsilon Y, U] = -\epsilon U(a)X - \epsilon U(b)U = \tilde{\alpha}(X + \epsilon Y) + \tilde{\beta}U,$$

where

$$\tilde{\alpha} = -\epsilon\frac{U(a)}{1+\epsilon a}, \quad \tilde{\beta} = -\epsilon U(b) + \epsilon^2\frac{bU(a)}{1+\epsilon a}.$$

By Theorem 3.5, we have

$$\begin{aligned} \frac{\partial T(s,\epsilon)}{\partial s} &= \int_0^{T_\epsilon(s)} \tilde{\alpha}(\mathbf{x}_\epsilon(t,s))e^{-\int_0^t \tilde{\beta}(\mathbf{x}_\epsilon(\tau,s))d\tau}dt \\ &= -\epsilon\int_0^{T_\epsilon(s)} \frac{U(a)}{1+a\epsilon}(\mathbf{x}_\epsilon(t,s))\left(1 + \epsilon\int_0^t U(b)(\mathbf{x}_\epsilon(\tau,s))d\tau\right)dt\bigg|_{\epsilon=0} + O(\epsilon^2) \\ &= -\epsilon\int_0^{T_0} U(a)(\mathbf{x}(t,s))dt + O(\epsilon^2) = \epsilon T_1'(s) + O(\epsilon^2), \end{aligned}$$

proving the theorem.

The proof of the second part is an immediate consequence of the Implicit Function Theorem. ∎

Let U be a local commutator of Z_ϵ at the origin. It follows that there exist two analytic functions α and β such that

$$Y_\epsilon = Z_\epsilon - Z_0 = \alpha Z_0 + \beta U, \tag{3.19}$$

where Y_ϵ is the vector field corresponding to the perturbation part of Z_ϵ. In fact, it is clear that

$$\alpha = \frac{Y_\epsilon \wedge U}{Z_0 \wedge U} \quad \text{and} \quad \beta = \frac{Z_0 \wedge Y_\epsilon}{Z_0 \wedge U}. \tag{3.20}$$

Let us denote by $j^k(Z_\epsilon) = \sum_{i=0}^{k} Z^i\epsilon^i$ the truncated Taylor development of Z_ϵ in $\epsilon = 0$ at ϵ^k. Particularly, $j^0(Z_\epsilon) = Z_0$. The following result is the one of Grau-Villadelprat about analytic perturbations.

Theorem 3.6. *Let us consider the period function $T(s;\epsilon)$ of the system* (3.13)*. Assume that, for some $k \in \mathbb{N}$, $j^k(Z_\epsilon)$ has an isochronous center at the origin for all $\epsilon \approx 0$. Then $T_0' \equiv T_1' \equiv \cdots \equiv T_k' \equiv 0$ and*

$$T'_{k+1}(s) = -\int_0^{T_0} U_\mu(\alpha_{k+1})|_{(x,y)=\varphi(t;s)}\, dt \quad \text{for all } s \in I, \tag{3.21}$$

where $\varphi(t;s)$ is the solution of Z_0 with $\varphi(0;s) = \xi(s)$ and α_{k+1} is the $(k+1)$th therm of the Taylor development in $\epsilon = 0$ of α.

Remark 3.5. The authors in [9], as they explain in Remark 3.3, did not know whether the characterization of isochronous as local linearizable centers was true for families of vector fields. This is the reason why in the statement of Theorem 3.2 [9], instead of assuming that $j^k(X_\epsilon)$ is an isochronous for all ϵ, they require the existence of an analytic family of diffeomorphisms $\{\Phi_\epsilon\}$ linearizing $j^k(X_\epsilon)$. Remark 1.3 allows us use isochronicity, that is easier to verify than linearizability in general. □

The following lemma provides the necessary information in order to apply the above results to the isochronous of Theorem 3.1. It gives the commutator and the first integral of each isochronous center.

Lemma 3.2. *(I_1^*) The vector field $X_{\mu_1} = -y\partial_x + (1 + x - (1+x)^{-3})\partial_y$ is conjugated to the system*

$$Z_0^1 = (-y + 2xy)\partial_x + (x - x^2 + y^2)\partial_y$$

by the map $\phi : (x,y) \mapsto ((1-2x)^{-\frac{1}{2}} - 1, 2y(1-2x)^{-\frac{1}{2}})$, has first integral

$$H(x,y) = \frac{x^2+y^2}{1-2x}$$

and commutator $U^1 = (x - x^2 + y^2)\partial_x + (y - 2xy)\partial_y$.

Moreover, in this coordinates,

$$Z_\epsilon^1 = (-y + 2xy)\partial_x + \frac{1}{4}((1-2x)^{-\frac{\psi_2(\epsilon)}{2}} - (1-2x)^{2-\frac{\psi_1(\epsilon)}{2}} + 4y^2)\partial_y.$$

(I_2^*) The vector field

$$Z_0^2 = -y\partial_x + \left(1 - \frac{1}{\sqrt{1+x}}\right)\partial_y$$

has first integral

$$H(x,y) = 2 + x - 2\sqrt{1+x} + \frac{y^2}{2}$$

and commutator $U^2 = (2 + 2x - 2\sqrt{1+x} + y^2)\partial_x + \left(\frac{y}{\sqrt{1+x}}\right)\partial_y$.

(I_3^*) The vector field

$$Z_0^3 = -y\partial_x + x\partial_y$$

has first integral $H(x,y) = \frac{x^2+y^2}{2}$ and commutator $U^3 = x\partial_x + y\partial_y$.

Proof. (I_1^*) The vector field X_{μ_1} is given by

$$Z_0^1 = -y\partial_x + \left(1 + x - \frac{1}{(1+x)^3}\right)\partial_y \tag{3.22}$$

and the perturbed system is

$$Z_\epsilon^1 = -y\partial_x + \left((1+x)^{1+\psi_2(\epsilon)} - (1+x)^{-3+\psi_1(\epsilon)}\right)\partial_y,$$

that comes from (3.13), with $\psi(\epsilon) = (\psi_1(\epsilon), \psi_2(\epsilon))$. The system (3.22) is a Quadratic Loud system through the change of variables

$$\begin{cases} x = (1-2u)^{-\frac{1}{2}} - 1 \\ y = 2v(1-2u)^{-\frac{1}{2}} \end{cases}$$

that can be found in [1] and [7]. In these variables, the system becomes

$$Z_0^1 = (-v + 2uv)\,\partial_u + \left(u - u^2 + v^2\right)\partial_v$$

which is $(\tilde{B}, \tilde{D}, \tilde{F}) = (2, -1, 1)$ in the notation of [7] and System S_1 in [1]. Since the name of the variables will be not important, we replace the name (u, v) by (x, y). With this coordinates, the perturbed vector field becomes

$$Z_\epsilon^1 = (-y + 2xy)\partial_x + \frac{1}{4}\left((1-2x)^{-\frac{\psi_2(\epsilon)}{2}} - (1-2x)^{2-\frac{\psi_1(\epsilon)}{2}} + 4y^2\right)\partial_y.$$

The commutator and the first integral are given in [7],

$$U^1 = (x - x^2 + y^2)\partial_x + (y - 2xy)\partial_y$$

and

$$H(x, y) = \frac{x^2 + y^2}{1 - 2x}.$$

(I_2^*) The vector field X_{μ_2} is given by

$$Z_0^2 = -y\partial_x + \left(1 - \frac{1}{\sqrt{1+x}}\right)\partial_y.$$

The first integral is easy to compute since $H(x, y) = y^2/2 + V(x)$, so

$$H(x, y) = 2 + x - 2\sqrt{1+x} + \frac{y^2}{2}.$$

In order to obtain a commutator, we will use Proposition 1.2. In this case, the involution associated to the potential is $\sigma(x) = 4 + x - 4\sqrt{1+x}$. Therefore, $h(x) = 2\left(\sqrt{1+x} - 1\right)$ and the inverse is $h^{-1}(x) = \frac{1}{4}x^2 + x$ so the commutator given by Proposition 1.2 is

$$U^2 = (2 + 2x - 2\sqrt{1+x} + y^2)\partial_x + \left(\frac{y}{\sqrt{1+x}}\right)\partial_y.$$

(I_3^*) The vector field X_{μ_3} is given by

$$Z_0^3 = -y\partial_x + x\partial_y,$$

which is the linear center. Therefore, it is immediate that the first integral is

$$H(x,y) = \frac{x^2+y^2}{2}$$

and the commutator is

$$U^3 = x\partial_x + y\partial_y.$$

■

Remark 3.6. As we mentioned before, the three commutators are well defined in the period annulus of their respective centers. Therefore, we will use them as a parametrization of the periodic orbits. □

Remark 3.7. There are two reasons for the change of variable of (I_1^*). The first one is because we have more information about the system in this form, for example its commutator is given in [7] and it is essentially the orthogonal vector field. The second one is that the computations that follow become easier than if we work with the potential system. □

The main result of this section will follow once we prove the next theorem.

Theorem 3.7. *Let Z_0 be one of the three isochronous centers in Lemma 3.2 and let U be the corresponding commutator. Consider the perturbation $Y_\epsilon = Z_\epsilon - Z_0$ where*

$$Z_\epsilon = X_{\mu_i+\psi(\epsilon)}$$

for each $i = 1,2,3$ described in (3.13) and where $\psi(\epsilon) = (\psi_1(\epsilon), \psi_2(\epsilon))$ is an analytic curve in a neighbourhood of $\epsilon = 0$ with $\psi_1(0) = \psi_2(0) = 0$. Let $\xi : I \to \mathbb{R}^2$ be a transversal section to $\mathscr{P}$ given by a solution of U and let $T(s;\epsilon)$ be the period of the periodic orbit of Z_ϵ passing through $\xi(s) \in \mathscr{P}$. If $T_0' \equiv T_1' \equiv \cdots \equiv T_k' \equiv 0$ and T_{k+1}' is not identically zero, then T_{k+1}' has at most one zero for $s \in I$ and it is reached.

In order to prove this result some definitions and lemmas are needed.

Definition 15. Let $f_0, f_1, \dots f_{n-1}$ be analytic function on an open interval L of $\mathbb{R}$.

(a) $\{f_0, f_1, \dots f_{n-1}\}$ is a *Chebyshev system* (for short, a T-system) on L if any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{n-1} f_{n-1}(x)$$

has at most $n-1$ isolated zeros on L.

(b) $(f_0, f_1, \dots f_{n-1})$ is a *complete Chebyshev system* (for short, a CT-system) on L if $(f_0, f_1, \dots, f_{k-1})$ is a T-system for all $k = 1, 2, \dots, n$.

(c) $(f_0, f_1, \dots f_{n-1})$ is a *extended complete Chebyshev system* (for short, a ECT-system) on L if, for all $k = 1, 2, \dots n$, any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \dots + \alpha_{k-1} f_{k-1}(x)$$

has at most $k-1$ isolated zeros on L counted with multiplicities.

(Let us mention that, in these abbreviations, "T" stands for Tchebycheff, which in some sources is the transcription of the Russian name Chebyshev).

$\triangle$

Remark 3.8. The use of "$\{\}$" instead of "()" in the first definition is because the first one does not need an order of the functions. However, the other two definitions need of an order because are defined by truncations. $\square$

It is clear that if $(f_0, f_1, \dots, f_{n-1})$ is an ECT-system on L, then $(f_0, f_1, \dots, f_{n-1})$ is a CT-system on L. However, the reverse implication is not true.

Definition 16. Let $f_0, f_1, \dots, f_{k-1}$ be analytic function on an open interval L of $\mathbb{R}$. The *continuous Wronskian* of $(f_0, f_1, \dots, f_{k-1})$ at $x \in L$ is

$$W[f_0, f_1, \dots, f_{k-1}](x) = \det\left(f_j^{(i)}(x)\right)_{0 \le i,j \le k-1} = \begin{vmatrix} f_0(x) & \dots & f_{k-1}(x) \\ f_0'(x) & \dots & f_{k-1}'(x) \\ & \vdots & \\ f_0^{(k-1)}(x) & \dots & f_{k-1}^{(k-1)}(x) \end{vmatrix}.$$

The *discrete Wronskian* of $(f_0, f_1, \dots, f_{k-1})$ at $(x_0, x_1, \dots, x_{k-1}) \in L^k$ is

$$\begin{aligned} D[f_0, f_1, \dots, f_{k-1}](x_0, x_1, \dots, x_{k-1}) &= \det\left(f_j(x_i)\right)_{0 \le i,j \le k-1} \\ &= \begin{vmatrix} f_0(x_0) & \dots & f_{k-1}(x_0) \\ f_0(x_1) & \dots & f_{k-1}(x_1) \\ & \vdots & \\ f_0(x_{k-1}) & \dots & f_{k-1}(x_{k-1}) \end{vmatrix}. \end{aligned}$$

For the sake of shortness, given any "letter" x and $k \in \mathbb{N}$ we use the notation

$$x_0, x_1, \dots, x_{k-1} = \mathbf{x_k}$$

Accordingly, we write

$$W[f_0, f_1, \dots, f_{k-1}] = W[\mathbf{f_k}](x)$$

and

$$D[f_0, f_1, \dots, f_{k-1}](x_0, x_1, \dots, x_{k-1}) = W[\mathbf{f_k}](\mathbf{x_k})$$

for the continuous and discrete Wronskian, respectively. $\triangle$

The following result is well known (see [8] and references therein).

Lemma 3.3. *The following equivalences hold:*

(a) $(f_0, f_1, \ldots, f_{k-1})$ *is a CT-system on* L *if, and only if, for each* $k = 1, 2, \ldots n$,

$$D[\boldsymbol{f_k}](\boldsymbol{x_k}) \neq 0 \text{ for all } \boldsymbol{x_k} \in L^k \text{ such that } x_i \neq x_j \text{ for } i \neq j.$$

(b) $(f_0, f_1, \ldots, f_{k-1})$ *is an ECT-system on* L *if, and only if, for each* $k = 1, 2, \ldots n$,

$$W[\boldsymbol{f_k}](x) \neq 0 \text{ for all } x \in L.$$

The next lemma establishes a formula that allows us to write the integrand of an Abelian integral as another which is suitable to apply the results.

Lemma 3.4. *Let* γ_h *be an oval inside the level curve* $\{A(x) + B(x)y^2 = h\}$, *and we consider a function* F *such that* F/A' *is analytic at* $x = 0$. *Then, for any* $k \in \mathbb{N}$,

$$\int_{\gamma_h} F(x)y^{k-2}dx = \int_{\gamma_h} G(x)y^k dx,$$

where $G(x) = \frac{2}{k}\left(\frac{BF}{A'}\right)'(x) - \left(\frac{B'F}{A'}\right)(x)$.

Proof. Let $(x, y) \in \gamma_h \subset \{A(x) + B(x)y^2 = h\}$ be a point on the oval onside the level curve of energy h, then $\frac{dy}{dx} = -\frac{A'(x)+B'(x)y^2}{2B(x)y}$ and accordingly,

$$\begin{aligned} d(g(x)y^k) &= g'(x)y^k dx + kg(x)y^{k-1}dy \\ &= \left(g'(x) - \frac{k}{2}\left(\frac{B'g}{B}\right)(x)\right)y^k dx - \frac{k}{2}\left(\frac{A'g}{B}\right)(x)y^{k-2}dx. \end{aligned}$$

We take $F(x) = \frac{k}{2}\left(\frac{A'g}{B}\right)(x)$ in the above equality and then $g(x) = \frac{2}{k}\left(\frac{FB}{A'}\right)(x)$ which is analytic at $x = 0$. Then, we use the fact that $\int_{\gamma_h} d(g(x)y^k) = 0$ since the integrand has no poles and the result follows. ■

Grau, Mañosas and Villadelprat in [8] give a criterion for a collection of Abelian integrals to have the Chebyshev property. This condition involves the functions in the integrand of the Abelian integrals in the way as the following result shows.

Definition 17. Let σ be an involution. We define the *balance* with respect to σ of a function g as

$$\mathscr{B}_\sigma(g)(x) := g(x) - g(\sigma(x)).$$

△

Theorem 3.8. *Let us consider the Abelian integrals*

$$I_i(h) = \int_{\gamma_h} f_i(x)y^{2s-1}dx, \; i = 0, 1, \ldots, n-1, \tag{3.23}$$

where, for each $h \in (0, h^*)$, γ_h *is the oval surrounding the origin inside the level curve* $\{A(x) + B(x)y^{2m} = h\}$. *Let* σ *be the involution associated to* A, *and we define*

$$l_i = \mathscr{B}_\sigma\left(\frac{f_i}{A'B^{\frac{2s-1}{2m}}}\right). \tag{3.24}$$

Then $(I_0, I_1, \ldots, I_{n-1})$ *is an ECT-system on* $(0, h^*)$ *if* $(l_0, l_1, \ldots, l_{n-1})$ *is a CT-system on* $(0, x_R)$ *and* $s > m(n-2)$.

Thanks to this result, the way of checking the Chebyshev condition is much more easier because in this sense the property to be Chebyshev is transfered from the integrands to the Abelian integrals.

Proposition 3.4. *Following the notation in the statement of Theorem 3.7, we write* $Y_\epsilon = Z_\epsilon - Z_0 = \alpha Z_0 + \beta U$ *and, setting* $\alpha = \sum_{i=0}^{\infty} \alpha_i \epsilon^i$*, we define*

$$R_k(s) = \int_0^{T_0} U(\alpha_k)|_{(x,y)=\varphi(t;s)}\, dt,$$

where $\varphi(t;s)$ *is the solution of* Z_0 *with* $\varphi(0;s) = \xi(s)$*. Let us assume that* $R_0 \equiv R_1 \equiv \cdots \equiv R_{k-1} \equiv 0$*. Then* $R_k(s) = b_k I_0(s) + a_k I_1(s)$*, where* (I_0, I_1) *is an ECT-system on* I*.*

Proof. We will prove the theorem by induction over $k \geq 1$. Let us first assume that $k = 1$.

As we said at the beginning of this section, $T(s;\epsilon)$ can be expressed as a series development at $\epsilon = 0$ as

$$T(s;\epsilon) = T_0 + \sum_{i=1}^{\infty} T_i(s)\epsilon^i,$$

where T_0 does not depend on s because of Z_0 is an isochronous. Therefore, $T_0' \equiv 0$. Moreover, $j^0(Z_\epsilon) = Z_0$ that has an isochronous center at the origin, so we are in the hypothesis for apply Theorem 3.6 to compute $T_{l+1}'(s)$ for $l = 0$:

$$T_1'(s) = -\int_0^{T_0} U(\alpha_1)|_{(x,y)=\varphi(t;s)}\, dt$$

for all $s \in I$ where $\varphi(t;s)$ is the solution of Z_0 with $\varphi(0;s) = \xi(s)$. Since $R_k(s) = -T_k'(s)$, we will prove verify the theorem for $T_1'(s)$.

Notice that we can write $T_k'(s)$ for all $k \geq 1$ as an Abelian integral, taking into account that the isochronous center Z_0 has a first integral H (see Lemma 3.2). Thus, if γ_s denotes the periodic orbit of Z_0 inside the energy level $\{H(x,y) = H(\xi(s))\}$ and $Z_0 = P_0\partial_x + Q_0\partial_y$, then we have

$$T_k'(s) = -\int_{\gamma_s} \frac{\nabla\alpha_k(x,y)\cdot U(x,y)}{P_0(x,y)} dx.$$

In fact, we can parametrize T_k' by the energy level of γ_s through using $\eta(s) := H(\xi(s))$ so

$$I^k(h) := R_k(\eta^{-1}(h)) = \int_{\gamma_h} \frac{\nabla\alpha_k(x,y)\cdot U(x,y)}{P_0(x,y)} dx,$$

where here γ_h is the oval inside the level curve $\{H(x,y) = h\}$. In the case $k = 1$, we have

$$I^1(h) = \int_{\gamma_h} \frac{\nabla\alpha_1(x,y)\cdot U(x,y)}{P_0(x,y)} dx.$$

Moreover, since $Y_\epsilon = Z_\epsilon - Z_0 = \alpha Z_0 + \beta U$, we have

$$\sum_{i=0}^{\infty} Y_i\epsilon^i = \left(\sum_{i=0}^{\infty} \alpha_i\epsilon^i\right) Z_0 + \left(\sum_{i=0}^{\infty} \beta_i\epsilon^i\right) U$$

so by collecting powers of ϵ we have

$$\alpha_1(x,y) = \frac{Y_1 \wedge U}{Z_0 \wedge U}.$$

Then, for each vector field of Lemma 3.2 we have:

(I_1^*) $Y_1 = \frac{1}{8}(a_1 - b_1 - 4a_1x + 4a_1x^2)\log(1-2x)\partial_y$ and therefore

$$\alpha_1(x,y) = \frac{(b_1 - a_1 + 4a_1x - 4a_1x^2)(-x + x^2 - y^2)\log(1-2x)}{8(x^2 - 2x^3 + x^4 + y^2 - 2xy^2 + 2x^2y^2 + y^4)}$$

Then, $I^1(h)$ can be written as a combination of Abelian Integrals

$$I^1(h) = b_1 I_0(h)dx + a_1 I_1(h)dx$$

where

$$I_0(h) = \int_{\gamma_h} \frac{-2(x-x^2+y^2)^2 + (1-2x)^2(x-1)x\log(1-2x) + (1-2x)^2y^2\log(1-2x)}{8(1-2x)^2(x^2+y^2)((x-1)^2+y^2)y} dx,$$

$$I_1(h) = \int_{\gamma_h} \frac{2(x-x^2+y^2)^2 - ((x-1)x + (1+12(x-1)x)y^2 - 4y^4)\log(1-2x)}{8(x^2+y^2)((x-1)^2+y^2)y} dx.$$

We will apply now Theorem 3.8 in order to prove that the system is an ECT-system. For this reason, we need to express $I_i(h)$ in the form

$$\int_{\gamma_h} f_i(x) y^s dx$$

at it will be useful to use the first integral of the vector field in order to see that in both integrands the denominator of the fraction decomposes in a product of a function of h, a function of x, and y. That is, $8((x-1)^2+y^2)(x^2+y^2)$ becomes $8h(h+1)(1-2x)^2$. That happens essentially because the denominator of α_1 is a first integral of the system. Moreover, using Lemma 3.4, the Abelian integral is written as

$$I^1(h) = \frac{1}{8h(h+1)}\left[b_1 \underbrace{\int_{\gamma_h} \frac{-16}{(1-2x)^4} y^3 dx}_{I_0(h)} + a_1 \underbrace{\int_{\gamma_h} \frac{-16}{3(1-2x)^2} y^3 dx}_{I_1(h)} \right].$$

The result of Grau, Mañosas, Villadelprat in [8], Theorem 3.8 of this work, tell us that if (l_0, l_1) is a CT-system then (I_0, I_1) is an ECT-system where $l_i(x) = \mathcal{B}_\sigma\left(\frac{f_i}{A'B^{3/2}}\right)(x)$ and $\mathcal{B}_\sigma$ is the balance with the involution associated to $A(x)$. This involution is given by $\sigma(x) = \frac{x}{2x-1}$ and the balanced integrands are

$$l_0(x) = -\frac{16(1-2x+4x^2)}{x\sqrt{1-2x}},$$
$$l_1(x) = -\frac{16\sqrt{1-2x}}{3x}.$$

We have to prove that the system (l_0, l_1) is a CT-system for x on the right side of the period annulus projection over the x-axis. In the main variables, this interval was $(0,\infty)$ that becomes in $(0,1/2)$ through the change of variables. It is easy to check that $l_0(x)$ has no roots in $(0,1/2)$ because $1-2x+4x^2$ has no real roots and then $l_0(x)$ is always negative in that interval. Then, we can divide by $l_0(x)$ and only is necessary to check that $(l_1/l_0)(x)$ is monotone. Dividing we have that

$$\frac{l_1}{l_0}(x) = \frac{1-2x}{3-6x+12x^2}$$

and we can compute the derivative easily, which is

$$\left(\frac{l_1}{l_0}\right)'(x) = \frac{8x(x-1)}{3(1-2x+4x^2)^2}$$

that has no roots in the interval $(0,1/2)$. Then, we proved that (l_0, l_1) is an ECT-system and, by the Theorem, (I_0, I_1) too.

(I_2^*) $Y_1 = \left(b_1 \log(1+x) - \frac{a_1 \log(1+x)}{\sqrt{1+x}}\right)\partial_y$ and therefore

$$\alpha_1 = \frac{(a_1 - b_1\sqrt{1+x})(-2-2x+2\sqrt{1+x}-y^2)\log(1+x)}{-4+4\sqrt{1+x}+2x(-2+\sqrt{1+x})+\sqrt{1+x}y^2}.$$

Using the same result as in the case before, $I_1'(h)$ can be written as

$$I^1(h) = \frac{1}{4\sqrt{2}h^2}\left(b_1 I_0(h) + a_1 I_1(h)\right)$$

where

$$I_0(h) = \int_{\gamma_h} -\frac{1+5\sqrt{1+x}}{5(1+x)^{3/2}} y^5 dx,$$
$$I_1(h) = \int_{\gamma_h} -\frac{4(2+2x+\sqrt{1+x})}{15(1+x)^{3/2}} y^5 dx.$$

The first integral of (I_2^*) is $H(x,y) = A(x)+B(x)y^2 = 2+x-2\sqrt{1+x}+\frac{1}{2}y^2$ and the involution associated to $A(x)$ is given by $\sigma(x) = 4+x-4\sqrt{1+x}$. By the balance associated to the involution, as in the case before, we can get another system (l_0, l_1) such that if it is a ECT-system at $(0, x_R) = (0,3)$ then (I_0, I_1) also is at $(0, h^*)$. Here,

$$l_i(x) = g_i(x+1) - g_i(\sigma(x)+1), \; i = 0,1$$

where

$$g_0(z) = \frac{-4\sqrt{2}\,(1+6\sqrt{z}+5z)}{5(z^2-z)},$$
$$g_1(z) = \frac{-16\sqrt{2}\,(3z+(2z+1)\sqrt{z})}{15z^2(z-1)}.$$

In order to prove that l_0 has no roots in the right side of the projection of the period annulus, we will see that $g_0(x+1)-g_0(\sigma(x)+1)$ has no roots. Notice that $1+6t+5t^2$ has no roots in $t>0$, so $g_0(z)$ does not vanish. Moreover, the denominator changes its sign in $z=1$ while the numerator not. Since for each $1<x+1<4$ we have exactly one $0<\sigma(x)+1<1$, it is not possible to have $g_0(x+1)=g_0(\sigma(x)+1)$ because one is positive and the other is negative.

Then, we divide by l_0 and we have to prove that $(l_1/l_0)(x)$ is monotone. In this case, we compute the derivative as follows

$$\begin{aligned} l_0^2\left(\frac{l_1}{l_0}\right)'(x) &= \left(g_1'(x+1) - g_1'(\sigma(x)+1)\sigma'(x)\right)\left(g_0(x+1) - g_0(\sigma(x)+1)\right) \\ &\quad - \left(g_0'(x+1) - g_0'(\sigma(x)+1)\sigma'(x)\right)\left(g_1(x+1) - g_1(\sigma(x)+1)\right) \end{aligned}$$

Then, we take the change of variable $z=+\sqrt{x+1}$ and as $\sigma'(x) = 1-\frac{2}{\sqrt{1+x}}$:

$$l_0^2\left(\frac{l_1}{l_0}\right)' = \frac{4096(1+8z-4z^3+z^4)}{75(z-2)^6(z-1)z^7}.$$

Then, we have to prove that this expression has no roots in the interval $(1,2)$ (which is the interval $(1,4)$ through the change of variables). It is easy to check that the 4 roots of this function are not inside the interval, so it is proved that (l_0,l_1) is an ECT-system and, therefore, also (I_0,I_1) is.

(I_3^*) $Y_1 = (b_1-a_1+b_1x)\log(1+x)\partial_y$ and therefore

$$\alpha_1 = \frac{x(b_1-a_1+b_1x)\log(1+x)}{x^2+y^2}.$$

In this case, the expression of $I^1(h)$ is

$$I^1(h) = b_1\int_{\gamma_h}\frac{x(\log(1+x)-x)}{y(x^2+y^2)}dx + a_1\int_{\gamma_h}\frac{x(x-(1+x)\log(1+x))}{(1+x)y(x^2+y^2)}dx.$$

Since $H(x,y)=\frac{1}{2}(x^2+y^2)=h$, then

$$I^1(h) = \frac{1}{2h}\left(b_1I_0(h)+a_1I_1(h)\right)$$

where

$$
\begin{aligned}
I_0(h) &= \int_{\gamma_h} x(\log(1+x) - x)y^{-1}dx \\
I_1(h) &= \int_{\gamma_h} \frac{x(x-(1+x)\log(1+x))}{(1+x)}y^{-1}dx
\end{aligned}
$$

Therefore, $I^1(h)$ is written as a linear combination of Abelian integrals like in Theorem 3.23. Although we can apply the Theorem because the conditions hold, we are going to use Lemma 3.4 to increase the exponent of y^{-1} to y because the function involved will be easier to manage. Since $V'(x) = x$ it is clear that the condition for apply the Lemma holds so

$$
\begin{aligned}
I_0(h) &= \int_{\gamma_h} \frac{-x}{x+1}ydx \\
I_1(h) &= \int_{\gamma_h} \frac{-x}{(x+1)^2}ydx
\end{aligned}
$$

In order to see that (I_0, I_1) is an ECT-system on $(0, h^*)$, we will check it for (l_0, l_1) in $(0, x_R) = (0, 1)$ by Theorem 3.8 where

$$
\begin{aligned}
l_0(x) &= -\frac{8\sqrt{2}x}{x^2-1}, \\
l_1(x) &= \frac{16\sqrt{2}x}{(x^2-1)^2}.
\end{aligned}
$$

are the balanced functions given in (3.24). Now (l_0, l_1) is easy to prove that is an ECT-system in the right side of the period annulus projection $(0, 1)$ because $l_0(x)$ does not vanish in this interval and

$$
\frac{l_1}{l_0}(x) = \frac{2}{1-x^2}
$$

is a monotone increasing function with image $(2, +\infty)$. Therefore, (l_0, l_1) is an ECT-system so (I_0, I_1) also is an ECT-system.

Thus, for each isochronous in Lemma 3.2, $I^1(h) = b_1 I_0(h) + a_1 I_1(h)$ for some I_0, I_1 such that (I_0, I_1) is an ECT-system on $(0, h^*)$. Then, since $R_1(s) = -b_1 I_0(s) - a_1 I_1(s)$ with (I_0, I_1) an ECT-system on I. That proves the Theorem if $k = 1$.

Let us assume know that it is true since $k-1$ and let us prove that is true for k. We have by hypothesis that $R_0 \equiv R_1 \equiv \cdots R_{k-1} \equiv 0$. Then, since each $R_i = i!b_i I_0 + i!a_i I_1$ with (I_0, I_1) an ECT-system for all $i = 1, \ldots, k-1$, we have that $a_1 = a_2 = \ldots a_{k-1} = b_1 = b_2 = \cdots = b_{k-1} = 0$ and the perturbation is

$$
\psi(\epsilon) = (\psi_1(\epsilon), \psi_2(\epsilon)) = \left(a_k\epsilon^k + o(\epsilon^k), b_k\epsilon^k + o(\epsilon^k)\right).
$$

Particularly, $\dfrac{d^k\psi}{d\epsilon^k} = (k!a_k, k!b_k)$. By the expression

$$\sum_{i=0}^{\infty} Y_i\epsilon^i = \left(\sum_{i=0}^{\infty} \alpha_i\epsilon^i\right) Z_0 + \left(\sum_{i=0}^{\infty} \beta_i\epsilon^i\right) U$$

we have

$$Y_k = \alpha_k Z_0 + \beta_k U$$

so the expression of α_k depends directly of Y_k. Notice that, since $Y_\epsilon = Z_\epsilon - Z_0$,

$$Y_k = \frac{1}{k!}\frac{d^k Y_\epsilon}{d\epsilon^k} = \frac{1}{k!}\frac{d^k Z_\epsilon}{d\epsilon^k}$$

at $\epsilon = 0$ because Z_0 does not depend on ϵ. Let us denote $Z(\psi(\epsilon)) := Z_\epsilon$. Then $Z(\psi(\epsilon)) = Z_1\partial_x + Z_2(\psi(\epsilon))\partial_y$ where Z_1 does not depend on ϵ. Since we have $\dfrac{d^i\psi}{d\epsilon^i} = 0$ for all $i = 1, 2, \ldots, k-1$ at $\epsilon = 0$, then

$$\frac{d^k Z_2(\psi(\epsilon))}{d\epsilon^k} = \nabla Z_2(\psi) \cdot \frac{d^k\psi(\epsilon)}{d\epsilon^k} \tag{3.25}$$

at $\epsilon = 0$. It is easy to check that

$$\nabla Z_2(\psi) = ((1+x)^{p_i}\log(1+x), -(1+x)^{q_i}\log(1+x))$$

so therefore,

$$\frac{d^k Z(\psi(\epsilon))}{d\epsilon^k} = (a_k(1+x)^{p_i} - b_k(1+x)^{q_i})k!\log(1+x)\partial_y$$

and

$$Y_k = (a_k(1+x)^{p_i} - b_k(1+x)^{q_i})\log(1+x)\partial_y.$$

Then, if $F(a,b) := (a(1+x)^{p_i} - b(1+x)^{q_i})\log(1+x)$, $Y_k = F(a_k, b_k)\partial_y$. That means Y_k is essentially the same vector field as Y_1 substituting (a_1, b_1) by (a_k, b_k). Therefore, all computations are valid for each isochronous changing (a_1, b_1) by (a_k, b_k) and the result hold. Moreover, the abelian integrals $I_0(h)$ and $I_1(h)$ do not depend on a_1, b_1 so there are the same integrals also for k. That is,

$$R_k(s) = b_k I_0(s) + a_k I_1(s)$$

where (I_0, I_1) an ECT-system in I. ■

Proof of Theorem 3.7. We are going to prove the result only for the perturbation of the isochronous (I_1^*) because the other cases follow exactly in the same way.

Consider the vector field Z_0 with the isochronous center (I_1^*) at the origin and let U be the commutator. We will prove the theorem by induction on k.

First of all, recall that, for each $k \geq 0$, Theorem 3.6 provides us a formula for T'_{k+1} and ensures that $T'_l \equiv 0$ for all $0 \leq l \leq k-1$ if $j^k(Z_\epsilon)$ is has an isochronous for all $\epsilon \approx 0$.

Let us assume $k = 0$. Then, $j^0(Z_\epsilon) = Z_0$ has an isochronous at the origin so using Theorem 3.6, $T_0 \equiv 0$ and

$$T'_1(s) = -\int_0^{T_0} U(\alpha_1)|_{(x,y)=\varphi(t;s)}\, dt.$$

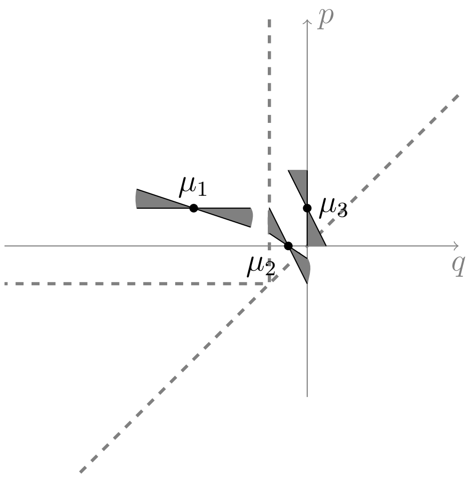


Figure 3.4: Regions of critical points near the isochronous centers.

Consequently, from Proposition 3.4, $T_1'(s)$ can be written as a combination of two Abelian integrals

$$T_1'(s) = -b_1 I_0(s) - a_1 I_1(s)$$

where (I_0, I_1) is an ECT-system on I. Therefore, if $T_1'(s)$ is not identically zero, $T_1'(s)$ can only have one root. More concretely, the root is given implicitly by

$$b_1 I_0(s^*) + a_1 I_1(s^*) = 0$$

so if $(I_1/I_0)(s^*) = -b_1/a_1$ with $s^* \in I$. That is, if $-b_1/a_1 \in \mathrm{Im}((I_1/I_0)(s))$ where $\mathrm{Im}(f(x))$ denotes the image range of $f(x)$, $x \in I$.

Now let us suppose that the statement is true since $k-1$ and let us prove it for k. By Proposition 3.4, the only we need to prove is that the hypothesis of Theorem 3.6 hold. Indeed, since for all $l = 1, \ldots, k$ we have $T_l' \equiv 0$, since (I_0, I_1) is an ECT-system then $a_l, b_l = 0$ for all $l = 1, \ldots, k$. Therefore, the perturbation is

$$\psi(\epsilon) = (a_{k+1}\epsilon^{k+1}, b_{k+1}\epsilon^{k+1})$$

so the truncated $j^k(Z_\epsilon) = Z_0$ has an isochronous center at the origin. Therefore, applying Theorem 3.6 and Proposition 3.4,

$$T_{k+1}'(s) = -b_k I_0(s) - a_k I_1(s)$$

where (I_0, I_1) is an ECT-system in I. Then, if T_{k+1}' is not identically zero, T_{k+1}' can only have one root. More concretely, the root is given implicitly by

$$b_k I_0(s^*) + a_k I_1(s^*) = 0$$

so if $(I_1/I_0)(s^*) = -b_k/a_k$ with $s^* \in I$. That is, if $-b_k/a_k \in \mathrm{Im}((I_1/I_0)(s))$. ■

Remark 3.9. Notice that Theorem 3.7 proves that from each isochronous at most one critical period can bifurcate from the period annulus by Remark 3.3. Moreover, in each case the image range of the quotient $(I_1/I_0)(s)$ give us the relation that a_k and b_k must satisfy to obtain a simple zero s^* of $T_k'(s)$ and, therefore, bifurcate a critical period from $s^* \in I$.

Although the property of being Chebyshev is translated through (l_0, l_1) to (I_0, I_1), the image range is not generally translated. This means that we are not able to know the directions where the critical periods bifurcate in the parameter space. However, in this cases we are able to compute explicitly the Abelian integrals $I_0(s)$ and $I_1(s)$.

(I_1^*) The Abelian integrals are

$$I_0(h) = \int_{\gamma_h} \frac{-16}{(1-2x)^4} y^3 dx$$
$$I_1(h) = \int_{\gamma_h} \frac{-16}{3(1-2x)^2} y^3 dx$$

Using the Hamiltonian, the integrals can be written as

$$I_0(h) = \int_{-h-\sqrt{h+h^2}}^{-h+\sqrt{h+h^2}} \frac{-16}{(1-2x)^4} \left(h - 2xh - x^2\right)^{3/2}$$
$$I_1(h) = \int_{-h-\sqrt{h+h^2}}^{-h+\sqrt{h+h^2}} \frac{-16}{3(1-2x)^2} \left(h - 2xh - x^2\right)^{3/2}$$

and one primitive of each integrand are

$$F_0(x) = \frac{2\sqrt{-2hx+h-x^2}\left(-6h^2-2(12h(h+1)+11)x^2+(2h(12h+5)+15)x+h-3\right)}{3(2x-1)^3} +$$
$$+ (4h^3+6h^2-1)\arctan\left(\frac{-2hx+h-x}{\sqrt{-2hx+h-x^2}}\right) - \arctan\left(\frac{h+x}{\sqrt{-2hx+h-x^2}}\right)$$
$$F_1(x) = \frac{2\sqrt{-2hx+h-x^2}(5h(2x-1)+x(2x+3)-3)}{6x-3} +$$
$$-(2h(h+1)+1)\arctan\left(\frac{h+x}{\sqrt{-2hx+h-x^2}}\right) + (2h+1)\arctan\left(\frac{h(2x-1)+x}{\sqrt{-2hx+h-x^2}}\right)$$

So evaluating at the endpoints of the integral we obtain

$$I_0(h) = \int_{-h-\sqrt{h+h^2}}^{-h+\sqrt{h+h^2}} \frac{-16}{(1-2x)^4} \left(h - 2xh - x^2\right)^{3/2} dx = -2h^2(3+2h)\pi$$
$$I_1(h) = \int_{-h-\sqrt{h+h^2}}^{-h+\sqrt{h+h^2}} \frac{-16}{3(1-2x)^2} \left(h - 2xh - x^2\right)^{3/2} dx = -2h^2\pi$$

and then

$$\frac{I_1}{I_0}(h) = \frac{1}{3+2h} = -\frac{b}{a},$$

that has image $(0, \frac{1}{3})$ for $h \in (0, +\infty)$.

(I_2^*) In order to compute the integrals for this case, we use the inverse procedure that we showed in Lemma 3.4 to take them more simple. That is, if

$$\int_{\gamma_h} F(x) y^{k-2} dx = \int_{\gamma_h} G(x) y^k dx,$$

where $G(x) = (2/k)(BF/A')'(x) - (B'F/A')(x)$, as $B' \equiv 0$ then

$$F(x) = (k/2) \int G(x)A'(x)/B(x).$$

So we obtain

$$I_0(h) = \frac{1}{4\sqrt{2}h^2} \int_{\gamma_h} \frac{(1-\sqrt{1+x})(-2+5\sqrt{1+x}\log(1+x))}{1+x} y^3 dx,$$
$$I_1(h) = \frac{1}{4\sqrt{2}h^2} \int_{\gamma_h} \frac{4(3+4x-3\sqrt{1+x})}{3(1+x)^{3/2}} y^3 dx.$$

Then, since γ_h is given by $2 + x - 2\sqrt{1+x} + y^2/2 = h$, we can isolate x and take $y = 0$ to obtain the limits of the integral: $-2\sqrt{h} + h$, $2\sqrt{h} + h$. Moreover, isolating y^2 from the same expression, we can express those integrals in terms of x and h. If we use the change of variable $z^2 = 1 + x$ we have,

$$I_0(h) = \frac{1}{4\sqrt{2}h^2} \int_{\sqrt{1-2\sqrt{h}+h}}^{\sqrt{1+2\sqrt{h}+h}} \frac{(1-z)(-2+5z\log(z^2))}{z^2} (2(-1+h-z^2+2z))^{3/2} 2z dz,$$
$$I_1(h) = \frac{1}{4\sqrt{2}h^2} \int_{\sqrt{1-2\sqrt{h}+h}}^{\sqrt{1+2\sqrt{h}+h}} \frac{4(-1+4z^2-3z)}{3z^3} (2(-1+h-z^2+2z))^{3/2} 2z dz.$$

These two integrals can be computed by a primitive and the solutions are

$$I_0(h) = \frac{1}{4\sqrt{2}h^2} \left(4\sqrt{2}h(2-2\sqrt{1-h}+(-3+2\sqrt{1-h})h)\pi\right),$$
$$I_1(h) = \frac{1}{4\sqrt{2}h^2} \left(8\sqrt{2}h(-2+\sqrt{1-h}+h)\pi\right).$$

Here $0 < h < 1$, $h = 0$ corresponding to the fixed point and $h = 1$ to the policycle. Then, $\mathrm{Im}(I_1/I_0) = \left(\frac{2}{3}, 2\right)$.

(I_3^*) In this case the Abelian integrals are

$$I_0(h) = -\int_{\gamma_h} \frac{x}{x+1} y dx = -\int_{-\sqrt{2h}}^{\sqrt{2h}} \frac{x}{x+1} \sqrt{2h - x^2},$$
$$I_1(h) = -\int_{\gamma_h} \frac{x}{(x+1)^2} y dx = -\int_{-\sqrt{2h}}^{\sqrt{2h}} \frac{x}{(x+1)^2} \sqrt{2h - x^2},$$

where we used $2h = x^2 + y^2$. Then, using a primitive of each integrand, we can compute them and

$$I_0(h) = -\int_{-\sqrt{2h}}^{\sqrt{2h}} \frac{x}{x+1} \sqrt{2h - x^2} = -\pi(-1 + \sqrt{1-2h} + h),$$
$$I_1(h) = -\int_{-\sqrt{2h}}^{\sqrt{2h}} \frac{x}{(x+1)^2} \sqrt{2h - x^2} = -\frac{2\pi(-1+\sqrt{1-2h}+h)}{\sqrt{1-2h}}.$$

Here $0 < h < 1/2$, $h = 0$ corresponding to the fixed point and $h = 1/2$ to the policycle. Then, $\mathrm{Im}(I_1/I_0) = (2 + \infty)$.

Therefore, a critical period bifurcates from μ_i if and only if $T_k'(s)$ is not identically zero and (a_k, b_k) satisfy in each case the following condition (see Figure 3.4),

- $\dfrac{b_k}{a_k} \in (-\frac{1}{3}, 0)$ for μ_1.
- $\dfrac{b_k}{a_k} \in (-2, -\frac{2}{3})$ for μ_2.
- $\dfrac{b_k}{a_k} \in (-\infty, -\frac{1}{2})$ for μ_3.

$\square$

Chapter 4

Period function near the outer boundary

In this section we will study the behaviour of the period function near the outer boundary. Let us recall that the period annulus $\mathscr{P}$ is defined as the maximum punctured neighbourhood of the origin $\mathcal{O}$ such that contains only periodic orbits surrounding $\mathcal{O}$. $\mathscr{P}$ has two different boundaries: the origin itself and the one given essentially by the other connected components of $\partial\mathscr{P}$. In the previous sections we studied the period function near to this first boundary: the critical point of the center. Now we will study $T(h)$ near the second boundary.

In the same way as the origin was identified by $h = 0$ through the energy levels of the first integral $H(x, y) = \frac{y^2}{2} + V(x)$, we can identify the outer boundary as $h = h^*$ defined by

$$\lim_{x\to x_L} H(x, 0) = \lim_{x\to x_R} H(x, 0) = h^*, \tag{4.1}$$

where x_L and x_R are respectively the left and right endpoints of the period annulus projection over the x-axis. Hence we are interested in the study of the function $T(h)$ where h tends to h^*. The first we will attempt is to study the value of $T(h)$ near the outer boundary. More concretely, we will study its finiteness. In order to do it, we will use the following two theorems that give us a general view of the problem.

Theorem 4.1. *Let $\mathcal{I} = (x_L, x_R)$ be the projection on the x-axis of the period annulus $\mathscr{P}$. Let us assume that $V \in \mathcal{C}^2(\mathcal{I})$, $h^* < \infty$ and $V''(x)$ does not have x_L or x_R as accumulation points of zeros. Then,*

(i) $\lim\limits_{x\to\pm\sqrt{h^*}} (g^{-1})'(x)$ *or it exists or it is infinity.*

(ii) Let $L = \sqrt{2}\displaystyle\int_{-\pi/2}^{\pi/2} (g^{-1})'(\sqrt{h^*}\sin\theta)d\theta$.

- *(a) If L diverges then* $\lim\limits_{h\to h^*} T(h) = \infty$.
- *(b) If L converges then* $\lim\limits_{h\to h^*} T(h) = L$.

Proof. Let us prove (i). Since $V''(x)$ does not have x_L or x_R as accumulation points of zeros, there exists $\epsilon_L > 0$ and $\epsilon_R > 0$ such that $V'(x)$ is monotone at $(x_L, x_L + \epsilon_L)$ and

$(x_R-\epsilon_R, x_R)$. Therefore, $(g^{-1})'$ is also monotone in $g((x_L, x_L+\epsilon_L)) = (-\sqrt{h^*}, -\sqrt{h^*}+\delta_L)$ and $g((x_R-\epsilon_R, x_R)) = (\sqrt{h^*}-\delta_R, \sqrt{h^*})$. Then, by monotonicity $\lim\limits_{x\to\pm\sqrt{h^*}}(g^{-1})'(x)$ exist or it is infinity.

Let us prove $(ii.a)$. By definition (see (1.6)), $(g^{-1})' > 0$. Then, we will have that

$$\lim_{\epsilon\to 0}\sqrt{2}\int_{-\frac{\pi}{2}+\epsilon}^{\frac{\pi}{2}-\epsilon}(g^{-1})'(\sqrt{h^*}\sin\theta)d\theta = +\infty.$$

Given $M>0$, let $\epsilon>0$ be fixed such that $\sqrt{2}\int_{-\frac{\pi}{2}+\epsilon}^{\frac{\pi}{2}-\epsilon}(g^{-1})'(\sqrt{h^*}\sin\theta)d\theta > M$.

Let $S(h) = \sqrt{2}\int_{-\frac{\pi}{2}+\epsilon}^{\frac{\pi}{2}-\epsilon}(g^{-1})'(\sqrt{h}\sin\theta)d\theta$. Since $S(h)$ is a continuous function in $[0,h^*]$ and $S(h^*)>M$ we have that there exists $\delta>0$ such that $S(h)>M$ for h satisfying $|h^*-h|<\delta$. Therefore,

$$T(h) = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(g^{-1})'(\sqrt{h}\sin\theta)d\theta \geq \sqrt{2}\int_{-\frac{\pi}{2}+\epsilon}^{\frac{\pi}{2}-\epsilon}(g^{-1})'(\sqrt{h}\sin\theta)d\theta = S(h) > M$$

for h satisfying $|h^*-h|<\delta$. This shows that, $\lim\limits_{h\to h^*}T(h) = \infty$.

Let us prove $(ii.b)$. Let $\epsilon>0$ be fixed. We will prove that there exists $\delta>0$ such that $|L-T(h)|<\epsilon$ if $|h^*-h|<\delta$.

If $(g^{-1})'$ has limit in both endpoints of the interval, then there are nothing to say. Let us suppose that the limit of $(g^{-1})'$ at $g(x_L) = -\sqrt{h^*}$ is infinity. The proof for the case when the limit at $g(x_R) = \sqrt{h^*}$ is infinity or with both at the same time are infinity is analogue.

By the assumption about $V''(x)$, since x_L is not an accumulation point of zeros of $V''(x)$, there exists a point $\widehat{x}<0$ such that $V'(x)$ is monotone at $(x_L,\widehat{x})$. Then, $(g^{-1})'(z)$ is monotone at $(-\sqrt{h^*},\widehat{z})$ since $2gg' = V'$. Moreover, since $(g^{-1})'$ tends to infinity at $z=-\sqrt{h^*}$, $(g^{-1})'$ is monotone decreasing for $z\in(-\sqrt{h^*},\widehat{z})$.

Let $\xi>0$ and $\delta_1>0$ small enough such that $\sqrt{h}\sin\theta < g(\widehat{x}) = \widehat{z}$ for all $\theta\in(-\frac{\pi}{2},-\frac{\pi}{2}+\xi)$ and $|h^*-h|<\delta_1$. Therefore, since $h^*>h$,

$$(g^{-1})'(\sqrt{h^*}\sin\theta) > (g^{-1})'(\sqrt{h}\sin\theta) > 0$$

for each $\theta\in(-\frac{\pi}{2},-\frac{\pi}{2}+\xi)$ and, as a consequence,

$$\left|\sqrt{2}\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+\xi}(g^{-1})'(\sqrt{h^*}\sin\theta)d\theta\right| > \left|\sqrt{2}\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+\xi}(g^{-1})'(\sqrt{h}\sin\theta)d\theta\right|.$$

Moreover, by assumption L is convergent so there exists $0<\chi<\xi$ small enough such that

$$\left|\sqrt{2}\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+\chi}(g^{-1})'(\sqrt{h^*}\sin\theta)d\theta\right| < \frac{\epsilon}{4}.$$

In addition, since we supposed x_L the only point where $(g^{-1})'$ tends to infinity, the function

$$S(h) = \sqrt{2}\int_{-\frac{\pi}{2}+\chi}^{\frac{\pi}{2}}(g^{-1})'(\sqrt{h}\sin\theta)d\theta$$

is continuous for all $h \in [0, h^*]$ and then there exists $\delta_2 > 0$ small enough such that

$$\left| \sqrt{2} \int_{-\frac{\pi}{2}+\chi}^{\frac{\pi}{2}} \left((g^{-1})'(\sqrt{h^*} \sin\theta) - (g^{-1})'(\sqrt{h} \sin\theta) \right) d\theta \right| < \frac{\epsilon}{2}$$

for $|h^* - h| < \delta_2$. Then, taking $\delta = \min\{\delta_1, \delta_2\}$, if $|h^* - h| < \delta$,

$$\begin{aligned}
|L - T(h)| &= \left| \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g^{-1})'(\sqrt{h^*} \sin\theta) d\theta - \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g^{-1})'(\sqrt{h} \sin\theta) d\theta \right| \\
&\leq \left| \sqrt{2} \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+\chi} (g^{-1})'(\sqrt{h^*} \sin\theta) d\theta \right| + \left| \sqrt{2} \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+\chi} (g^{-1})'(\sqrt{h} \sin\theta) d\theta \right| \\
&\quad + \left| \sqrt{2} \int_{-\frac{\pi}{2}+\chi}^{\frac{\pi}{2}} \left((g^{-1})'(\sqrt{h^*} \sin\theta) - (g^{-1})'(\sqrt{h} \sin\theta) \right) d\theta \right| \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

■

Theorem 4.2. *Let us assume $h^* = \infty$ and that there exist $a, b \in \mathbb{R} \cup \{\infty\}$ such that $\lim_{x \to -\infty} (g^{-1})'(x) = a$ and $\lim_{x \to +\infty} (g^{-1})'(x) = b$. Then,*

$$\lim_{h \to \infty} T(h) = \frac{(a+b)\pi}{\sqrt{2}}.$$

Proof. We will prove that $\lim_{h \to \infty} \sqrt{2} \int_{-\frac{\pi}{2}}^{0} (g^{-1})'(\sqrt{h} \sin\theta) d\theta = \frac{a\pi}{\sqrt{2}}$. Let us assume $a < \infty$. Notice that, since $(g^{-1})' > 0$, then $a > 0$. Since $\lim_{x \to -\infty} (g^{-1})'(x) = a$ then there exists $M > a$ such that $(g^{-1})'(x) < M$ for all $x \in (-\infty, 0]$. Let us consider $\epsilon > 0$, take $\epsilon' = \epsilon/\sqrt{2}$ and let $z < 0$ such that $|(g^{-1})'(y) - a| < \epsilon'/\pi$ for all $y < z$. Finally, let h_0 be such that $\sqrt{h_0} \sin\left(-\frac{\epsilon'}{4M}\right) < z$. Then if $h > h_0$ we have

$$\begin{aligned}
\left| \sqrt{2} \int_{-\frac{\pi}{2}}^{0} (g^{-1})'(\sqrt{h} \sin\theta) d\theta - \frac{a\pi}{\sqrt{2}} \right| &\leq \left| \sqrt{2} \int_{-\frac{\pi}{2}}^{-\epsilon'/4M} \left((g^{-1})'(\sqrt{h} \sin\theta) - a \right) d\theta \right| \\
&\quad + \left| \sqrt{2} \int_{-\epsilon'/4M}^{0} \left((g^{-1})'(\sqrt{h} \sin\theta) - a \right) d\theta \right| \\
&\leq \sqrt{2} \left(\frac{\epsilon'}{\pi} \frac{\pi}{2} + \frac{2M\epsilon'}{4M} \right) = \sqrt{2}\epsilon' = \epsilon.
\end{aligned}$$

Let us consider now $a = \infty$ and let $K > 0$. Let z be such that $(g^{-1})'(y) > K$ for all $y < z$ and let h_0 such that $\sqrt{h_0} \sin\left(-\frac{\pi}{4}\right) < z$. Then, if $h > h_0$ we have

$$\sqrt{2} \int_{-\frac{\pi}{2}}^{0} (g^{-1})'(\sqrt{h} \sin\theta) d\theta \geq \sqrt{2} \int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} (g^{-1})'(\sqrt{h} \sin\theta) d\theta \geq K\sqrt{2}\frac{\pi}{4} > K.$$

Then, $\lim_{h\to\infty}\sqrt{2}\int_{-\frac{\pi}{2}}^{0}(g^{-1})'(\sqrt{h}\sin\theta)d\theta=\infty$ in this case.
In the same way can be proved that

$$\lim_{h\to\infty}\sqrt{2}\int_{0}^{\frac{\pi}{2}}(g^{-1})'(\sqrt{h}\sin\theta)d\theta=\frac{b\pi}{\sqrt{2}}$$

so the result holds. ■

Now we will apply Theorem 4.1 and Theorem 4.2 in order to show the behaviour of $T(h)$ near the outer boundary.

Theorem 4.3. *Let $T(h)$ be the period function defined in* (1.13) *parametrized by the energy level h. Let h^* be the associated energy level of the outer boundary and let $\mathcal{I}=(x_L,x_R)$ the projection on the x-axis of the period annulus $\mathscr{P}$. Let us assume the parameter space divided in $\mathscr{R}_1$, $\mathscr{R}_2$, $\mathscr{R}_3$ (see 1.17 and Figure 1.3). Then,*

(i) For $\mu\in\mathscr{R}_1$,

(a) If $q\geq 1$, $\lim_{h\to h^*}T(h)=+\infty$.

(b) If $q<1$, $\lim_{h\to h^*}T(h)=L<\infty$.

(ii) For $\mu\in\mathscr{R}_2$,

(a) If $p>1$, $\lim_{h\to h^*}T(h)=0$.

(b) If $p=1$, $\lim_{h\to h^*}T(h)=\pi$.

(c) If $p<1$, $\lim_{h\to h^*}T(h)=+\infty$.

(iii) For $\mu\in\mathscr{R}_3$, $\lim_{h\to h^*}T(h)=+\infty$.

Proof. Let us prove (i). Since $\mu\in\mathscr{R}_1$, $\mathcal{I}=(-1,\rho)$ and $h^*<\infty$. By Theorem 4.1 we only need to compute

$$L=\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(g^{-1})'(\sqrt{h^*}\sin\theta)d\theta=\int_{-1}^{\rho}\frac{dx}{\sqrt{h^*-V(x)}}.$$

Therefore,

$$L=\int_{-1}^{\rho}\frac{dx}{\sqrt{h^*-V(x)}}=\int_{-1}^{\rho}\frac{dx}{\sqrt{\frac{(x+1)^{q+1}}{q+1}-\frac{(x+1)^{p+1}}{p+1}}}=\int_{-1}^{\rho}\frac{dx}{\sqrt{(x+1)^{q+1}}\sqrt{\frac{1}{q+1}-\frac{(x+1)^{p-q}}{p+1}}}$$

which is an improper integral that tends to infinity only at $x=-1$. Therefore, if $q\geq 1$ then $L=\infty$ and if $q<1$ then $L<\infty$.

Let us prove (ii). Since $\mu\in\mathscr{R}_2$, $\mathcal{I}=(-1,\infty)$ and $h^*=\infty$. By Theorem 4.2 we only need to compute the limits of $(g^{-1})'$ in $a=g(-1)=-\infty$ and $b=g(\infty)=\infty$. Notice that,

$$\lim_{x\to -1} g'(x) = \lim_{x\to -1} \frac{V'(x)}{2g(x)} = \lim_{x\to -1} \frac{(x+1)^p-(x+1)^q}{2\mathrm{sgn}(x)\sqrt{\frac{(x+1)^{p+1}}{p+1}-\frac{(x+1)^{q+1}}{q+1}+\frac{p-q}{(p+1)(q+1)}}}$$
$$= \lim_{z\to 0} \frac{-1}{2}\frac{z^p-z^q}{\sqrt{\frac{z^{p+1}}{p+1}-\frac{z^{q+1}}{q+1}+\frac{p-q}{(p+1)(q+1)}}} = \lim_{z\to 0} \frac{1}{2}\frac{z^q}{\sqrt{-\frac{z^{q+1}}{q+1}+\frac{p-q}{(p+1)(q+1)}}}$$
$$= \lim_{z\to 0} \frac{1}{2\sqrt{-(q+1)}}\frac{z^q}{z^{\frac{q+1}{2}}} = \lim_{z\to 0} \frac{1}{2\sqrt{-(q+1)}} z^{\frac{q-1}{2}} = +\infty.$$

Therefore, $\lim_{x\to -\infty}(g^{-1})'(x)=0$. On the other way,

$$\lim_{x\to +\infty} g'(x) = \lim_{x\to +\infty} \frac{V'(x)}{2g(x)} = \lim_{x\to +\infty} \frac{(x+1)^p-(x+1)^q}{2\mathrm{sgn}(x)\sqrt{\frac{(x+1)^{p+1}}{p+1}-\frac{(x+1)^{q+1}}{q+1}+\frac{p-q}{(p+1)(q+1)}}}$$
$$= \lim_{z\to +\infty} \frac{1}{2}\frac{z^p-z^q}{\sqrt{\frac{z^{p+1}}{p+1}-\frac{z^{q+1}}{q+1}+\frac{p-q}{(p+1)(q+1)}}} = \lim_{z\to +\infty} \frac{1}{2}\frac{z^p}{\sqrt{\frac{z^{p+1}}{p+1}}}$$
$$= \lim_{z\to +\infty} \frac{\sqrt{p+1}}{2}\frac{z^p}{z^{\frac{p+1}{2}}} = \lim_{z\to +\infty} \frac{\sqrt{p+1}}{2} z^{\frac{p-1}{2}} = \begin{cases} 0 & \text{if } p<1 \\ \frac{\sqrt{2}}{2} & \text{if } p=1 \\ +\infty & \text{if } p>1 \end{cases}.$$

Therefore, $\lim_{x\to +\infty}(g^{-1})'(x) = \begin{cases} \infty & \text{if } p<1 \\ \sqrt{2} & \text{if } p=1 \\ 0 & \text{if } p>1 \end{cases}$. Then, by Theorem 4.2 we have that

$$\lim_{h\to h^*} T(h) = \begin{cases} \infty & \text{if } p<1 \\ \pi & \text{if } p=1 \\ 0 & \text{if } p>1 \end{cases}.$$

Finally let us prove (iii). Since $\mu \in \mathscr{R}_3$, $\mathcal{I}=(\rho,\infty)$ and $h^*<\infty$. In this case by Theorem 4.1 the limit of the period function is given by

$$L = \int_\rho^\infty \frac{dx}{\sqrt{h^*-V(x)}}.$$

Notice that, since $\lim_{x\to +\infty} V(x) = h^*$, the integrand of L tends to infinity if x tends to infinity. Then the result is immediate. ■

Remark 4.1. Previous result particularly shows us that if $0<q<1$ the limit of the period function at the outer boundary exist. The system X_μ with $0<q<1$ is a $\mathcal{C}^0$ vector field that can be extended to the whole plane. Particularly, it can be extended to the point $(x,y)=(-1,0)$ which is a critical point of X_μ. Let us assume that the system has uniqueness of solutions. Therefore, since the vector field is continuous, the orbits are continuous with respect to initial conditions. The orbits that tend to the outer boundary particularly tend to the point $(-1,0)$. Since $(-1,0)$ is a critical point, there exist orbits close enough to $(-1,0)$ spending an arbitrary large time in a neighbourhood of $(-1,0)$. However, this fact contradicts the result of Theorem 4.3 that ensures $\lim_{h\to h^*} T(h) = L < \infty$ for this case. Then, the system X_μ does not have uniqueness of solutions in the whole plane.

However, and by the existence of a Hamiltonian function, although there are no uniqueness of solutions, there are uniqueness in the trace of the solutions. $\square$

Chapter 5

Overall view and future developments

Through all previous chapters we have seen many different dynamical properties of the period function of the family $\{X_\mu\}_{\mu \in \mathscr{R}}$. These properties can be summarized in the following sentences.

We have proved that there exist some regions where the period function is monotone increasing or decreasing (Figure 2.2). Also we proved the existence of three isochronous centers and a collection of weak centers of order one such that from all of them there are at most one critical period that bifurcates from the center. Moreover, the infinitesimal region where we can ensure the existence of a critical period that bifurcates from the origin at the finite weak centers is a one-side band of the curve described by the vanishing of the first period constant; and this band changes of side with respect to the curve when the curve crosses an isochronous center.

In Remark 3.9, we showed that the region where a critical period bifurcates from the period annulus is a conic neighbourhood of the isochronous center with an specific tangent lines. One of the tangents corresponds to the tangent line of the curve that describes the vanishing of the first period constant. The other can be interpreted as the tangent of the curve described by the vanishing of the corresponding first period constant near the outer boundary. That is, if we were able to compute the series expression of the period function at the outer boundary, then this curve would be the vanishing of the first non-constant coefficient.

That last idea caused us to confect a conjecture of a global behaviour of the periodic function in $\mathscr{R}$, using all the features obtained.

Conjecture 1. *The period function of the family X_μ has three monotone regions and five regions with one critical period. Moreover, the character of the critical periods and of the monotonicity is the one showed at Figure 5.1.*

There are more information about the period function that comes from this work that the showed explicitly. For instance, one can use the information given by Theorem 4.3 to show that if $p > 1$ and $q < -1$ then we can compare the value of the period function near the outer boundary with the value on the center and then take conclusions about the existence of critical periods in a region. More concretely, since the value at the center is $\frac{2\pi}{\sqrt{p-q}} > 0$ and the value near the outer boundary is 0, then we can ensure that there is

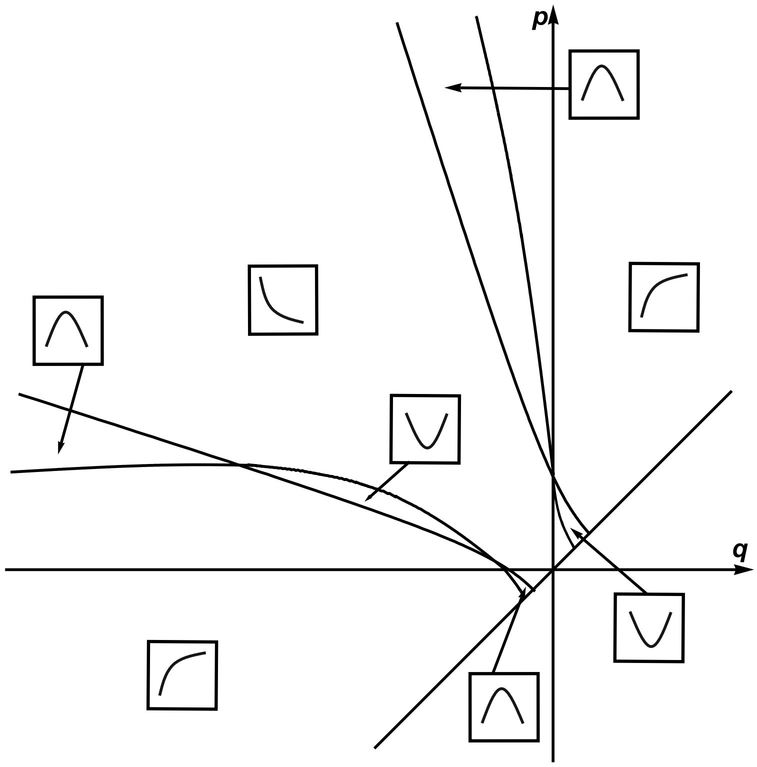


Figure 5.1: Conjecture of the period function behaviour.

a critical period for all the parameters with $p > 1$, $q < -1$ and $\Delta_1(q,p) > 0$. By the same argument, we can ensure a critical period for all the parameter with $p < 1$, $q < -1$ and $\Delta_1(q,p) < 0$.

For future developments, the first approach is to generalize the family and the whole study for $p = -1$ and $q = -1$. In fact, it is only necessary to include a compensator since qualitatively the potential function in this cases is the same as the one in region $\mathscr{R}_2$. Indeed, for $p = -1$ we have

$$V_q(x) = \log(x+1) - \frac{(x+1)^{q+1}}{q+1} + \frac{1}{q+1}$$

with $q + 1 < 0$. Then, $\lim\limits_{x\to+\infty} V_q(x) = \lim\limits_{x\to-1} V_q(x) = +\infty$. Also, for $q = -1$,

$$V_p(x) = \frac{(x+1)^{p+1}}{p+1} - \log(x+1) - \frac{1}{p+1}$$

with $p + 1 < 0$, so $\lim\limits_{x\to+\infty} V_p(x) = \lim\limits_{x\to-1} V_p(x) = +\infty$. Therefore, it is expected that essentially all the result for $\mathscr{R}_2$ can be extended to $p = -1$ and $q = -1$.

Finally, another way to study is trying to develop more theory about the period function near the outer boundary, which is the less studied because of its difficulty. A good way for begin to prove the previous conjecture is computing the respective first period constant at the outer boundary.

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