SIMULTANEOUS PERIODIC ORBITS BIFURCATING FROM TWO ZERO–HOPF EQUILIBRIA IN A TRITROPHIC FOOD CHAIN MODEL

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ABSTRACT. We are interested in the coexistence of three species forming a tritrophic food chain model. Considering a linear grow for the lowest trophic species or prey, and a type III Holling functional response for the middle and highest trophic species (first and second predator respectively). We prove that this model exhibits two small amplitud periodic solutions bifurcating simultaneously each one from one of the two zero-Hopf equilibrium points that the model has for adequate values of its parameters. As far as we know this is the first time that this phenomena appear in the literature related with food chain models.

1. INTRODUCTION

In general, the Hopf bifurcation is a useful tool to analyse the existence of limit cycles in predator–prey interaction models. For instance, in [27] the authors proved the existence, uniqueness and nonexistence of limit cycles in a predator–prey model considering a strong Allee effect in a prey. In [1] it is considered a model of three species competing for three resources and it is proved the existence of two limit cycles evolving the coexistence equilibrium point, other example is [20]. In a food web the Hopf bifurcation is also the principal tool for proving the coexistence of species that compose the food chain. In this direction Freedman and Waltman [11] studied the persistence of species in a three–level food chain model, they introduce a relative general model, and criteria for the boundedness and stability are established. They consider a Lotka–Volterra predation with a carrying capacity at the lowest level via a logistic map and with a Holling functional response type II predation at the level of the first predator. They gave sufficient conditions for persistence of all three species. Later on in [12] Freedman and So established criteria for which a simple food–chain model has a globally stable positive equilibrium and also develop criteria in order that such a food chain model exhibits uniform persistence (see also [13]). In these articles the possibility of existence of limit cycles is important, however it was not studied.

Recently Françoise and Llibre analyse a model representing a tritrophic food chain composed of a logistic prey, a Holling type II predator and a Holling type II top–predator in [10]. Using the averaging theory (see [5, 23, 26]) they prove the existence of a stable periodic orbit contained in the region of coexistence of the three species in a tritrophic chain. For some values of the parameters three limit cycles born via a triple Hopf bifurcation. One is contained in the plane where the top–predator is absent. Another one is not contained in the domain of interest where all variables are positive and the third one is contained where the three species coexist. In the literature there are many paper dedicated to find these type of limit cycles which came from a Hopf bifurcation, but in all these paper the existence of a triple Hopf bifurcation was not proved analytically, see for instance [7, 8, 9, 18, 21, 22].

In this paper we analyse a tritrophic food chain model considering Holling functional response of type III for middle and top trophic level and linear grow for the lowest tropic level.

Accordingly with the previous works a general tritrophic food chain model has the form

$$\dot{x} = xh(x) - f(x)y,$$
$$\dot{y} = y\left(-d_1 + f(x)\right) - g(y)z,$$
$$\dot{z} = z\left(g(y) - d_2\right).$$

Here x represents the number of lowest trophic species or prey, y is the number of the middle trophic level species or first predator (called also as predator), and z is the number of highest trophic level species or second predator (super–predator). The parameters d_1 and d_2 are positives. The function $h(x)$ represents the specific growth rate of the prey and must always satisfy

$$h(0) = \alpha > 0, \quad \frac{\partial h(x)}{\partial x} \leq 0 \quad \text{for all } x \geq 0.$$

The function $f(x)$ is the functional response of predator (second consumer or first predator) and must satisfy

$$f(0) = 0, \quad \frac{\partial f(x)}{\partial x} \leq 0 \quad \text{for all } x \geq 0.$$

Finally, the function $g(y)$ is the functional response of the super–predator (tertiary consumer or second predator) and satisfy the conditions

$$g(0) = 0, \quad \frac{\partial g(y)}{\partial y} \leq 0 \quad \text{for all } y \geq 0.$$

There are many functions that satisfy the above conditions, for example the functional responses of predation include the usual functions found in the literature (see, e.g., [17]). In this paper we will consider linear growth without environmental carrying capacity for the prey and Holling functional response type III for the predator and the super–predator. So we consider

the functions

$$h(x) = \rho, \quad f(x) = \frac{a_1 x^2}{x^2 + b_1} \quad \text{and} \quad g(y) = \frac{a_2 y^2}{y^2 + b_2},$$

where ρ, a_1, a_2, b_1 and b_2 are positive constants. Consequently, the tritrophic food chain model that we shall study is

$$\dot{x} = x\left(\rho - \frac{a_1 xy}{b_1 + x^2}\right),$$

(1)
$$\dot{y} = y\left(-d_1 + \frac{a_1 x^2}{b_1 + x^2} - \frac{a_2 yz}{b_2 + y^2}\right),$$

$$\dot{z} = z\left(-d_2 + \frac{a_2 y^2}{b_2 + y^2}\right).$$

For ecological restrictions the analysis is in the positive octant of $\mathbb{R}^3$, i.e. in the region $x > 0$, $y > 0$ and $z > 0$.

We give necessary conditions on the parameters to guarantee the existence of two equilibrium points of the differential system (1) in the region of interest. At these equilibrium points we find two families of parameters for which these equilibrium are zero–Hopf, see Proposition 1. The main result shows that only one of these families of parameters produce a double simultaneously zero–Hopf bifurcation, appearing at the same time two small amplitude periodic orbits bifurcating simultaneous of the two different equilibria of the system, see Theorem 2.

2. Equilibrium points in the positive octant

As we mention above, the tritrophic food chain model (1) has two equilibrium points in the positive octant of $\mathbb{R}^3$ when the parameters satisfy the following three conditions:

(i) $a_2 - d_2 \neq 0$,

(ii) $a_1^2 b_2 d_2 + 4 b_1 (d_2 - a_2)\rho^2 \geq 0$,

(iii) $\dfrac{a_1^2 b_2 d_2 + 2 b_1 (d_2 - a_2)\rho^2 \pm a_1 \sqrt{b_2 d_2 \left(a_1^2 b_2 d_2 + 4 b_1 (d_2 - a_2)\rho^2\right)}}{(a_2 - d_2)\rho^2} \geq 0.$

These conditions are necessary because in the coordinates of these two equilibrium points appear the expression

$$\sqrt{\frac{a_1^2 b_2 d_2 + 2 b_1 (d_2 - a_2)\rho^2 \pm a_1 \sqrt{b_2 d_2 \left(a_1^2 b_2 d_2 + 4 b_1 (d_2 - a_2)\rho^2\right)}}{(a_2 - d_2)\rho^2}}.$$

In order that the expression of the equilibrium points become easier we change the parameter b_2 for the new parameter $k > 0$ defined through

$$\frac{a_1^2 b_2 d_2 + 2 b_1 (d_2 - a_2)\rho^2 - a_1 \sqrt{b_2 d_2 \left(a_1^2 b_2 d_2 + 4 b_1 (d_2 - a_2)\rho^2\right)}}{(a_2 - d_2)\rho^2} = k^2.$$

Solving b_2 in terms of k from the above expression we obtain

$$b_2 = \frac{(a_2 - d_2)(2b_1 + k^2)^2 \rho^2}{2a_1^2 d_2 k^2}.$$

Therefore we need that $a_2 > d_2$, otherwise b_2 would be negative. Hence the condition (i) becomes

  (i) $a_2 - d_2 > 0$.

  Now the equilibrium points in the positive octant are

$$p_1 = \left(\frac{k}{\sqrt{2}}, \frac{(2b_1 + k^2)\rho}{\sqrt{2}a_1 k}, \frac{((a_1 - d_1)k^2 - 2b_1 d_1)\rho}{\sqrt{2}a_1 d_2 k} \right),$$

$$p_2 = \left(\frac{2\sqrt{2}b_1}{k}, \frac{(2b_1 + k^2)\rho}{\sqrt{2}a_1 k}, \frac{(2b_1 a_1 - d_1(2b_1 + k^2))\rho}{\sqrt{2}a_1 d_2 k} \right).$$

Our first interest is to analyse when of these two equilibrium points are of type zero–Hopf.

3. Zero–Hopf equilibrium points and bifurcation

We recall that an equilibrium point is a *zero–Hopf equilibrium* of a 3–dimensional autonomous differential equation, if it has a zero real eigenvalue and a pair of purely imaginary eigenvalues. We know that a *zero–Hopf bifurcation* is a two–parameter unfolding (or family) of a 3-dimensional autonomous differential system with a zero–Hopf equilibrium. The unfolding has an isolated equilibrium point with a zero eigenvalue and a pair of purely imaginary eigenvalues if the two parameters take zero values, and the unfolding has different topological type of dynamics in the small neighbourhood of this isolated equilibrium as the two parameters vary in a small neighbourhood of the origin. This theory of zero–Hopf bifurcation has been analysed by Guckenheimer, Han, Holmes, Kuznetsov, Marsden and Scheurle in [14, 15, 16, 19, 24]. In particular they shown that some complicated invariant sets of the unfolding could bifurcate from the isolated zero–Hopf equilibrium under some conditions. Hence in some cases the zero–Hopf bifurcation could imply a local birth of "chaos" see for instance the articles [2, 3, 4, 6, 25] of Baldomá and Seara, Broer and Vegter, Champneys and Kirk, Scheurle and Marsden.

In the next result we characterize when the equilibrium points p_1 or p_2 of our tritrophic system (1) are zero–Hopf equilibrium.

Proposition 1. *The equilibrium points p_1 and p_2 are zero–Hopf equilibrium points simultaneously if $b_1 = k^2/2$ and one of the following two conditions holds:*

  (a) $a_2 = 2d_2$ *and* $2d_1 d_2 - a_1(d_2 + \rho) < 0$.
  (b) $a_1 = 2d_1$.

Proof. The proof is made computing directly the eigenvalues at each equilibrium point. First, the characteristic polynomial of the linear approximation of the tritropic system (1) at the equilibrium p_1 is

$$p(\lambda) = -\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0,$$

where,

$$A_0 = \frac{2(a_2 - d_2)d_2 \left(2b_1 - k^2\right)\left(2b_1d_1 + (d_1 - a_1)k^2\right)\rho}{a_2 \left(2b_1 + k^2\right)^2},$$

$$A_1 = \frac{a_2B_2 - 2d_2B_1}{a_2 \left(2b_1 + k^2\right)^2},$$

$$A_2 = \frac{(a_2 - 2d_2)(2b_1d_1 + (d_1 - a_1)k^2) + a_2(k^2 - 2b_1)\rho}{a_2(2b_1 + k^2)},$$

$$B_1 = \left(2b_1d_1 + (d_1 - a_1)k^2\right)\left(k^2(d_2 - \rho) + 2b_1(d_2 + \rho)\right),$$

$$B_2 = k^4(d_1 - a_1)(2d_2 - \rho) + 4b_1^2 d_1(2d_2 + \rho) - 2b_1k^2(2a_1d_2 - 4d_1d_2 + 3a_1\rho).$$

Imposing the condition that $p(\lambda) = -\lambda(\lambda - \varepsilon + \omega I)(\lambda - \varepsilon - \omega I)$, we obtain a system of three equations, that correspond to the coefficients of the terms of degree 0, 1 and 2 in λ of the polynomial. So the solutions of this system in terms of the variables ω, ε, b_1, d_2 and a_1 are the next three group of solutions:

(s1) $\omega = \omega_1,$ $\varepsilon = -\dfrac{(a_1 - 2d_1)(a_2 - 2d_2)}{4a_2},$ $b_1 = \dfrac{k^2}{2};$

(s2) $\omega = \omega_2,$ $\varepsilon = \dfrac{k^2(a_1 - d_1 + \rho) - 2b_1(d_1 + \rho)}{2\left(2b_1 + k^2\right)},$ $d_2 = a_2;$

(s3) $\omega = \omega_3,$ $\varepsilon = \dfrac{\left(k^2 - 2b_1\right)\rho}{2\left(2b_1 + k^2\right)},$ $a_1 = d_1 + \dfrac{2b_1d_1}{k^2}.$

Here each ω_i for $i = 1, 2, 3$ is a funciton in the parameters of the system that it is not necessary to provide explicitly. We must omit solution (s2) because it does not satisfy condition (i).

As we want that the eigenvalues of the linear approximation at p_1 are 0 and $\pm\omega i$, we need that $\varepsilon = 0$ to conclude that p_1 is a zero–Hopf equilibrium point.

(1) When ε is zero we have two cases for (s1).
    (1.1) $\varepsilon = 0$ and $a_1 = 2d_1$. Then we have that the eigenvalues are 0 and $\pm i\sqrt{d_1\rho}$. Then p_1 is a zero–Hopf equilibrium. This corresponds to statement (b) for p_1.
    (1.2) $\varepsilon = 0$ and $a_2 = 2d_2$. In this case the eigenvalues are 0 and $\pm i\sqrt{(-2d_1d_2 + a_1(d_2 + \rho))/2}$. So in order to obtain purely imaginary conjugate eigenvalues it is necessary that $a_1(d_2 + \rho) - 2d_1d_2 > 0$. Then p_1 is a zero–Hopf equilibrium. This corresponds to statement (a) for p_1.

(2) In (s3) we have that $\varepsilon = 0$ if and only if $-2b_1 + k^2 = 0$, which implies that $b_1 = k^2/2$. So the eigenvalues at the point p_1 are

$$0 \quad \text{and} \quad \pm i\sqrt{d_1\rho}.$$

Then we have two pure imaginary conjugate eigenvalues and then p_1 is zero–Hopf equilibrium. Since $b_1 = k^2/2$ we again obtain statement (b) for p_1.

In a similar way we study the eigenvalues of the linear approximation at the equilibrium point p_2 to complete the proof of the proposition. Thus, the set of solutions of the corresponding system of equations determined from the coefficients of degree 0, 1 and 2 in λ of the equality $p(\lambda) = -\lambda(\lambda - \varepsilon + \omega I)(\lambda - \varepsilon - \omega I)$, where $p(\lambda)$ is the characteristic polynomial of the linear part at the point p_2, in terms of variables ω, ε, b_1, d_2 and a_1, are

(s4) $\omega = \omega_1$, $\quad \varepsilon = -\dfrac{(a_1 - 2d_1)(a_2 - 2d_2)}{4a_2}$, $\qquad b_1 = \dfrac{k^2}{2}$;

(s5) $\omega = \omega_2$, $\quad \varepsilon = -\dfrac{-2a_1b_1 + 2b_1(d_1 - \rho) + k^2(d_1 + \rho)}{2(2b_1 + k^2)}$, $\quad d_2 = a_2$;

(s6) $\omega = \omega_3$, $\quad \varepsilon = -\dfrac{(-2b_1 + k^2)\rho}{2(2b_1 + k^2)}$, $\qquad a_1 = d_1 + \dfrac{d_1 k^2}{2b_1}$.

Also here each ω_i for $i = 1, 2, 3$ has an expression in function of the parameters that it is not necessary to write. Again we must omit the solution (s5) because it does not satisfy condition (i).

If we made the analysis using the set of solutions (s4) and (s6), we obtain again the statements (a) and (b) for the equilibrium point p_2. This completes the proof of the proposition. $\qquad\square$

4. The main result

Proposition 1 guarantees the existence of three–dimensional parameter families for which the equilibrium points p_1 and p_2 are of zero–Hopf type simultaneously. Therefore it is possible to have simultaneously two zero–Hopf bifurcations, one on each equilibria. The following theorem establishes that one of these two families of parameters gives rise to a simultaneously zero–Hopf bifurcation in each equilibria, in the sense that a small amplitude periodic orbit borns simultaneously at p_1 and p_2. For the other family of simultaneous zero–Hopf equilibria it is not possible, using the averaging theory, to show that small amplitude periodic orbits borns from those equilibria simultaneously.

Theorem 2. *Assume that the parameters satisfy:*
  (c1) $b_1 = k^2/2$,
  (c2) $a_2 = 2d_2 + \mu$ *where μ is a small parameter,*
  (c3) $a_1(d_2 + \rho) - 2d_1d_2 > 0$, *and*
  (c4) $a_1 - 2d_1 \neq 0$.

Then for $\mu > 0$ sufficiently small two small amplitude periodic orbits born simultaneously one at the equilibrium point p_1 and the other at the equilibrium point p_2 when $\mu = 0$.

Proof. We prove this theorem using the averaging theory of first order, a summary of this theory is given in the appendix. This summary facilitates to follow the computations necessary for proving this theorem.

The hypotheses of the theorem imply that the equilibrium points p_1 and p_2 are zero–Hopf when $\mu = 0$ (see statement (a) of Proposition 1). First, we prove that at the point p_1 there is a zero–Hopf bifurcation. We translate the equilibrium point $p_1 = (x_1, y_1, z_1)$ to the origin of coordinates and we substitute $b_1 = k^2/2$ and $a_2 = 2d_2 + \mu$ with μ a small paramete. Then the differential system (1) becomes

$$\dot{x} = -\frac{\left(\sqrt{2}k + 2x\right)\left(a_1\left(\sqrt{2}k + 2x\right)y - 2x^2\rho\right)}{4\left(k^2 + \sqrt{2}kx + x^2\right)},$$

(2) $$\dot{y} = \left(y + \frac{\sqrt{2}k\rho}{a_1}\right)\left(-d_1 + \frac{a_1\left(\sqrt{2}k + 2x\right)^2}{4\left(k^2 + \sqrt{2}kx + x^2\right)} - E\right),$$

$$\dot{z} = \frac{y(d_2 + \mu)\left(a_1y + 2\sqrt{2}k\rho\right)\left(2a_1d_2z + \sqrt{2}(a_1 - 2d_1)k\rho\right)}{2a_1^2d_2y^2 + 4\sqrt{2}a_1d_2ky\rho + 4k^2(2d_2 + \mu)\rho^2},$$

where

$$E = \frac{(2d_2 + \mu)\left(a_1y + \sqrt{2}k\rho\right)\left(2a_1d_2z + \sqrt{2}(a_1 - 2d_1)k\rho\right)}{2a_1^2d_2y^2 + 4\sqrt{2}a_1d_2ky\rho + 4k^2(2d_2 + \mu)\rho^2}.$$

The matrix of the linear approximation of system (2) at the origin is

$$\begin{pmatrix} 0 & -\dfrac{a_1}{2} & 0 \\ \rho & -\dfrac{(a_1 - 2d_1)\mu}{2(2d_2 + \mu)} & -d_2 \\ 0 & \dfrac{(a_1 - 2d_1)(d_2 + \mu)}{2d_2 + \mu} & 0 \end{pmatrix},$$

and the eigenvalues when $\mu = 0$ are

$$0 \quad \text{and} \quad \pm\frac{M}{\sqrt{2}}i,$$

where $M = \sqrt{a_1(d_2 + \rho) - 2d_1d_2}$. Then the origin of coordinates is a zero–Hopf equilibrium point of (2) when $\mu = 0$.

Now we apply a rescaling of the variables through the change of coordinates $(x, y, z) \to (\mu X, \mu Y, \mu Z)$ obtaining the new differential system

$$\dot{X} = -\frac{a_1 Y}{2} + \mu \frac{X(-a_1 Y + X\rho)}{\sqrt{2}k} + \mathcal{O}(\mu^2),$$

(3)
$$\dot{Y} = -d_2 Z + X\rho + \mu \frac{E_1}{8 d_2 k \rho} + \mathcal{O}(\mu^2),$$

$$\dot{Z} = \frac{1}{2}(a_1 - 2d_1)Y + \mu \frac{Y E_2}{8 d_2 k \rho} + \mathcal{O}(\mu^2),$$

where

$$
\begin{aligned}
E_1 = \ & \sqrt{2}a_1^2 d_2 Y^2 + 4\rho\left(d_1 k Y - \sqrt{2}d_2 X^2 \rho\right) - \\
& 2a_1 Y\left(k\rho + \sqrt{2}d_2(d_1 Y + 2d_2 Z - 2X\rho)\right), \\
E_2 = \ & -\sqrt{2}a_1(a_1 - 2d_1)d_2 Y + 4\sqrt{2}a_1 d_2^2 Z + 2a_1 k\rho - 4d_1 k\rho.
\end{aligned}
$$

Now we shall write the linear part at the origin of the differential system (2) when $\mu = 0$ into its real Jordan normal form, i.e. as

$$
\begin{pmatrix}
0 & -\dfrac{M}{\sqrt{2}} & 0 \\[2mm]
\dfrac{M}{\sqrt{2}} & 0 & 0 \\[2mm]
0 & 0 & 0
\end{pmatrix}.
$$

To do this, we apply a change of variables $(X, Y, Z) \to (u, v, w)$, given by

(4) $\quad X = \dfrac{\sqrt{2}a_1 d_2 w + a_1 u M}{\sqrt{2}M^2}, \quad Y = v, \quad Z = \dfrac{\sqrt{2}a_1 w\rho + (2d_1 - a_1)\, uM}{\sqrt{2}M^2}.$

In the new variables (u, v, w) the differential system (3) writes

$$
\begin{aligned}
\dot{u} \ = \ & -\frac{M}{\sqrt{2}}v + \frac{\mu}{kM}\left(\frac{(a_1 d_2 w\rho)^2}{M^4} + \frac{\sqrt{2}a_1^2 d_2 uw\rho^2}{M^3} + \frac{(a_1 u\rho)^2}{2M^2}\right. \\
& -\frac{a_1^2 d_2 vw(d_2 + \rho)}{M^2} + \frac{a_1 uv\left((a_1 - 2d_1)d_2^2 - a_1 \rho^2\right)}{\sqrt{2}\rho M} \\
& \left. -\frac{k(a_1 - 2d_1)v}{2\sqrt{2}} + \frac{a_1(a_1 - 2d_1)d_2 v^2}{4\rho}\right) + \mathcal{O}(\mu^2), \\
(5) \\
\dot{v} \ = \ & \frac{M}{\sqrt{2}}u + \frac{\mu}{8}\left(\frac{4d_1 v}{d_2} - \frac{2a_1 v\left(\sqrt{2}d_1 d_2 v + k\rho - 2d_2 uM\right)}{d_2 k\rho}\right. \\
& + \frac{a_1^2}{k\rho M^4}\left(\sqrt{2}v^2 M^4 + 2\rho^2\left(\sqrt{2}u^2 M^2\right.\right. \\
& \left.\left.\left. -2\sqrt{2}d_2^2 w^2 - 4d_2 uwM\right)\right)\right) + \mathcal{O}(\mu^2),
\end{aligned}
$$

$$\dot{w} = \frac{\mu}{8}\left(\frac{2(a_1 - 2d_1)v}{d_2} - \frac{\sqrt{2}a_1(a_1 - 2d_1)v^2}{k\rho}\right.$$
$$+\frac{8\sqrt{2}a_1d_1d_2vw}{kM^2} + \frac{4\sqrt{2}a_1(a_1 - 2d_1)d_2^2w^2\rho}{kM^4}$$
$$+\frac{8a_1(a_1 - 2d_1)d_2uw\rho}{kM^3} + \frac{2\sqrt{2}a_1(a_1 - 2d_1)u^2\rho}{kM^2}$$
$$\left. -\frac{4a_1(a_1 - 2d_1)uv(d_2 + \rho)}{k\rho M}\right) + \mathcal{O}(\mu^2),$$

and this system has its linear part at the origin in the real Jordan normal form.

To apply the averaging theory we need to write the differential system (5) in cylindrical coordinates (r, θ, ω). Then we do the change of variables defined by $u = r\cos\theta$, $v = r\sin\theta$ $w = w$, and system (5) becomes

$$\frac{dr}{d\theta} = \frac{\mu}{4\sqrt{2}d_2k\rho M^6}\left[\left(4a_1^2d_2r^2\rho^3M^2\cos^3\theta\right.\right.$$
$$+2\sqrt{2}a_1d_2rM\cos^2\theta\left(4a_1d_2w\rho^3 + r\left(2(a_1 - 2d_1)d_2^2\right.\right.$$
$$\left.-3a_1\rho^2\right)M^2\sin\theta\right) + 2d_2\cos\theta$$
$$\left(4a_1^2d_2^2w^2\rho^3 - r\rho M^2\left(4\sqrt{2}d_1^2d_2k - 2\sqrt{2}a_1d_1k(2d_2 + \rho)\right.\right.$$
$$\left.+a_1^2\left(\sqrt{2}d_2k + 4d_2^2w + \sqrt{2}k\rho + 8d_2w\rho\right)\right)\sin\theta$$
$$\left.+a_1r^2(3a_1d_2 - 6d_1d_2 + 2a_1\rho)M^4\sin^2\theta\right)$$
$$+M\sin(\theta)\left(-4\sqrt{2}a_1^2d_2^3w^2\rho^2 + (a_1 - 2d_1)rM^4\sin\theta\right.$$
$$\left.\left.\left.\left(-2k\rho + \sqrt{2}a_1d_2r\sin\theta\right)\right)\right)\right] + \mathcal{O}(\mu^2)$$
$$\text{(6)} \qquad = \mu F_{1,1}(\theta, r, w) + \mathcal{O}(\mu^2),$$
$$\frac{dw}{d\theta} = \frac{\mu}{4\sqrt{2}k\rho M^5}\left[4\sqrt{2}a_1(a_1 - 2d_1)d_2^2\rho w^2\right.$$
$$+2\sqrt{2}a_1(a_1 - 2d_1)r^2\rho M^2\cos^2\theta$$
$$-\sqrt{2}a_1(a_1 - 2d_1)r^2M^4\sin^2\theta$$
$$+\frac{8\sqrt{2}r\left(kM^2(a_1 - 2d_1) + 4\sqrt{2}a_1d_1d_2^2w\right)\rho M^3\sin\theta}{d_2}$$
$$-4a_1(a_1 - 2d_1)rM\cos\theta\left(-2d_2w\rho^2\right.$$
$$\left.\left.+r(d_2 + \rho)M^2\sin\theta\right)\right] + \mathcal{O}(\mu^2)$$
$$= \mu F_{1,2}(\theta, r, w) + \mathcal{O}(\mu^2).$$

Using the notation of the appendix we have $t = \theta$, $T = 2\pi$, $\mathbf{x} = (r, w)^T$,

$$F_1(\theta, r, w) = \begin{pmatrix} F_{1,1}(\theta, r, w) \\ F_{1,2}(\theta, r, w) \end{pmatrix}, \quad \text{and} \quad f_1(r, w) = \begin{pmatrix} f_{1,1}(r, w) \\ f_{1,2}(r, w) \end{pmatrix}.$$

It is immediate to check that system (6) satisfies all the assumptions of Theorem 3.

Now we compute the integrals (10), i.e.

$$f_{1,1}(r, w) = \frac{1}{2\pi} \int_0^{2\pi} F_{1,1}(\theta, r, w) dT$$
$$= \frac{r \left(8a_1^2 d_2^2 w\rho^2 - \sqrt{2}(a_1 - 2d_1)kM^4\right)}{8kd_2 M^5},$$

$$f_{1,2}(r, w) = \frac{1}{2\pi} \int_0^{2\pi} F_{1,2}(\theta, r, w) dT$$
$$= \frac{a_1(a_1 - 2d_1) \left(8(d_2 w\rho)^2 + (rM)^2 \left(2\rho^2 - M^2\right)\right)}{8k\rho M^5}.$$

The system $f_{1,1}(r, w) = f_{1,2}(r, w) = 0$ has a unique solution (r^*, w^*), namely

$$r^* = \frac{(a_1 - 2d_1)kM^3}{2a_1^2 d_2 \rho \sqrt{M^2 - 2\rho^2}}, \quad w^* = \frac{(a_1 - 2d_1)kM^4}{4\sqrt{2}(a_1 d_2 \rho)^2}.$$

Finally, the Jacobian (11) at the point (r^*, w^*) takes the value

$$\frac{(a_1 - 2d_1)^3}{16a_1 d_2 \rho M^2},$$

that by assumptions it is not zero. Then by the averaging theorem (Theorem 3) we have a periodic solution $(r(\theta, \mu), w(\theta, \mu))$ of system (6) for $\mu > 0$ sufficiently small such that $(r(0, \mu), w(0, \mu)) \to (r^*, w^*)$ when $\mu \to 0$. Hence, the differential system (5) has the periodic solution

$$(7) \qquad \begin{pmatrix} u(\theta, \mu) \\ v(\theta, \mu) \\ w(\theta, \mu) \end{pmatrix} = \begin{pmatrix} r(\theta, \mu) \cos \theta \\ r(\theta, \mu) \sin \theta \\ w(\theta, \mu) \end{pmatrix},$$

considering $\mu > 0$ sufficiently small. Consequently, the differential system (3) has a periodic orbit

$$\left(X(\theta), Y(\theta), Z(\theta)\right)$$

obtained from (7) through the change of variables (4). To finish, the differential system (2) has a periodic solution

$$\left(x(\theta), y(\theta), z(\theta)\right) = \left(\mu X(\theta), \mu Y(\theta), \mu Z(\theta)\right),$$

for $\mu > 0$ sufficiently small. Clearly, this periodic orbit tends to the origen of coordinates when $\mu \to 0$. Therefore, it is a small amplitude periodic solution starting at the zero–Hopf equilibrium point located at the origin of coordinates when $\mu = 0$ which correspond to the zero–Hopf equilibrium point p_1.

Following exactly the same computations we prove that at the equilibrium point p_2 also there exists a small amplitud periodic solution bifurcating from the equilibrium point p_2. This concludes the proof of the theorem.   □

Appendix: The averaging theory of first order

In this section we present some basic results related with the averaging theory that we will use in the proof of our main result.

The next theorem establish the existence and stability or instability of the periodic solutions for a periodic differential system. The proof of this theorem can be found in Theorems 11.5 and 11.6 of Verhulst [26].

Consider the differential systems

$$\dot{\mathbf{x}} = \mu F_1(t, \mathbf{x}) + \mu^2 F_2(t, \mathbf{x}, \mu), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{8}$$

with $\mathbf{x} \in D$, where D is an open subset of $\mathbb{R}^n$, $t \geq 0$ and μ is a small parameter. Moreover we assume that both $F_1(t, \mathbf{x})$ and $F_2(t, \mathbf{x}, \mu)$ are $T-$periodic in t. Now we also consider in D the averaged differential equation

$$\dot{\mathbf{y}} = \mu f_1(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{9}$$

where

$$f_1(\mathbf{y}) = \frac{1}{T} \int_0^T F_1(t, \mathbf{y}) dt. \tag{10}$$

Under certain conditions the equilibrium solutions of the averaged equation (9) correspond to $T-$periodic solutions of equation (8).

Theorem 3. *Consider the two initial value problems (8) and (9) and suppose:*

    (i) *F_1, its Jacobian $\partial F_1/\partial x$, its Hessian $\partial^2 F_1/\partial x^2$, F_2 and its Jacobian $\partial F_2/\partial x$ are defined, continuous and bounded by a constant independent of μ in $[0, \infty) \times D$ and $\mu \in (0, \mu_0]$.*

    (ii) *F_1 and F_2 are $T-$periodic in t (T independent of μ).*

Then the following statements hold.

    (a) *If p is an equilibrium point of the averaged equation (9) and*

$$\det \left(\frac{\partial f_1}{\partial \mathbf{y}} \right) \bigg|_{\mathbf{y}=p} \neq 0, \tag{11}$$

        then there exists a $T-$periodic solution $\varphi(t, \mu)$ of the differential equation (8) such that $\varphi(0, \mu) \to p$ as $\mu \to 0$.

    (b) *The stability or instability of the periodic solution $\varphi(t, \mu)$ is given by the stability or instability of the equilibrium point p of the averaged system (9). In fact the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\varphi(t, \mu)$.*

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