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# A SURVEY ON THE MINIMAL SETS OF LEFSCHETZ PERIODS FOR MORSE–SMALE DIFFEOMORPHISMS ON SOME CLOSED MANIFOLDS

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*Dedicated to Jorge Lewowicz in occasion of his* Doctorado Honoris Causa *conferred by* Universidad de La República.

Abstract. We present the actual state of the study of the minimal sets of Lefschetz periods $\mathrm{MPer}_L(f)$ for the Morse–Smale diffeomorphisms on some closed manifolds, as the connected compact surfaces (orientable or not) without boundary, the $n$–dimensional torus and some other manifolds. The results on $\mathrm{MPer}_L(f)$ are valid for $C^1$ self–maps on the mentioned closed manifolds with finitely many periodic points all of them hyperbolic such that all the eigenvalues of the induced maps on homology are roots of unity. This class of maps includes the Morse–Smale diffeomorphisms.

## 1. Introduction

In the study of the discrete dynamical systems and, in particular in the study of the orbits of self–maps defined on a given compact manifold, the periodic orbits play an important role. These last forty years there was many results showing that some simple assumptions force qualitative and quantitative properties (like the set of periods) of a map. One of the first results in this direction was the famous paper *Period three implies chaos* for the interval continuous self–maps, see [24].

One of the most used tool for studying the existence of fixed points and periodic points, for continuous self maps on compact manifolds, and more generally topological spaces which are retract of finite simplicial complexes, is the Lefschetz fixed point theorem and its improvements (*cf.* [1, 2, 7, 8, 9, 11, 18, 19, 25, 30]). The Lefschetz zeta function $\zeta_f(t)$ simplifies the study of the periodic points of $f$. It is a generating function for the Lefschetz numbers of all iterates of $f$. All these notions are defined in Section 3.

The Morse-Smale diffeormorphisms have simple dynamic behaviour, however they are an important class of discrete dynamical systems. Our objective is to describe the periodic structure of these systems, in particular their minimal sets of periods. The results that we present here are valid for a class of maps that includes the Morse-Smale diffeomorphisms, i.e. $C^1$ maps having finitely many periodic points all of them hyperbolic and with the same action on the homology as the Morse–Smale diffeomorphisms.

Many papers have been published analyzing the relationships between the dynamics of the Morse–Smale diffeomorphisms and the topology of the manifold where they are defined, see for instance [3, 4, 5, 11, 12, 13, 14, 15, 31, 33, 35, 36, 37]. The Morse–Smale diffeomorphisms have a relatively simple orbit structure. In fact, their set of periodic orbits is finite, and their structure is preserved under small $C^1$ perturbations.

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Let $X$ be a topological space. Let $f : X \to X$ be a continuous map. A point $x \in M$ is *nonwandering* of $f$ if for any neighborhood $\mathcal{U}$ of $x$ there exists some positive integer $m$ such that $f^m(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$. The set of nonwandering points of $f$ is denoted by $\Omega(f)$.

We say that $x$ is a *periodic point* of $f$ of period $p$ if $f^p(x) = x$ and $f^j(x) \neq x$ for all $0 \leq j < p$. The set $\{x, f(x), \ldots, f^{p-1}(x)\}$ is called the *periodic orbit* of the periodic point $x$. If $X = M$ is a $C^1$ manifold and $f$ a $C^1$ map, we say that $x$ a periodic point of period $p$, is *hyperbolic* if the eigenvalues of $Df^p(x)$ have modulus different from 1.

If $x$ is a hyperbolic periodic point of $f$ of period $p$, the *stable manifold* of $x$ is

$$W^s(x) = \{y \in M \, : \, d(x, f^{pm}(y)) \to 0 \text{ as } m \to \infty\}$$

and the *unstable manifold* of $x$ is

$$W^u(x) = \{y \in M \, : \, d(x, f^{pm}(y)) \to 0 \text{ as } m \to -\infty\},$$

where $d$ is the distance on $M$ induced by the supremum norm.

We say that $M$ is a *closed manifold* if it is a connected compact manifold without boundary. A diffeomorphism $f : M \to M$ is *Morse–Smale* if

(i) $\Omega(f)$ is finite,
(ii) all periodic points are hyperbolic, and
(iii) for each $x, y \in \Omega(f)$, $W^s(x)$ and $W^u(y)$ have transversal intersections.

The first condition implies that $\Omega(f)$ is the set of all periodic points of $f$.

Two diffeomorphisms $f, g \in \mathrm{Diff}(M)$ are $C^1$ *equivalent* if and only if there exists a $C^1$ homeomorphism $h : M \to M$ such that $h \circ f = g \circ h$. A diffeomorphism $f$ is *structurally stable* provided that there exists a neighborhood $\mathcal{U}$ of $f$ in $\mathrm{Diff}(M)$ such that each $g \in \mathcal{U}$ is topologically equivalent to $f$. Since the class of Morse–Smale diffeomorphisms is structurally stable inside the class of all diffeomorphisms (see [33, 34, 32]), to understand the dynamics of this class is an interesting problem.

Let

$$\mathrm{Per}(f) = \{m \in \mathbb{N} \, : \, f \text{ has a periodic orbit of period } m\},$$

i.e. $\mathrm{Per}(f)$ is the *set of periods* of $f$.

The Lefschetz zeta function $\zeta_f(t)$ for a $C^1$ Morse–Smale diffeomorphism $f$ on a closed surface $M$ is introduced in Section 3. Using this function we define the minimal set of Lefschetz periods $\mathrm{MPer}_L(f)$ for a such diffeomorphism $f$ in Section 4. As we shall see the study of this set is important because any other $C^1$ Morse–Smale diffeomorphism $g$ on a manifold $M$ in the same homology class than $f$ satisfies

$$\mathrm{MPer}_L(f) \subseteq \mathrm{Per}(g).$$

The set $\mathrm{MPer}_L(f)$ is computable from the Lefschetz zeta function of $f$, and it consists of odd positive integers, see Proposition 7. In Section 5 we mention the results related to the $\mathrm{MPer}_L(f)$ for maps on orientable closed surfaces, in Section 6 on non-orientable closed surfaces, and in Section 7 on the $n$-dimensional torus.

The results are of two different types, some give an explicit description of the $\mathrm{MPer}_L(f)$ for all Morse-Smale diffeomorphisms on a given manifold, see Theorems 8, 13 and 15. Other results describe what type of subsets of odd positive integers can be $\mathrm{MPer}_L(f)$ for some Morse-Smale diffeomorphisms $f$, see Theorems 9, 10, 11, 12 14, 16 and 17.

## 2. Cyclotomic polynomials

In this section we describe some basic properties of the cyclotomic polynomials which we shall use in our study of the Lefschetz zeta function.

Let $n$ denote an integer. The $n$–th cyclotomic polynomial is given by

$$c_n(t) = \frac{1-t^n}{\prod\limits_{\substack{d|n \\ d<n}} c_d(t)}, \tag{1}$$

for $n > 1$ and $c_1(t) = 1 - t$. An alternative way to express $c_n(t)$ is

$$c_n(t) = \prod_k (w_k - t),$$

for $n \neq 2$, where $w_k = e^{2\pi ik/n}$ and $k$ runs over the relative primes to $n$ and smaller than $n$, for $c_2(t) = -(w_2 - t)$. For more details about these polynomials see [23].

Let $\varphi(n)$ be the degree of $c_n(t)$. Then $n = \sum_{d|n} \varphi(d)$. So $\varphi(n)$ is the *Euler function*, which satisfies

$$\varphi(n) = n \prod_{\substack{p|n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right).$$

Therefore if the prime decomposition of $n$ is $p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then

$$\varphi(n) = \prod_{j=1}^{k} p_j^{\alpha_j - 1}(p_j - 1).$$

From the formula (1), we have

$$c_n(t) = \prod_{d|n} (1 - t^d)^{\mu(n/d)}$$

where $\mu$ is the *Möbius function*, i.e.

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } k^2 | m \text{ for some } k \in \mathbb{N}, \\ (-1)^r & \text{if } m = p_1 \cdots p_r \text{ has distinct primes factors.} \end{cases}$$

**Lemma 1** (Gauss)**.** *Irreducible polynomials whose roots are roots of unity are precisely the collection of cyclotomic polynomials.*

Here are some elementary properties of the cyclotomic polynomials (cf. [23]).

(p1) If $p > 1$ is prime then $c_p(t) = (1 - t^p)/(1 - t)$.
(p2) If $p = 2r$ with $r$ odd then $c_{2r}(t) = c_r(-t)$.
(p3) If $p = 2^\alpha$ with $\alpha$ a positive integer, then $c_p(t) = 1 + t^{2^{\alpha-1}}$.
(p4) If $p = r^\alpha$ with $r > 2$ prime and $\alpha$ a positive integer,, then $c_p(t) = c_r(t^{r^{\alpha-1}}) = (1 - t^{r^\alpha})/(1 - t^{r^{\alpha-1}})$.
(p5) If $p = 2^n r$ with $r$ odd and $n > 1$, then $c_n(t) = c_{2r}(t^{2^{n-1}})$.
(p6) For all $n$ we have that $c_n(0) = 1$, and the leading term of $c_n(t)$ is 1 if $n \geq 2$.
(p7) The degree of $c_n(t)$ is even for $n > 2$.

## 3. Lefschetz Zeta Function

Let $X$ be a topological space which is a retract of a finite simplicial complex [20]. The compact manifolds, the CW complexes are spaces of this type. Let $n$ be the topological dimension of $X$. If $f : X \to X$ is a continuous map on $X$, it induces a homomorphism on the $k$–th rational homology group of $X$ for $0 \leq k \leq n$, i.e. $f_{*k} : H_k(X, \mathbb{Q}) \to H_k(X, \mathbb{Q})$. The $H_k(X, \mathbb{Q})$ is a finite dimensional vector space over $\mathbb{Q}$ and it is torsion free, because it is a vector space over $\mathbb{Q}$. The map $f_{*k}$ is linear given by a matrix with integer entries, then the *Lefschetz number* of $f$ defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \text{trace}(f_{*k}),$$

TABLE 1. The first thirty cyclotomic polynomials.

| | | |
|---|---|---|
| $c_1(t) = 1 - t$ | $c_2(t) = 1 + t$ | $c_3(t) = \dfrac{1-t^3}{1-t}$ |
| $c_4(t) = 1 + t^2$ | $c_5(t) = \dfrac{1-t^5}{1-t}$ | $c_6(t) = \dfrac{1+t^3}{1+t}$ |
| $c_7(t) = \dfrac{1-t^7}{1-t}$ | $c_8(t) = 1 + t^4$ | $c_9(t) = \dfrac{1-t^9}{1-t^3}$ |
| $c_{10}(t) = \dfrac{1+t^5}{1+t}$ | $c_{11}(t) = \dfrac{1-t^{11}}{1-t}$ | $c_{12}(t) = \dfrac{1+t^6}{1+t^2}$ |
| $c_{13}(t) = \dfrac{1-t^{13}}{1-t}$ | $c_{14}(t) = \dfrac{1+t^7}{1+t}$ | $c_{15}(t) = \dfrac{(1-t^{15})(1-t)}{(1-t^3)(1-t^5)}$ |
| $c_{16}(t) = 1 + t^8$ | $c_{17}(t) = \dfrac{1-t^{17}}{1-t}$ | $c_{18}(t) = \dfrac{1+t^9}{1+t^3}$ |
| $c_{19}(t) = \dfrac{1-t^{19}}{1-t}$ | $c_{20}(t) = \dfrac{1+t^{10}}{1+t^2}$ | $c_{21}(t) = \dfrac{(1-t^{21})(1-t)}{(1-t^3)(1-t^7)}$ |
| $c_{22}(t) = \dfrac{1+t^{11}}{1+t}$ | $c_{23}(t) = \dfrac{1-t^{23}}{1-t}$ | $c_{24}(t) = \dfrac{1+t^{12}}{1+t^4}$ |
| $c_{25}(t) = \dfrac{1-t^{25}}{1-t^5}$ | $c_{26}(t) = \dfrac{1+t^{13}}{1+t}$ | $c_{27}(t) = \dfrac{1-t^{27}}{1-t^9}$ |
| $c_{28}(t) = \dfrac{1+t^{14}}{1+t^2}$ | $c_{29}(t) = \dfrac{1-t^{29}}{1-t}$ | $c_{30}(t) = \dfrac{(1+t^{15})(1+t)}{(1+t^3)(1+t^5)}$ |

is always an integer number.

The Lefschetz Fixed Point Theorem states that if $L(f) \neq 0$ then $f$ has a fixed point (cf. [7]). If we consider the Lefschetz number of $f^m$, then in general is not true that $L(f^m) \neq 0$ implies that $f$ has a periodic point of period $m$; it only implies the existence of a periodic point with period a divisor of $m$.

The technique of using Lefschetz numbers to obtain information about the periods of a map is also used in many other papers, see for instance the book of Jezierski and Marzantowicz [22], the article of Gierzkiewicz and Wójcik [17] and the references quoted in both.

The *Lefschetz zeta function* of $f$ is defined as

$$\zeta_f(t) = \exp\left( \sum_{m \geq 1} \frac{L(f^m)}{m}\, t^m \right).$$

This function keeps the information of the Lefschetz number for all the iterates of $f$, so this function gives information about the set of periods of $f$. There is an alternative way to compute it:

$$\zeta_f(t) = \prod_{k=0}^{n} \det(Id_{*k} - t f_{*k})^{(-1)^{k+1}}, \tag{2}$$

where $n = \dim X$, $n_k = \dim H_k(X, \mathbb{Q})$, $Id := Id_{*k}$ is the identity map on $H_k(X, \mathbb{Q})$, and by convention $\det(Id_{*k} - tf_{*k}) = 1$ if $n_k = 0$ (cf. [11]).

A rational linear transformation is called *quasi–unipotent* if their eigenvalues are roots of unity. We say that a continuous map $f : X \to X$ is *quasi–unipotent* if the maps $f_{*k}$ are quasi–unipotent for $0 \le k \le n$.

**Proposition 2** (Shub [35]). *Let $M$ be a compact manifold. If $f : M \to M$ is a Morse–Smale diffeomorphism, then $f$ is quasi–unipotent.*

The following result shows that the class of $C^1$ quasi-unipotent maps are more general that the Morse-Smale diffeomorphisms.

**Theorem 3** ([29]). *Let $M$ be a $C^1$ closed manifold of dimension $n$ and $f : M \to M$ be a $C^1$ map with finitely many periodic points all of then hyperbolic. Then the eigenvalues of $f_{*k}$ are zero or roots of unity for $0 \le k \le n$.*

When $X$ is a surface, i.e. a 2-dimensional manifold, we can compute the Lefschetz zeta function of a quasi-unipotent self-map on $X$. If $X = M_g$ is an orientable surface of genus $g$ without boundary then $H_0(X, \mathbb{Q}) = \mathbb{Q}$, $H_2(X, \mathbb{Q}) = \mathbb{Q}$ and

$$H_1(X, \mathbb{Q}) = \underbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}_{2g}.$$

And if $X = N_g$ a non-orientable surface without boundary of genus $g$, i.e., $X$ is a connected sum of $g$ real projective planes, then $H_0(X, \mathbb{Q}) = \mathbb{Q}$, $H_2(X, \mathbb{Q}) \approx 0$ and

$$H_1(X, \mathbb{Q}) = \underbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}_{g-1}.$$

If the rational linear transformation $f_{*k}$ is quasi-unipotent. Then its characteristic polynomial is in $\mathbb{Z}[x]$ and its factors over $\mathbb{Z}$ are irreducible polynomials whose roots are roots of unity. So these factors are cyclotomic polynomials.

The following result is used to compute the Lefschetz zeta functions of $f$.

**Proposition 4** ([26]). *If $f_{*1}$ is quasi–unipotent, then*

$$\det(Id_{*1} - tf_{*1}) = (-1)^{1+\det(f_{*1})} \det(f_{*1} - tId_{*1}).$$

**Proposition 5.** *Let $X$ be a closed surface and $f : X \to X$ be a continuous map such that $f_{*1}$ is quasi-unipotent and $p(t)$ its characteristic polynomial .*

(1) *If $X = N_g$ is a non–orientable closed surface of genus $g$, then*

$$\zeta_f(t) = \frac{p(t)}{1-t}, \tag{3}$$

*being $p(t)$ a product of cyclotomic polynomials of degree $g - 1$.*

(2) *If $X = M_g$ is an orientable closed surface of genus $g$, then*

$$\zeta_f(t) = \begin{cases} \dfrac{p(t)}{(1-t)^2} & \text{if } f \text{ is orientation preserving,} \\ \dfrac{p(t)}{(1-t)(1+t)} & \text{if } f \text{ is orientation reversing} \end{cases} \tag{4}$$

*being $p(t)$ a product of cyclotomic polynomials of degree $2g$.*

We say that a rational function $\zeta_f(t)$ is a *possible zeta function*, if $\zeta_f(t)$ satisfies formula either (3), or (4) for a map $f : X \to X$ satisfying the hypothesis of Proposition 5.

Proposition 5 allows us to describe explicitly the possible Lefschetz zeta functions for quasi-unipotent maps on $M_g$ and $N_g$. By finding all possible products of cyclotomic polynomials of total degree $2g$ or $g - 1$, in the orientable or non-orientable cases respectively. In the following paragraphs, we present the possible Lefschetz zeta functions for small $g$, see [26, 28] for details.

If $X = M_0 = \mathbb{S}^2$, then $\zeta_f(t) = (1-t)^{-2}$ when $f$ is orientation preserving, and $\zeta_f(t) = (1-t^2)^{-1}$ when $f$ is orientation reversing.

Let $X = M_1 = \mathbb{T}^2$. If $f$ preserves the orientation, then the possible characteristic polynomials of $f_{*1}$ are:

$$c_1^2(t), \quad c_2^2(t), \quad c_3(t), \quad c_4(t), \quad c_6(t).$$

So the possible zeta functions are:

$$1, \quad \frac{(1+t)^2}{(1-t)^2}, \quad \frac{1+t^2}{(1-t)^2}, \quad \frac{1-t^3}{(1-t)^3}, \quad \frac{1+t^3}{(1+t)(1-t)^2}.$$

If $f$ reverses the orientation, then the only possible characteristic polynomial of $f_{*1}$ is $c_1(t)c_2(t)$. Then $\zeta_f(t) = 1$ is the only possible zeta function.

Let $X = M_2$. If $f$ preserves the orientation, then the possible $\zeta_f(t)$ are:

$$\frac{1-t^5}{(1-t)^3}, \quad \frac{1+t^5}{(1+t)(1-t)^2}, \quad \frac{1+t^4}{(1-t)^2}, \quad \frac{1+t^6}{(1+t^2)(1-t)^2},$$

$$\frac{(1-t^3)^2}{(1-t)^4}, \quad \frac{(1-t^3)(1+t^3)}{(1-t)^3(1+t)}, \quad \frac{(1+t^2)(1-t^3)}{(1-t)^3}, \quad \frac{(1+t^2)^2}{(1-t)^2},$$

$$\frac{(1+t^2)(1+t^3)}{(1-t)^2(1+t)}, \quad \frac{(1+t^3)^2}{(1+t)^2(1-t)^2}, \quad \frac{(1-t^3)(1+t)^2}{(1-t)^3}, \quad \frac{(1+t^2)(1+t)^2}{(1-t)^2},$$

$$\frac{(1+t^3)(1+t)}{(1-t)^2}, \quad \frac{1-t^3}{1-t}, \quad 1+t^2, \quad \frac{1+t^3}{1+t},$$

$$\frac{(1+t)^4}{(1-t)^2}, \quad (1+t)^2, \quad (1-t)^2.$$

If $f$ reverses the orientation, then the possible $\zeta_f(t)$ are:

$$\frac{1-t^3}{1-t}, \quad 1+t^2, \quad \frac{1+t^3}{1+t}, \quad (1-t)^2, \quad (1+t)^2.$$

In Tables 2 and 3 are listed the possible Lefschetz zeta functions for quasi-unipotent maps on $M_3$.

Now we consider non–orientable surfaces. If $X = N_1$, i.e. the real projective plane, then the only possible zeta function is $\zeta_f(t) = (1-t)^{-1}$. When $X = N_2$ the Klein bottle, the possible zeta functions are $\zeta_f(t) = 1$ or $\zeta_f(t) = (1+t)(1-t)^{-1}$. When $X = N_3$ the possible Lefschetz zeta functions are

$$1-t, \quad 1+t, \quad \frac{(1+t)^2}{1-t}, \quad \frac{1-t^3}{(1-t)^2}, \quad \frac{1+t^3}{(1-t)(1+t)}, \quad \frac{1+t^2}{1-t}. \tag{5}$$

In Table 4 and 5 the possible Lezfchetz zeta functions on $N_4$ and $N_5$ are listed, respectively. For higher $g$, see [28].

The case of $X = \mathbb{D}_n$, the closed disc with $n$-holes, is similar to $N_{n+1}$; since they have the same homology groups, see [16].

The case of $X = \mathbb{T}^n$ is studied in [14] and [6]. The homology spaces of $\mathbb{T}^n$ with rational coefficients are

$$H_k(\mathbb{T}^n, \mathbb{Q}) = \underbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}_{n_k},$$

where $n_k = \binom{n}{k}$. Since the homology spaces of $\mathbb{T}^n$ form an exterior algebra, then the map $f_{*1}$ determines all the other $f_{*k}$, for $2 \le k \le n$, in the following way (*cf.* [38]): Let $p_1(t)$ be the characteristic polynomial of $f_{*1}$, then

$$p_1(t) = \prod_{j=1}^{n} (t - \lambda_j),$$

where the $\lambda_j$ are the eigenvalues of $f_{*1}$. Then the other $p_k(t)$ are expressed as:

$$\begin{aligned} p_2(t) &= \prod_{i<j}(t-\lambda_i\lambda_j), \\ p_3(t) &= \prod_{i<j<l}(t-\lambda_i\lambda_j\lambda_l), \\ &\vdots \\ p_n(t) &= t-(\lambda_1\lambda_2\cdots\lambda_n). \end{aligned}$$

Using this information, Proposition 4 and formula (2), we can compute explicitly some of the possible possible Lefschetz zeta functions for quasi-unipotent maps on $\mathbb{T}^n$ (*cf.* [6]). In [14] all possible $\zeta_f(t)$ for quasi-unipotent maps on $\mathbb{T}^3$ and $\mathbb{T}^4$ are listed.

## 4. The minimal set of Lefschetz periods $\mathrm{MPer}_L(f)$

In this section we assume that $X$ is a $C^1$ compact manifold. and let $f : X \to X$ be a $C^1$ map. Let $x$ be a hyperbolic periodic point of period $p$ of $f$ and $E^u_x$ its unstable space, i.e. the subspace of the tangent space $T_xX$ generated by the eigenvectors of $Df^p(x)$ of modulus larger than 1. Let $\gamma$ be the orbit of $x$, the *index* $u$ of $\gamma$ is the dimension of $E^u_x$. We define the *orientation* type $\Delta$ of $\gamma$ as $+1$ if $Df^p(x) : E^u_x \to E^u_x$ preserves orientation and $-1$ if reverses the orientation. The collection of the triples $(p, u, \Delta)$ belonging to all periodic orbits of $f$ is called the *periodic data* of $f$. The same triple can appear more than once if it corresponds to different periodic orbits.

**Theorem 6** (Franks [10])**.** *Let $f$ be a $C^1$ a map on a closed manifold having finitely many periodic points all of them hyperbolic, and let $\Sigma$ be the periodic data of $f$. Then the Lefschetz zeta function $\zeta_f(t)$ of $f$ satisfies*

$$\zeta_f(t) = \prod_{(p,u,\Delta)\in\Sigma} (1-\Delta t^p)^{(-1)^{u+1}}. \tag{6}$$

Clearly the Morse–Smale diffeomorphisms on orientable and non–orientable closed manifolds satisfy the hypotheses of this theorem.

We remark, this theorem is also true when $X$ is a $C^1$ compact manifold with boundary and $f : X \to X$ a $C^1$ map such that it does not have periodic points on the boundary of $X$, see [10].

Theorem 6 allows to define the minimal set of Lefschetz periods of a $C^1$ map on a compact manifold having finitely many periodic points all of them hyperbolic. Such a map has a Lefschetz zeta function of the form (6). Note that in general the expression of one of these Lefschetz zeta functions is not unique as product of elements of the form $(1 \pm t^p)^{\pm 1}$. For instance the following possible Lefschetz zeta function can be written in four different ways in the form given by (6):

$$\zeta_f(t) = \frac{(1-t^3)(1+t^3)}{(1-t)^2(1+t)} = \frac{1-t^6}{(1-t)^2(1+t)} = \frac{1-t^6}{(1-t)(1-t^2)} = \frac{(1-t^3)(1+t^3)}{(1-t)(1-t^2)}.$$

According with Theorem 6, the first expression will provide the periods $\{1, 3\}$ for $f$, the second the periods $\{1, 6\}$, the third the period $\{1, 2, 6\}$, and finally the fourth the periods $\{1, 2, 3\}$. Then for this Lefschetz zeta function $\zeta_f(t)$ we will define its *minimal set of Lefschetz periods* as

$$\mathrm{MPer}_L(f) = \{1, 3\} \cap \{1, 6\} \cap \{1, 2, 6\} \cap \{1, 2, 3\} = \{1\}.$$

If $\zeta_f(t) \neq 1$ then it can be written as

$$\zeta_f(t) = \prod_{i=1}^{N_\zeta} (1+\Delta_i t^{r_i})^{m_i}, \tag{7}$$

TABLE 2. Possible Lefschetz zeta functions for orientation preserving quasi-unipotent maps on $M_3$

| | | | |
|---|---|---|---|
| $\frac{1-t^7}{(1-t)^3}$ | $\frac{1-t^9}{(1-t)^2(1-t^3)}$ | $\frac{1+t^7}{(1-t)^2(1+t)}$ | $\frac{1+t^9}{(1+t^3)(1-t)^2}$ |
| $\frac{(1-t^5)(1-t^3)}{(1-t)^4}$ | $\frac{(1-t^5)(1+t^2)}{(1-t)^3}$ | $\frac{(1-t^5)(1+t^3)}{(1-t)^3(1+t)}$ | $\frac{(1+t^5)(1-t^3)}{(1+t)(1-t)^3}$ |
| $\frac{(1+t^5)(1+t^2)}{(1+t)(1-t)^2}$ | $\frac{(1+t^5)(1+t^3)}{(1-t)^2(1+t)^2}$ | $\frac{(1+t^4)(1-t^3)}{(1-t)^3}$ | $\frac{(1+t^4)(1+t^2)}{(1-t)^2}$ |
| $\frac{(1+t^4)(1+t^3)}{(1-t)^2(1+t)}$ | $\frac{(1+t^6)(1-t^3)}{(1+t^2)(1-t)^3}$ | $\frac{1+t^6}{(1-t)^2}$ | $\frac{(1+t^6)(1+t^3)}{(1+t^2)(1-t)^2(1+t)}$ |
| $\frac{(1-t^3)^3}{(1-t)^5}$ | $\frac{(1-t^3)^2(1+t^2)}{(1-t)^4}$ | $\frac{(1-t^3)^2(1+t^3)}{(1-t)^4(1+t)}$ | $\frac{(1-t^3)(1+t^3)(1+t^2)}{(1-t)^3(1+t)}$ |
| $\frac{(1-t^3)(1+t^3)^2}{(1-t)^3(1+t)^2}$ | $\frac{(1+t^2)^2(1-t^3)}{(1-t)^3}$ | $\frac{(1+t^2)^3}{(1-t)^2}$ | $\frac{(1+t^2)^2(1+t^3)}{(1-t)^2(1+t)}$ |
| $\frac{(1+t^2)(1+t^3)^2}{(1-t)^2(1+t)^2}$ | $\frac{(1+t^3)^3}{(1+t)^3(1-t)^2}$ | $\frac{(1-t^3)^2(1+t)^2}{(1-t)^4}$ | $\frac{(1-t^3)(1+t)^2(1+t^2)}{(1-t)^3}$ |
| $\frac{(1-t^3)(1+t)(1+t^3)}{(1-t)^3}$ | $\frac{(1+t^2)^2(1+t)^2}{(1-t)^2}$ | $\frac{(1+t^2)(1+t)(1+t^3)}{(1-t)^2}$ | $\frac{(1+t^3)^2}{(1-t)^2}$ |
| $\frac{(1-t^3)^2}{(1-t)^2}$ | $\frac{(1-t^3)(1+t^2)}{1-t}$ | $\frac{(1-t^3)(1+t^3)}{(1-t)(1+t)}$ | $(1+t^2)^2$ |
| $\frac{(1+t^2)(1+t^3)}{1+t}$ | $\frac{(1+t^3)^2}{(1+t)^2}$ | $\frac{(1+t)^4(1-t^3)}{(1-t)^3}$ | $\frac{(1+t)^4(1+t^2)}{(1-t)^2}$ |
| $\frac{(1+t)^3(1+t^3)}{(1-t)^2}$ | $\frac{(1-t^3)(1+t)^2}{1-t}$ | $(1+t^2)(1+t)^2$ | $(1+t^3)(1+t)$ |
| $(1-t^3)(1-t)$ | $(1+t^2)(1-t)^2$ | $\frac{(1+t^3)(1-t)^2}{1+t}$ | $\frac{1-t^5}{1-t}$ |
| $\frac{(1-t^5)(1+t)^2}{(1-t)^3}$ | $\frac{1+t^5}{1+t}$ | $\frac{(1+t^5)(1+t)}{(1-t)^2}$ | $1+t^4$ |
| $\frac{(1+t^4)(1+t)^2}{(1-t)^2}$ | $\frac{1+t^6}{1+t^2}$ | $\frac{(1+t^6)(1+t)^2}{(1+t^2)(1-t)^2}$ | $\frac{(1-t^3)^2(1+t)^2}{(1-t)^4}$ |
| $\frac{(1+t^2)(1+t)^4}{(1-t)^2}$ | $(1+t)^4$ | $\frac{(1+t)^6}{(1-t)^2}$ | $(1-t)^2(1+t)^2$ |
| $(1-t)^4$ | | | |

where $\Delta_i = \pm 1$, the $r_i$'s are positive integers, $m_i$'s are nonzero integers and $N_\zeta$ is a positive integer depending on $f$.

If $\zeta_f(t) \neq 1$ the *minimal set of Lefschetz periods* of $f$ is defined as

$$\mathrm{MPer}_L(f) := \bigcap \{r_1, \ldots, r_{N_\zeta}\},$$

where the intersection is considered over all the possible expressions (7) of $\zeta_f(t)$. If $\zeta_f(t) = 1$, then we define $\mathrm{MPer}_L(f) := \emptyset$. Roughly speaking the minimal set of Lefschetz periods of $f$ is the intersection of all the sets of periods forced by the finitely many different representations of $\zeta_f(t)$ as product and quotient of elements of the form

TABLE 3. Possible Lefschetz zeta functions for orientation reversing quasi-unipotent maps on $M_3$.

| | | | |
|---|---|---|---|
| $\frac{(1-t^3)^2}{(1-t)^2}$ | $\frac{(1-t^3)(1+t^2)}{1-t}$ | $\frac{(1-t^3)(1+t^3)}{(1-t)(1+t)}$ | $(1+t^2)^2$ |
| $\frac{(1+t^2)(1+t^3)}{1+t}$ | $\frac{(1+t^3)^2}{(1+t)^2}$ | $\frac{(1-t^3)(1+t)^2}{(1-t)}$ | $(1-t^3)(1-t)$ |
| $(1+t^2)(1+t)^2$ | $(1+t^2)(1-t)^2$ | $(1+t^3)(1+t)$ | $\frac{(1+t^3)(1-t)^2}{1+t}$ |
| $(1+t)^4$ | $(1+t)^2(1-t)^2$ | $(1-t)^4$ | |

TABLE 4. Possible Lefschetz zeta functions for quasi-unipotent maps on $N_4$.

| | | | | |
|---|---|---|---|---|
| $(1-t)^2$, | $(1-t)(1+t)$, | $(1+t)^2$, | $\frac{(1+t)^3}{1-t}$, | $\frac{1-t^3}{1-t}$, |
| $\frac{1+t^3}{1+t}$, | $1+t^2$, | $\frac{(1+t)(1-t^3)}{(1-t)^2}$, | $\frac{(1+t)(1+t^2)}{1-t}$. | $\frac{1+t^3}{1-t}$ |

TABLE 5. Possible Lefschez zeta functions for quasi-unipotent maps on $N_5$.

| | | | |
|---|---|---|---|
| $\frac{1-t^5}{(1-t)^2}$, | $\frac{1+t^5}{(1-t)(1+t)}$, | $\frac{1+t^4}{1-t}$, | $\frac{(1-t^3)^2}{(1-t)^3}$, |
| $\frac{(1-t^3)(1+t^3)}{(1+t)(1-t)^2}$, | $\frac{(1-t^3)(1+t^2)}{(1-t)^2}$, | $\frac{(1+t^3)^2}{(1+t)^2(1-t)}$, | $\frac{(1+t^3)(1+t^2)}{(1+t)(1-t)}$, |
| $\frac{(1+t^2)^2}{1-t}$, | $\frac{1+t^6}{(1-t)(1+t^2)}$ | $\frac{(1-t^3)(1+t)}{1-t}$, | $\frac{(1-t^3)(1+t)^2}{(1-t)^2}$, |
| $\frac{(1+t^3)(1-t)}{1+t}$, | $1+t^3$, | $\frac{(1+t^3)(1+t)}{1-t}$, | $1-t^3$, |
| $(1+t^2)(1-t)$, | $(1+t^2)(1+t)$, | $\frac{(1+t^2)(1+t)^2}{1-t}$, | $(1-t)^3$, |
| $(1+t)(1-t)^2$, | $(1+t)^2(1-t)$, | $(1+t)^3$, | $\frac{(1+t)^4}{1-t}$. |

$(1 \pm t^p)^{\pm 1}$. Clearly

$$\mathrm{MPer}_L(f) \subseteq \mathrm{Per}(f).$$

If $M$ is a closed $C^1$ manifold we denote by $\mathcal{F}(M)$ the set of all $C^1$ self-maps on $M$ having finitely many periodic points all of them hyperbolic, and such that the induced maps on homology $f_{*k}$ are quasi-unipotent. By Proposition 2 the Morse-Smale diffeomorphisms on $M$ belong to $\mathcal{F}(M)$. It can be checked that there are elements in $\mathcal{F}(M)$, which are not Morse-Smale diffeormorphisms.

The results of this article apply to maps $f$ in $\mathcal{F}(M)$. Moreover the techniques used in the proofs of the main theorems of the present paper, can be used when the map $f$ satisfies the hypothesis of Theorem 3, i.e. $f$ has finitely many periodic points all of them hyperbolic. Therefore it is possible to obtain similar results with this weaker hypothesis.

**Proposition 7** ([21, 27])**.** *There are no even numbers in $MPer_L(f)$.*

*Proof.* If the number $2d$ is in $\mathrm{MPer}_L(f)$ then $(1 \pm t^{2d})^m$ is a factor of the Lefschetz zeta function $\zeta_f(t)$, for some $m \neq 0$. So if the factor is $(1 - t^{2d})^m$ it can be written as $(1 - t^d)^m(1 + t^d)^m$. Then due to the fact that the intersection of the exponents is taken over all possible expressions (7) of $\zeta_f(t)$, the number $2d$ is not in $\mathrm{MPer}_L(f)$.

If the factor is $(1 + t^{2d})^m$, then it can be written as

$$(1 + t^{2d})^m = \frac{(1 + t^{2d})^m(1 - t^{2d})^m}{(1 - t^{2d})^m} = \frac{(1 - t^{4d})^m}{(1 - t^d)^m(1 + t^d)^m}.$$

Therefore, again $2d \notin \mathrm{MPer}_L(f)$. □

## 5. $\mathrm{MPer}_L(f)$ for maps in $\mathcal{F}(M_g)$

In this section we summarize the results related to the minimal set of Lefschetz periods for maps on orientable closed surfaces.

**Theorem 8** ([26])**.** *Let $f \in \mathcal{F}(M_g)$.*

(a) *If $g = 0$ then $MPer_L(f) = \{1\}$ if $f$ preserves orientation, and $MPer_L(f) = \emptyset$ if $f$ reverses orientation.*

(b) *If $g = 1$ then $MPer_L(f) = \emptyset$ if $f$ reverses orientation, and*

$$MPer_L(f) = \begin{cases} \emptyset & \text{if} \quad \zeta_f(t) = 1, \\ \{1\} & \text{if} \quad \zeta_f(t) = \dfrac{(1+t)^2}{(1-t)^2}, \ \text{or} \ \zeta_f(t) = \dfrac{1+t^2}{(1-t)^2}, \\ \{1,3\} & \text{if} \quad \zeta_f(t) = \dfrac{1-t^3}{(1-t)^3}, \ \text{or} \ \dfrac{1+t^3}{(1-t)(1-t^2)}. \end{cases}$$

*if $f$ preserves orientation.*

(c) *If $g = 2$ and $f$ is orientation reversing then*

$$MPer_L(f) = \begin{cases} \emptyset & \text{if} \quad \zeta_f(t) = 1 + t^2, \\ \{1\} & \text{if} \quad \zeta_f(t) = (1-t)^2, \ \text{or} \ \zeta_f(t) = (1+t)^2, \\ \{1,3\} & \text{if} \quad \zeta_f(t) = \dfrac{1-t^3}{(1-t)}, \ \text{or} \ \dfrac{1+t^3}{(1+t)}. \end{cases}$$

(d) *If $g = 2$ and $f$ is orientation preserving then its $MPer_L(f)$ is one of the following ones:*

$\{1\}$ *if* $\zeta_f(t)$ *is* $\dfrac{(1-t^3)(1+t^3)}{(1-t)^3(1+t)}$, $\dfrac{(1+t)^4}{(1-t)^2}$, $(1+t)^2$, $(1-t)^2$, $\dfrac{(1+t^2)(1+t)^2}{(1-t)^2}$, $\dfrac{(1+t^2)^2}{(1-t)^2}$, $\dfrac{1+t^4}{(1-t)^2}$ *or* $\dfrac{1+t^6}{(1+t^2)(1-t)^2}$;

$\{3\}$ *if* $\zeta_f(t)$ *is* $\dfrac{(1+t^3)^2}{(1+t)^2(1-t)^2}$;

$\{1,3\}$ *if* $\zeta_f(t)$ *is* $\dfrac{(1-t^3)^2}{(1-t)^4}$, $\dfrac{(1+t^3)(1+t)}{(1-t)^2}$, $\dfrac{1-t^3}{1-t}$, $\dfrac{1+t^3}{1+t}$, $\dfrac{(1-t^3)(1+t)^2}{(1-t)^2}$, $\dfrac{(1+t^2)(1-t^3)}{(1-t)^3}$ *or* $\dfrac{(1+t^2)(1+t^3)}{(1-t)^2(1+t)}$;

$\{1,5\}$ *if* $\zeta_f(t)$ *is* $\dfrac{1-t^5}{(1-t)^3}$ *or* $\dfrac{1+t^5}{(1+t)(1-t)^2}$;

$\emptyset$ *if* $\zeta_f(t)$ *is* $1+t^2$.

(e) *If $g = 3$ and $f$ is orientation reversing then $MPer_L(f)$ is $\emptyset$, $\{1\}$, or $\{1,3\}$.*

(f) *If $g = 3$ and $f$ is orientation preserving then $MPer_L(f)$ is $\emptyset$, $\{1\}$, $\{1,3\}$, $\{1,5\}$, $\{1,7\}$, $\{3\}$, $\{3,5\}$, $\{1,3,5\}$ or $\{1,3,9\}$.*

The proof of Theorem 8 follows from the calculation of all possible Lefschetz zeta functions, as it was shown in Section 3, and after using the definition of the Lefschetz periods.

From the properties of the cyclotomic polynomials we obtain Theorems 9, 10, 11 and 12.

**Theorem 9** ([26]). *The following statements hold.*

(a) *If $2g-1$ is an odd prime, then $\{1, 2g-1\}$ is a possible $MPer_L(f)$ for some orientation preserving map $f \in \mathcal{F}(M_g)$.*

(b) *If $g = p^{\alpha-1}(p-1)/2 + 1$ with $p$ odd prime then $p^\alpha$ and $p^{\alpha-1}$ are contained in the $MPer_L(f)$ for some orientation preserving map $f \in \mathcal{F}(M_g)$. If $p = 2$ then $MPer_L(f) = \emptyset$.*

(c) *If $g$ is an odd prime number, then $MPer_L(f) = \emptyset$ for some orientation preserving map $f \in \mathcal{F}(M_g)$.*

We would like to know what kind of subsets of the positive odd integers can be realized as $\mathrm{MPer}_L(f)$ for some maps $f \in \mathcal{F}(M_g)$, for some $g$. The following result gives a partial answer to this question.

**Theorem 10** ([21]). *Let $p_1, \ldots, p_k$ be distinct odd primes and $\alpha_{i,j}$ integers, such that $\alpha_{i,j} > \alpha_{i,j+1}$ for $1 \le i \le k$, $1 \le j \le l_i$. Let $S$ be one of the following sets*

(1) $S = \{p_1, \ldots, p_k\}$,

(2) $S = \{p_i^{\alpha_{i,1}}, \ldots, p_i^{\alpha_{i,l_i}}\}$, *for* $1 \le i \le k$,

(3) $S = \left\{p_1^{\alpha_{1,1}}, \ldots, p_1^{\alpha_{1,l_1}}, \ldots, p_k^{\alpha_{k,1}}, \ldots, p_k^{\alpha_{k,l_k}}\right\}$,

(4) $S = \{1, p_1, \ldots, p_k\}$,

(5) $S = \{1, p_i^{\alpha_{i,1}}, \ldots, p_i^{\alpha_{i,l_i}}\}$, *for* $1 \le i \le k$, *or*

(6) $S = \left\{1, p_1^{\alpha_{1,1}}, \ldots, p_1^{\alpha_{1,l_1}}, \ldots, p_k^{\alpha_{k,1}}, \ldots, p_k^{\alpha_{k,l_k}}\right\}$.

*Then there is a possible Lefschetz zeta function $\zeta_f(t)$, with $f \in \mathcal{F}(M_g)$ for some $g$, such that $MPer_L(f) = S$.*

Theorems 11 and 12 give a complete characterization when the number 1 belongs to the minimal set of Lefschetz periods for a map $f \in \mathcal{F}(M_g)$. This characterization is given in terms of the arithmetic properties of the cyclotomic polynomials which are factors of the characteristic polynomial of the induced map on the first homology space.

**Theorem 11** ([21]). *Let $f \in \mathcal{F}(M_g)$ an orientation reversing map. Let $q(t)$ be the characteristic polynomial of $f_{*1}$. Then $q(t)$ can be written as*

$$q(t) = (c_{n_1}(t))^{\gamma_1} \cdots (c_{n_k}(t))^{\gamma_k} (c_{2^{\alpha_1} m_1}(t))^{\beta_1} \cdots (c_{2^{\alpha_l} m_l}(t))^{\beta_l} (c_1(t))^{2n+1} (c_2(t))^{2m+1},$$

*where $n_1, \dots, n_k, m_1, \dots, m_l$ are positive odd integers, greater than 1, and $\alpha_i \geq 1$, $\beta_j \geq 0$, $\gamma_i \geq 0$, $n, m \geq 0$, such that*

$$\left( \sum_{i=1}^{k} \varphi(n_i)\gamma_i \right) + \left( \sum_{i=1}^{l} \varphi(m_i) 2^{\alpha_i - 1} \beta_i \right) + 2n + 2m + 2 = 2g,$$

*with $\varphi(m)$ the Euler function of $m$. Moreover $1 \notin MPer_L(f)$ if and only if*

$$\sum_{\{j\,:\,\alpha_j = 1\}} \mu(m_j)\beta_j + 2m = \sum_{i=1}^{k} \mu(n_i)\gamma_i + 2n,$$

*where $\mu(m)$ is the Möbius function of $m$*

**Theorem 12** ([21]). *Let $f \in \mathcal{F}(M_g)$ an orientation reversing map. Let $q(t)$ be the characteristic polynomial of $f_{*1}$. Then $q(t)$ can be written as Then*

$$q(t) = (c_{n_1}(t))^{\gamma_1} \cdots (c_{n_k}(t))^{\gamma_k} (c_{2^{\alpha_1} m_1}(t))^{\beta_1} \cdots (c_{2^{\alpha_l} m_l}(t))^{\beta_l} (c_1(t))^{2n} (c_2(t))^{2m},$$

*where $n_1, \dots, n_k, m_1, \dots, m_l$ are positive odd integers greater than 1, and $\alpha_i \geq 1$, $\beta_j \geq 0$, $\gamma_i \geq 0$, $n, m \geq 0$, such that*

$$\left( \sum_{i=1}^{k} \varphi(n_i)\gamma_i \right) + \left( \sum_{i=1}^{l} \varphi(m_i) 2^{\alpha_i - 1} \beta_i \right) + 2n + 2m = 2g,$$

*with $\varphi$ the Euler function. Furthermore $1 \notin MPer_L(f)$ if and only if*

$$\sum_{\{j\,:\,\alpha_j = 1\}} \mu(m_j)\beta_j + 2m = \sum_{i=1}^{k} \mu(n_i)\gamma_i + 2(n-1).$$

## 6. MPer$_L(f)$ for maps in $\mathcal{F}(N_g)$

In this section we summarize the results related to the minimal set of Lefschetz periods for maps on non-orientable closed surfaces.

**Theorem 13** ([28]). *Let $f$ be a Morse–Smale diffeomorphism on $N_g$, or more generally a map belonging to $\mathcal{F}(N_g)$.*

(a) *If $g = 1$ then $MPer_L(f)$ is $\{1\}$.*
(b) *If $g = 2$ then $MPer_L(f)$ is $\emptyset$ or $\{1\}$.*
(c) *If $g = 3$ then $MPer_L(f)$ is $\{1\}$, $\{3\}$ or $\{1, 3\}$.*
(d) *If $g = 4$ then $MPer_L(f)$ is $\emptyset$, $\{1\}$ or $\{1, 3\}$.*
(e) *If $g = 5$ then $MPer_L(f)$ is $\{1\}$, $\{3\}$, $\{5\}$, $\{1, 3\}$ or $\{1, 5\}$.*
(f) *If $g = 6$ then $MPer_L(f)$ is $\emptyset$, $\{1\}$, $\{3\}$, $\{1, 3\}$ or $\{1, 5\}$.*
(g) *If $g = 7$ then $MPer_L(f)$ is $\{1\}$, $\{3\}$, $\{5\}$, $\{7\}$, $\{1, 3\}$, $\{1, 5\}$ $\{1, 7\}$, $\{1, 3, 5\}$ or $\{1, 3, 9\}$.*
(h) *If $g = 8$ then $MPer_L(f)$ is $\emptyset$, $\{1\}$, $\{3\}$, $\{1, 3\}$, $\{1, 5\}$, $\{1, 7\}$, $\{3, 5\}$, $\{3, 9\}$, $\{1, 3, 5\}$ or $\{1, 3, 9\}$.*
(i) *If $g = 9$ then $MPer_L(f)$ is $\{1\}$, $\{3\}$, $\{5\}$, $\{7\}$,$\{9\}$, $\{1, 3\}$, $\{1, 5\}$ $\{1, 7\}$, $\{1, 9\}$, $\{3, 9\}$, $\{1, 3, 5\}$, $\{1, 3, 7\}$, $\{1, 3, 9\}$, $\{3, 5, 15\}$ or $\{1, 3, 5, 15\}$.*

**Theorem 14** ([28]). *The following statement hold.*

(a) *If $g$ is an odd prime, then the sets $\{1, g\}$ and $\{g\}$ are possible $MPer_L(f)$ for some $f \in \mathcal{F}(N_g)$.*
(b) *If $g = p^{\alpha-1}(p-1)+1$ with $p$ an odd prime and $\alpha > 1$, then the set $\{1, p^{\alpha-1}, p^{\alpha}\}$ is a possible $MPer_L(f)$ for some $f \in \mathcal{F}(N_g)$.*
(c) *If $g$ is even, then the empty set is a possible $MPer_L(f)$ for some $f \in \mathcal{F}(N_g)$.*
(d) *If $g$ is odd, then $MPer_L(f) \neq \emptyset$ for all $f \in \mathcal{F}(N_g)$.*
(e) *For every positive integer $g$, $\{1\}$ is a possible $MPer_L(f)$ for some $f \in \mathcal{F}(N_g)$.*
(f) *If $g$ is odd and $p$ is an odd prime such that $1 < p \leq g$, then $\{p\}$ is a possible $MPer_L(f)$ for some $f \in \mathcal{F}(N_g)$.*
(g) *Let $p_1, \ldots, p_k$ be different odd primes larger than 1, and let $\alpha_1, \ldots, \alpha_k$ be positive integers.*
  (g.1) *Then the set $\{p_1, \ldots, p_k\}$ is a possible $MPer_L(f)$ for some $f \in \mathcal{F}(N_g)$, where $g = \left(\sum_{i+1}^{k} p_i\right) - (k-1)$ if $k$ is odd, and $g = \left(\sum_{i+1}^{k} p_i\right) - (k-2)$ if $k$ is even.*
  (g.2) *Then the set $\{p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}\}$ is a possible $MPer_L(f)$ for some $f \in \mathcal{F}(N_g)$ and for some convenient genus $g$.*
  (g.3) *Let $p$ be an odd prime number. Then the set $\{p^{\alpha_1}, \ldots, p^{\alpha_k}\}$ is a possible $MPer_L(f)$ for some $f \in \mathcal{F}(N_g)$ and for some convenient genus $g$.*
  (g.4) *Let $\alpha_{i,j}$ positive integers such that $\alpha_{i,j} > \alpha_{i,j+1}$ for $1 \leq i \leq k$ and $1 \leq j \leq l_i$. Then the set $\{p_1^{\alpha_{1,1}}, \ldots, p_1^{\alpha_{1,l_1}}, \ldots, p_k^{\alpha_{k,1}}, \ldots, p_k^{\alpha_{k,l_k}}\}$ is a possible $MPer_L(f)$ for some $f \in \mathcal{F}(N_g)$ and for some convenient genus $g$.*

Statement (d) of Theorem 14 shows an important difference in the periodic structure between the $C^1$ Morse–Smale diffeomorphisms on orientable and non–orientable compact surfaces without boundary. Statement (c) of Theorem 9 states that for all orientable compact surfaces without boundary there are $C^1$ Morse–Smale diffeomorphisms having their minimal set of Lefschetz periods empty, and statement (d) of Theorem 14 shows that this is never the case for $C^1$ Morse–Smale diffeomorphisms on the non–orientable closed surfaces.

The results of statements (f) and (g) of Theorem 14 also for $C^1$ Morse–Smale diffeomorphisms on orientable closed surfaces, and their proof are similar to the proof of Theorem 10.

The description of the $\mathrm{MPer}_L(f)$ for Morse-Smale diffeomorphisms on $\mathbb{D}_g$ (and maps in $\mathcal{F}(\mathbb{D}_g)$), which does not have periodic points on the boundary is the same of the $\mathrm{MPer}_L(f)$ of Morse-Smale diffeomorphisms on $N_{g+1}$ (maps in $\mathcal{F}(N_{g+1})$), since the possible Lefschetz zeta functions are the same, see [16].

**Open question**. *Can any finite set of odd positive integers be the minimal set of Lefschetz periods for a $C^1$ Morse–Smale diffeomorphism on some orientable/non–orientable compact surface without boundary with a convenient genus?*

We think that the answer to this open question is positive. In [21] this open question was stated as a conjecture for the $C^1$ Morse–Smale diffeomorphisms on orientable compact surface without boundary.

## 7. $\mathrm{MPer}_L(f)$ for maps in $\mathcal{F}(\mathbb{T}^n)$

In this section we summarize the results related to the minimal set of Lefschetz periods for maps on the $n$-dimensional torus.

**Theorem 15** ([14]). *Let $f \in \mathcal{F}(\mathbb{T}^n)$.*

(a) *If $n = 3$ and $f$ is orientation preserving then $MPer_L(f) = \emptyset$ .*
(b) *If $n = 3$ and $f$ is orientation reversing then $MPer_L(f) = \emptyset$, $\{1\}$, or $\{1, 3\}$.*
(c) *If $n = 4$ and $f$ is orientation reversing then $MPer_L(f) = \emptyset$ .*

(d) *If* $n = 4$ *and* $f$ *is orientation preserving then* $MPer_L(f) = \emptyset$, $\{1\}$,$\{1,3\}$ *or* $\{1,5\}$.

**Theorem 16** ([14]). *If* $n$ *is even then* $MPer_L(f) = \emptyset$, *for* $f \in \mathcal{F}(\mathbb{T}^n)$ *and orientation reversing. If* $n$ *is odd then* $MPer_L(f) = \emptyset$, *for* $f \in \mathcal{F}(\mathbb{T}^n)$ *and orientation preserving.*

**Theorem 17** ([6]). *Let* $f$ *be an orientation preserving map in* $\mathcal{F}(\mathbb{T}^n)$, *with* $n = p - 1$ *and* $p$ *and odd prime, such that the characteristic polynomial of* $f_{*1}$ *is* $c_p(t)$, *Then* $MPer_L(f) = \{1, p\}$.

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