

PERIODIC SOLUTIONS INDUCED BY AN UPRIGHT POSITION OF SMALL OSCILLATIONS OF A SLEEPING SYMMETRICAL GYROSTAT

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ABSTRACT. The aim of this paper is to provide sufficient conditions for the existence of periodic solutions emerging from an upright position of small oscillations of a sleeping symmetrical gyrostat with equations of motion

$$\begin{aligned}\ddot{x} + \alpha\dot{y} - \beta x &= \varepsilon F_1(t, x, \dot{x}, y, \dot{y}), \\ \ddot{y} - \alpha\dot{x} - \beta y &= \varepsilon F_2(t, x, \dot{x}, y, \dot{y}).\end{aligned}$$

being α and β parameters satisfying $\Delta = \alpha^2 - 4\beta > 0$ and $\beta - \frac{\alpha^2}{2} \pm \frac{\alpha\sqrt{\Delta}}{2} < 0$, ε a small parameter and, F_1 and F_2 smooth periodic maps in the variable t in resonance $p:q$ with some of the periodic solutions of the system for $\varepsilon = 0$, where p and q are positive integers relatively prime. The main tool used is the averaging theory.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The equations of motion of a sleeping gyrostat on the upright position under the influence of small periodic momenta are

$$(1) \quad \begin{aligned}\ddot{x} + \alpha\dot{y} - \beta x &= \varepsilon F_1(t, x, \dot{x}, y, \dot{y}), \\ \ddot{y} - \alpha\dot{x} - \beta y &= \varepsilon F_2(t, x, \dot{x}, y, \dot{y}),\end{aligned}$$

for more details on them see Appendix II or papers [6] and [10]. Here the dot denotes derivative with respect to the time t . The parameter ε is small and the smooth functions F_1 and F_2 define the perturbed torques which, in general, are periodic functions in the variable t and in resonance $p:q$ with some of the periodic solutions of the sleeping symmetrical gyrostat for $\varepsilon = 0$, being p and q positive integers relatively prime.

Recall that a gyrostat is a mechanical system S composed by a rigid solid S_1 to which other bodies S_2 are connected; these other bodies may be variable or rigid, but the key property is that they must not be rigidly connected to S_1 , so that the movements of S_2 with respect to S_1 do not modify the distribution of mass within the compound system S .

For instance, we can consider a rigid main body S_1 , designated as the *platform*, supporting additional bodies S_2 , which possess axial symmetry and are designated as *rotors*. These rotors may rotate with respect to the platform in such a way that the mass distribution within the system as a whole is not altered; this will produce an internal angular momentum, designated as *gyrostat momentum*, which will be

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normally regarded as a constant. Note that when this constant vector is zero, the motion of the system is reduced to the motion of a rigid solid, see for instance Figure 1 where a gyrostat in the frame of the three body problem is presented, and [3, 4] or [5] for more details on this class of mechanical systems.

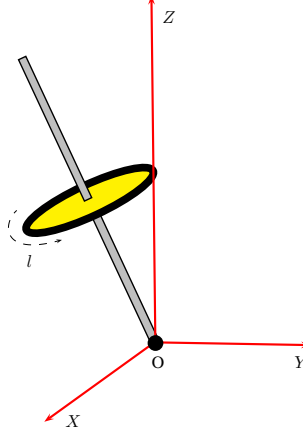


FIGURE 1. Gyrostat with a fixed point

The objective of this paper is to provide, using the averaging theory, a system of nonlinear equations whose simple zeros provide periodic solutions of the differential system (1). In order to present our results we need some preliminary definitions and notation. Periodic orbits of other kind of gyrostat have been studied for several authors, see for instance the article [1] and the references therein.

The unperturbed system with two differential equations of second order

$$(2) \quad \begin{aligned} \ddot{x} + \alpha\dot{y} - \beta x &= 0, \\ \ddot{y} - \alpha\dot{x} - \beta y &= 0, \end{aligned}$$

written as a differential system of first order with four in the variables (x, \dot{x}, y, \dot{y}) has a unique singular point at the origin with eigenvalues

$$\pm\omega_1 i, \quad \pm\omega_2 i,$$

which are the roots of the polynomial $\omega^4 - (\alpha^2 - 2\beta)\omega^2 + \beta^2$ when the parameters α and β verifies that $\Delta = \alpha^2 - 4\beta > 0$. The frequencies ω_i are given by

$$\omega_1 = \sqrt{\left| \beta - \frac{\alpha^2}{2} - \frac{\alpha\sqrt{\Delta}}{2} \right|}, \quad \omega_2 = \sqrt{\left| \beta - \frac{\alpha^2}{2} + \frac{\alpha\sqrt{\Delta}}{2} \right|} \quad \text{since} \quad \beta - \frac{\alpha^2}{2} \pm \frac{\alpha\sqrt{\Delta}}{2} < 0.$$

As usual we define that the ratio of the two frequencies is *non-resonant* with π if $\omega_2\pi/\omega_1$ is not a rational number.

System (2) in the phase space (x, \dot{x}, y, \dot{y}) has two planes passing through the origin filled of periodic solutions with the exception of the origin. These periodic solutions have periods $T_1 = 2\pi/\omega_1$ and $T_2 = 2\pi/\omega_2$, according they belong to the plane associated to the eigenvectors with eigenvalues $\pm\omega_1 i$ or $\pm\omega_2 i$, respectively. We shall study which of these periodic solutions persist for the perturbed system

(1) when the parameter ε is sufficiently small and the perturbed functions F_i for $i = 1, 2$ have period either pT_1/q , or pT_2/q , where p and q are positive integers relatively prime.

We define the constants Φ_1 and Φ_2 by

$$\Phi_1 = 2 \left(\alpha + \sqrt{\Delta} \right)^{-1}, \quad \Phi_2 = 2 \left(\alpha - \sqrt{\Delta} \right)^{-1},$$

and the functions:

$$(3) \quad \begin{aligned} \mathcal{G}_1(X_0, Y_0) &= \int_0^{pT_1} (\cos(\omega_1 t) F_1^*(t) - \sin(\omega_1 t) F_2^*(t)) dt, \\ \mathcal{G}_2(X_0, Y_0) &= \int_0^{pT_1} (\sin(\omega_1 t) F_1^*(t) + \cos(\omega_1 t) F_2^*(t)) dt, \end{aligned}$$

where

$$\begin{aligned} F_1^*(t) &= \frac{\Phi_2}{\Phi_2 - \Phi_1} F_2(t, \mathcal{A}_1(t), \mathcal{B}_1(t), \mathcal{C}_1(t), \mathcal{D}_1(t)), \\ F_2^*(t) &= -\frac{\omega_1}{\Phi_2 \omega_2^2 - \Phi_1 \omega_1^2} F_1(t, \mathcal{A}_1(t), \mathcal{B}_1(t), \mathcal{C}_1(t), \mathcal{D}_1(t)), \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_1(t) &= \Phi_1 (X_0 \cos(\omega_1 t) + Y_0 \sin(\omega_1 t)), \\ \mathcal{B}_1(t) &= \omega_1 \Phi_1 (Y_0 \cos(\omega_1 t) - X_0 \sin(\omega_1 t)), \\ \mathcal{C}_1(t) &= (X_0 \sin(\omega_1 t) - Y_0 \cos(\omega_1 t)) / \omega_1, \\ \mathcal{D}_1(t) &= X_0 \cos(\omega_1 t) + Y_0 \sin(\omega_1 t). \end{aligned}$$

A zero (X_0^*, Y_0^*) of the nonlinear system

$$(4) \quad \mathcal{G}_1(X_0, Y_0) = 0, \quad \mathcal{G}_2(X_0, Y_0) = 0,$$

such that

$$\det \left(\frac{\partial(\mathcal{G}_1, \mathcal{G}_2)}{\partial(X_0, Y_0)} \Big|_{(X_0, Y_0) = (X_0^*, Y_0^*)} \right) \neq 0,$$

is called a *simple zero* of system (4).

The statement of our main result on the periodic solutions of the differential system (1) which bifurcate from the periodic solutions of period T_1 of the unperturbed system traveled p times is the following.

Theorem 1. *Let p and q be positive integers relatively prime and assume that the smooth functions F_1 and F_2 of the equations of motion of (1) are periodic in the variable t of period pT_1/q . We assume that the ratio of the frequencies ω_2/ω_1 is not resonant with π . Then for $\varepsilon \neq 0$ sufficiently small and for every simple zero $(X_0^*, Y_0^*) \neq (0, 0)$ of the nonlinear system (4), the perturbed system (1) has a periodic solution $(x(t, \varepsilon), y(t, \varepsilon))$ tending to the periodic solution $(x(t), y(t)) = (\mathcal{A}_1(t), \mathcal{C}_1(t))|_{(X_0, Y_0) = (X_0^*, Y_0^*)}$ of the unperturbed system (2) traveled p times.*

Theorem 1 is proved in section 2. Its proof is based in the averaging theory for computing periodic solutions, see Appendix I for more details on this technique.

An application of Theorem 1 is presented in the following corollary, which will be proved in section 3.

Corollary 2. *Assume that $F_1(t, x, \dot{x}, y, \dot{y}) = 0$, $F_2(t, x, \dot{x}, y, \dot{y}) = \sin(\omega_1 t)(1 - \dot{x}^2)y^2$ and that the ratio of the frequencies ω_2/ω_1 is not resonant with π . Then the system (1) for $\varepsilon \neq 0$ sufficiently small has two periodic solutions $(x(t, \varepsilon), y(t, \varepsilon))$ tending to the two periodic solutions $(x(t), y(t)) = (\mathcal{A}_1(t), \mathcal{C}_1(t))|_{(X_0, Y_0) = (X_k^*, Y_k^*)}$ of (2) when $\varepsilon \rightarrow 0$, given by $(X_1^*, Y_1^*) = (\sqrt{30}/(5\Phi_1\omega_1), 0)$ and $(X_2^*, Y_2^*) = (0, \sqrt{2}/(\Phi_1\omega_1))$.*

Now we define the functions:

$$(5) \quad \begin{aligned} \mathcal{G}_3(Z_0, W_0) &= \int_0^{pT_2} (\cos(\omega_2 t) F_3^*(t) - \sin(\omega_2 t) F_4^*(t)) dt, \\ \mathcal{G}_4(Z_0, W_0) &= \int_0^{pT_2} (\sin(\omega_2 t) F_3^*(t) + \cos(\omega_2 t) F_4^*(t)) dt, \end{aligned}$$

with

$$\begin{aligned} F_3^*(t) &= -\frac{\Phi_1}{\Phi_2 - \Phi_1} F_2(t, \mathcal{A}_2(t), \mathcal{B}_2(t), \mathcal{C}_2(t), \mathcal{D}_2(t)), \\ F_4^*(t) &= \frac{\omega_2}{\Phi_2\omega_2^2 - \Phi_1\omega_1^2} F_1(t, \mathcal{A}_2(t), \mathcal{B}_2(t), \mathcal{C}_2(t), \mathcal{D}_2(t)), \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_2(t) &= \Phi_2(Z_0 \cos(\omega_2 t) + W_0 \sin(\omega_2 t)), \\ \mathcal{B}_2(t) &= \omega_2 \Phi_2(W_0 \cos(\omega_2 t) - Z_0 \sin(\omega_2 t)), \\ \mathcal{C}_2(t) &= (Z_0 \sin(\omega_2 t) - W_0 \cos(\omega_2 t)) / \omega_2, \\ \mathcal{D}_2(t) &= Z_0 \cos(\omega_2 t) + W_0 \sin(\omega_2 t). \end{aligned}$$

Consider the nonlinear system

$$(6) \quad \mathcal{G}_3(Z_0, W_0) = 0, \quad \mathcal{G}_4(Z_0, W_0) = 0.$$

The statement of our main result on the periodic solutions of the differential equations (1) which bifurcate from the periodic solutions of the unperturbed system (2) with period T_2 traveled p times is the following.

Theorem 3. *Let p and q be positive integers relatively prime and assume that the smooth functions F_1 and F_2 of the equations of motion of (1) are periodic in the variable t of period pT_2/q . Assume that the ratio of the frequencies ω_2/ω_1 is not resonant with π . Then for $\varepsilon \neq 0$ sufficiently small and for every simple zero $(Z_0^*, W_0^*) \neq (0, 0)$ of the nonlinear system (6), the perturbed system (1) has a periodic solution $(x(t, \varepsilon), y(t, \varepsilon))$ tending to the periodic solution $(x(t), y(t)) = (\mathcal{A}_2(t), \mathcal{C}_2(t))|_{(Z_0, W_0) = (Z_0^*, W_0^*)}$ of the unperturbed system (2) traveled p times.*

Theorem 3 will be proved in section 2.

In the next corollary an application of Theorem 3 is given.

Corollary 4. *Assume that $F_1(t, x, \dot{x}, y, \dot{y}) = \sin(\omega_2 t)(1 - x\dot{x})$, $F_2(t, x, \dot{x}, y, \dot{y}) = \cos(\omega_2 t)y$, and that the ratio of the frequencies ω_2/ω_1 is not resonant with π . Then system (1) for $\varepsilon \neq 0$ sufficiently small has one periodic solution $(x(t, \varepsilon), y(t, \varepsilon))$ tending to the periodic solutions $(x(t), y(t)) = (\mathcal{A}_2(t), \mathcal{C}_2(t))|_{(Z_0, W_0) = (Z^*, W^*)}$ of the unperturbed system (2) when $\varepsilon \rightarrow 0$, where $(Z^*, W^*) = (\sqrt{2}/(\Phi_2\sqrt{\omega_2}))(1, -1)$.*

Corollary 4 will be proved in section 3.

2. PROOF OF THEOREMS 1 AND 3

Introducing the variables $(X_1, Y_1, Z_1, W_1) = (x, \dot{x}, y, \dot{y})$ we can write the differential system (1) as a first-order differential system defined in \mathbb{R}^4 in the following form

$$(7) \quad \begin{aligned} \dot{X}_1 &= Y_1, \\ \dot{Y}_1 &= \beta X_1 - \alpha W_1 + \varepsilon F_1(t, X_1, Y_1, Z_1, W_1), \\ \dot{Z}_1 &= W_1, \\ \dot{W}_1 &= \alpha Y_1 + \beta Z_1 + \varepsilon F_2(t, X_1, Y_1, Z_1, W_1). \end{aligned}$$

Note that the differential system (7) when $\varepsilon = 0$ is equivalent to the differential system (2), called simply in what follows the *unperturbed system*. When $\varepsilon \neq 0$ we called it the *perturbed system*.

The change of variables $(X, Y, Z, W) \rightarrow (X_1, Y_1, Z_1, W_1)$ given by

$$(8) \quad \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{pmatrix} = \begin{pmatrix} \Phi_1 & 0 & \Phi_2 & 0 \\ 0 & \omega_1 \Phi_1 & 0 & \omega_2 \Phi_2 \\ 0 & -1/\omega_1 & 0 & -1/\omega_2 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix},$$

writes the linear part of the differential system (7) in its real Jordan normal form, and this system in the new variables (X, Y, Z, W) becomes

$$(9) \quad \begin{aligned} \dot{X} &= \omega_1 Y + \varepsilon F_1^*, \\ \dot{Y} &= -\omega_1 X + \varepsilon F_2^*, \\ \dot{Z} &= \omega_2 W + \varepsilon F_3^*, \\ \dot{W} &= -\omega_2 Z + \varepsilon F_4^*, \end{aligned}$$

where

$$\begin{aligned} F_1^*(t, X, Y, Z, W) &= \frac{\Phi_2}{\Phi_2 - \Phi_1} F_2(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}), \\ F_2^*(t, X, Y, Z, W) &= -\frac{\omega_1}{\Phi_2 \omega_2^2 - \Phi_1 \omega_1^2} F_1(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}), \\ F_3^*(t, X, Y, Z, W) &= -\frac{\Phi_1}{\Phi_2 - \Phi_1} F_2(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}), \\ F_4^*(t, X, Y, Z, W) &= \frac{\omega_2}{\Phi_2 \omega_2^2 - \Phi_1 \omega_1^2} F_1(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \end{aligned}$$

with

$$\mathcal{A} = \Phi_1 X + \Phi_2 Z, \quad \mathcal{B} = \omega_1 \Phi_1 Y + \omega_2 \Phi_2 W, \quad \mathcal{C} = -\frac{Y}{\omega_1} - \frac{W}{\omega_2}, \quad \mathcal{D} = X + Z.$$

Now, in the following lemma we characterize the periodic orbits of the unperturbed system as a first step for proving Theorems 1 and 3.

Lemma 5. *The periodic solutions $(X(t), Y(t), Z(t), W(t))$ of the differential system (9) with $\varepsilon = 0$ are*

$$(10) \quad (X_0 \cos(\omega_1 t) + Y_0 \sin(\omega_1 t), Y_0 \cos(\omega_1 t) - X_0 \sin(\omega_1 t), 0, 0),$$

of period T_1 , and

$$(11) \quad (0, 0, Z_0 \cos(\omega_2 t) + W_0 \sin(\omega_2 t), W_0 \cos(\omega_2 t) - Z_0 \sin(\omega_2 t)),$$

of period T_2 .

Proof. Since (9) for $\varepsilon = 0$ is a linear differential system the proof follows easily. \square

Proof of Theorem 1. Assume that the functions F_1 and F_2 of (1) are periodic in t of period pT_1/q with p and q positive integers relatively prime. Then, we can consider that the differential system (9) and the periodic solutions (10) have the same period pT_1 .

We apply Theorem 6 of Appendix I to the differential system (9), and we use the notation introduced there. Note that system (9) can be written in the form of system (12) taking

$$\mathbf{x} = \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix}, G_0(t, \mathbf{x}) = \begin{pmatrix} \omega_1 Y \\ -\omega_1 X \\ \omega_2 W \\ -\omega_2 Z \end{pmatrix}, G_1(t, \mathbf{x}) = \begin{pmatrix} F_1^* \\ F_2^* \\ F_3^* \\ F_4^* \end{pmatrix}, G_2(t, \mathbf{x}, \varepsilon) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now we shall study what periodic solutions of the unperturbed system (9) with $\varepsilon = 0$ of the type (10) persist as periodic solutions for the perturbed one for $\varepsilon \neq 0$ sufficiently small.

We start with the description of the different elements which appear in the statement of Theorem 6 for the particular case of the differential system (9). Thus, we have that $\Omega = \mathbb{R}^4$, $k = 2$ and $n = 4$. Now, let $r_1 > 0$ be arbitrarily small and let $r_2 > 0$ be arbitrarily large. Let V be the open and bounded subset of the plane $Z = W = 0$ of the form $V = \{(X_0, Y_0, 0, 0) \in \mathbb{R}^4 : r_1 < \sqrt{X_0^2 + Y_0^2} < r_2\}$. As usual $\text{Cl}(V)$ denotes the closure of V . If $\alpha = (X_0, Y_0)$, then we identify V with the set $\{\alpha \in \mathbb{R}^2 : r_1 < \|\alpha\| < r_2\}$, being $\|\cdot\|$ the Euclidean norm in \mathbb{R}^2 . The function $\beta : \text{Cl}(V) \rightarrow \mathbb{R}^2$ is $\beta(\alpha) = (0, 0)$. Therefore, for our system we have

$$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)), \alpha \in \text{Cl}(V)\} = \{(X_0, Y_0, 0, 0) \in \mathbb{R}^4 : r_1 \leq \sqrt{X_0^2 + Y_0^2} \leq r_2\}.$$

We consider for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the periodic solution $\mathbf{x}(t, \mathbf{z}_\alpha) = (X(t), Y(t), 0, 0)$ given by (10) of period pT_1 .

Computing the fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of the linear differential system (9) with $\varepsilon = 0$ associated to the pT_1 -periodic solution $\mathbf{z}_\alpha = (X_0, Y_0, 0, 0)$ such that $M_{\mathbf{z}_\alpha}(0)$ be the identity of \mathbb{R}^4 , we get

$$M_{\mathbf{z}_\alpha}(t) = M(t) = \begin{pmatrix} \cos(\omega_1 t) & \sin(\omega_1 t) & 0 & 0 \\ -\sin(\omega_1 t) & \cos(\omega_1 t) & 0 & 0 \\ 0 & 0 & \cos(\omega_2 t) & \sin(\omega_2 t) \\ 0 & 0 & -\sin(\omega_2 t) & \cos(\omega_2 t) \end{pmatrix}.$$

Note that the matrix $M_{\mathbf{z}_\alpha}(t)$ does not depend of the particular periodic solution $\mathbf{x}(t, \mathbf{z}_\alpha, 0)$. The matrix

$$M^{-1}(0) - M^{-1}(pT_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sin^2\left(\frac{p\pi\omega_2}{\omega_1}\right) & \sin\left(\frac{2p\pi\omega_2}{\omega_1}\right) \\ 0 & 0 & -\sin\left(\frac{2p\pi\omega_2}{\omega_1}\right) & 2\sin^2\left(\frac{p\pi\omega_2}{\omega_1}\right) \end{pmatrix},$$

satisfies the assumptions of statement (ii) of Theorem 6 because the determinant

$$\begin{vmatrix} 2 \sin^2 \left(\frac{p\pi\omega_2}{\omega_1} \right) & \sin \left(\frac{2p\pi\omega_2}{\omega_1} \right) \\ -\sin \left(\frac{2p\pi\omega_2}{\omega_1} \right) & 2 \sin^2 \left(\frac{p\pi\omega_2}{\omega_1} \right) \end{vmatrix} = 4 \sin^2 \left(\frac{p\pi\omega_2}{\omega_1} \right) \neq 0,$$

because the ratio of the frequencies is non-resonant with π . In short, all the assumptions of Theorem 6 are satisfied by the system (9).

For our system the map $\xi : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ has the form $\xi(X, Y, Z, W) = (X, Y)$. Calculating the function

$$\mathcal{G}(X_0, Y_0) = \mathcal{G}(\alpha) = \xi \left(\frac{1}{pT_1} \int_0^{pT_1} M_{\mathbf{z}_\alpha}^{-1}(t) G_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0)) dt \right),$$

we obtain that

$$\mathcal{G}(X_0, Y_0) = (\mathcal{G}_1(X_0, Y_0), \mathcal{G}_2(X_0, Y_0)),$$

where the functions \mathcal{G}_k for $k = 1, 2$ are the ones given in (3). Then, by Theorem 6 we have that for every simple zero $(X_0^*, Y_0^*) \in V$ of the system of nonlinear equations (4) we have a periodic solution $(X, Y, Z, W)(t, \varepsilon)$ of system (9) such that

$$(X, Y, Z, W)(0, \varepsilon) \rightarrow (X_0^*, Y_0^*, 0, 0) \quad \text{when } \varepsilon \rightarrow 0.$$

Going back through the change of coordinates (8) we get a periodic solution $(X_1, Y_1, Z_1, W_1)(t, \varepsilon)$ of system (9) such that

$$\begin{pmatrix} X_1(t, \varepsilon) \\ Y_1(t, \varepsilon) \\ Z_1(t, \varepsilon) \\ W_1(t, \varepsilon) \end{pmatrix} \rightarrow \begin{pmatrix} \Phi_1 (X_0^* \cos(\omega_1 t) + Y_0^* \sin(\omega_1 t)) \\ \omega_1 \Phi_1 (Y_0^* \cos(\omega_1 t) - X_0^* \sin(\omega_1 t)) \\ (X_0^* \sin(\omega_1 t) - Y_0^* \cos(\omega_1 t)) / \omega_1 \\ X_0^* \cos(\omega_1 t) + Y_0^* \sin(\omega_1 t) \end{pmatrix} \quad \text{when } \varepsilon \rightarrow 0.$$

Consequently we obtain a periodic solution $(x, y)(t, \varepsilon)$ of system (1) such that

$$(x, y)(t, \varepsilon) \rightarrow \begin{pmatrix} \Phi_1 (X_0^* \cos(\omega_1 t) + Y_0^* \sin(\omega_1 t)) \\ (X_0^* \sin(\omega_1 t) - Y_0^* \cos(\omega_1 t)) / \omega_1 \end{pmatrix} \quad \text{when } \varepsilon \rightarrow 0.$$

This completes the proof of the theorem. \square

Proof of Theorem 3. The proof is analogous to the proof of Theorem 1 changing the roles of T_1 for T_2 . \square

3. PROOF OF THE TWO COROLLARIES

Proof of Corollary 2. Under the assumptions of Corollary 2 the nonlinear system (4) becomes

$$\begin{aligned} \mathcal{G}_1(X_0, Y_0) &= \frac{\Phi_2 X_0 Y_0 (1 - (X_0^2 + Y_0^2) \omega_1^2 \Phi_1^2)}{4(\Phi_1 - \Phi_2) \omega_1^2} = 0, \\ \mathcal{G}_2(X_0, Y_0) &= \frac{\Phi_2 ((5X_0^4 + 6X_0^2 Y_0^2 + Y_0^4) \omega_1^2 \Phi_1^2 - 2(3X_0^2 + Y_0^2))}{16(\Phi_1 - \Phi_2) \omega_1^2} = 0. \end{aligned}$$

This system has the following four solutions

$$(X_0^*, Y_0^*) = \left(\pm \frac{\sqrt{30}}{5\Phi_1\omega_1}, 0 \right) \quad \text{and} \quad (X_0^*, Y_0^*) = \left(0, \pm \frac{\sqrt{2}}{\Phi_1\omega_1} \right).$$

Note that the solutions which differs in a sign are different initial conditions of the same periodic solution of the system (2). Moreover, since

$$\det \left(\frac{\partial(\mathcal{G}_1, \mathcal{G}_2)}{\partial(X_0, Y_0)} \Big|_{(X_0, Y_0) = \left(\pm \frac{\sqrt{30}}{5\Phi_1\omega_1}, 0 \right)} \right) = \frac{9\Phi_2^2}{200\Phi_1^2(\Phi_1 - \Phi_2)\omega_1^6} \neq 0$$

and

$$\det \left(\frac{\partial(\mathcal{G}_1, \mathcal{G}_2)}{\partial(X_0, Y_0)} \Big|_{(X_0, Y_0) = \left(0, \pm \frac{\sqrt{2}}{\Phi_1\omega_1} \right)} \right) = -\frac{\Phi_2^2}{8\Phi_1^2(\Phi_1 - \Phi_2)\omega_1^6} \neq 0,$$

these solutions are simple. Finally, by Theorem 1 we only have two periodic solutions for the system of this corollary. \square

Proof of Corollary 4. Under the assumptions of Corollary 4 the nonlinear system (6) becomes

$$\mathcal{G}_3(Z_0, W_0) = \frac{\omega_2(2 + \omega_2\Phi_2^2W_0Z_0)}{4(\omega_1^2\Phi_1 - \omega_2^2\Phi_2)} = 0, \quad \mathcal{G}_4(Z_0, W_0) = \frac{\Phi_2^2\omega_2^2(W_0^2 - Z_0^2)}{8(\omega_1^2\Phi_1 - \omega_2^2\Phi_2)} = 0.$$

This system only has the two real solutions

$$(Z_0^*, W_0^*) = \left(\frac{\sqrt{2}}{\Phi_2\sqrt{\omega_2}}, -\frac{\sqrt{2}}{\Phi_2\sqrt{\omega_2}} \right) \quad \text{and} \quad (Z_0^*, W_0^*) = \left(-\frac{\sqrt{2}}{\Phi_2\sqrt{\omega_2}}, \frac{\sqrt{2}}{\Phi_2\sqrt{\omega_2}} \right),$$

both providing different initial conditions of the same periodic orbit of the unperturbed system. Since

$$\det \left(\frac{\partial(\mathcal{G}_3, \mathcal{G}_4)}{\partial(Z_0^*, W_0^*)} \Big|_{(Z_0^*, W_0^*) = \left(-\frac{\sqrt{2}}{\Phi_2\sqrt{\omega_2}}, \frac{\sqrt{2}}{\Phi_2\sqrt{\omega_2}} \right)} \right) = \frac{\Phi_2^2\omega_2^3}{4(\omega_1^2\Phi_1 - \omega_2^2\Phi_2)^2} \neq 0,$$

by Theorem 3 we only have one periodic solution of the differential system of this corollary. \square

APPENDIX I: BASIC RESULTS ON AVERAGING THEORY

In this appendix we present the basic result from the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of T -periodic solutions from a differential system of the form

$$(12) \quad \dot{\mathbf{x}}(t) = G_0(t, \mathbf{x}) + \varepsilon G_1(t, \mathbf{x}) + \varepsilon^2 G_2(t, \mathbf{x}, \varepsilon),$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $G_0, G_1 : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^n$ and $G_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^n$ are \mathcal{C}^2 functions, T -periodic in the first variable, and Ω is an open subset of \mathbb{R}^n . The main assumption is that the unperturbed system

$$(13) \quad \dot{\mathbf{x}}(t) = G_0(t, \mathbf{x}),$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ be the solution of the system (13) such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ as

$$(14) \quad \dot{\mathbf{y}} = D_{\mathbf{x}}G_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}.$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (14), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \rightarrow \mathbb{R}^k$ the projection of \mathbb{R}^n onto its first k coordinates; i.e. $\xi(x_1, \dots, x_n) = (x_1, \dots, x_k)$.

We assume that there exists a k -dimensional submanifold \mathcal{Z} of Ω filled with T -periodic solutions of (13). Then an answer to the problem of bifurcation of T -periodic solutions from the periodic solutions contained in \mathcal{Z} for system (12) is given in the following result.

Theorem 6. *Let V be an open and bounded subset of \mathbb{R}^k , and let $\beta : \text{Cl}(V) \rightarrow \mathbb{R}^{n-k}$ be a C^2 function. We assume that*

- (i) $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)), \alpha \in \text{Cl}(V)\} \subset \Omega$ and that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha)$ of (13) is T -periodic;
- (ii) for each $\mathbf{z}_\alpha \in \mathcal{Z}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (14) such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ has in the upper right corner the $k \times (n-k)$ zero matrix, and in the lower right corner a $(n-k) \times (n-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$.

We consider the function $\mathcal{G} : \text{Cl}(V) \rightarrow \mathbb{R}^k$

$$(15) \quad \mathcal{G}(\alpha) = \xi \left(\frac{1}{T} \int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) G_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0)) dt \right).$$

If there exists $a \in V$ with $\mathcal{G}(a) = 0$ and $\det((d\mathcal{G}/d\alpha)(a)) \neq 0$, then there is a T -periodic solution $\mathbf{x}(t, \varepsilon)$ of system (12) such that $\mathbf{x}(0, \varepsilon) \rightarrow \mathbf{z}_a$ as $\varepsilon \rightarrow 0$.

For a proof of Theorem 6 see Malkin [7] and Roseau [9], or [2] for shorter proof.

4. APPENDIX II: EQUATIONS OF MOTION

Let $OXYZ$ be an inertial reference system and $Oxyz$ a mobile one with origin at the fixed point of the gyrost. Let us denote by A and C the inertia momenta, by $\omega = (p, q, r)$ the angular velocity, by $\mathbf{l} = (Ap, Aq, Cr)$ the angular momentum, by $\mathbf{l}_r = (0, 0, l)$ the gyrostatic momentum and by \mathcal{M} the total momentum induced by the axial forces of a symmetric potential $V(k_3)$. The Poisson vector is denoted by $\mathbf{k} = (k_1, k_2, k_3)$.

The equations of motion on the non-inertial reference system $Oxyz$ are given by

$$\dot{\mathbf{l}} + \omega \times (\mathbf{l} + \mathbf{l}_r) = \mathcal{M}, \quad d\mathbf{k} = \omega \times \mathbf{k},$$

see for more details [6]. More precisely

$$(16) \quad \begin{aligned} A\dot{p} &= (A - C)qr - lq + k_2 V'(k_3), \\ A\dot{q} &= (C - A)pr + lp - k_1 V'(k_3), \\ \dot{r} &= 0, \\ \dot{k}_1 &= rk_2 - qk_3, \\ \dot{k}_2 &= pk_3 - rk_1, \\ \dot{k}_3 &= qk_1 - pk_2. \end{aligned}$$

Clearly the points $(p = 0, q = 0, r = r_0, k_1 = 0, k_2 = 0, k_3 = 1)$ and $(p = 0, q = 0, r = r_0, k_1 = 0, k_2 = 0, k_3 = -1)$ are equilibria solution of the equations (16). The first of them corresponds to the rotation around the vertical axis in ascendent sense of the gyrostat, the second one corresponds to rotation in descendent sense.

To derive the equations of motion close to the upright solution we consider the Lagrangian of the system given by

$$\mathcal{L} = \frac{1}{2} (A(p^2 + q^2) + C(r + l)^2) - V(k_3)$$

We introduce the generalized Eulerian angular variables $(\theta_1, \theta_2, \varphi)$. The following relations are verified.

$$p = -\frac{d\theta_1}{dt}, q = \frac{d\theta_2}{dt} \cos \theta_1, r = \frac{d\varphi}{dt} + \frac{d\theta_2}{dt} \sin \theta_1, k_1 = \cos \theta_1 \sin \theta_2, \\ k_2 = \sin \theta_1, k_3 = \cos \theta_1 \cos \theta_2$$

and the Lagrangian of the problem is

$$\mathcal{L} = \frac{A}{2} \left(\left(\frac{d\theta_1}{dt} \right)^2 + \left(\frac{d\theta_2}{dt} \right)^2 \cos^2 \theta_1 \right) + \frac{C}{2} \left(\frac{d\varphi}{dt} + \frac{d\theta_2}{dt} \sin \theta_1 + l \right)^2 - V(\cos \theta_1 \cos \theta_2).$$

The variable φ is a cyclic coordinate of \mathcal{L} and $k_\varphi = C \left(\frac{d\varphi}{dt} + \frac{d\theta_2}{dt} \sin \theta_1 + l \right)$ is an integral of the system. The constant k_φ are given by

$$k_\varphi = C \left(r_0 + l + \frac{d\theta_2}{dt} \Big|_{t=0} \sin(\theta_1(0)) \right)$$

being $\frac{d\varphi}{dt}(0) = r_0$.

For Lagrangians with a cyclic variable it is possible to construct the so called *Routh's map* given by $\mathcal{R} = \mathcal{L} - k_\varphi \frac{d\theta}{dt}$ removing the cyclic variable of the motion equations. Thus, performing this Lagrangian reduction to \mathcal{L} we obtain the Routhian of our system given by

$$\mathcal{R} = \frac{A}{2} \left(\left(\frac{d\theta_1}{dt} \right)^2 + \left(\frac{d\theta_2}{dt} \right)^2 \cos^2 \theta_1 \right) + k_\varphi \frac{d\theta_2}{dt} \sin \theta_1 - \frac{k_\varphi^2}{2C} - V(\cos \theta_1 \cos \theta_2).$$

The equations of motion associated to \mathcal{R} are

$$A \left(\frac{d^2\theta_1}{dt^2} + \left(\frac{d\theta_2}{dt} \right)^2 \cos \theta_1 \sin \theta_1 \right) - k_\varphi \frac{d\theta_2}{dt} \cos \theta_1 - V'(\cos \theta_1 \cos \theta_2)(\sin \theta_1 \cos \theta_2) = 0 \\ A \left(\frac{d^2\theta_2}{dt^2} \cos^2 \theta_1 - 2\theta_1 \frac{d\theta_2}{dt} \sin \theta_1 \cos \theta_1 \right) + k_\varphi \frac{d\theta_1}{dt} \cos \theta_1 \\ - V'(\cos \theta_1 \cos \theta_2)(\cos \theta_1 \sin \theta_2) = 0$$

When θ_1 y θ_2 are small via the change of variables $\theta_1 = \epsilon y$, $\theta_2 = \epsilon x$ and expanding in Taylor series to second order in ϵ we obtain that equations (16) are reduced to the unperturbed equations (2)

$$\ddot{x} + \alpha \dot{y} - \beta x = 0, \\ \ddot{y} - \alpha \dot{x} - \beta y = 0,$$

with $\alpha = (Cr_0 + l)/A$ y $\beta = V'(1)/A$.

Remark 1. We remark that system (2) models the perturbations given by the maps $F_{11}\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}, t\right)$ and $F_{21}\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}, t\right)$ of the small oscillations around the vertical position of a symmetrical gyrost.

Remark 2. Our results provides sufficient conditions for the existence of periodic solutions of system (2).

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