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CONFIGURATIONS OF SINGULARITES FOR QUADRATIC DIFFERENTIAL SYSTEMS WITH TOTAL FINITE MULTIPLICITY LOWER THAN 2

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Abstract. In [3] we classified globally the configurations of singularities at infinity of quadratic differential systems, with respect to the *geometric equivalence relation*. The global classification of configurations of finite singularities was done in [2] modulo the coarser *topological equivalence relation* for which no distinctions are made between a focus and a node and neither are they made between a strong and a weak focus or between foci of different orders. These distinctions are however important in the production of limit cycles close to the foci in perturbations of the systems. The notion of *geometric equivalence relation* of configurations of singularities allows us to incorporates all these important purely algebraic features. This equivalence relation is also finer than the *qualitative equivalence relation* introduced in [19]. In this article we initiate the joint classification of configurations of singularities, finite and infinite, using the finer *geometric equivalence relation*, for the subclass of quadratic differential systems possessing finite singularities of total multiplicity $m_f \leq 1$. We obtain 84 *geometrically distinct* configurations of singularities for this family. We also give here the global bifurcation diagram, with respect to the *geometric equivalence relation*, of configurations of singularities, both finite and infinite, for this class of systems. This bifurcation set is algebraic. The bifurcation diagram is done in the 12-dimensional space of parameters and it is expressed in terms of polynomial invariants. The results can therefore be applied for any family of quadratic systems, given in any normal form. Determining the configurations of singularities for any family of quadratic systems, becomes thus a simple task using computer algebra calculations.

Résumé. Dans [3] nous avons classifié globalement les configurations de singularités à l'infini des systèmes différentiels quadratiques, par rapport à la *relation d'équivalence géométrique*. La classification globale des configurations de singularités finies a été donée dans [2], modulo la *relation d'équivalence topologique*, pour laquelle aucune distinction n'a pas été faite entre un foyer ou un neud, ou entre un foyer fort et un foyer faible ou bien entre foyers d'ordre différents. Ces distinctions sont pourtant importantes dans la production de cycles limites proches des foyers, dans les perturbations de ces systèmes. La *relation d'équivalence geometrique* de configurations de singularités nous permet d'incorporer toutes ces charactéristiques importantes purement algebriques. Cette relation d'équivalence est aussi plus fine que la *relation d'équivalence qualitative* introduite dans [19]. Dans cet article nous initions la classification globale de toutes les configurations de singularités, finies et infinies, utilisant la relation plus fine d'*équivalence géométrique*, pour la sous-classe de systèmes differentiels quadratiques possédant des singularités finies de multiplicité totale $m_f \leq 1$. Nous obtenons 84 configurations *distinctes géométriquement* de singularités pour cette famille. Nous donnons aussi le diagramme global de bifurcation, par rapport à la *relation d'équivalence géométrique* des configurations de singularités, finies et infinies, pour cette famille de systèmes. L'ensemble des points de bifurcation est algébrique. Le diagramme de bifurcation est réalisé dans l'espace 12-dimensionnel des paramètres et il est exprimé en termes de polynômes invariants. Les resultats peuvent donc être appliqués à toute famille de systèmes différentiels quadratiques avec $m_f \leq 1$, par rapport à n'importe quelle form normale dans laquelle les systèmes nous sont présentés. Détérminer les configurations de singularités pour cette famille devient alors une simple tâche utilisant le calcul symbolique à l'ordinateur.

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1. Introduction and statement of main results

We consider here differential systems of the form

$$\frac{dx}{dt} = p(x, y), \qquad \frac{dy}{dt} = q(x, y), \tag{1}$$

where $p,\ q \in \mathbb{R}[x, y]$, i.e. $p,\ q$ are polynomials in $x,\ y$ over $\mathbb{R}$. We call *degree* of a system (1) the integer $m = \max(\deg p, \deg q)$. In particular we call *quadratic* a differential system (1) with $m = 2$. We denote here by **QS** the whole class of real quadratic differential systems.

The study of the class **QS** has proved to be quite a challenge since hard problems formulated more than a century ago, are still open for this class. The complete characterization of the phase portraits for real quadratic vector fields is not known and attempting to topologically classify these systems, which occur rather often in applications, is a very complex task. This is partly due to the elusive nature of limit cycles and partly to the rather large number of parameters involved. This family of systems depends on twelve parameters but due to the group action of real affine transformations and time homotheties, the class ultimately depends on five parameters, still a rather large number of parameters. For the moment only subclasses depending on at most three parameters were studied globally, including global bifurcation diagrams (for example [2]) . On the other hand we can restrict the study of the whole quadratic class by focusing on specific global features of the systems in this family. We may thus focus on the global study of singularities and their bifurcation diagram. The singularities are of two kinds: finite and infinite. The infinite singularities are obtained by compactifying the differential systems on the sphere or on the Poincaré disk as they are defined in Section 6.1 (see also [16]).

The global study of quadratic vector fields in the neighborhood of infinity was initiated by Nikolaev and Vulpe in [22] where they classified topologically the singularities at infinity in terms of invariant polynomials. Schlomiuk and Vulpe used geometrical concepts defined in [28], and also introduced some new geometrical concepts in [29] in order to simplify the invariant polynomials and the classification. To reduce the number of phase portraits in half, in both cases the topological equivalence relation was taken to mean the existence of a homeomorphism carrying orbits to orbits and *preserving or reversing* the orientation. In [4] the authors classified topologically (adding also the distinction between nodes and foci) the whole quadratic class, according to configurations of their finite singularities.

In the topological classification no distinction was made among the various types of foci or saddles, strong or weak of various orders. However these distinctions, of algebraic nature, are very important in the study of perturbations of systems possessing such singularities. Indeed, the maximum number of limit cycles which can be produced close to the weak foci in perturbations depends on the orders of the foci.

The distinction among weak saddles is also important since for example when a loop is formed using two separatrices of one weak saddle, the maximum number of limit cycles that can be obtained close to the loop in perturbations is the order of weak saddle.

There are also three kinds of simple nodes as we can see in Figure 1 below where the local phase portraits around the singularities are given.

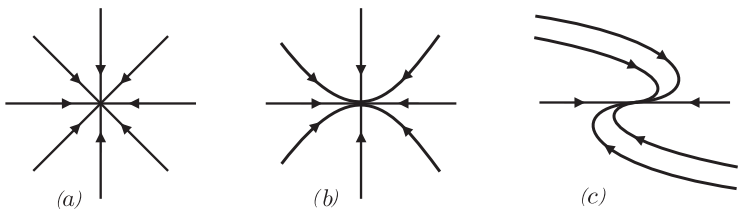

Figure 1. Different types of nodes

In the three phase portraits of FIGURE 1 the corresponding three singularities are stable nodes. These portraits are topologically equivalent but the solution curves do not arrive at the nodes in the same way. In the first case, any two distinct non-trivial phase curves arrive at the node with distinct slopes. Such a node is called a star node. In the second picture all non-trivial solution curves excepting two of them arrive at the node with the same slope but the two exception curves arrive at the node with a different slope. This is the generic node with two directions. In the third phase portrait all phase curves arrive at the node with the same slope.

We recall that the first and the third types of nodes could produce foci in perturbations and the first type of nodes is also involved in the existence of invariant straight lines of differential systems. For example it can easily be shown that if a quadratic differential system has two finite star nodes then necessarily the system possesses invariant straight lines of total multiplicity 6.

Furthermore, a generic node may or may not have the two exceptional curves lying on the line at infinite. This leads to two different situations for the phase portraits. For this reason we split the generic nodes at infinity in two types.

The distinctions among the nilpotent and linearly zero singularities finite or infinite can also be refined, as it will be seen in Section 4. Such singularities are usually called *degenerate* singularities so here too we call them degenerate.

The geometric equivalence relation for finite or infinite singularities, introduced in [3], takes into account such distinctions.

This equivalence relation is finer than the *qualitative equivalence relation* introduced by Jiang and Llibre in [19] since it distinguishes among the foci of different orders and among the various types of nodes. This equivalence relation also induces a finer distinction among the more complicated degenerate singularities.

To distinguish among the foci (or saddles) of various orders we use the algebraic concept of Poincaré-Lyapunov constants. We call strong focus (or strong saddle) a focus with non–zero trace of the linearization matrix at this point. Such a focus (or saddle) will be considered to have the order zero. A focus (or saddle) with trace zero is called a weak focus (weak saddle). For details on Poincaré-Lyapunov constants and weak foci we refer to [20].

Algebraic means, i.e. the linearization matrices at these nodes and their eigenvalues, distinguish the nodes in FIGURE 1.

The finer distinctions of singularities are also algebraic in nature. In fact the whole bifurcation diagram of the global configurations of singularities, finite and infinite, in quadratic vector fields and more generally in polynomial vector fields can be obtained by using only algebraic means, among them, the algebraic tool of polynomial invariants.

Algebraic information may not be significant for the local (topological) phase portrait around a singularity. For example, topologically there is no distinction between a focus and a node or between a weak and a strong focus. However, as indicated before, algebraic information plays a fundamental role in the study of perturbations of systems possessing such singularities.

In [13] Coppel wrote:

"Ideally one might hope to characterize the phase portraits of quadratic systems by means of algebraic inequalities on the coefficients. However, attempts in this direction have met with very limited success..."

This proved to be impossible to realize. Indeed, Dumortier and Fiddelers [15] and Roussarie [25] exhibited examples of families of quadratic vector fields which have non-algebraic bifurcation sets. However, the following is a legitimate question:

How far can we go in the global theory of quadratic (or more generally polynomial) vector fields by using mainly algebraic means?

For certain subclasses of quadratic vector fields the full description of the phase portraits as well as of the bifurcation diagrams can be obtained using only algebraic tools. Examples of such classes are:

- the quadratic vector fields possessing a center [37, 26, 40, 23];
- the quadratic Hamiltonian vector fields [1, 5];
- the quadratic vector fields with invariant straight lines of total multiplicity at least four [30, 31];
- the planar quadratic differential systems possessing a line of singularities at infinity [32];
- the quadratic vector fields possessing an integrable saddle [6].
- the family of Lotka-Volterra systems [33, 34], once we assume Bautin's analytic result saying that such systems have no limit cycles;

In the case of other subclasses of the quadratic class **QS**, such as the subclass of systems with a weak focus of order 3 or 2 (see [20, 2]) the bifurcation diagrams were obtained by using an interplay of algebraic, analytic and numerical methods. These subclasses were of dimensions 2 and 3 modulo the action of the affine group and time rescaling. So far no 4-dimensional subclasses of **QS** were studied globally so as to produce also bifurcation diagrams and such problems are very difficult due to the number of parameters as well as the increased complexities of these classes.

Although we now know that in trying to understand these systems, there is a limit to the power of algebraic methods, these methods have not been used far enough. For example the global classification of singularities, finite and infinite, using the *geometric equivalence relation*, which is finer than the *qualitative equivalence relation*, can be done by using only algebraic methods. The first step in this direction was done in [3] where the study of *the whole class* **QS**, according to the configurations of the singularities at infinity was obtained by using only algebraic methods. This classification was done with respect to the *geometric equivalence relation*. Our work in [3] can be extended by incorporating also the finite singularities. In this way we can obtain the global *geometric classification* of all possible configurations of singularities, finite and infinite, of quadratic differential systems, by purely algebraic means.

Our goal in this work is to take the first step in this direction by joining the results for infinite singularities in [3] with finite singularities of total multiplicity $mf \leq 1$, of quadratic differential systems.

We extend here below the notion of *configuration of singularities* defined in [3] only for infinite singularities, to all singularities, both finite and infinite. We distinguish two cases:

1) If we have a finite number of infinite singular points and a finite number of finite singularities we call *configuration of singularities*, finite and infinite, the set of all these singularities each endowed with its own multiplicity together with their local phase portraits endowed with additional geometric structure involving the concepts of tangent, order and blow–up equivalences defined in Section 4 and using the notations described in Section 5.

2) If the line at infinity $Z = 0$ is filled up with singularities, in each one of the charts at infinity $X \neq 0$ and $Y \neq 0$, the system is degenerate and we need to do a rescaling of an appropriate degree of the system, so that the degeneracy be removed. The resulting systems have only a finite number of singularities on the line $Z = 0$. In this case we call *configuration of singularities*, finite and infinite, the union of the set of all points at infinity (they are all singularities) with the set of finite singularities - taking care of singling out the singularities of the "reduced" system at infinity -, taken together with the local phase portraits of finite singularities endowed with additional geometric structure as above and of the infinite singularities of the reduced system.

We continue to use here **ISP**s as a shorthand for "infinite singular points".

We obtain the following

Main Theorem. (*A*) *The configurations of singularities, finite and infinite, of all quadratic vector fields with finite singularities of total multiplicity* $\boldsymbol{m_f \leq 1}$ *are classified in* Diagrams *1 and 2 according to the geometric equivalence relation. We have 84 geometric distinct configurations of singularities, finite and infinite. More precisely 32 configurations with* $\boldsymbol{m_f = 0}$ *and 52 with* $\boldsymbol{m_f = 1}$.

(B) *For* $\boldsymbol{m_f = 1}$ *we have only two configurations with a center but 5 configurations with a finite integrable saddle, and the maximum order of a weak focus (or of a weak saddle) is one.*

(C) *For* $\boldsymbol{m_f = 1}$ *we have: 4 configurations with a weak focus of order one but only 2 configurations with a weak finite saddle of order one; 6 configurations with a strong focus but 7 configurations with a strong finite saddle.*

(D) *Necessary and sufficient conditions for each one of the 84 different equivalence classes can be assembled from* Diagrams *1 and 2 in terms of 30 invariant polynomials with respect to the action of the affine group and time rescaling, given in Section 7.*

(E) *The* Diagrams *1 and 2 actually contain the global bifurcation diagram in the 12-dimensional space of parameters, of the global configurations of singularities, finite and infinite, of this family (*$\boldsymbol{m_f \leq 1}$*) of quadratic differential systems.*

(F) *The phase portraits in the neighborhood of the line at infinity corresponding to* $\boldsymbol{m_f = 0}$ *and to* $\boldsymbol{m_f = 1}$ *are given in* Figure *2. More precisely we have:*

$\boldsymbol{m_f = 0}$*: Configs - 3; 4; 5; 30; 18; 28; 17; 13; 8; 24; 11; 15; 36; 35; 32; 46;*

$\boldsymbol{m_f = 1}$*: Configs - 2; 6; 31; 20; 14; 26; 25; 9; 23; 16; 12; 21; 39; 37; 33; 38; 45.*

The invariants and comitants of differential equations used for proving our main results are obtained following the theory of algebraic invariants of polynomial differential systems, developed by Sibirsky and his disciples (see for instance [35, 38, 24, 7, 12]).

2. Some geometrical concepts

In this section we use the same concepts we considered in [3] such as *orbit* γ *tangent to a semi–line* L *at* p, *well defined angle at* p, *characteristic orbit at a singular point* p, *characteristic angle at a singular point*, *characteristic direction at* p. Since these are basic concepts for the notion of *geometric equivalence relation* we recall here these notions.

We assume that we have an isolated singularity p. Suppose that in a neighborhood U of p there is no other singularity. Consider an orbit γ in U defined by a solution $\Gamma(t) = (x(t), y(t))$ such that $\lim_{t\to+\infty} \Gamma(t) = p$ (or $\lim_{t\to-\infty} \Gamma(t) = p$). For a fixed t consider the unit vector $C(t) = \overrightarrow{(\Gamma(t) - p)}/\|\overrightarrow{\Gamma(t) - p}\|$. Let L be a semi–line ending at p. We shall say that *the orbit* γ *is tangent to a semi–line* L *at* p if $\lim_{t\to+\infty} C(t)$ (or $\lim_{t\to-\infty} C(t)$) exists and L contains this limit point on the unit circle centered at p. In this case we call *well defined angle of* Γ *at* p the angle between the positive x–axis and the semi–line L measured in the counter–clockwise sense. We may also say that *the solution curve* $\Gamma(t)$ *tends to* p *with a well defined angle.* A *characteristic orbit at a singular point* p is the orbit of a solution curve $\Gamma(t)$ which tends to p with a well defined angle. We call *characteristic angle at the singular point* p a well defined angle of a solution curve $\Gamma(t)$. The line through p extending the semi-line L is called a *characteristic direction.*

If a singular point has an infinite number of characteristic directions, we will call it a *star–like* point.

It is known that the neighborhood of any isolated singular point of a polynomial vector field, which is not a focus, a center or a star-like point, is formed by a finite number of sectors which could only be of three types: parabolic, hyperbolic and elliptic (see [16]). It is also known that any degenerate singular point can be desingularized by means of a finite number of changes of variables, called blow–up's, into elementary singular points (for more details see Section 3 or [16]).

Consider the three singular points given in Figure 3. All three are topologically equivalent and their neighborhoods can be described as having two elliptic sectors and two parabolic ones. But we can easily detect some geometric features which distinguish them. For example (a) and (b) have three characteristic directions and (c) has only two. Moreover in (a) the solution curves of the parabolic sectors are tangent

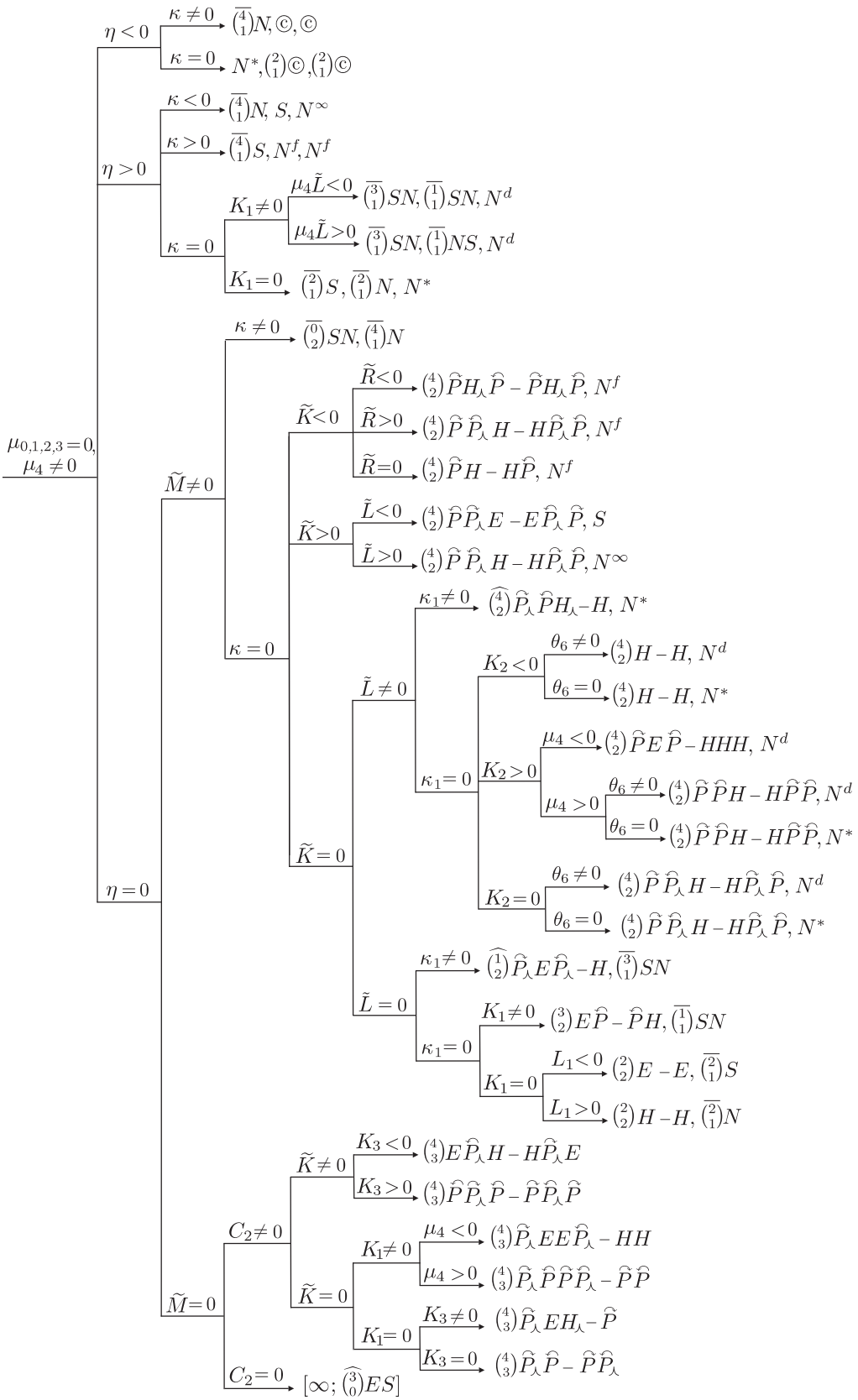

DIAGRAM 1. **Global configurations: case $\mu_0=\mu_1=\mu_2=\mu_3=0$, $\mu_4\neq 0$**

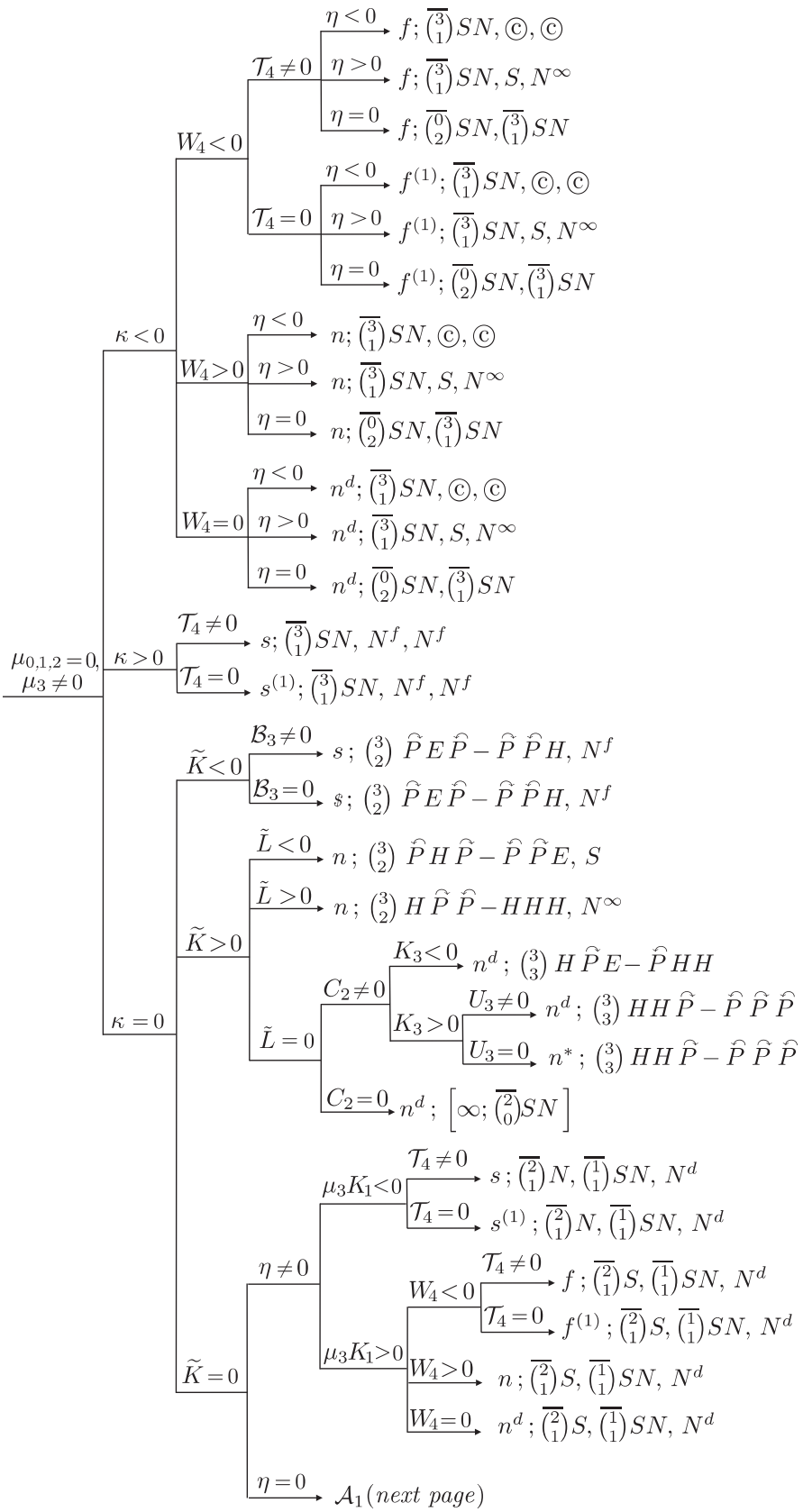

DIAGRAM 2. **Global configurations: case $\mu_0 = \mu_1 = \mu_2 = 0$, $\mu_3 \neq 0$**

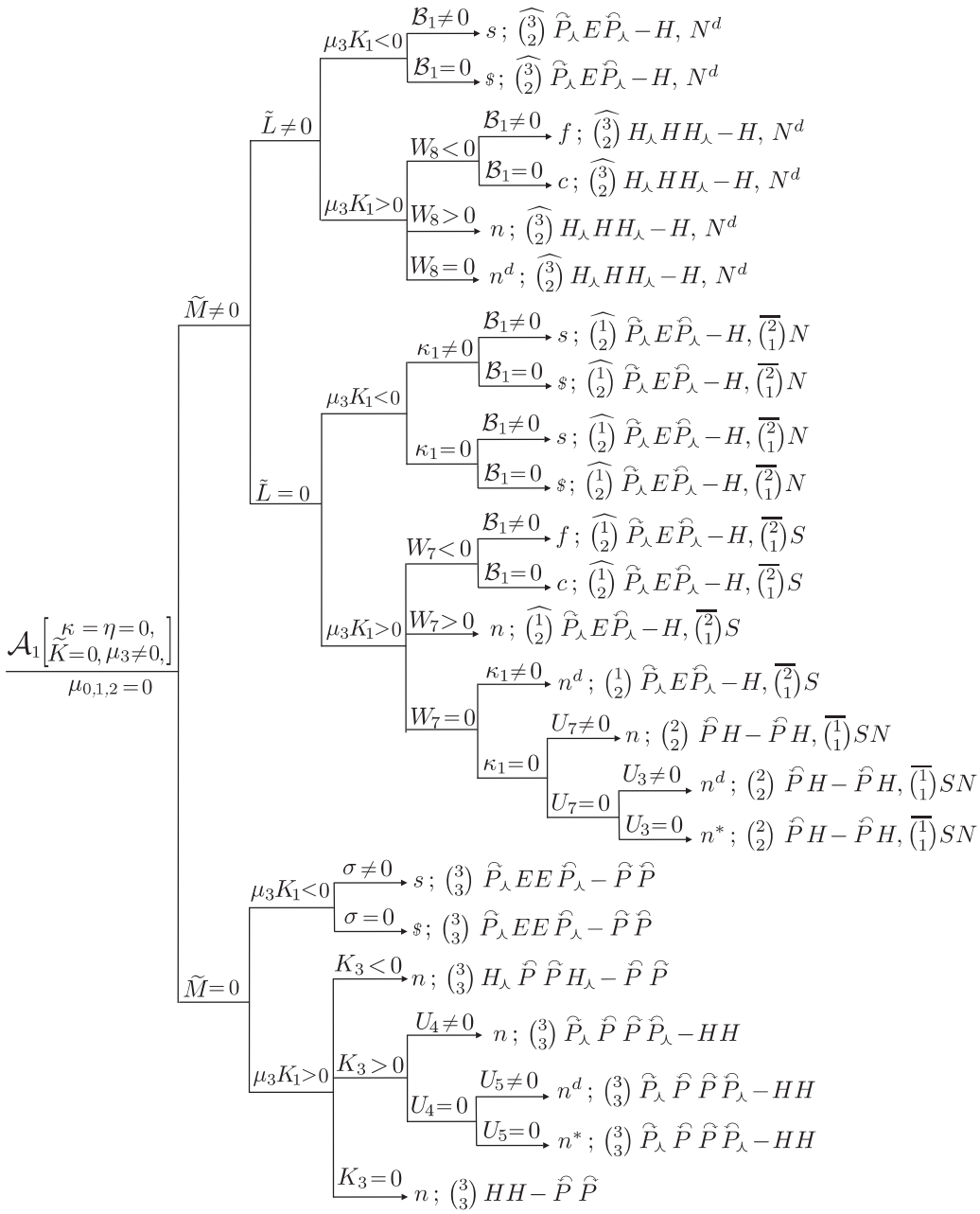

DIAGRAM 2 (*continued*). **Global configurations: case $\mu_0=\mu_1=\mu_2=0$, $\mu_3\neq 0$**

to only one characteristic direction and in (b) they are tangent to two characteristic directions. All these properties can be determined algebraically.

The usual definition of a sector is of a topological nature and it is local with respect to a neighborhood around the singular point. We work with a new definition of local sector, introduced in [3] which is of an algebraic nature and which distinguishes the systems of FIGURE 3.

We call *borsec* (contraction of border and sector) any orbit of the original system which carried through consecutive stages of the desingularization ends up as an orbit of the phase portrait in the final stage which is either a separatrix or a representative orbit of a characteristic angle of a node or a of saddle–node in the final desingularized phase portrait.

Using the concept of borsec, we define a *geometric local sector* with respect to a neighborhood V as a region in V delimited by two consecutive borsecs. For example, a semi–elementary saddle–node can be topologically

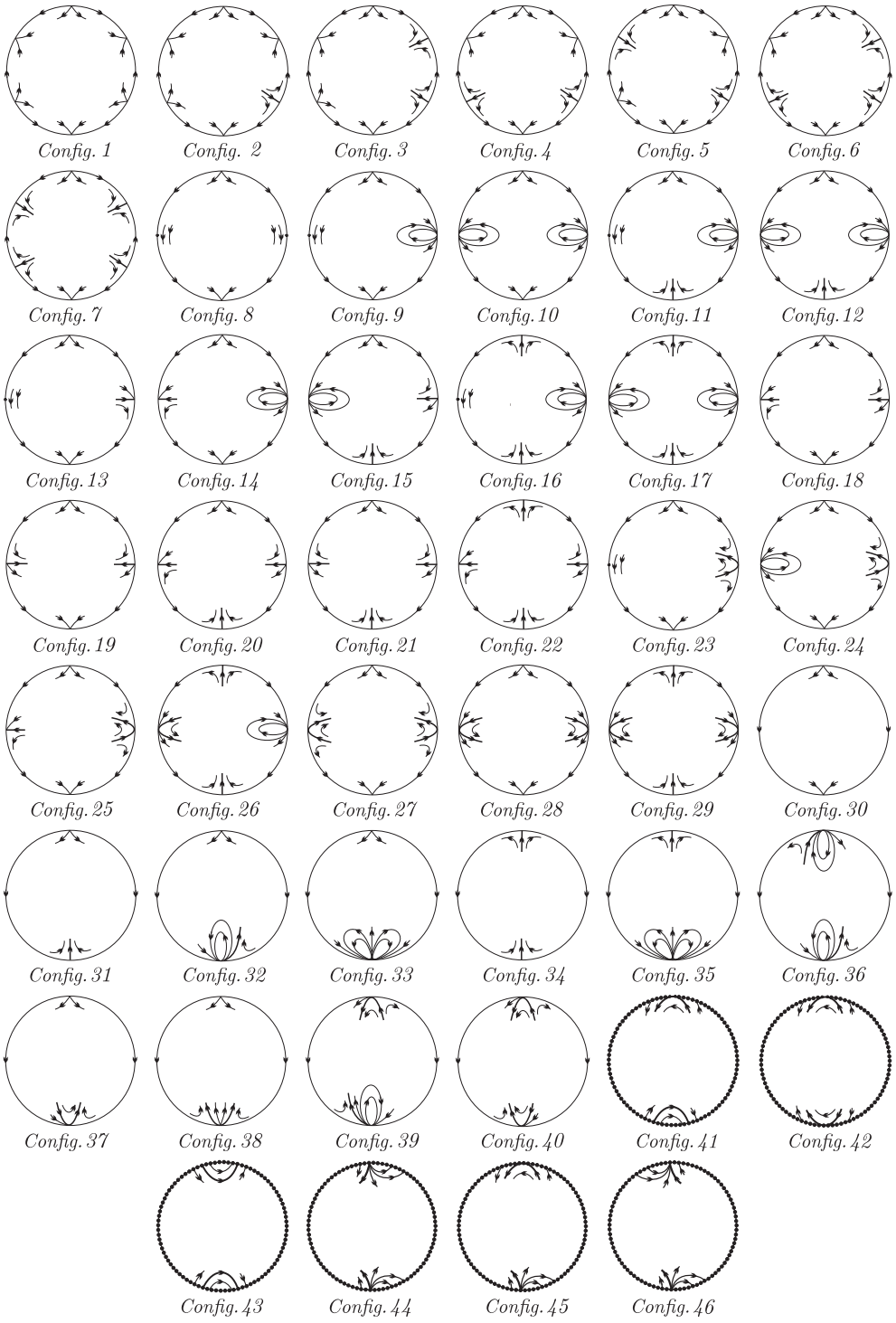

FIGURE 2. Topologically distinct local configurations of **ISP**s ([29],[32])

described as a singular point having two hyperbolic sectors and a single parabolic one. But if we add the borsec which is any orbit of the parabolic sector, then the description would consist of two hyperbolic sectors and two parabolic ones. This distinction will be critical when trying to describe a singular point like the one in FIGURE 4 which topologically is a saddle–node but qualitatively (in the sense of [20]) is different from a semi–elementary saddle–node.

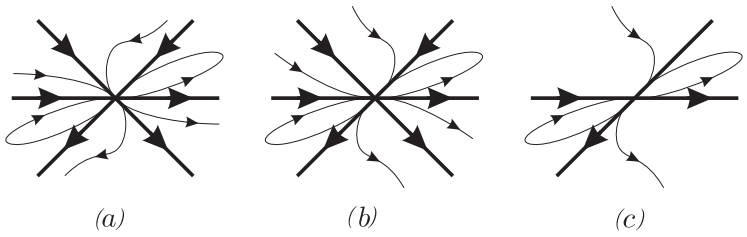

FIGURE 3. Some topologically equivalent singular points

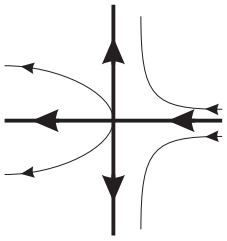

FIGURE 4. Local phase portrait of a saddle-node.

Generically, a *geometric local sector* is defined by two borsecs arriving at the singular point with two different well defined angles and which are consecutive. If this sector is parabolic, then the solutions can arrive at the singular point with one of the two characteristic angles and this is a geometrical information than can be revealed with the blow–up. In this case it may also happen that orbits arrive at the singular point in every angle inside the sector. We will call such a sector a *star–like parabolic sector* and we will be denoted by P^*.

If the sector is elliptic, then generically the solutions inside the sector will depart from and arrive at the singular point in both characteristic angles. It may also happen that orbits arrive at the singular point in every angle inside the sector. Such a sector will be called *star–like elliptic sector* and will be denoted by E^*.

There is also the possibility that two borsecs defining a geometric local sector tend to the singular point with the same well defined angle. Such a sector will be called a *cusp–like sector* which can either be hyperbolic, elliptic or parabolic respectively denoted by $H_\curlywedge$, $E_\curlywedge$ and $P_\curlywedge$.

Moreover, in the case of parabolic sectors we want to include the information as to whether the orbits arrive tangent to one or to the other borsec. We distinguish the two cases writing by $\widehat{P}$ if they arrive tangent to the borsec limiting the previous sector in clock–wise sense or $\widehat{\widehat{P}}$ if they arrive tangent to the borsec limiting the next sector. In the case of a cusp–like parabolic sector, all orbits must arrive with only one well determined angle, but the distinction between $\widehat{P}$ and $\widehat{\widehat{P}}$ is still valid because it occurs at some stage of the desingularization and this can be algebraically determined. Thus, complicated degenerate singular points like the two we see in FIGURE 5 may be described as $\widehat{\widehat{P}}E\,\widehat{P}\,HHH$ (case (a)) and $E\widehat{\widehat{P}}_\curlywedge H H\widehat{P}_\curlywedge E$ (case (b)), respectively.

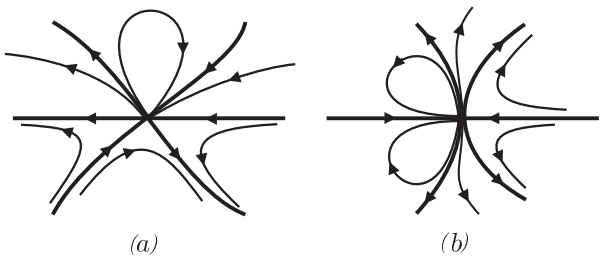

FIGURE 5. Two phase portraits of degenerate singular points.

A star–like point can either be a node or something much more complicated with elliptic and hyperbolic sectors included. In case there are hyperbolic sectors, they must be cusp–like. Elliptic sectors can either be cusp–like or star–like. So, some special angles will be relevant. We will call *special characteristic angle* any well defined angle in which either none or more than one solution curve tends to p within this well defined angle. We will call *special characteristic direction* any line such that at least one of the two angles defining it, is a special characteristic angle.

3. The blow–up technique

To draw the phase portrait around an elementary hyperbolic singularity of a smooth planar vector field we just need to use the Hartman-Grobman theorem. For an elementary non-hyperbolic singularity the system can be brought by an affine change of coordinates and time rescaling to the form $dx/dt = -y + ...$, $dy/dt = x + ...$ and it is well known that in this case the singularity is either a center or a focus. One way to see this is by the Poincaré-Lyapounov theory. In the quadratic case we can actually determine using the Poincaré-Lyapounov constants if it is a focus or a center and then the local phase portrait is known (see [37], [26]). For higher order systems we have the center-focus problem: we can only say that the phase portrait around the singularity is of a center or of a focus but we cannot determine with certainty which one of the two it is.

In case of a more complicated singularity, such as a degenerate one, we need to use the *blow–up technique*. This is a well known technique but since it plays such a crucial role in this work, we shall briefly describe it here. We are using this technique in a slightly modified (actually simplified) way to lighten the calculations. This slightly modified way is in complete agreement with the usual blow–up procedure.

The idea behind the blow–up technique is to replace a singular point p by a circle or by a line on which the "composite" degenerate singularity decomposes (ideally) into a finite number of simpler singularities p_i. For this idea to work we need to construct a new surface, on which we have a diffeomorpic copy of our vector field on $\mathbb{R}^2\backslash\{p\}$ or at least on the complement of a line passing through p, and whose associated foliation with singularities extends also to the circle (or to a line) which replaces the point p on the new surface.

One way to do this is to use polar coordinates. Clearly we may assume that the singularity is placed at the origin. Consider the map $\phi : \mathbb{S}^1 \times \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by $\phi(\theta, r) \mapsto (r\cos\theta, r\sin\theta)$. Restrictions of this map ϕ on $\mathbb{S}^1 \times (0, \infty)$ and on $\mathbb{S}^1 \times (-\infty, 0)$ are diffeomeorphisms, mapping the upper, respectively lower part of the cylinder on $\mathbb{R}^2\backslash\{(0, 0)\}$. But $\phi^{-1}(0, 0)$ is the circle $\mathbb{S}^1 \times \{0\}$. This application defines a diffeomorphic vector field on the upper part of the cylinder $\mathbb{S}^1 \times \mathbb{R}$. In fact this is the passing to polar coordinates. The resulting smooth vector field extends to the whole cylinder just by allowing r to be negative or zero. This full vector field on the cylinder has either a finite number of singularities on the circle (this occurs when the initial singular point is nilpotent) or the circle is filled up with singularities (when we start with a point for which the linear part of the system at this point vanishes). In this latter case we need to work with the reduced system obtained by dividing the right hand side of the equations by a factor r^s with an adequate s to obtain a finite number of singularities. Since $\mathbb{R}^2 \setminus \{(0, 0)\}$ is diffeomorphic to the upper part of the cylinder we only need to consider $r > 0$ for which this factor r^s is also positive. Removing this factor does not affect the nature of the orbits and their orientation. The map ϕ collapses the circle on the cylinder (and hence the singularities located on this circle) to the origin of coordinates in the plane. In case the phase portraits around the singularities on the circle can be drawn then the inverse process of *blowing down* the upper side of the cylinder completed with the circle allows us to draw the portrait around the origin of $\mathbb{R}^2$. In case the singularities on the circle are still degenerate, we need to repeat the process a finite number of times. This is guaranteed by the theorem of desingularization of singularities (see [10] and [14])

The blow–up by polar coordinates is simple, leading to a simple surface (the cylinder), on which a diffeomorphic copy of our vector field on $\mathbb{R}^2\backslash\{(0, 0)\}$ extends to a vector field on the full cylinder. The origin of the plane "blows-up" to the circle $\phi^{-1}(0, 0)$ on which the singularity splits into several simpler singularities.

The visualization of this blow–up is easy. But this process has the disadvantage of using the transcendental functions: *cos* and *sin* and in case several such blow–ups are needed this is computationally very inconvenient.

It would be more advantageous to use a construction involving rational functions. More difficult to visualize, this algebraic blow–up is computationally simpler, using only rational transformations. The blow-up in this case starts with a *directional blow–up* of a point of the plane, by this meaning that in this case to replace the point with a line sitting on a manifold playing the role of the cylinder in the preceding case.

Consider the algebraic surface S in $\mathbb{R}^3$ defined by the equation $y = xz$. We may think of this surface as being here the analogue of the cylinder in the polar blow-up. Like the cylinder, S is a differentiable manifold. Indeed, $\pi_{1,3} : S \to \mathbb{R}^2$, is a global chart for this manifold. We observe that the line $L_z = \{(0, 0, z)|z \in \mathbb{R}\}$ (the z-axis in $\mathbb{R}^3$), lies on S. The projection $\pi_{1,2} : S \to \mathbb{R}^2$, $\pi_{1,2}(x, xz, z) = (x, xz)$ collapses the z-axis to the point $(0, 0)$. The line L_z may be thought here as the analogue of the circle in the polar blow-up construction. The restriction

$$\psi = \pi_{1,2}\big|_{S\setminus L_z} : S \setminus L_z \longrightarrow \mathbb{R}^2 \setminus \{x = 0\}$$

of $\pi_{1,2}$ to $S\setminus L_z$ is a diffeomorphism with inverse $\psi^{-1}(x, y) = (x, y, y/x)$ transferring our vector field restricted to the open set $x \neq 0$ of the plane (x, y) to a diffeomorphic vector field on $S\setminus L_z$. The map $\pi_{1,3} \circ \psi^{-1}$ carries our vector field on the plane (x, y), restricted to $x \neq 0$, to a diffeomorphic vector field on the open set $x \neq 0$ of the plane (x, z). This is actually the vector field on $S \setminus L_z$ calculated in the chart given by $\pi_{1,3}$.

We now compute this vector field on the plane (x, z). We start with a polynomial differential system of the form (1) with a degenerate singular point at the origin $(0, 0)$. We have $p(x, y) = p_1(x, y) + \ldots + p_n(x, y)$ and $q(x, y) = q_1(x, y) + \ldots + q_n(x, y)$ where $p_i(x, y)$ and $q_i(x, y)$ (for $i = 1 \ldots, n$) are the sums of the homogeneous terms involving $x^r y^l$ with $r + l = i$ of p and q. We call the starting degree of (1) the positive integer m such that $(p_m(x, y), q_m(x, y)) \neq (0, 0)$ but $(p_i(x, y), q_i(x, y)) = (0, 0)$ for $i = 0, 1, \ldots, m - 1$.

This differential system when transferred on S and calculated in the chart $\pi_{1,3}$ by using $y = xz$ becomes:

$$dx/dt = x^m(p_m(1, z) + \ldots + x^{n-m}p_n(1, z)),$$
$$dz/dt = x^{m-1}[q_m(1, z) + \ldots + x^{n-m}q_n(1, z) - zx(p_m(1, z) + \ldots + x^{n-m}p_n(1, z))],$$

because $dy/dt = d(xz)/dt = zdx/dt + xdz/dt$. This system is defined over the whole plane (x, z) and in case $m > 1$ the line $x = 0$ (the z-axis in the plane (x, z)), is filled up with singularities. If $m = 1$ then $p_1(x, y)$ and $q_1(x, y)$ cannot be both identically zero. If $q_1(x, y) \equiv 0$ then $q_1(1, z) \equiv 0$ and again we must have the z-axis filled up with singularities. But if $q_1(x, y) = ax + by$ is not identically zero, then $(a, b) \neq (0, 0)$. If $b \neq 0$ then $q_1(1, z) = a + bz$ and $(0, -a/b)$ is the unique singular point on the z-axis. If however $b = 0$ then $q_1(x, y) = ax$ and hence $q_1(1, z) = a \neq 0$ and we have no singular point on the z-axis. So for a nilpotent point with $m = 1$ we either get an infinite number of singularities or a unique singularity or no singularity on the z-axis.

Just like in the polar blow-up when we eliminated the common factor r^s, here we eliminate the common factor x^{m-1} (or x^m in case $q_m(x, y) \equiv 0$ but p_m is not identically zero. But in doing so we need to take some precautions which we explain below. Consider the system above and its associated "reduced" system

$$\begin{aligned} dx/dt &= x\big[p_m(1, z) + \ldots + p_n(1, z)\big], \\ dz/dt &= q_m(1, z) + \ldots + x^{n-m}q_n(1, z) - z\big[p_m(1, z) + \ldots + x^{n-m}p_n(1, z)\big], \end{aligned} \tag{2}$$

obtained by removing the common factor x^{m-1} on the right side of the equations. We observe that for $x > 0$ the two systems have the same orbits and their orbits have the same orientations, but the orbits are described by the solutions of the two systems with different speeds so we have a time change (rescaling). If m is even then $m - 1$ is odd and hence x^{m-1} is negative for $x < 0$ and the orbits of the two systems for $x < 0$ are described by the solutions of the two differential systems with opposite orientations. We need to take care of this when at the end we blow down the line to the point $(0, 0)$. At the points on the z-axis $(x = 0)$ for which $q_m(1, z) = 0$ we have singularities. The finite number of singularities obtained in this way for the reduced system is analogous to the finite number of singularities on the circle we obtained in the reduced system in

the polar blow-up. Thus the singular point at the origin is blown-up to a finite number of singularities on the z-axis of the plane (x, z). We call this the *directional blow-up in the direction of y-axis* of the plane (x, y).

In this blow-up construction the y-axis was excluded. Indeed, the surface S does not contain the y-axis and we have a copy of our vector field on S only for the complement in the plane (x, y) of the y-axis, i.e only on the open set $x \neq 0$. However, by doing an analogous blow-up in the direction of x-axis, the y-axis can be included. The two blow-ups can then be glued so as to obtain a complete blow-up on a Möbius band which will in this case be the full analogue of the cylinder in the polar blow-up. The circle at the center of the Möbius band is then viewed as the space $P_1(\mathbb{R})$ of all directions in the plane (x, y). To see here the need of this twisting on the Möbius band we observe that the map $\pi_{1,3} \circ \psi^{-1}$ sends the left side of the y-axis in the (x, y) plane to the left side of the z-axis on the (x, z) plane. While sending the semi-line $y = 0$ and $x < 0$ to the semi-line $z = 0$ and $x < 0$ this map flips the second and third quadrant in the (x, z) plane. Indeed, the second (respectively third) quadrant in the (x, y) plane are sent to the third (respectively second) quadrant in the (x, z) plane. In this work we use a procedure, a sort of shortcut, to be explained further below which enables us to manage without the Möbius band.

The equation giving us the singular points on the z-axis in the (x, z) plane according to (2) is $zp_m(1, z) - q_m(1, z) = 0$ and going back to the (x, y) coordinates by replacing $z = y/x$ (for $x \neq 0$) we get the equation $yp_m(x, y) - xq_m(x, y) = 0$.

The polynomial $PCD(x, y) = yp_m(x, y) - xq_m(x, y)$, where m is the starting degree of a system of the form (1), is called the *Polynomial of Characteristic Directions* of (1). In case $PCD(x, y) \not\equiv 0$ the factorization of $PCD(x, y)$ gives the characteristic directions at the origin. So, in order to be sure that the y–axis is not a characteristic direction we only need to show that x is not a factor of $PCD(x, y)$. In case it is, we need to do a linear change of variables which moves this direction out of the vertical axis and does not place any other characteristic direction on this axis. If all the directions are characteristic, i.e. $PCD(x, y) \equiv 0$, then the degenerate point will be star–like and at least two blow–ups must be done to obtain the desingularization. Anyway, in quadratic systems there are no degenerate star–like singular points which are degenerate. So, the number of characteristic directions is finite and there exists the possibility to make such a linear change. We will use changes of the type $(x, y) \to (x + ky, y)$ where k is some number (usually 1). It seems natural to call this linear change a *k–twist* as the y–axis gets twisted with some angle depending on k. It is obvious that the phase portrait of the degenerate point which is studied cannot depend on the set of k's used in the desingularization process.

Once we are sure that we have no characteristic direction on the y–axis we do the directional blow–up $(x, y) = (x, xz)$. This change sends the x axis of the (x, y) plane to the X axis of the (x, z) plane and replaces the singular point $(0, 0)$ with a whole vertical axis in the (x, z) plane. The old orbits which arrived at $(0, 0)$ with a well defined slope s now arrive at the singular point $(0, s)$ of the new system. Studying these new singular points, one can determine the local behavior around them and their separatrices which after the blow–down describe the behavior of the orbits around the original singular point up to geometrical equivalence (for definition see next section). Often one needs to do a tree of blow–up's (combined with some translation and/or twists) if some of the singular points which appear on $x = 0$ after the first blow–up are also degenerate.

4. Equivalence relations for singularities of planar polynomial vector fields

We first recall the *topological equivalence relation* as it is used in most of the literature. Two singularities p_1 and p_2 are *topologically equivalent* if there exist open neighborhoods N_1 and N_2 of these points and a homeomorphism $\Psi : N_1 \to N_2$ carrying orbits to orbits and preserving their orientations. To reduce the number of cases, by *topological equivalence* we shall mean here that the homeomorphism Ψ *preserves or reverses* the orientation. We observe that this second notion which is usually used in the literature on classification problems of polynomial vector fields (see [19, 2]), does dot conserve stability.

In [19] Jiang and Llibre introduced another equivalence relation for singularities which is finer than the topological equivalence:

We say that p_1 and p_2 are *qualitatively equivalent* if i) they are topologically equivalent through a local homeomorphism Ψ; and ii) two orbits are tangent to the same straight line at p_1 if and only if the corresponding two orbits are also tangent to the same straight line at p_2.

We say that two simple finite nodes, with the respective eigenvalues λ_1, λ_2 and σ_1, σ_2, of a planar polynomial vector field are *tangent equivalent* if and only if they satisfy one of the following three conditions: a) $(\lambda_1 - \lambda_2)(\sigma_1 - \sigma_2) \neq 0$; b) $\lambda_1 - \lambda_2 = 0 = \sigma_1 - \sigma_2$ and both linearization matrices at the two singularities are diagonal; c) $\lambda_1 - \lambda_2 = 0 = \sigma_1 - \sigma_2$ and the corresponding linearization matrices are not diagonal.

We say that two infinite simple nodes P_1 and P_2 are *tangent equivalent* if and only if their corresponding singularities on the sphere are tangent equivalent and in addition, in case they are generic nodes, we have $(|\lambda_1| - |\lambda_2|)(|\sigma_1| - |\sigma_2|) > 0$ where λ_1 and σ_1 are the eigenvalues of the eigenvectors tangent to the line at infinity.

Finite and infinite singular points may either be real of complex. In case we have a complex singular point we will specify this with the symbols © and Ⓒ for finite and infinite points respectively. We point out that the sum of the multiplicities of all singular points of a quadratic system with a finite number of singular points, is always 7. (Here of course we refer to the compactification on the complex projective plane $P_2(\mathbb{C})$ of the foliation with singularities associated to the complexification of the vector field, see Section 6.1). The sum of the multiplicities of the infinite singular points is always at least 3, more precisely it is always 3 plus the sum of the multiplicities of the finite points disappeared at infinity.

We use here the following terminology for singularities:

We call *elemental* a singular point with its both eigenvalues not zero;
We call *semi–elemental* a singular point with exactly one of its eigenvalues equal to zero;
We call *nilpotent* a singular point with both its eigenvalues zero but with its Jacobian matrix at that point not identically zero;
We call *intricate* a singular point with its Jacobian matrix identically zero.

The *intricate* singularities are usually called in the literature *linearly zero*. We use here the term *intricate* to indicate the rather complicated behavior of phase curves around such a singularity.

Roughly speaking a singular point p of an analytic differential system χ is a *multiple singularity of multiplicity* m if p generates m singularities, as closed to p as we wish, in analytic perturbations χ_ε of this system and m is the maximal such number. In polynomial differential systems of fixed degree n we have several possibilities for obtaining multiple singularities. i) A finite singular point splits into several finite singularities in n-degree polynomial perturbations. ii) An infinite singular point splits into some finite and some infinite singularities in n-degree polynomial perturbations. iii) An infinite singularity splits only in infinite singular points of the systems in n-degree perturbations. To all these cases we can give a precise mathematical meaning using the notion of intersection multiplicity at a point p of two algebraic curves (see [27], [28]).

We will say that two foci (or saddles) are *order equivalent* if their corresponding orders coincide.

Semi–elemental saddle–nodes are always topologically equivalent.

To define the notion of *geometric equivalence relation* of singularities we first define for nilpotent and intricate singular points, the notion of *blow–up equivalence*. We start by having a degenerate singular point p_1 at the origin of the plane of coordinates (x_0, y_0), such that p_1 has a finite number of characteristic directions. We define an ε-twist as a k-twist with k small enough so that no characteristic direction (or special characteristic direction in case of a star point) with negative slope is moved to positive slope. Then if $x_0 = 0$ is a characteristic direction, we do an ε-twist. After the blow–up $(x_0, y_0) = (x_1, y_1 x_1)$ the singular point is replaced by the

straight line $x_1 = 0$ in the plane (x_1, y_1). The neighborhood of the straight line $x_1 = 0$ in the projective plane obtained identifying the opposite infinite points of the Poincaré disk is a Möebius band M_1.

The straight line $x_1 = 0$ will be invariant and may be formed by a continuum of singular points. In that case, with a time change, this degeneracy may be removed and the y_1–axis will remain invariant.

Now we have a number k_1 of singularities located on the affine axis $x_1 = 0$. We do not include the infinite singular point which is the origin of the local chart U_2 at infinity ($Y \neq 0$) because we already know that it does not play any role in understanding the local phase portrait of the singularity p_1. We can then list the k_1 singularities as $p_{1,1}, p_{1,2}, ..., p_{1,k_1}$ with decreasing order of the y_1 coordinate. The $p_{1,i}$ is adjacent to $p_{1,i+1}$ in the usual sense and p_{1,k_1} is also adjacent to $p_{1,1}$ on the Möebius band.

Assume now that we have a degenerate singular point p_1 at the origin of the plane (x_0, y_0) with an infinite number of characteristic directions. Then if $x_0 = 0$ is a special characteristic direction, we do an ε-twist. After the blow–up $(x_0, y_0) = (x_1, y_1 x_1)$ the singular point is replaced by the straight line $x_1 = 0$ in the plane (x_1, y_1). The neighborhood of the straight line $x_1 = 0$ in the projective plane obtained identifying the opposite infinite points of the Poincaré disk is a Möebius band M_1.

The straight line $x_1 = 0$ will be invariant and formed by a continuum of singular points. In that case, with a time change, this degeneracy may be removed and the y_1–axis will no longer be invariant.

Now we have a set of cardinality k_1 formed by singularities located on the axis $x_1 = 0$ plus contact points of the flow with the axis $x_1 = 0$. Again we do not include the infinite singular point at the origin of the local chart U_2 at infinity ($Y \neq 0$) because we already know that it does not play any role in understanding the local phase portrait of the singularity p_1. We list again the k_1 points as $p_{1,1}, p_{1,2}, ..., p_{1,k_1}$ with decreasing order of the y_1 coordinate. The $p_{1,i}$ is adjacent to $p_{1,i+1}$ in the usual sense and p_{1,k_1} is also adjacent to $p_{1,1}$ by the Möebius band.

Let p_2 be a degenerate singularity of another polynomial vector field and suppose that it is located at the origin of the plane $(\bar{x}_0, \bar{y}_0)$.

The next definition works whether the singular points are star–like or not.

We say that p_1 and p_2 are *one step blow–up equivalent* if modulus a rotation with center p_2 (before the blow–up) and a reflection (if needed) we have:

(i) the cardinality k_1 from p_1 equals the cardinality k_2 from p_2;
(ii) we can construct a homeomorphism $\phi^1_{p_1} : M_1 \to M_2$ such that $\phi^1_{p_1}(\{x_1 = 0\}) = \{\bar{x}_1 = 0\}$, $\phi^1_{p_1}$ sends the points $p_{1,i}$ to $p_{2,i}$ and the phase portrait in a neighborhood U of the axis $x_1 = 0$ is topologically equivalent to the phase portrait on $\phi^1_{p_1}(U)$;
(iii) $\phi^1_{p_1}$ sends an elemental (respectively semi–elemental, nilpotent or intricate) singular point to an elemental (respectively semi–elemental, nilpotent or intricate) singular point;
(iv) $\phi^1_{p_1}$ sends a contact point to a contact point.

Assuming $p_{1,j}$ and $\phi^1_{p_1}(p_{1,j}) = p_{2,j}$ are both intricate or both nilpotent, then the process of desingularization (blow–up) must be continued.

We do exactly the same study we did before for p_1 and p_2 now for $p_{1,j}$ and $p_{2,j}$. We move them to the respective origins of the planes (x_1, y_1) and $(\bar{x}_1, \bar{y}_1)$ and we determine whether they are one step blow–up equivalent or not.

If successive degenerate singular points appear from desingularization of p_1 we do the same kind of changes that we did for $p_{1,j}$ and apply the corresponding definition of one step blow–up equivalence. This is repeated until after a finite number of blow–up's all the singular points that appear are elemental or semi–elemental.

We say that two singularities p_1 and p_2, both nilpotent or both intricate, of two polynomial vector fields χ_1 and χ_2, are *blow–up equivalent* if and only if

(i) they are one step blow–up equivalent;

(ii) at each level j in the process of desingularization of p_1 and of p_2, two singularities which are related via the corresponding homeomorphism are one step blow–up equivalent.

Definition 1. *Two singularities p_1 and p_2 of two polynomial vector fields are locally geometrically equivalent if and only if they are topologically equivalent, they have the same multiplicity and one of the following conditions is satisfied:*

- *p_1 and p_2 are order equivalent foci (or saddles);*
- *p_1 and p_2 are tangent equivalent simple nodes;*
- *p_1 and p_2 are both centers;*
- *p_1 and p_2 are both semi–elemental singularities;*
- *p_1 and p_2 are blow–up equivalent nilpotent or intricate singularities.*

We say that two infinite isolated singularities P_1 and P_2 of two polynomial vector fields are *blow–up equivalent* if they are blow–up equivalent finite singularities in the corresponding infinite local charts and the number, type and ordering of sectors on each side of the line at infinity of P_1 coincide with those of P_2.

Definition 2. *Let χ_1 and χ_2 be two polynomial vector fields each having a finite number of singularities. We say that χ_1 and χ_2 have geometric equivalent configurations of singularities if and only if we have a bijection ϑ carrying the singularities of χ_1 to singularities of χ_2 and for every singularity p of χ_1, $\vartheta(p)$ is geometric equivalent with p.*

5. Notations for singularities of polynomial differential systems

In this work we encounter all the possibilities we have for the geometric features of both the finite and the infinite singularities in the whole quadratic class as well as the way they assemble in systems of this class. Since we want to describe precisely these geometric features and in order to facilitate understanding, it is important to have a clear, compact and congenial notation which conveys easily the information. The notation we use, even though it is used here to describe finite and infinite singular points of quadratic systems, can easily be extended to general polynomial systems.

We describe the finite and infinite singularities, denoting the first ones with lower case letters and the second with capital letters. When describing in a sequence both finite and infinite singular points, we will always place first the finite ones and only later the infinite ones, separating them by a semicolon';'.

Elemental points: We use the letters 's','S' for "saddles"; 'n', 'N' for "nodes"; 'f' for "foci"; 'c' for "centers" and © (respectively Ⓒ) for complex finite (respectively infinite) singularities. In order to augment the level of precision we will distinguish the finite nodes as follows:

- 'n' for a node with two distinct eigenvalues (generic node);
- 'n^d' (a one–direction node) for a node with two identical eigenvalues whose Jacobian matrix is not diagonal;
- 'n^*' (a star–node) for a node with two identical eigenvalues whose Jacobian matrix is diagonal.

Moreover, in the case of an elemental infinite generic node, we want to distinguish whether the eigenvalue associated to the eigenvector directed towards the affine plane is, in absolute value, greater or lower than the eigenvalue associated to the eigenvector tangent to the line at infinity. This is relevant because this determines if all the orbits except one on the Poincaré disk arrive at infinity tangent to the line at infinity or transversal to this line. We will denote them as 'N^∞' and 'N^f' respectively.

Finite elemental foci and saddles are classified as strong or weak foci, respectively strong or weak saddles. When the trace of the Jacobian matrix evaluated at those singular points is not zero, we call them strong saddles and strong foci and we maintain the standard notations 's' and 'f.' But when the trace is zero, except for centers and saddles of infinite order (i.e. saddles with all their Poincaré-Lyapounov constants equal to

zero), it is known that the foci and saddles, in the quadratic case, may have up to 3 orders. We denote them by '$s^{(i)}$' and '$f^{(i)}$' where $i = 1, 2, 3$ is the order. In addition we have the centers which we denote by 'c' and saddles of infinite order (integrable saddles) which we denote by '$\mathfrak{s}$'.

Foci and centers cannot appear as singular points at infinity and hence there is no need to introduce their order in this case. In case of saddles, we can have weak saddles at infinity but the maximum order of weak singularities in cubic systems is not yet known. For this reason, a complete study of weak saddles at infinity cannot be done at this stage. Due to this, in this work we shall not even distinguish between a saddle and a weak saddle at infinity.

All non–elemental singular points are multiple points, in the sense that there are perturbations which have at least two elemental singular points as close as we wish to the multiple point. For finite singular points we denote with a subindex their multiplicity as in '$\overline{s}_{(5)}$' or in '$\widehat{es}_{(3)}$' (the notation '$\overline{\ }$' indicates that the saddle is semi–elemental and '$\widehat{es}_{(3)}$' indicates that the singular point is nilpotent). In order to describe the various kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [29]. Thus we denote by '$\binom{a}{b}...$' the maximum number a (respectively b) of finite (respectively infinite) singularities which can be obtained by perturbation of the multiple point. For example '$\overline{\binom{1}{1}}SN$' means a saddle–node at infinity produced by the collision of one finite singularity with an infinite one; '$\overline{\binom{0}{3}}S$' means a saddle produced by the collision of 3 infinite singularities.

Semi–elemental points: They can either be nodes, saddles or saddle–nodes, finite or infinite. We will denote the semi–elemental ones always with an overline, for example '$\overline{sn}$', '$\overline{s}$' and '$\overline{n}$' with the corresponding multiplicity. In the case of infinite points we will put '$\overline{\ }$' on top of the parenthesis with multiplicities.

Moreover, in cases that will be explained later (see page 18), an infinite saddle–node may be denoted by '$\overline{\binom{1}{1}}NS$' instead of '$\overline{\binom{1}{1}}SN$'. Semi–elemental nodes could never be 'n^d' or 'n^*' since their eigenvalues are always different. In case of an infinite semi–elemental node, the type of collision determines whether the point is denoted by 'N^f' or by 'N^∞' where '$\overline{\binom{2}{1}}N$' is an 'N^f' and '$\overline{\binom{0}{3}}N$' is an 'N^∞'.

Nilpotent points: They can either be saddles, nodes, saddle–nodes, elliptic–saddles, cusps, foci or centers. The first four of these could be at infinity. We denote the nilpotent singular points with a hat '$\widehat{\ }$' as in $\widehat{es}_{(3)}$ for a finite nilpotent elliptic–saddle of multiplicity 3 and $\widehat{cp}_{(2)}$ for a finite nilpotent cusp point of multiplicity 2. In the case of nilpotent infinite points, we will put the '$\widehat{\ }$' on top of the parenthesis with multiplicity, for example $\widehat{\binom{1}{2}}PEP-H$ (the meaning of $PEP-H$ will be explained in next paragraph). The relative position of the sectors of an infinite nilpotent point, with respect to the line at infinity, can produce topologically different phase portraits. This forces us to use a notation for these points similar to the notation which we will use for the intricate points.

Intricate points: It is known that the neighborhood of any singular point of a polynomial vector field (except for foci and centers) is formed by a finite number of sectors which could only be of three types: parabolic, hyperbolic and elliptic (see [16]). Then, a reasonable way to describe intricate and nilpotent points is to use a sequence formed by the types of their sectors. The description we give is the one which appears in the clock–wise direction (starting anywhere) once the blow–down of the desingularization is done. Thus in non degenerate quadratic systems, we have just seven possibilities for finite intricate singular points of multiplicity four (see [4]) which are the following ones:

- a) $phpphp_{(4)}$;
- b) $phph_{(4)}$;
- c) $hh_{(4)}$;
- d) $hhhhhh_{(4)}$;
- e) $peppep_{(4)}$;
- f) $pepe_{(4)}$;
- g) $ee_{(4)}$.

We use lower case letters because of the finite nature of the singularities and add the subindex (4) since they are all of multiplicity 4.

For infinite intricate and nilpotent singular points, we insert a dash (hyphen) between the sectors to split those which appear on one side or the other of the equator of the sphere. In this way we will distinguish between $\binom{2}{2}PHP - PHP$ and $\binom{2}{2}PPH - PPH$.

Whenever we have an infinite nilpotent or intricate singular point, we will always start with a sector bordering the infinity (to avoid using two dashes). When one needs to describe a configuration of singular points at infinity, then the relative positions of the points, is relevant in some cases. In [3] this situation only occurs once for systems having two semi–elemental saddle–nodes at infinity and a third singular point which is elemental. In this case we need to write NS instead of SN for one of the semi–elemental points in order to have coherence of the positions of the parabolic (nodal) sector of one point with respect to the hyperbolic (saddle) of the other semi–elemental point. More concretely, *Figure* 3 from [29] (which corresponds to *Config. 3* in FIGURE 2) must be described as $\overline{\binom{1}{1}}SN$, $\overline{\binom{1}{1}}SN$, N since the elemental node lies always between the hyperbolic sectors of one saddle–node and the parabolic ones of the other. However, *Figure* 4 from [29] (which corresponds to *Config. 4* in FIGURE 2) must be described as $\overline{\binom{1}{1}}SN$, $\overline{\binom{1}{1}}NS$, N since the hyperbolic sectors of each saddle–node lie between the elemental node and the parabolic sectors of the other saddle–node. These two configurations have exactly the same description of singular points but their relative position produces topologically (and geometrically) different portraits.

For the description of the topological phase portraits around the isolated singular points the information described above is sufficient. However we are interested in additional geometrical features such as the number of characteristic directions which figure in the final global picture of the desingularization. In order to add this information we need to introduce more notation. If two borsecs (the limiting orbits of a sector) arrive at the singular point with the same slope and direction, then the sector will be denoted by $H_{\curlywedge}$, $E_{\curlywedge}$ or $P_{\curlywedge}$. The index in this notation refers to the cusp–like form of limiting trajectories of the sectors. Moreover, in the case of parabolic sectors we want to make precise whether the orbits arrive tangent to one borsec or to the other. We distinguish the two cases by $\widehat{P}$ if they arrive tangent to the borsec limiting the previous sector in clock–wise sense or $\widehat{\widehat{P}}$ if they arrive tangent to the borsec limiting the next sector. Clearly, a parabolic sector denoted by P^* would correspond to a sector in which orbits arrive with all possible slopes between the those of the borsecs. In the case of a cusp–like parabolic sector, all orbits must arrive with only one slope, but the distinction between $\widehat{P}$ and $\widehat{\widehat{P}}$ is still valid if we consider the different desingularizations we obtain from them. Thus, complicated intricate singular points like the two we see in FIGURE 5 may be described as $\binom{4}{2}\ \widehat{P}E\,\widehat{P} - HHH$ (case (a)) and $\binom{4}{3}\,E\widehat{P}_{\curlywedge}H - H\widehat{P}_{\curlywedge}E$ (case (b)), respectively.

The lack of finite singular points will be encapsulated in the notation $\emptyset$. In the cases we need to point out the lack of an infinite singular point, we will use the symbol $\emptyset\!\!\!/$.

Finally there is also the possibility that we have an infinite number of finite or of infinite singular points. In the first case, this means that the polynomials defining the differential system are not coprime. Their common factor may produce a line or conic with real coefficients filled up with singular points.

Line at infinity filled up with singularities: It is known that any such system has in a sufficiently small neighborhood of infinity one of 6 topological distinct phase portraits (see [32]). The way to determine these portraits is by studying the reduced systems on the infinite local charts after removing the degeneracy of the systems within these charts. In case a singular point still remains on the line at infinity we study such a point. In [32] the tangential behavior of the solution curves was not considered in the case of a node. If after the removal of the degeneracy in the local charts at infinity a node remains, this could either be of the type N^d, N and $N^\star$ (this last case does not occur in quadratic systems as it was shown in [3]). Since no eigenvector of such a node N (for quadratic systems) will have the direction of the line at infinity we do not need to distinguish N^f and N^∞. Other types of singular points at infinity of quadratic systems, after removal of the degeneracy, can be saddles, centers, semi–elemental saddle–nodes or nilpotent elliptic–saddles. We also have

the possibility of no singularities after the removal of the degeneracy. To convey the way these singularities were obtained as well as their nature, we use the notation $[\infty;\, \emptyset]$, $[\infty;\, N]$, $[\infty;\, N^d]$, $[\infty;\, S]$, $[\infty;\, C]$, $[\infty;\, \overline{\binom{1}{0}}SN]$ or $[\infty;\, \widehat{\binom{3}{0}}ES]$.

Degenerate systems: We will denote with the symbol $\ominus$ the case when the polynomials defining the system have a common factor. This symbol stands for the most generic of these cases which corresponds to a real line filled up with singular points. The degeneracy can also be produced by a common quadratic factor which defines a conic. It is well known that by an affine transformation any conic over $\mathbb{R}$ can be brought to one of the following forms: $x^2+y^2-1=0$ (real ellipse), $x^2+y^2+1=0$ (complex ellipse), $x^2-y^2=1$ (hyperbola), $y-x^2=0$ (parabola), $x^2-y^2=0$ (pair of intersecting real lines), $x^2+y^2=0$ (pair of intersecting complex lines), $x^2-1=0$ (pair of parallel real lines), $x^2+1=0$ (pair of parallel complex lines), $x^2=0$ (double line).

We will indicate each case by the following symbols:

- $\ominus[|]$ for a real straight line;
- $\ominus[\circ]$ for a real ellipse;
- $\ominus[\copyright]$ for a complex ellipse;
- $\ominus[\,)(\,]$ for an hyperbola;
- $\ominus[\cup]$ for a parabola;
- $\ominus[\times]$ for two real straight lines intersecting at a finite point;
- $\ominus[\cdot\,]$ for two complex straight lines which intersect at a real finite point.
- $\ominus[\|]$ for two real parallel lines;
- $\ominus[\|^c]$ for two complex parallel lines;
- $\ominus[|_2]$ for a double real straight line.

Moreover, we also want to determine whether after removing the common factor of the polynomials, singular points remain on the curve defined by this common factor. If the reduced system has no finite singularity on this curve, we will use the symbol $\emptyset$ to describe this situation. If some singular points remain we will use the corresponding notation of their types. As an example we complete the notation above as follows:

- $\left(\ominus\,[|];\emptyset\right)$ denotes the presence of a real straight line filled up with singular points such that the reduced system has no singularity on this line;
- $\left(\ominus\,[|];f\right)$ denotes the presence of the same straight line such that the reduced system has a strong focus on this line;
- $\left(\ominus\,[\cup];\emptyset\right)$ denotes the presence of a parabola filled up with singularities such that no singular point of the reduced system is situated on this parabola.

Degenerate systems with non–isolated singular points at infinity, which are however isolated on the line at infinity: The existence of a common factor of the polynomials defining the differential system also affects the infinite singular points. We point out that the projective completion of a real affine line filled up with singular points has a point on the line at infinity which will then be also a non–isolated singularity.

In order to describe correctly the singularities at infinity, we must mention also this kind of phenomena and describe what happens to such points at infinity after the removal of the common factor. To show the existence of the common factor we will use the same symbol $\ominus$ as before, and for the type of degeneracy we use the symbols introduced above. We will use the symbol $\emptyset$ to denote the non–existence of real infinite singular points after the removal of the degeneracy. We will use the corresponding capital letters to describe the singularities which remain there. Let us take note that a simple straight line, two parallel lines (real or complex), one double line or one parabola defined by the common factor (all taken over the reals) imply the existence of one real non–isolated singular point at infinity in the original degenerate system. However a hyperbola and two real straight lines intersecting at a finite point imply the presence of two real non–isolated singular points at infinity in the original degenerate system. Finally, a complex ellipse and two complex straight lines which intersect at a real finite point imply the presence of two complex non–isolated singular

points at infinity in the original degenerate system. Thus, in the reduced system these points may disappear as singularities and in case they remain, they must be described. For the first four cases mentioned above we will give the description of the corresponding infinite point. In the next four cases we will give the description of the corresponding two singular points. As agreed, we will use capital letters to denote them since they are on the line at infinity. We give below some examples:

- $N^f, S, \left(\ominus\, [|]; \emptyset\right)$ means that the system has a node at infinity such that an infinite number of orbits arrive tangent to the eigenvector in the affine part, a saddle, and one non–isolated singular point which belongs to a real affine straight line filled up with singularities, and that the reduced linear system has no infinite singular points in that position;
- $S, \left(\ominus\, [|]; N^*\right)$ means that the system has a saddle at infinity, and one non–isolated singular point which belongs to a real affine straight line filled up with singularities, and that the reduced linear system has a star node in that position;
- $S, \left(\ominus\, [\,)(\,]; \emptyset, \emptyset\right)$ means that the system has a saddle at infinity, and two non–isolated singular points which belong to a hyperbola filled up with singularities, and that the reduced constant system has no singularities in those positions;
- $\left(\ominus\, [\times]; N^*, \emptyset\right)$ means that the system has two non–isolated singular points at infinity which belong to two real intersecting straight lines filled up with singularities, and that the reduced constant system has a star node in one of those positions and no singularities in the other;
- $S, \left(\ominus\, [\circ]; \emptyset, \emptyset\right)$ means that the system has a saddle at infinity, and two non–isolated (complex) singular points which are located on the complexification of a real ellipse which has no real points at infinity, and the reduced constant system has no singularities in those positions.

When there is a non–isolated infinite singular point such that the reduced system has a singularity at that position, it may happen that one or several characteristic directions at this point, directed towards the affine plane, could coincide with a tangent line to the curve of singularities at this point. This situation could produce many different geometrical (or even topological) combinations but in the quadratic case we only have a few of them for which we introduce a coherent notation. This notation can be further developed for higher degree systems. In quadratic systems we only need to distinguish among some situations in which, after the removal of the degeneracy, a characteristic direction of the infinite singular point may coincide or may not coincide with a tangent line to the curve of singularities at this point. We show in FIGURE 6 two cases that need to be distinguished (case (a) and (b)). Here we will use a numerical subscript which denotes the cardinal number $\mathcal{K}$ of the union of the set of characteristic directions, together with the set of tangent lines to the curve of singularities at this point, all of them considered in a neighborhood of the point at infinity on the Poincaré sphere. The singularities at infinity of examples (a) and (b) of FIGURE 6 would then be denoted by $S, \left(\ominus\, [|]; N_3^\infty\right)$ (case (a)) and $S, \left(\ominus\, [|]; N_2^\infty\right)$ (case (b)).

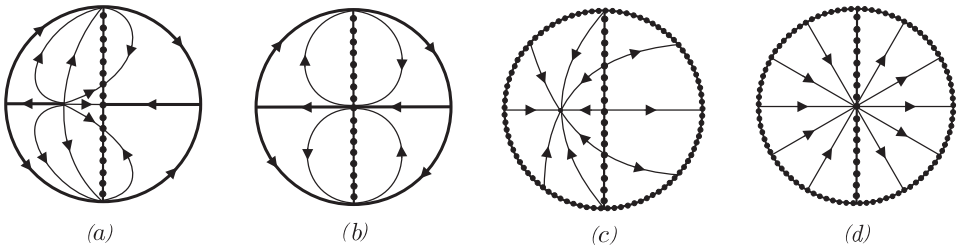

FIGURE 6

Degenerate systems with the line at infinity filled up with singularities: For a quadratic system this implies that the polynomials must have a common linear factor and there are only two possible phase portraits, which can be seen in FIGURE 6 (the portraits (c) and (d)). In order to be consistent with our

notation and considering generalization to higher degree systems, we describe the two cases in a way coherent with what we have done up to now.

The case (c) is denoted by $[\infty; \big(\ominus [|]; \emptyset_3\big)]$ which means:

- the line at infinity is filled up with singular points;
- the reduced quadratic system has on one of the infinite local charts a non–isolated singular point on the line at infinity due to the affine line of degeneracy;
- once the original system at infinity is reduced to a linear one by removing the common factor, the infinity continues to be filled up with singular points;
- once the system on a local chart around the singularity which is common to both lines filled up with singular points, is reduced by completely removing the degeneracy, there is no singular point on that intersection;
- the cardinal number $\mathcal{K}$ is 3. This means that apart from the line of singularities and the line at infinity, we have another characteristic direction pointing towards the affine plane.

The second case is denoted by $[\infty; \big(\ominus [|]; \emptyset_2\big)]$, which means exactly the same items as above with the exception that cardinal number $\mathcal{K}$ is 2. That is, beyond the line of singularities and the line at infinity, we have no other characteristic direction.

6. Assembling multiplicities for global configurations of singularities at infinity using divisors

The singular points at infinity belong to compactifications of planar polynomial differential systems, defined on the affine plane. We begin this section by briefly recalling these compactifications.

6.1. **Compactifications associated to planar polynomial differential systems.**

6.1.1. *Compactification on the sphere and on the Poincaré disk.* Planar polynomial differential systems (1) can be compactified on the sphere. For this we consider the affine plane of coordinates (x, y) as being the plane $Z = 1$ in $\mathbb{R}^3$ with the origin located at $(0, 0, 1)$, the x–axis parallel with the X–axis in $\mathbb{R}^3$, and the y–axis parallel to the Y–axis. We use central projection to project this plane on the sphere as follows: for each point $(x, y, 1)$ we consider the line joining the origin with $(x, y, 1)$. This line intersects the sphere in two points $P_1 = (X, Y, Z)$ and $P_2 = (-X, -Y, -Z)$ where $(X, Y, Z) = (1/\sqrt{x^2 + y^2 + 1})(x, y, 1)$. The applications $(x, y) \mapsto P_1$ and $(x, y) \mapsto P_2$ are bianalytic and associate to a vector field on the plane (x, y) an analytic vector field Ψ on the upper hemisphere and also an analytic vector field Ψ'on the lower hemisphere. A theorem stated by Poincaré and proved in [17] says that there exists an analytic vector field Θ on the whole sphere which simultaneously extends the vector fields on the two hemispheres. By the *Poincaré compactification on the sphere* of a planar polynomial vector field we mean the restriction $\bar{\Psi}$ of the vector field Θ to the union of the upper hemisphere with the equator. For more details we refer to [20]. The vertical projection of $\bar{\Psi}$ on the plane $Z = 0$ gives rise to an analytic vector field Φ on the unit disk of this plane. By the *compactification on the Poincaré disk* of a planar polynomial vector field we understand the vector field Φ. By a *singular point at infinity* of a planar polynomial vector field we mean a singular point of the vector field $\bar{\Psi}$ which is located on the equator of the sphere, respectively a singular point of the vector field Φ located on the circumference of the Poincaré disk.

6.1.2. *Compactification on the projective plane.* To a polynomial system (1) we can associate a differential equation $\omega_1 = q(x, y)dx - p(x, y)dy = 0$. Assuming the differential system (1) is with real coefficients, we may associate to it a foliation with singularities on the real, respectively complex, projective plane as indicated below. The equation $\omega_1 = 0$ defines a foliation with singularities on the real or complex plane depending if we consider the equation as being defined over the real or complex affine plane. It is known that we can compactify these foliations with singularities on the real respectively complex projective plane. In the study of real planar

polynomial vector fields, their associated complex vector fields and their singularities play an important role. In particular such a vector field could have complex, non-real singularities, by this meaning singularities of the associated complex vector field. We briefly recall below how these foliations with singularities are defined.

The application $\Upsilon : \mathbb{K}^2 \longrightarrow P_2(\mathbb{K})$ defined by $(x, y) \mapsto [x : y : 1]$ is an injection of the plane $\mathbb{K}^2$ over the field $\mathbb{K}$ into the projective plane $P_2(\mathbb{K})$ whose image is the set of $[X : Y : Z]$ with $Z \neq 0$. If $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ this application is an analytic injection. If $Z \neq 0$ then $(\Upsilon)^{-1}([X : Y : Z]) = (x, y)$ where $(x, y) = (X/Z, Y/Z)$. We obtain a map $i : \mathbb{K}^3 - \{Z = 0\} \longrightarrow \mathbb{K}^2$ defined by $[X : Y : Z] \mapsto (X/Z, Y/Z)$.

Considering that $dx = d(X/Z) = (ZdX - XdZ)/Z^2$ and $dy = (ZdY - YdZ)/Z^2$, the pull-back of the form ω_1 via the map i yields the form $i * (\omega_1) = q(X/Z, Y/Z)(ZdX - XdZ)/Z^2 - p(X/Z, Y/Z)(ZdY - YdZ)/Z^2$ which has poles on $Z = 0$. Then the form $\omega = Z^{m+2} i * (\omega_1)$ on $K^3 - \{Z = 0\}$, K being $\mathbb{R}$ or $\mathbb{C}$ and m being the degree of systems (1) yields the equation $\omega = 0$:

$$A(X, Y, Z)dX + B(X, Y, Z)dY + C(X, Y, Z)dZ = 0$$

on $K^3 - \{Z = 0\}$ where A, B, C are homogeneous polynomials over K whith $A(X, Y, Z) = ZQ(X, Y, Z)$, $Q(X, Y, Z) = Z^m q(X/Z, Y/Z)$, $B(X, Y, Z) = ZP(X, Y, Z)$, $P(X, Y, Z) = Z^m p(X/Z, Y/Z)$ and $C(X, Y, Z) = YP(X, Y, Z) - XQ(X, Y, Z)$.

The equation $AdX + BdY + CdZ = 0$ defines a foliation F with singularities on the projective plane over K with K either $\mathbb{R}$ or $\mathbb{C}$. The *points at infinity* of the foliation defined by $\omega_1 = 0$ on the affine plane are the points $[X : Y : 0]$ and the line $Z = 0$ is called the *line at infinity* of the foliation with singularities generated by $\omega_1 = 0$.

The singular points of the foliation F are the solutions of the three equations $A = 0$, $B = 0$, $C = 0$. In view of the definitions of A, B, C it is clear that the singular points at infinity are the points of intersection of $Z = 0$ with $C = 0$.

6.2. **Assembling data on infinite singularities in divisors of the line at infinity.** In the previous sections we have seen that there are two types of multiplicities for a singular point p at infinity: one expresses the maximum number m of infinite singularities which can split from p, in small perturbations of the system and the other expresses the maximum number m' of finite singularities which can split from p, in small perturbations of the system. In Section 2 we mentioned that we shall use a column $(m, m')^t$ to indicate this situation.

We are interested in the global picture which includes *all* singularities at infinity. Therefore we need to assemble the data for individual singularities in a convenient, precise way. To do this we use for this situation the notion of *cycle* on an algebraic variety as indicated in [23] and which was used in [20] as well as in [29].

We briefly recall here the definition of this notion. Let V be an irreducible algebraic variety over a field $\mathbb{K}$. A *cycle* of dimension r or $r - cycle$ on V is a formal sum $\sum_W n_W W$, where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $n_W \in \mathbb{Z}$, and only a finite number of the coefficients n_W are non-zero. The degree $\deg(J)$ of a cycle J is defined by $\sum_W n_W$. An $(n-1)$-cycle is called a divisor on V. These notions were used for classification purposes of planar quadratic differential systems in [23, 20, 29].

To a system (1) we can associate two divisors on the line at infinity $Z = 0$ of the complex projective plane: $D_S(P, Q; Z) = \sum_w I_w(P, Q)w$ and $D_S(C, Z) = \sum_w I_w(C, Z)w$ where $w \in \{Z = 0\}$ and where by $I_w(F, G)$ we mean the intersection multiplicity at w of the curves $F(X, Y, Z) = 0$ and $G(X, Y, Z) = 0$, with F and G homogeneous polynomials in X, Y, Z over $\mathbb{C}$. For more details see [20].

Following [29] we assemble the above two divisors on the line at infinity into just one but with values in the ring $\mathbb{Z}^2$:

$$D_S = \sum_{\omega \in \{Z=0\}} \begin{pmatrix} I_w(P, Q) \\ I_w(C, Z) \end{pmatrix} w.$$

This divisor encodes for us the total number of singularities at infinity of a system (1) as well as the two kinds of multiplicities which each singularity has. The meaning of these two kinds of multiplicities are described in the definition of the two divisors $D_S(P,Q;Z)$ and $D_S(C,Z)$ on the line at infinity.

7. Invariant polynomials and preliminary results

Consider real quadratic systems of the form:

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y), \\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y) \end{aligned} \tag{3}$$

with homogeneous polynomials p_i and q_i $(i = 0,1,2)$ of degree i in x, y:

$$p_0 = a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2,$$
$$q_0 = b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2.$$

Let $\tilde{a} = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ be the 12-tuple of the coefficients of systems (3) and denote $\mathbb{R}[\tilde{a}, x, y] = \mathbb{R}[a_{00}, \ldots, b_{02}, x, y]$.

7.1. **Affine invariant polynomials associated to infinite singularities.** It is known that on the set **QS** of all quadratic differential systems (3) acts the group $Aff(2,\mathbb{R})$ of the affine transformation on the plane (cf. [29]). For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **QS**. We can identify the set **QS** of systems (3) with a subset of $\mathbb{R}^{12}$ via the map $\mathbf{QS} \longrightarrow \mathbb{R}^{12}$ which associates to each system (3) the 12–tuple $(a_{00}, \ldots, b_{02})$ of its coefficients.

For the definitions of a GL–comitant and invariant as well as for the definitions of a T–comitant and a CT–comitant we refer the reader to the paper [29] (see also [35]). Here we shall only construct the necessary T–comitants and CT–comitants associated to configurations of infinite singularities (including multiplicities) of quadratic systems (3).

Consider the polynomial $\Phi_{\alpha,\beta} = \alpha P^* + \beta Q^* \in \mathbb{R}[\tilde{a}, X, Y, Z, \alpha, \beta]$ where $P^* = Z^2 P(X/Z, Y/Z)$, $Q^* = Z^2 Q(X/Z, Y/Z)$, P, $Q \in \mathbb{R}[\tilde{a}, x, y]$ and $\max(\deg_{(x,y)} P, \deg_{(x,y)} Q) = 2$. Then

$$\Phi_{\alpha,\beta} = s_{11}(\tilde{a},\alpha,\beta)X^2 + 2s_{12}(\tilde{a},\alpha,\beta)XY + s_{22}(\tilde{a},\alpha,\beta)Y^2 + 2s_{13}(\tilde{a},\alpha,\beta)XZ + 2s_{23}(\tilde{a},\alpha,\beta)YZ + s_{33}(\tilde{a},\alpha,\beta)Z^2$$

and we denote

$$\begin{aligned} \widetilde{D}(\tilde{a},x,y) &= 4\det ||s_{ij}(\tilde{a},y,-x)||_{i,j\in\{1,2,3\}}, \\ \widetilde{H}(\tilde{a},x,y) &= 4\det ||s_{ij}(\tilde{a},y,-x)||_{i,j\in\{1,2\}}. \end{aligned}$$

We consider the polynomials

$$\begin{aligned} C_i(\tilde{a},x,y) &= yp_i(\tilde{a},x,y) - xq_i(\tilde{a},x,y), \\ D_i(\tilde{a},x,y) &= \frac{\partial}{\partial x}p_i(\tilde{a},x,y) + \frac{\partial}{\partial y}q_i(\tilde{a},x,y), \end{aligned} \tag{4}$$

in $\mathbb{R}[\tilde{a}, x, y]$ for $i = 0, 1, 2$ and $i = 1, 2$ respectively. Using the so–called *transvectant of order k* (see [18], [21]) of two polynomials f, $g \in \mathbb{R}[\tilde{a}, x, y]$

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h} \frac{\partial^k g}{\partial x^h \partial y^{k-h}},$$

we construct the following GL—comitants of the second degree with the coefficients of the initial system

$$\begin{aligned} &T_1 = (C_0, C_1)^{(1)}, \quad T_2 = (C_0, C_2)^{(1)}, \quad T_3 = (C_0, D_2)^{(1)}, \\ &T_4 = (C_1, C_1)^{(2)}, \quad T_5 = (C_1, C_2)^{(1)}, \quad T_6 = (C_1, C_2)^{(2)}, \\ &T_7 = (C_1, D_2)^{(1)}, \quad T_8 = (C_2, C_2)^{(2)}, \quad T_9 = (C_2, D_2)^{(1)}. \end{aligned} \tag{5}$$

Using these GL—comitants as well as the polynomials (4) we construct the additional invariant polynomials (see also [29])

$$
\begin{aligned}
\widetilde{M}(\tilde{a},x,y) =&(C_2,C_2)^{(2)} \equiv 2\text{Hess}\left(C_2(\tilde{a},x,y)\right);\\
\eta(\tilde{a}) =&(\widetilde{M},\widetilde{M})^{(2)}/384 \equiv \text{Discrim}\left(C_2(\tilde{a},x,y)\right);\\
\widetilde{K}(\tilde{a},x,y) =&\text{Jacob}\left(p_2(\tilde{a},x,y),q_2(\tilde{a},x,y)\right);\\
K_1(\tilde{a},x,y) =&p_1(\tilde{a},x,y)q_2(\tilde{a},x,y)-p_2(\tilde{a},x,y)q_1(\tilde{a},x,y);\\
K_2(\tilde{a},x,y) =&4(T_2,\widetilde{M}-2\widetilde{K})^{(1)}+3D_1(C_1,\widetilde{M}-2\widetilde{K})^{(1)}-(\widetilde{M}-2\widetilde{K})\left(16T_3-3T_4/2+3D_1^2\right);\\
K_3(\tilde{a},x,y) =&C_2^2(4T_3+3T_4)+C_2(3C_0\widetilde{K}-2C_1T_7)+2K_1(3K_1-C_1D_2);\\
\tilde{L}(\tilde{a},x,y) =&4\widetilde{K}+8\widetilde{H}-\widetilde{M};\\
L_1(\tilde{a},x,y) =&(C_2,\widetilde{D})^{(2)};\\
\widetilde{R}(\tilde{a},x,y) =&\tilde{L}+8\widetilde{K};\\
\kappa(\tilde{a}) =&(\widetilde{M},\widetilde{K})^{(2)}/4;\\
\kappa_1(\tilde{a}) =&(\widetilde{M},C_1)^{(2)};\\
\widetilde{N}(\tilde{a},x,y) =&\widetilde{K}(\tilde{a},x,y)+\widetilde{H}(\tilde{a},x,y);\\
\theta_6(\tilde{a},x,y) =&C_1T_8-2C_2T_6.
\end{aligned}
$$

The geometrical meaning of the invariant polynomials C_2, $\widetilde{M}$ and η is revealed in the next lemma (see [29]).

Lemma 1. *The form of the divisor $D_S(C,Z)$ for systems (3) is determined by the corresponding conditions indicated in Table 1, where we write $w_1^c+w_2^c+w_3$ if two of the points, i.e. w_1^c, w_2^c, are complex but not real. Moreover, for each form of the divisor $D_S(C,Z)$ given in Table 1 the quadratic systems (3) can be brought via a linear transformation to one of the following canonical systems $(\mathbf{S}_I)-(\mathbf{S}_V)$ corresponding to their behavior at infinity.*

Table 1

Case	Form of $D_S(C,Z)$	Necessary and sufficient conditions on the comitants
1	$w_1+w_2+w_3$	$\eta>0$
2	$w_1^c+w_2^c+w_3$	$\eta<0$
3	$2w_1+w_2$	$\eta=0, \quad \widetilde{M}\neq 0$
4	$3w$	$\widetilde{M}=0, \quad C_2\neq 0$
5	$D_S(C,Z)$ undefined	$C_2=0$

$$\begin{cases} \dot{x} &= a+cx+dy+gx^2+(h-1)xy, \\ \dot{y} &= b+ex+fy+(g-1)xy+hy^2; \end{cases} \quad (\mathbf{S}_I)$$

$$\begin{cases} \dot{x} &= a+cx+dy+gx^2+(h+1)xy, \\ \dot{y} &= b+ex+fy-x^2+gxy+hy^2; \end{cases} \quad (\mathbf{S}_{II})$$

$$\begin{cases} \dot{x} &= a+cx+dy+gx^2+hxy, \\ \dot{y} &= b+ex+fy+(g-1)xy+hy^2; \end{cases} \quad (\mathbf{S}_{III})$$

$$\begin{cases} \dot{x} &= a+cx+dy+gx^2+hxy, \\ \dot{y} &= b+ex+fy-x^2+gxy+hy^2, \end{cases} \quad (\mathbf{S}_{IV})$$

$$\begin{cases} \dot{x} &= a+cx+dy+x^2, \\ \dot{y} &= b+ex+fy+xy. \end{cases} \quad (\mathbf{S}_V)$$

7.2. **Affine invariant polynomials associated to finite singularities.** Consider the differential operator $\mathcal{L} = x\cdot\mathbf{L}_2 - y\cdot\mathbf{L}_1$ acting on $\mathbb{R}[a,x,y]$ constructed in [9], where

$$\begin{aligned} \mathbf{L}_1 = &\ 2a_{00}\tfrac{\partial}{\partial a_{10}} + a_{10}\tfrac{\partial}{\partial a_{20}} + \tfrac{1}{2}a_{01}\tfrac{\partial}{\partial a_{11}} + 2b_{00}\tfrac{\partial}{\partial b_{10}} + b_{10}\tfrac{\partial}{\partial b_{20}} + \tfrac{1}{2}b_{01}\tfrac{\partial}{\partial b_{11}}, \\ \mathbf{L}_2 = &\ 2a_{00}\tfrac{\partial}{\partial a_{01}} + a_{01}\tfrac{\partial}{\partial a_{02}} + \tfrac{1}{2}a_{10}\tfrac{\partial}{\partial a_{11}} + 2b_{00}\tfrac{\partial}{\partial b_{01}} + b_{01}\tfrac{\partial}{\partial b_{02}} + \tfrac{1}{2}b_{10}\tfrac{\partial}{\partial b_{11}}. \end{aligned}$$

Using this operator and the affine invariant $\mu_0 = \operatorname{Res}_x\big(p_2(\tilde{a},x,y), q_2(\tilde{a},x,y)\big)/y^4$ we construct the following polynomials

$$\mu_i(\tilde{a},x,y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ i=1,..,4,$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$.

These polynomials are in fact comitants of systems (3) with respect to the group $GL(2,\mathbb{R})$ (see [9]). Their geometrical meaning is revealed in Lemmas 2 and 3 below.

Lemma 2. ([8]) *The total multiplicity of all finite singularities of a quadratic system* (3) *equals k if and only if for every $i\in\{0,1,\dots,k-1\}$ we have $\mu_i(\tilde{a},x,y)=0$ in $\mathbb{R}[x,y]$ and $\mu_k(\tilde{a},x,y)\neq 0$. Moreover a system* (3) *is degenerate (i.e.* $\gcd(P,Q)\neq$ *constant) if and only if $\mu_i(\tilde{a},x,y)=0$ in $\mathbb{R}[x,y]$ for every $i=0,1,2,3,4$.*

Lemma 3. ([9]) The point $M_0(0,0)$ is a singular point of multiplicity k ($1\le k\le 4$) for a quadratic system (3) if and only if for every $i\in\{0,1,\dots,k-1\}$ we have $\mu_{4-i}(\tilde{a},x,y)=0$ in $\mathbb{R}[x,y]$ and $\mu_{4-k}(\tilde{a},x,y)\neq 0$.

We denote

$$\sigma(\tilde{a},x,y) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \sigma_0(\tilde{a}) + \sigma_1(\tilde{a},x,y)\ (\equiv D_1(\tilde{a}) + D_2(\tilde{a},x,y))$$

and observe that the polynomial $\sigma(\tilde{a},x,y)$ is an affine comitant of systems (3). It is known, that if (x_i,y_i) is a singular point of a system (3) then for the trace of its respective linear matrix we have $\rho_i = \sigma(x_i,y_i)$.

Applying the differential operators $\mathcal{L}$ and $(*,*)^{(k)}$ (i.e. transvectant of index k) we shall define the following polynomial function which governs the values of the traces for finite singularities of systems (3).

Definition 3 ([36])**.** We call *trace polynomial* $\mathfrak{T}(w)$ over the ring $\mathbb{R}[\tilde{a}]$ the polynomial defined as follows:

$$\mathfrak{T}(w) = \sum_{i=0}^{4} \frac{1}{(i!)^2}\left(\sigma_1^i,\ \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0)\right)^{(i)} w^{4-i} = \sum_{i=0}^{4}\mathcal{G}_i(\tilde{a})\,w^{4-i}, \quad (6)$$

where the coefficients $\mathcal{G}_i(\tilde{a}) = \dfrac{1}{(i!)^2}(\sigma_1^i,\ \mu_i)^{(i)} \in \mathbb{R}[\tilde{a}]$, $i=0,1,2,3,4$ $\big(\mathcal{G}_0(\tilde{a})\equiv\mu_0(\tilde{a})\big)$ are GL–invariants.

Using the polynomial $\mathfrak{T}(w)$ we could construct the following four affine invariants $\mathcal{T}_4$, $\mathcal{T}_3$, $\mathcal{T}_2$, $\mathcal{T}_1$, which are responsible for the weak singularities:

$$\mathcal{T}_{4-i}(\tilde{a}) = \frac{1}{i!}\frac{d^i\mathfrak{T}}{dw^i}\Big|_{w=\sigma_0}, \in \mathbb{R}[\tilde{a}] \quad i=0,1,2,3 \quad \big(\mathcal{T}_4 \equiv \mathfrak{T}(\sigma_0)\big).$$

The geometric meaning of these invariants is revealed by the next lemma (see [36]).

Lemma 4. *Consider a non-degenerate system* (3) *and let* $\boldsymbol{a} \in \mathbb{R}^{12}$ *be its 12-tuple of coefficients. Denote by* ρ_s *the trace of the linear part of this system at a finite singular point* M_s, $1 \le s \le 4$ *(real or complex, simple or multiple). Then the following relations hold, respectively:*

(i) For $\mu_0(\boldsymbol{a}) \neq 0$ *(total multiplicity 4):*

$$\begin{aligned}
&\mathcal{T}_4(\boldsymbol{a}) = \mathcal{G}_0(\boldsymbol{a})\rho_1\rho_2\rho_3\rho_4,\\
&\mathcal{T}_3(\boldsymbol{a}) = \mathcal{G}_0(\boldsymbol{a})(\rho_1\rho_2\rho_3 + \rho_1\rho_2\rho_4 + \rho_1\rho_3\rho_4 + \rho_2\rho_3\rho_4),\\
&\mathcal{T}_2(\boldsymbol{a}) = \mathcal{G}_0(\boldsymbol{a})(\rho_1\rho_2 + \rho_1\rho_3 + \rho_1\rho_4 + \rho_2\rho_3 + \rho_2\rho_4 + \rho_3\rho_4),\\
&\mathcal{T}_1(\boldsymbol{a}) = \mathcal{G}_0(\boldsymbol{a})(\rho_1 + \rho_2 + \rho_3 + \rho_4);
\end{aligned} \tag{7}$$

(ii) For $\mu_0(\boldsymbol{a}) = 0,\ \mu_1(\boldsymbol{a}, x, y) \neq 0$ *(total multiplicity 3):*

$$\begin{aligned}
&\mathcal{T}_4(\boldsymbol{a}) = \mathcal{G}_1(\boldsymbol{a})\rho_1\rho_2\rho_3, && \mathcal{T}_3(\boldsymbol{a}) = \mathcal{G}_1(\boldsymbol{a})(\rho_1\rho_2 + \rho_1\rho_3 + \rho_2\rho_3),\\
&\mathcal{T}_2(\boldsymbol{a}) = \mathcal{G}_1(\boldsymbol{a})(\rho_1 + \rho_2 + \rho_3), && \mathcal{T}_1(\boldsymbol{a}) = \mathcal{G}_1(\boldsymbol{a});
\end{aligned} \tag{8}$$

(iii) For $\mu_0(\boldsymbol{a}) = \mu_1(\boldsymbol{a}, x, y) = 0,\ \mu_2(\boldsymbol{a}, x, y) \neq 0$ *(total multiplicity 2):*

$$\begin{aligned}
&\mathcal{T}_4(\boldsymbol{a}) = \mathcal{G}_2(\boldsymbol{a})\rho_1\rho_2, && \mathcal{T}_3(\boldsymbol{a}) = \mathcal{G}_2(\boldsymbol{a})(\rho_1 + \rho_2),\\
&\mathcal{T}_2(\boldsymbol{a}) = \mathcal{G}_2(\boldsymbol{a}), && \mathcal{T}_1(\boldsymbol{a}) = 0;
\end{aligned} \tag{9}$$

(iv) For $\mu_0(\boldsymbol{a}) = \mu_1(\boldsymbol{a}, x, y) = \mu_2(\boldsymbol{a}, x, y) = 0,\ \mu_3(\boldsymbol{a}, x, y) \neq 0$ *(one singularity):*

$$\mathcal{T}_4(\boldsymbol{a}) = \mathcal{G}_3(\boldsymbol{a})\rho_1, \quad \mathcal{T}_3(\boldsymbol{a}) = \mathcal{G}_3(\boldsymbol{a}), \quad \mathcal{T}_2(\boldsymbol{a}) = \mathcal{T}_1(\boldsymbol{a}) = 0. \tag{10}$$

In order to be able to calculate the values of the needed invariant polynomials directly for every canonical system we shall define here a family of T–comitants (see [29] for detailed definitions) expressed through C_i

$(i = 0, 1, 2)$ and D_j $(j = 1, 2)$:

$$
\begin{aligned}
\hat{A} =& \left(C_1, T_8 - 2T_9 + D_2^2\right)^{(2)}/144,\\
\widehat{D} =& \Big[2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(6T_7 - T_6 - (C_1, T_5)^{(1)} + 6D_1(C_1D_2 - T_5) - 9D_1^2C_2\Big]/36,\\
\widehat{E} =& \Big[D_1(2T_9 - T_8) - 3\,(C_1, T_9)^{(1)} - D_2(3T_7 + D_1D_2)\Big]/72,\\
\widehat{F} =& \left[6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7)\right. + 48C_0\,(D_2, T_9)^{(1)} - 9D_2^2T_4 + 288D_1\widehat{E}\\
&- 24\left(C_2, \widehat{D}\right)^{(2)} + 120\left(D_2, \widehat{D}\right)^{(1)} - 36C_1\,(D_2, T_7)^{(1)} + 8D_1\,(D_2, T_5)^{(1)}\Big]/144,\\
\widehat{B} =& \Big\{16D_1\,(D_2, T_8)^{(1)}\,(3C_1D_1 - 2C_0D_2 + 4T_2) + 32C_0\,(D_2, T_9)^{(1)}\,(3D_1D_2 - 5T_6 + 9T_7)\\
&+ 2\,(D_2, T_9)^{(1)}\left(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32\,(C_0, T_5)^{(1)}\right)\\
&+ 6\,(D_2, T_7)^{(1)}\,[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5) + C_2(9T_4 + 96T_3)]\\
&+ 6\,(D_2, T_6)^{(1)}\,[32C_0T_9 - C_1(12T_7 + 52D_1D_2) - 32C_2D_1^2] + 48D_2\,(D_2, T_1)^{(1)}\left(2D_2^2 - T_8\right)\\
&- 32D_1T_8\,(D_2, T_2)^{(1)} + 9D_2^2T_4\,(T_6 - 2T_7) - 16D_1\,(C_2, T_8)^{(1)}\left(D_1^2 + 4T_3\right)\\
&+ 12D_1\,(C_1, T_8)^{(2)}\,(C_1D_2 - 2C_2D_1) + 6D_1D_2T_4\left(T_8 - 7D_2^2 - 42T_9\right)\\
&+ 12D_1\,(C_1, T_8)^{(1)}\,(T_7 + 2D_1D_2) + 96D_2^2\left[D_1\,(C_1, T_6)^{(1)} + D_2\,(C_0, T_6)^{(1)}\right]\\
&- 16D_1D_2T_3\left(2D_2^2 + 3T_8\right) - 4D_1^3D_2\left(D_2^2 + 3T_8 + 6T_9\right) + 6D_1^2D_2^2\,(7T_6 + 2T_7)\\
&- 252D_1D_2T_4T_9\}/(2^8 3^3).\\
\widehat{K} =& (T_8 + 4T_9 + 4D_2^2)/72 \equiv \widetilde{K}/4,\\
\widehat{H} =& (8T_9 - T_8 + 2D_2^2)/72 \equiv -\widetilde{H}/4,\\
\widehat{M} =& T_8.
\end{aligned}
$$

These polynomials in addition to (4) and (5) will serve as bricks in constructing affine invariant polynomials for systems (3).

The following 42 affine invariants $A_1, \ldots, A_{42}$ form the minimal polynomial basis of affine invariants up to degree 12. This fact was proved in [11] by constructing $A_1, \ldots, A_{42}$ using the above bricks.

$$
\begin{aligned}
A_1 &= \hat{A}, & A_{22} &= \tfrac{1}{1152}\left[C_2, \widehat{D})^{(1)}, D_2\right)^{(1)}, D_2\right)^{(1)}, D_2\right)^{(1)}D_2\Big)^{(1)},\\
A_2 &= (C_2, \widehat{D})^{(3)}/12, & A_{23} &= \left[\widehat{F}, \widehat{H})^{(1)}, \widehat{K}\right)^{(2)}/8,\\
A_3 &= \left[C_2, D_2)^{(1)}, D_2\right)^{(1)}, D_2\right)^{(1)}/48, & A_{24} &= \left[C_2, \widehat{D})^{(2)}, \widehat{K}\right)^{(1)}, \widehat{H}\right)^{(2)}/32,\\
A_4 &= (\widehat{H}, \widehat{H})^{(2)}, & A_{25} &= \left[\widehat{D}, \widehat{D})^{(2)}, \widehat{E}\right)^{(2)}/16,\\
A_5 &= (\widehat{H}, \widehat{K})^{(2)}/2, & A_{26} &= (\widehat{B}, \widehat{D})^{(3)}/36,\\
A_6 &= (\widehat{E}, \widehat{H})^{(2)}/2, & A_{27} &= \left[\widehat{B}, D_2)^{(1)}, \widehat{H}\right)^{(2)}/24,\\
A_7 &= \left[C_2, \widehat{E})^{(2)}, D_2\right)^{(1)}/8, & A_{28} &= \left[C_2, \widehat{K})^{(2)}, \widehat{D}\right)^{(1)}, \widehat{E}\right)^{(2)}/16,\\
A_8 &= \left[\widehat{D}, \widehat{H})^{(2)}, D_2\right)^{(1)}/8, & A_{29} &= \left[\widehat{D}, \widehat{F})^{(1)}, \widehat{D}\right)^{(3)}/96,\\
A_9 &= \left[\widehat{D}, D_2)^{(1)}, D_2\right)^{(1)}, D_2\right)^{(1)}/48, & A_{30} &= \left[C_2, \widehat{D})^{(2)}, \widehat{D}\right)^{(1)}, \widehat{D}\right)^{(3)}/288,\\
A_{10} &= \left[\widehat{D}, \widehat{K})^{(2)}, D_2\right)^{(1)}/8, & A_{31} &= \left[\widehat{D}, \widehat{D})^{(2)}, \widehat{K}\right)^{(1)}, \widehat{H}\right)^{(2)}/64,
\end{aligned}
$$

$$
\begin{aligned}
A_{11} &= (\widehat{F},\widehat{K})^{(2)}/4, & A_{32} &= \big[\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},\widehat{H}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{12} &= (\widehat{F},\widehat{H})^{(2)}/4, & A_{33} &= \big[\widehat{D},D_2)^{(1)},\widehat{F}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128,\\
A_{13} &= \big[C_2,\widehat{H})^{(1)},\widehat{H}\big)^{(2)},D_2\big)^{(1)}/24, & A_{34} &= \big[\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},\widehat{K}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{14} &= (\widehat{B},C_2)^{(3)}/36, & A_{35} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{E}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128,\\
A_{15} &= (\widehat{E},\widehat{F})^{(2)}/4, & A_{36} &= \big[\widehat{D},\widehat{E})^{(2)},\widehat{D}\big)^{(1)},\widehat{H}\big)^{(2)}/16,\\
A_{16} &= \big[\widehat{E},D_2)^{(1)},C_2\big)^{(1)},\widehat{K}\big)^{(2)}/16, & A_{37} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{D}\big)^{(1)},\widehat{D}\big)^{(3)}/576,\\
A_{17} &= \big[\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},D_2\big)^{(1)}/64, & A_{38} &= \big[C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(2)},\widehat{D}\big)^{(1)},\widehat{H}\big)^{(2)}/64,\\
A_{18} &= \big[\widehat{D},\widehat{F})^{(2)},D_2\big)^{(1)}/16, & A_{39} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{F}\big)^{(1)},\widehat{H}\big)^{(2)}/64,\\
A_{19} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{H}\big)^{(2)}/16, & A_{40} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{F}\big)^{(1)},\widehat{K}\big)^{(2)}/64,\\
A_{20} &= \big[C_2,\widehat{D})^{(2)},\widehat{F}\big)^{(2)}/16, & A_{41} &= \big[C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(2)},\widehat{F}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{21} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{K}\big)^{(2)}/16, & A_{42} &= \big[\widehat{D},\widehat{F})^{(2)},\widehat{F}\big)^{(1)},D_2\big)^{(1)}/16.
\end{aligned}
$$

In the above list, the bracket "[" is used in order to avoid placing the otherwise necessary up to five parenthesizes "(".

Using the elements of the minimal polynomial basis given above we construct the affine invariants

$$
\begin{aligned}
\mathcal{F}_1(\tilde{a}) =& A_2,\\
\mathcal{F}_2(\tilde{a}) =& -2A_1^2A_3+2A_5(5A_8+3A_9)+A_3(A_8-3A_{10}+3A_{11}+A_{12})-\\
&-A_4(10A_8-3A_9+5A_{10}+5A_{11}+5A_{12}),\\
\mathcal{F}_3(\tilde{a}) =& -10A_1^2A_3+2A_5(A_8-A_9)-A_4(2A_8+A_9+A_{10}+A_{11}+A_{12})+\\
&+A_3(5A_8+A_{10}-A_{11}+5A_{12}),\\
\mathcal{F}_4(\tilde{a}) =& 20A_1^2A_2-A_2(7A_8-4A_9+A_{10}+A_{11}+7A_{12})+A_1(6A_{14}-22A_{15})-4A_{33}+4A_{34},\\
\mathcal{F}(\tilde{a}) =& A_7,\\
\mathcal{B}(\tilde{a}) =& -(3A_8+2A_9+A_{10}+A_{11}+A_{12}),\\
\mathcal{H}(\tilde{a}) =& -(A_4+2A_5),
\end{aligned}
$$

as well as the CT-comitants:

$$
\begin{aligned}
\mathcal{B}_1(\tilde{a}) =& \Big\{\big(T_7,D_2\big)^{(1)}\big[12D_1T_3+2D_1^3+9D_1T_4+36\big(T_1,D_2\big)^{(1)}\big]\\
&-2D_1\big(T_6,D_2\big)^{(1)}\big[D_1^2+12T_3\big]+D_1^2\big[D_1\big(T_8,C_1\big)^{(2)}+6\big(\big(T_6,C_1\big)^{(1)},D_2\big)^{(1)}\big]\Big\}/144,\\
\mathcal{B}_2(\tilde{a}) =& \Big\{\big(T_7,D_2\big)^{(1)}\big[8T_3\big(T_6,D_2\big)^{(1)}-D_1^2\big(T_8,C_1\big)^{(2)}-4D_1\big(\big(T_6,C_1\big)^{(1)},D_2\big)^{(1)}\big]+\\
&+\Big[\big(T_7,D_2\big)^{(1)}\Big]^2\big(8T_3-3T_4+2D_1^2\big)\Big\}/384,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_3(\tilde{a},x,y) =& -D_1^2(4D_2^2+T_8+4T_9)+3D_1D_2(T_6+4T_7)-24T_3(D_2^2-T_9),\\
\mathcal{B}_4(\tilde{a},x,y) =& D_1(T_5+2D_2C_1)-3C_2(D_1^2+2T_3).
\end{aligned}
$$

We note that the invariant polynomials $\mathcal{T}_i$, $\mathcal{F}_i$, $\mathcal{B}_i$ (i=1,2,3,4), and $\mathcal{B}$, $\mathcal{F}$, $\mathcal{H}$ and σ are responsible for weak singularities of the family of quadratic systems (see [36, Main Theorem]).

Now we need also the invariant polynomials which are responsible for the types of the finite singularities. These were constructed in [4]. Here we need only the following ones (we keep the notations from [4]):

$$\begin{aligned}
W_4(\tilde a) =&\big[1512A_1^2(A_{30}-2A_{29})-648A_{15}A_{26}+72A_1A_2(49A_{25}+39A_{26})\\
&+6A_2^2(23A_{21}-1093A_{19})-87A_2^4+4A_2^2(61A_{17}+52A_{18}+11A_{20})\\
&-6A_{37}(352A_3+939A_4-1578A_5)-36A_8(396A_{29}+265A_{30})\\
&+72A_{29}(17A_{12}-38A_9-109A_{11})+12A_{30}(76A_9-189A_{10}-273A_{11}-651A_{12})\\
&-648A_{14}(23A_{25}+5A_{26})-24A_{18}(3A_{20}+31A_{17})+36A_{19}(63A_{20}+478A_{21})\\
&+18A_{21}(2A_{20}+137A_{21})-4A_{17}(158A_{17}+30A_{20}+87A_{21})\\
&-18A_{19}(238A_{17}+669A_{19})\big]/81,\\
W_7(\tilde a) =&12A_{26}(A_{26}-2A_{25})+(2A_{29}-A_{30})(A_2^2-20A_{17}-12A_{18}+6A_{19}+6A_{21})\\
&+48A_{37}(A_1^2-A_8-A_{12}),\\
W_8(\tilde a) =&64D_1\big[\big((T_6,C_1)^{(1)},D_2\big)^{(1)}\big]^2\big[16\big(C_0,T_6\big)^{(1)}-37\big(D_2,T_1\big)^{(1)}+12D_1T_3\big]+4(108D_1^4-3T_4^2\\
&-128T_3T_4+42D_1^2T_4)\big[\big((T_6,C_1)^{(1)},D_2\big)^{(1)}\big]^2+36D_1\big((T_6,C_1)^{(1)},D_2\big)^{(1)}\big[4D_1\big(C_0,T_6\big)^{(1)}\\
&-D_1^2(4T_3+T_4)++24T_3^2\big]\big(C_1,T_8\big)^{(2)}+64\big[\big((T_6,C_1)^{(1)},D_2\big)^{(1)}\big]^2\big[27T_3^2+\\
&+16\big((T_6,C_1)^{(1)},C_0\big)^{(1)}\big]-54\big[8D_1^4+D_1^2T_4-8D_1\big(C_0,T_6\big)^{(1)}+8D_1^2T_3+8T_3^2\big]\\
&\times\big((T_6,C_1)^{(1)},T_6\big)^{(1)}\big(C_1,T_8\big)^{(2)}+108D_1T_3\big[\big(C_1,T_8\big)^{(2)}\big]^2\big[D_1T_3-2\big(C_0,T_6\big)^{(1)}\big]\\
&+576\big((T_6,C_1)^{(1)},D_2\big)^{(1)}\big((T_6,C_1)^{(1)},T_6\big)^{(1)}\big[2\big(D_2,T_1\big)^{(1)}-5D_1T_3\big]\\
&-27\big[\big(C_1,T_8\big)^{(2)}\big]^2\big[T_4^4/8+\big(C_0,T_1\big)^{(1)}\big],\\
F_4(\tilde a,x,y) =&\ \mu_3(\tilde a,x,y),\\
F_5(\tilde a,x,y) =&\ T_5+2C_1D_2-3C_2D_1,\\
G_3(\tilde a) =&\ A_2.
\end{aligned}$$

Finally we need the invariant polynomials which are responsible for the existence of one (or two) star node(s) arbitrarily located on the phase plane of a system (3). We have the following lemma (see [39]):

Lemma 5. *A quadratic system* (3) *possesses one star node if and only if one of the following set of conditions hold:*

$$(i)\quad U_1\neq 0,\ U_2\neq 0,\ U_3=Y_1=0;$$
$$(ii)\quad U_1=U_4=U_5=U_6=0,\quad Y_2\neq 0;$$

and it possesses two star nodes if and only if

$$(iii)\quad U_1=U_4=U_5=0,\quad U_6\neq 0,\quad Y_2>0,$$

where

$$\begin{aligned}
&U_1=\widetilde N,\quad U_2=(C_1,\widetilde H-\widetilde K)^{(1)}-2D_1\widetilde N,\\
&U_3=3\widetilde D(D_2^2-16\widetilde K)+C_2\big[(C_2,\widetilde D)^{(2)}-5(D_2,\widetilde D)^{(1)}+6\,\widetilde F\big],\\
&U_4=2T_5+C_1D_2,\quad U_5=3C_1D_1+4T_2-2C_0D_1,\\
&U_6=\widetilde H,\quad Y_1=A_1,\quad Y_2=2D_1^2+8T_3-T_4.
\end{aligned}$$

We base our work here on results obtained in [3] and [4].

8. The proof of the Main Theorem

8.1. **The family of systems without finite singularities.** The total multiplicity m_f of finite singularities of every system in this family is zero. In [3] we gave the full global geometric classification of the whole class

of quadratic systems according to their singularities at infinity. Since only infinite singularities occur in this family ($m_f = 0$), we can extract from [3] the classification of the configurations of singularities of this family. In fact from [3] we obtain more. Indeed, we extract from [3] the part of the *global bifurcation diagram* of configurations of singularities at infinity of **QS**, the fragment covering the case we need here, i.e. $m_f = 0$. We obtain the bifurcation diagram (see DIAGRAM 1) of configurations of singularities of this class, done in the 12-parameter space of coefficients and obtained with the help of invariant polynomials. The proof for this diagram is completely covered in [3] and thus there is no need for a proof here. We shall only give here examples, one for each kind of distinct geometric configurations occurring in this family.

1) *Systems with* $\eta < 0$;

- $\overline{\binom{4}{1}}N$, ©, © : *Example* $\Rightarrow (\dot{x} = 1 + xy;\ \dot{y} = -x^2)$;
- N^*, $\binom{2}{1}$ ©, $\binom{2}{1}$ © : *Example* $\Rightarrow (\dot{x} = 1;\ \dot{y} = -x^2 - y^2)$.

2) *Systems with* $\eta > 0$;

- $\overline{\binom{4}{1}}N$, S, N^∞ : *Example* $\Rightarrow (\dot{x} = -1 + xy;\ \dot{y} = 1 - xy + 2y^2)$;
- $\overline{\binom{4}{1}}S$, N^f, N^f : *Example* $\Rightarrow (\dot{x} = 1 - xy;\ \dot{y} = 2 - 2xy + y^2)$;
- $\overline{\binom{3}{1}}SN$, $\overline{\binom{1}{1}}SN$, N^d : *Example* $\Rightarrow (\dot{x} = 1 + x - xy;\ \dot{y} = 1 - xy)$;
- $\overline{\binom{3}{1}}SN$, $\overline{\binom{1}{1}}NS$, N^d : *Example* $\Rightarrow (\dot{x} = 1 - x + xy;\ \dot{y} = 1 + xy)$;
- $\overline{\binom{2}{1}}S$, $\overline{\binom{2}{1}}N$, N^* : *Example* $\Rightarrow (\dot{x} = 1 - xy;\ \dot{y} = -xy)$.

3) *Systems with* $\eta = 0$, $\widetilde{M} \neq 0$;

- $\overline{\binom{0}{2}}SN$, $\overline{\binom{4}{1}}N$: *Example* $\Rightarrow (\dot{x} = 1 + xy;\ \dot{y} = -1 - xy + y^2)$;
- $\binom{4}{2}$ $\widehat{P}\,H_{\curlywedge}\widehat{P} - \widehat{P}\,H_{\curlywedge}\widehat{P}$, N^f : *Example* $\Rightarrow (\dot{x} = x^2/4;\ \dot{y} = 1 - 3xy/4)$;
- $\binom{4}{2}$ $\widehat{P}\,\widehat{P}_{\curlywedge}H - H\,\widehat{P}_{\curlywedge}\widehat{P}$, N^f : *Example* $\Rightarrow (\dot{x} = 2x^2/3;\ \dot{y} = 1 - xy/3)$;
- $\binom{4}{2}$ $\widehat{P}\,H - H\,\widehat{P}$, N^f : *Example* $\Rightarrow (\dot{x} = x^2/2;\ \dot{y} = 1 - xy/2)$;
- $\binom{4}{2}$ $\widehat{P}\,\widehat{P}_{\curlywedge}E - E\,\widehat{P}_{\curlywedge}\,\widehat{P}$, S : *Example* $\Rightarrow (\dot{x} = -x^2;\ \dot{y} = 1 - 2xy)$;
- $\binom{4}{2}$ $\widehat{P}\,\widehat{P}_{\curlywedge}H - H\,\widehat{P}_{\curlywedge}\widehat{P}$, N^∞ : *Example* $\Rightarrow (\dot{x} = 2x^2;\ \dot{y} = 1 + xy)$;
- $\widehat{\binom{4}{2}}$ $\widehat{P}_{\curlywedge}\,\widehat{P}\,H_{\curlywedge} - H$, N^* : *Example* $\Rightarrow (\dot{x} = y + x^2;\ \dot{y} = 1)$;
- $\binom{4}{2}$ $H - H$, N^d : *Example* $\Rightarrow (\dot{x} = 1 + x^2;\ \dot{y} = x)$;
- $\binom{4}{2}$ $H - H$, N^* : *Example* $\Rightarrow (\dot{x} = 1 + x^2;\ \dot{y} = 1)$;
- $\binom{4}{2}$ $\widehat{P}E\,\widehat{P} - HHH$, N^d : *Example* $\Rightarrow (\dot{x} = -2 + x^2;\ \dot{y} = 1 + x)$;
- $\binom{4}{2}$ $\widehat{P}\,\widehat{P}H - H\widehat{P}\,\widehat{P}$, N^d : *Example* $\Rightarrow (\dot{x} = -1 + x^2;\ \dot{y} = 2 + x)$;
- $\binom{4}{2}$ $\widehat{P}\,\widehat{P}H - H\widehat{P}\,\widehat{P}$, N^* : *Example* $\Rightarrow (\dot{x} = -1 + x^2;\ \dot{y} = 1)$;
- $\binom{4}{2}$ $\widehat{P}\,\widehat{P}_{\curlywedge}H - H\,\widehat{P}_{\curlywedge}\widehat{P}$, N^d : *Example* $\Rightarrow (\dot{x} = x^2;\ \dot{y} = 1 + x)$;
- $\binom{4}{2}$ $\widehat{P}\,\widehat{P}_{\curlywedge}H - H\,\widehat{P}_{\curlywedge}\widehat{P}$, N^* : *Example* $\Rightarrow (\dot{x} = x^2;\ \dot{y} = 1)$;
- $\widehat{\binom{1}{2}}$ $\widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge} - H$, $\overline{\binom{3}{1}}SN$: *Example* $\Rightarrow (\dot{x} = y;\ \dot{y} = 1 - xy)$;
- $\binom{3}{2}$ $E\,\widehat{P} - \widehat{P}\,H$, $\overline{\binom{1}{1}}SN$: *Example* $\Rightarrow (\dot{x} = x;\ \dot{y} = 1 - xy)$;
- $\binom{2}{2}$ $E - E$, $\overline{\binom{2}{1}}S$: *Example* $\Rightarrow (\dot{x} = -1;\ \dot{y} = 1 - xy)$;

- $\binom{2}{2}\, H-H, \overline{\binom{2}{1}}N:$ *Example* $\Rightarrow (\dot{x}=1;\ \dot{y}=1-xy)$;

4) *Systems with* $\eta = \widetilde{M} = 0$;

- $\binom{4}{3}\, E\widehat{P}_{\curlywedge}H-H\widehat{P}_{\curlywedge}E:$ *Example* $\Rightarrow (\dot{x}=x^2;\ \dot{y}=1-x^2+xy)$;
- $\binom{4}{3}\, \widehat{P}\,\widehat{P}_{\curlywedge}\,\widehat{P}-\widehat{P}\,\widehat{P}_{\curlywedge}\widehat{P}:$ *Example* $\Rightarrow (\dot{x}=x^2;\ \dot{y}=-1-x^2+xy)$;
- $\binom{4}{3}\, \widehat{P}_{\curlywedge}EE\widehat{P}_{\curlywedge}-HH:$ *Example* $\Rightarrow (\dot{x}=x;\ \dot{y}=1-x^2)$;
- $\binom{4}{3}\, \widehat{P}_{\curlywedge}\widehat{P}\,\widehat{P}\,\widehat{P}_{\curlywedge}-\widehat{P}\,\widehat{P}:$ *Example* $\Rightarrow (\dot{x}=1+x;\ \dot{y}=-x^2)$;
- $\binom{4}{3}\, \widehat{P}_{\curlywedge}EH_{\curlywedge}-\widehat{P}:$ *Example* $\Rightarrow (\dot{x}=1;\ \dot{y}=y-x^2)$;
- $\binom{4}{3}\, \widehat{P}_{\curlywedge}\widehat{P}-\widehat{P}\,\widehat{P}_{\curlywedge}:$ *Example* $\Rightarrow (\dot{x}=1;\ \dot{y}=-x^2)$;
- $\left[\infty;\ \widehat{\binom{3}{0}}\, ES\right]:$ *Example* $\Rightarrow (\dot{x}=x^2;\ \dot{y}=1+xy)$.

8.2. **The family of quadratic differential systems with only one finite singularity which in addition is elemental.** In this subsection we consider all quadratic vector fields with total multiplicity m_f of finite singularities equal to 1. Since we have only one finite singular point, this point is of course real. To obtain the full global classification of configurations of singularities with respect to the *geometric equivalence relation* for this family, we need to: i) deepen the *topological* classification of all configurations of finite singularities done in [2] by using the finer *geometric equivalence relation*; ii) to integrate this with the *geometric* classification of infinite singularities done in [3] and iii) to search for a minimal set of invariants allowing us to obtain for this family the bifurcation diagram with respect to the *geometric equivalence relation* of configurations of singularities, finite and infinite, in the 12-dimensional space of parameters.

According to [36] in this case the conditions $\mu_0 = \mu_1 = \mu_2 = 0$ and $\mu_3 \neq 0$ must be satisfied and according to [3] the following lemma is valid.

Lemma 6. *The configurations of singularities at infinity of the family of quadratic systems possessing one elemental (real) finite singularity (i.e.* $\mu_0 = \mu_1 = \mu_2 = 0$ *and* $\mu_3 \neq 0$*) are classified in* Diagram *3 according to the geometric equivalence relation. Necessary and sufficient conditions for each one of the 22 different equivalence classes can be assembled from this diagrams in terms of 14 invariant polynomials with respect to the action of the affine group and time rescaling, given in Section 7.*

According to [36] the family of quadratic systems with one elemental finite singularity could be brought via an affine transformation to one of the two canonical forms in [36], governed by invariant polynomial $\widetilde{K} \neq 0$. In what follows we consider two cases: $\widetilde{K} \neq 0$ and $\widetilde{K} = 0$.

8.2.1. *Systems with* $\widetilde{K} \neq 0$. In this case by [36] via an affine transformation quadratic systems in this family could be brought to the systems

$$\begin{aligned} \dot{x} &= cx + dy + (2c+d)x^2 + 2dxy, \\ \dot{y} &= ex + fy + (2e+f)x^2 + 2fxy \end{aligned} \tag{11}$$

possessing the singular points $M_1(0,0)$. For these systems calculations yield

$$\mu_0 = \mu_1 = \mu_2 = 0, \quad \mu_3 = (cf-de)^2x^3, \quad \kappa = 256d^2(de-cf). \tag{12}$$

We remark that for the systems above we have $\mu_3 \neq 0$ and therefore in what follows we assume that the condition $cf - de \neq 0$ holds (i.e. the singular point $M_1(0,0)$ is elemental).

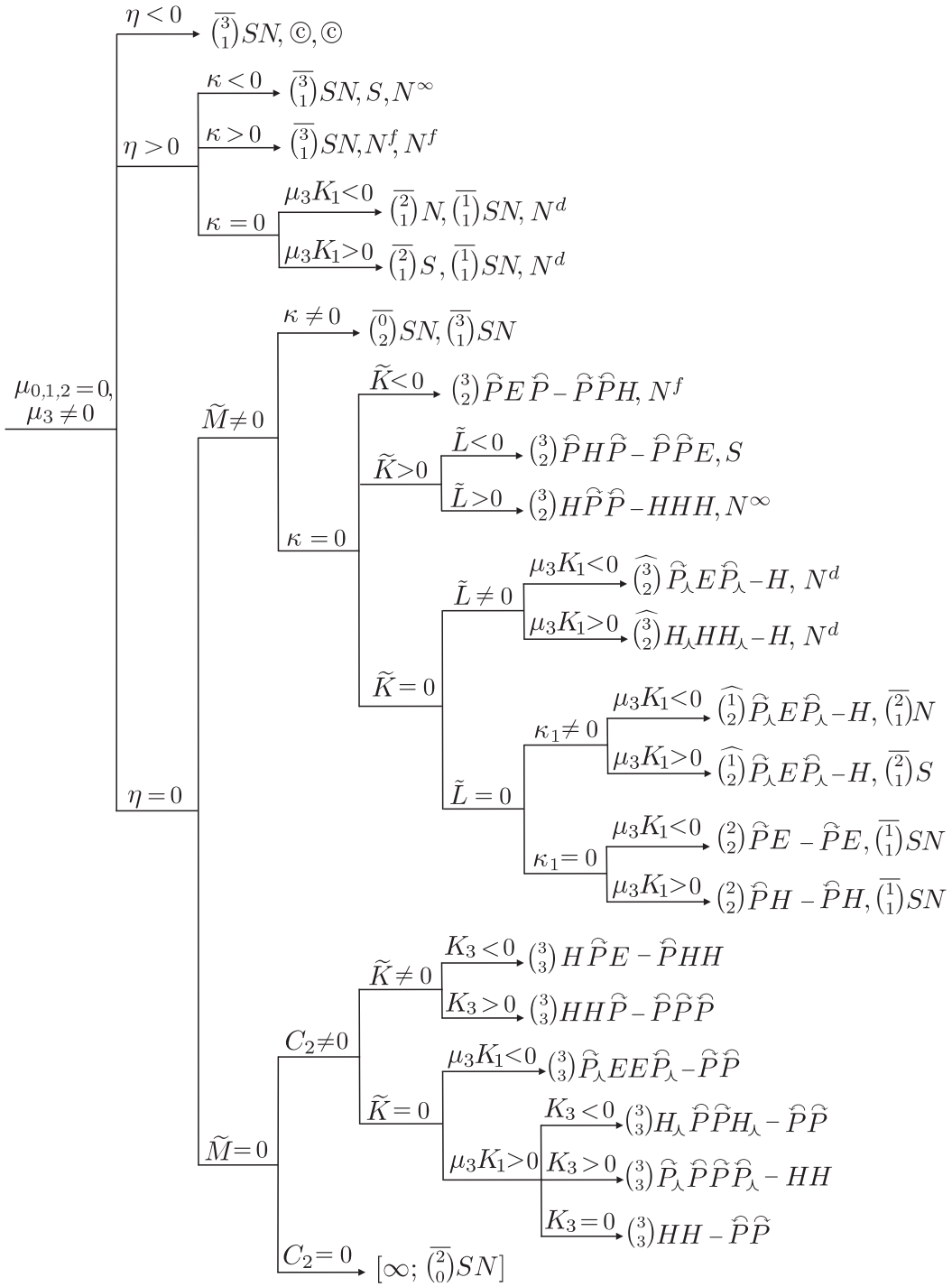

DIAGRAM 3. **The case $\mu_0=\mu_1=\mu_2=0$, $\mu_3\neq 0$**

8.2.1.1. **The case $\kappa\neq 0$.** Then $d\neq 0$ and due to a time rescaling we may assume $d=1$. So we consider the 3-parameter family of systems:

$$\begin{aligned}\dot{x}&=cx+y+(2c+1)x^2+2xy,\\ \dot{y}&=ex+fy+(2e+f)x^2+2fxy,\quad cf-e\neq 0,\end{aligned}\tag{13}$$

for which calculations yield

$$\begin{gathered}\mu_0=\mu_1=\mu_2=0,\ \ \mu_3=(cf-e)^2x^3,\ \ \widetilde{K}=8(cf-e)x^2,\\ \eta=4\big[(2c+1+2f)^2+16(e-cf)\big],\ \ \kappa=256(e-cf),\\ \mathcal{T}_4=-8(c+f)(cf-e)^2,\ \ \mathcal{T}_3=-8(cf-e)^2,\ \ \mathcal{F}_1=6(e-cf),\\ W_4=64(cf-e)^4\big[(c-f)^2+4e\big]=64(cf-e)^4\big[(c+f)^2+4(e-cf)\big],\\ \widetilde{M}=-8\big[(1+2c-2f)^2+6(2e+f)\big]x^2-16(1+2c-2f)xy-32y^2.\end{gathered}\tag{14}$$

Considering (14) we make the remark:

Remark 1. *Assume that the condition $\kappa\neq 0$ holds. Then*

(i) $\mathcal{T}_3\mathcal{F}_1\neq 0$ *and* $\operatorname{sign}(\widetilde{K})=-\operatorname{sign}(\kappa)$*;*
(ii) *the condition $\kappa>0$ implies $\eta>0$ and $W_4>0$;*
(iii) *in the case $\mathcal{T}_4=0$ we have $W_4\neq 0$ and* $\operatorname{sign}(W_4)=\operatorname{sign}(\kappa)$.

The first two statements follow obviously from (14). In the case $\mathcal{T}_4=0$ we get $f=-c$ and then $\kappa=256(c^2+e)$, $W_4=256(c^2+e)^5$ and this proves the last assertion.

8.2.1.1.1. **The subcase $\kappa<0$.** Then by the Remark above we obtain $\widetilde{K}>0$.

1) *The possibility $W_4<0$.* In this case considering the condition $\widetilde{K}>0$, according to [4] (see Table 1, line 184) the finite singularity is a focus.

a) *Assume first $\mathcal{T}_4\neq 0$.* Then by [36] the focus is strong. As $\kappa<0$ according to Lemma 6 we get the following three global configurations of singularities:

- $f;\overline{\binom{3}{1}}SN,ⓒ,ⓒ$: *Example* $\Rightarrow (c=0,\ e=-1,\ f=1)$ (if $\eta<0$);
- $f;\overline{\binom{3}{1}}SN,S,N^{\infty}$: *Example* $\Rightarrow (c=0,\ e=-1,\ f=7/4)$ (if $\eta>0$);
- $f;\overline{\binom{0}{2}}SN,\overline{\binom{3}{1}}SN$: *Example* $\Rightarrow (c=0,\ e=-1,\ f=3/2)$ (if $\eta=0$).

b) *Suppose now $\mathcal{T}_4=0$.* Then $f=-c$ and since by Remark 1 we have $\mathcal{T}_3\mathcal{F}_1\neq 0$, then by [36] the finite singularity is a first order weak focus. Considering the types of the infinite singularities mentioned above we obtain the following three configurations:

- $f^{(1)};\overline{\binom{3}{1}}SN,ⓒ,ⓒ$: *Example* $\Rightarrow (c=0,\ e=-1,\ f=0)$ (if $\eta<0$);
- $f^{(1)};\overline{\binom{3}{1}}SN,S,N^{\infty}$: *Example* $\Rightarrow (c=0,\ e=-1/18,\ f=0)$ (if $\eta>0$);
- $f^{(1)};\overline{\binom{0}{2}}SN,\overline{\binom{3}{1}}SN$: *Example* $\Rightarrow (c=0,\ e=-1/16,\ f=0)$ (if $\eta=0$).

2) *The possibility $W_4>0$.* Since $\widetilde{K}>0$, according to [4] systems (13) possess a node which is generic (due to $W_4\neq 0$). So considering Lemma 6 we have the configurations

- $n;\overline{\binom{3}{1}}SN,ⓒ,ⓒ$: *Example* $\Rightarrow (c=0,\ e=-1,\ f=-9/4)$ (if $\eta<0$);
- $n;\overline{\binom{3}{1}}SN,S,N^{\infty}$: *Example* $\Rightarrow (c=0,\ e=-1,\ f=-3)$ (if $\eta>0$);
- $n;\overline{\binom{0}{2}}SN,\overline{\binom{3}{1}}SN$: *Example* $\Rightarrow (c=0,\ e=-1,\ f=-5/2)$ (if $\eta=0$).

3) *The possibility $W_4=0$.* Then the singular point $M_1(0,0)$ of systems (13) is a node with coinciding eigenvalues which could not be a star node (due to the respective linear matrix). Considering the types of the infinite singularities given by Lemma 6 we get the next three configurations:

- $n^d;\overline{\binom{3}{1}}SN,ⓒ,ⓒ$: *Example* $\Rightarrow (c=0,\ e=-1/4,\ f=-1)$ (if $\eta<0$);
- $n^d;\overline{\binom{3}{1}}SN,S,N^{\infty}$: *Example* $\Rightarrow (c=0,\ e=-1/4,\ f=1)$ (if $\eta>0$);
- $n^d;\overline{\binom{0}{2}}SN,\overline{\binom{3}{1}}SN$: *Example* $\Rightarrow (c=0,\ e=-1/64,\ f=-1/4)$ (if $\eta=0$).

8.2.1.1.2. **The subcase $\kappa > 0$.** According to Remark 1 we obtain $\widetilde{K} < 0$ and according to [4] (see Table 1, line 178) the finite singularity is a saddle. By Remark 1, in this case we have $\eta > 0$ and considering Lemma 6 we have the unique configuration of infinite singularities: $\overline{\binom{3}{1}}SN,\ N^f, N^f$.

1) *Assume first* $\mathcal{T}_4 \neq 0$. In this case by [36] the saddle is strong and we arrive at the configuration

- $s; \overline{\binom{3}{1}}SN,\ N^f, N^f$: *Example* $\Rightarrow (c = 0,\ e = 1,\ f = 1)$.

2) *Suppose now* $\mathcal{T}_4 = 0$. Then $f = -c$ and as by Remark 1, we have $\mathcal{T}_3\mathcal{F}_1 \neq 0$. Considering [36] we deduce that the finite singularity is a weak saddle of the first order. So we obtain the configuration

- $s^{(1)}; \overline{\binom{3}{1}}SN,\ N^f, N^f$: *Example* $\Rightarrow (c = 0,\ e = 1,\ f = 0)$.

8.2.1.2. **The case $\kappa = 0$.** Then by (13) we have $d = 0$ and considering the condition $\mu_3 = c^2 f^2 x^3 \neq 0$ we obtain $cf \neq 0$. So doing a time rescaling we may assume $f = 1$ and we consider the 2-parameter family of systems:

$$\dot{x} = cx + 2cx^2, \quad \dot{y} = ex + y + (2e+1)x^2 + 2xy, \quad c \neq 0, \tag{15}$$

for which calculations yield

$$\begin{gathered} \mu_0 = \mu_1 = \mu_2 = 0,\ \ \mu_3 = c^2x^3,\ \ \widetilde{K} = 8cx^2,\ \ \eta = \kappa = 0,\ \ \widetilde{M} = -32(c-1)^2x^2, \\ C_2 = -(1+2e)x^3 + 2(c-1)x^2y,\ \ \sigma = 1 + c + 2(1+2c)x, \\ \mathcal{T}_i = 0,\ i = 1,2,3,4,\ \ \mathcal{F}_1 = \mathcal{H} = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = 0, \\ \mathcal{B}_3 = -288c^3(1+c)x^2\ \ W_4 = 0,\ \ \tilde{L} = 32c(c-1)x^2. \end{gathered} \tag{16}$$

Remark 2. *We observe that the corresponding matrix for the singular point* $M_1(0,0)$ *is* $\begin{pmatrix} c & 0 \\ e & 1 \end{pmatrix}$ *and hence this singular point is i) a saddle if* $c < 0$*; ii) a node with two direction if* $c > 0$ *and* $c \neq 1$*; iii) a node with one direction if* $c = 1$ *and* $e \neq 0$*; iv) a star node if* $c = 1$ *and* $e = 0$.

8.2.1.2.1. **The subcase $\widetilde{K} < 0$.** Then $c < 0$ and by the remark above the finite singularity is a saddle. Considering (16) according to [36] the saddle is weak if and only if $\mathcal{B}_3 = 0$ (see the statement $e_3[\gamma]$ of Main Theorem. Moreover in this case we have an integrable saddle.

Since $c < 0$ we have $\widetilde{M} \neq 0$. Then according to Lemma 6 at infinity we get the unique configuration of singularities given by $\binom{3}{2}\ \widehat{P}\,E\,\widehat{P} - \widehat{P}\,\widehat{P}\,H,\ N^f$. So we arrive at the next two global configurations of singularities

- $s\,;\ \binom{3}{2}\ \widehat{P}\,E\,\widehat{P} - \widehat{P}\,\widehat{P}\,H,\ N^f$: *Example* $\Rightarrow (c = -2,\ e = 0)$ (if $\mathcal{B}_3 \neq 0$);
- $\$\,;\ \binom{3}{2}\ \widehat{P}\,E\,\widehat{P} - \widehat{P}\,\widehat{P}\,H,\ N^f$: *Example* $\Rightarrow (c = -1,\ e = 0)$ (if $\mathcal{B}_3 = 0$).

8.2.1.2.2. **The subcase $\widetilde{K} > 0$.** Then $c > 0$ and by Remark 2 the finite singularity is a node. We observe that due to $\mu_3 \neq 0$ the condition $c = 1$ is equivalent to $\tilde{L} = 0$.

1) *The possibility* $\widetilde{L} \neq 0$. Then $\widetilde{M} \neq 0$ and by Remark 2 we have a generic node. On the other hand as $\widetilde{K} > 0$ we obtain $\operatorname{sign}(\tilde{L}) = \operatorname{sign}(c-1)$ and considering Lemma 6 we get the following two configurations:

- $n\,;\ \binom{3}{2}\ \widehat{P}\,H\,\widehat{P} - \widehat{P}\,\widehat{P}\,E,\ S$: *Example* $\Rightarrow (c = 1/2,\ e = 0)$ (if $\tilde{L} < 0$);
- $n\,;\ \binom{3}{2}\ H\,\widehat{P}\,\widehat{P} - HHH,\ N^\infty$: *Example* $\Rightarrow (c = 2,\ e = 0)$ (if $\tilde{L} > 0$).

2) *The possibility* $\tilde{L} = 0$. Then $c = 1$ and this implies $\widetilde{M} = 0$. By Remark 2 we have a node with coinciding eigenvalues. On the other hand for $c = 1$ we obtain $C_2 = -(1+2e)x^3$, $U_3 = -24ex^5$.

a) *Assume first* $C_2 \neq 0$. Then we have a single real infinite singularity of multiplicity six and according to Lemma 6 the type of this singularity depends on the sign of the invariant polynomial $K_3 = 6(1+2e)x^6$, which is nonzero due to $C_2 \neq 0$.

Thus taking into consideration Remark 2 and Lemma 6 we arrive at the next configurations

- n^d; $\binom{3}{3}\,H\,\widehat{P}\,E-\widehat{P}\,HH$: *Example* $\Rightarrow$ $(c=1,\ e=-1)$ (if $K_3<0$);
- n^d; $\binom{3}{3}\,HH\,\widehat{P}-\widehat{P}\,\widehat{P}\,\widehat{P}$: *Example* $\Rightarrow$ $(c=1,\ e=1)$ (if $K_3>0$, $U_3\neq 0$);
- n^*; $\binom{3}{3}\,HH\,\widehat{P}-\widehat{P}\,\widehat{P}\,\widehat{P}$: *Example* $\Rightarrow$ $(c=1,\ e=0)$ (if $K_3>0$, $U_3=0$).

b) *Suppose now* $C_2=0$. Then $e=-1/2$ and we get the system

$$\dot{x}=x(1+2x),\quad \dot{y}=-x/2+y+2xy$$

possessing a node n^d and the infinite line filled up with singularities. Considering Lemma 6 we obtain the configuration

- n^d; $\Big[\infty;\overline{\binom{2}{0}}SN\Big]$: *Example* $\Rightarrow$ $(c=1,\ e=-1/2)$.

8.2.2. *Systems with* $\widetilde{K}=0$. In this case, according to [36] we consider the following family of systems

$$\begin{aligned}&\dot{x}=x+dy,\\ &\dot{y}=ex+fy+lx^2+2mxy-d(dl-2m)y^2\end{aligned}\tag{17}$$

possessing the singular points $M_1(0,0)$. For these systems calculations yield

$$\eta=4d^2(dl-m)^2(dl-2m)^2,\quad \tilde{L}=8d(2m-dl)(x+dy)\big[lx-(dl-2m)y\big].\tag{18}$$

We consider two cases: $\eta\neq 0$ and $\eta=0$.

8.2.2.1. **The case** $\eta\neq 0$**.** Then $d(dl-m)(dl-2m)\neq 0$ and we may assume $d=l=1$ and $m=0$ due to the transformation

$$x_1=(dl-2m)x,\quad y_1=-\frac{(dl-2m)m}{dl-m}x+\frac{d(dl-2m)^2}{dl-m}y,\quad t_1=\frac{dl-m}{dl-2m}\,t.$$

So we consider the 2-parameter family of systems

$$\dot{x}=x+y,\quad \dot{y}=ex+fy+x^2-y^2\tag{19}$$

for which calculations yield

$$\begin{gathered}\mu_0=\mu_1=\mu_2=0,\ \ \mu_3=(f-e)(x-y)(x+y)^2,\ \ \widetilde{K}=\kappa=0,\\ \eta=4,\ \ K_1=(x-y)(x+y)^2,\ \ F_4F_5=6(f-e)(x-y)^2(x+y)^4,\\ G_3=2(e-f),\ \ W_4=64(e-f)^2\big[(f-1)^2+4e\big],\\ \mathcal{T}_4=8(f-e)(1+f),\ \ \mathcal{T}_3=8(f-e),\ \ \mathcal{F}_1=2(e-f).\end{gathered}\tag{20}$$

Remark 3. *In the case* $\eta\neq 0$ *the condition* $\mu_3\neq 0$ *implies* $\mathcal{T}_3\mathcal{F}_1F_4F_5G_3\neq 0$ *and* $\operatorname{sign}(\mu_3K_1)=\operatorname{sign}(F_4F_5)$.

8.2.2.1.1. **The subcase** $\mu_3K_1<0$**.** By the remark above we have $F_4F_5<0$ and according to [4] (see Table 1, line 179) the finite singularity is a saddle. Clearly this saddle is weak if and only if $f=-1$ and this is equivalent to $\mathcal{T}_4=0$. On the other hand by Remark 3 we have $\mathcal{T}_3\mathcal{F}_1\neq 0$ and according to [36] the weak saddle could be only of the first order. So considering Lemma 6 we get the following two global configurations of singularities:

- s; $\overline{\binom{2}{1}}N$, $\overline{\binom{1}{1}}SN$, N^d: *Example* $\Rightarrow$ $(e=2,\ f=1)$ (if $\mathcal{T}_4\neq 0$);
- $s^{(1)}$; $\overline{\binom{2}{1}}N$, $\overline{\binom{1}{1}}SN$, N^d: *Example* $\Rightarrow$ $(e=2,\ f=-1)$ (if $\mathcal{T}_4=0$).

8.2.2.1.2. **The subcase $\mu_3 K_1 > 0$.** In this case we have $F_4F_5 > 0$ and as $G_3 \neq 0$ by [4] we have a focus or a center if $W_4 < 0$ and a node if $W_4 \geq 0$.

1) *The possibility* $W_4 < 0$. Than we have a focus which is strong if $\mathcal{T}_4 \neq 0$ and it is weak of the first order if $\mathcal{T}_4 = 0$ (due to [36] and $\mathcal{T}_3\mathcal{F}_1 \neq 0$, see Remark 3). Considering Lemma 6 we arrive at the next two configurations:

- $f\,;\,\overline{\binom{2}{1}}S, \overline{\binom{1}{1}}SN,\ N^d$: *Example* $\Rightarrow (e=-2,\ f=1)$ (if $\mathcal{T}_4 \neq 0$);
- $f^{(1)}\,;\,\overline{\binom{2}{1}}S, \overline{\binom{1}{1}}SN,\ N^d$: *Example* $\Rightarrow (e=-2,\ f=-1)$ (if $\mathcal{T}_4 = 0$).

2) *The possibility* $W_4 > 0$. In this case we have a generic node (as $W_4 \neq 0$) and hence we get

- $n\,;\,\overline{\binom{2}{1}}S, \overline{\binom{1}{1}}SN,\ N^d$: *Example* $\Rightarrow (e=0,\ f=2)$.

3) *The possibility* $W_4 = 0$. Then we have a node with coinciding eigenvalues and due to the linearization matrix at the singularity $M_1(0,0)$ this is a one-direction node:

- $n^d\,;\,\overline{\binom{2}{1}}S, \overline{\binom{1}{1}}SN,\ N^d$: *Example* $\Rightarrow (e=-1/4,\ f=0)$.

8.2.2.2. **The case $\eta = 0$.** Then $d(dl-m)(dl-2m) \neq 0$ and we consider two subcases: $\tilde{L} \neq 0$ and $\tilde{L} = 0$.

8.2.2.2.1. **The subcase $\tilde{L} \neq 0$.** Considering (18) we obtain $d(dl-2m) \neq 0$ and then the condition $\eta = 0$ gives $m = dl$. In this case we have $\tilde{L} = 8d^2l^2(x+dy)^2 \neq 0$ and then via the rescaling $(x,y) \mapsto \big(x/(dl), y/(d^2l)\big)$ we obtain the following 2-parameter family of systems:

$$\dot{x} = x + y, \quad \dot{y} = ex + fy + (x+y)^2. \tag{21}$$

For these systems calculations yield

$$\begin{gathered}
\mu_0 = \mu_1 = \mu_2 = \eta = 0, \quad \mu_3 = (f-e)(x+y)^3, \quad \tilde{L} = 8(x+y)^2 = -\widetilde{M}, \\
\widetilde{K} = \widetilde{N} = \kappa = 0, \quad K_1 = (x+y)^3, \quad F_4F_5 = 6(f-e)(x+y)^6, \\
G_3 = 0, \quad W_8 = 2^{14}3^3(e-f)^4\big[(f-1)^2+4e\big], \quad \mathcal{T}_i = 0,\ i = 1,2,3,4, \\
\sigma = 1+f+2x+2y, \quad \mathcal{F}_1 = \mathcal{H} = 0, \quad \mathcal{B}_1 = 4(e-f)^2(1+f), \quad \mathcal{B}_2 = 4(e-f)^3
\end{gathered} \tag{22}$$

and we again have $\operatorname{sign}(\mu_3 K_1) = \operatorname{sign}(F_4F_5)$.

1) *The possibility* $\mu_3 K_1 < 0$. Then we have $F_4F_5 < 0$ and according to [4] (see Table 1, line 179) the finite singularity is a saddle. Clearly this saddle is weak if and only if $f = -1$ and this is equivalent to $\mathcal{B}_1 = 0$. On the other hand considering (22) we obtain $\mathcal{B}_2 > 0$ and according to [36] the weak saddle is an integrable one. So considering Lemma 6 we get the following two configurations:

- $s\,;\,\widehat{\binom{3}{2}}\,\hat{P}_{\curlywedge}E\hat{P}_{\curlywedge}-H,\ N^d$: *Example* $\Rightarrow (e=2,\ f=1)$ (if $\mathcal{B}_1 \neq 0$);
- $\$\,;\,\widehat{\binom{3}{2}}\,\hat{P}_{\curlywedge}E\hat{P}_{\curlywedge}-H,\ N^d$: *Example* $\Rightarrow (e=2,\ f=-1)$ (if $\mathcal{B}_1 = 0$).

2) *The possibility* $\mu_3 K_1 > 0$ In this case we have $F_4F_5 > 0$ and as $G_3 = \widetilde{N} = 0$ by [4] we have a focus or a center if $W_8 < 0$ and a node if $W_8 \geq 0$.

a) *The case* $W_8 < 0$. Than we have a focus which is strong if $\mathcal{B}_1 \neq 0$. Considering (22) we have $\mathcal{B}_2 < 0$ and according to [36] in the case $\mathcal{B}_1 = 0$ we have a center. So considering Lemma 6 we arrive at the configurations

- $f\,;\,\widehat{\binom{3}{2}}\,H_{\curlywedge}HH_{\curlywedge}-H,\ N^d$: *Example* $\Rightarrow (e=-2,\ f=1)$ (if $\mathcal{B}_1 \neq 0$);
- $c\,;\,\widehat{\binom{3}{2}}\,H_{\curlywedge}HH_{\curlywedge}-H,\ N^d$: *Example* $\Rightarrow (e=-2,\ f=-1)$ (if $\mathcal{B}_1 = 0$).

b) *The case* $W_8 > 0$. In this case we have a generic node (as the condition $W_8 \neq 0$ implies $\delta_1 = (f-1)^2+4e \neq 0$) and hence we get

- $n\,;\,\widehat{\binom{3}{2}}\,H_{\curlywedge}HH_{\curlywedge}-H,\ N^d$: *Example* $\Rightarrow (e=0,\ f=2)$.

c) *The case* $W_8 = 0$. Then we have a node with coinciding eigenvalues and due to the matrix of the linearization of the system at the singularity $M_1(0,0)$, this is a one-direction node:

• n^d; $\widehat{\binom{3}{2}}\, H_\lambda H H_\lambda - H$, N^d: *Example* $\Rightarrow$ $(e = -1/4,\ f = 0)$.

8.2.2.2.2. **The subcase** $\tilde{L} = 0$**.** Considering (18) we obtain $d(dl - 2m) = 0$ and as for systems (17) we have

$$\widetilde{M} = -32m^2x^2 - 8d(dl - 2m)(3lx^2 - 2mxy + d^2ly^2 - 2dmy^2)$$

the condition above gives $\widetilde{M} = -32m^2x^2$. We consider two possibilities: $\widetilde{M} \neq 0$ and $\widetilde{M} = 0$.

1) *The possibility* $\widetilde{M} \neq 0$. Then $m \neq 0$ and as the condition $d(dl - 2m) = 0$ holds, applying the transformation

$$x_1 = dlx, \quad y_1 = dl(x + dy)$$

when $d \neq 0$ (then $m = dl/2 \neq 0$ due to $\widetilde{M} \neq 0$), or the transformation

$$x_1 = 2mx, \quad y_1 = lx/(2m) + y$$

when $d = 0$, we arrive at the following family of systems

$$\begin{aligned} &\dot{x} = \varepsilon_1 x + \varepsilon_2 y, \qquad \varepsilon_1\varepsilon_2 = 0,\\ &\dot{y} = ex + fy + xy, \quad \varepsilon_1 + \varepsilon_2 = 1. \end{aligned} \tag{23}$$

For these systems calculations yield

$$\begin{gathered} \mu_0 = \mu_1 = \mu_2 = 0,\ \ \mu_3 = (\varepsilon_1 f - \varepsilon_2 e)xy(\varepsilon_1 x + \varepsilon_2 y),\ \ \widetilde{K} = \kappa = \tilde{L} = 0,\ \ \widetilde{N} = -x^2,\\ \eta = 0,\ \ \widetilde{M} = -8x^2,\ \ \kappa_1 = -32\varepsilon_2,\ \ K_1 = xy(\varepsilon_1 x + \varepsilon_2 y),\ \ G_3 = 0,\\ F_4F_5 = 6(\varepsilon_1 f - \varepsilon_2 e)x^2y^2(\varepsilon_1 x + \varepsilon_2 y)^2, \quad W_7 = 3\varepsilon_2 e^2(4\varepsilon_2 e + f^2)/16,\\ \mathcal{T}_i = 0,\ i = 1,2,3,4,\ \ \sigma = \varepsilon_1 + f + x,\ \ \mathcal{F}_1 = \mathcal{H} = 0,\ \ \mathcal{B}_1 = -\varepsilon_2 ef,\ \ \mathcal{B}_2 = \varepsilon_2 e/4. \end{gathered} \tag{24}$$

So we obtain again $\operatorname{sign}(\mu_3 K_1) = \operatorname{sign}(F_4F_5)$ and we consider two cases: $\mu_3K_1 < 0$ and $\mu_3K_1 > 0$.

a) *Assume first* $\mu_3K_1 < 0$. Then we have $F_4F_5 < 0$ and according to [4] (see Table 1, line 179) the finite singularity is a saddle. Clearly this saddle is weak if and only if $\varepsilon_1 + f = 0$.

α) *The case* $\kappa_1 \neq 0$. Then by (24) we have $\varepsilon_2 = 1$, $\varepsilon_1 = 0$ and the condition $\mu_3K_1 < 0$ yields $e > 0$. So $\mathcal{B}_2 > 0$ and we have $\mathcal{B}_1 = 0$ if and only if $f = 0$. In this case according to [36] we have an integrable saddle. Therefore considering the condition $\kappa_1 \neq 0$ and Lemma 6 we get the following two global configurations of singularities:

• s; $\widehat{\binom{1}{2}}\, \widehat{P}_\lambda E \widehat{P}_\lambda - H, \overline{\binom{2}{1}} N$: *Example* $\Rightarrow$ $(\varepsilon_2 = 1,\ e = 1,\ f = 1)$ (if $\mathcal{B}_1 \neq 0$);

• $\$$; $\widehat{\binom{1}{2}}\, \widehat{P}_\lambda E \widehat{P}_\lambda - H, \overline{\binom{2}{1}} N$: *Example* $\Rightarrow$ $(\varepsilon_2 = 1,\ e = 1,\ f = 0)$ (if $\mathcal{B}_1 = 0$).

β) *The case* $\kappa_1 = 0$. Then we have $\varepsilon_2 = 0$, $\varepsilon_1 = 1$ and the condition $\mu_3K_1 < 0$ yields $f < 0$. We observe that in this case the saddle is a weak one if and only if $f + 1 = 0$. On the other hand calculations yield

$$\mathcal{F}_1 = \mathcal{H} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = 0,\ \ \mathcal{B}_4 = 6(1 + f)x^2y. \tag{25}$$

So according to [36] in the case of weak saddle (i.e. $f = -1$) we have an integrable saddle. Therefore considering the condition $\kappa_1 = 0$ and Lemma 6 we obtain the configurations

• s; $\binom{2}{2}\, \widehat{P} E - \widehat{P} E, \overline{\binom{1}{1}} SN$: *Example* $\Rightarrow$ $(\varepsilon_2 = 0,\ e = 0,\ f = -2)$ (if $\mathcal{B}_4 \neq 0$);

• $\$$; $\binom{2}{2}\, \widehat{P} E - \widehat{P} E, \overline{\binom{1}{1}} SN$: *Example* $\Rightarrow$ $(\varepsilon_2 = 0,\ e = 0,\ f = -1)$ (if $\mathcal{B}_4 = 0$).

b) *Suppose now* $\mu_3K_1 > 0$. In this case we have $F_4F_5 > 0$ and as $G_3 = 0$ and $\widetilde{N} \neq 0$, according to [4] (see Table 1, lines 182, 186, 188) we have a focus or a center if $W_7 < 0$ and a node if $W_7 \geq 0$.

α) *The case* $W_7 < 0$. Then we have have $\varepsilon_2 = 1$, $\varepsilon_1 = 0$ (i.e. $\kappa_1 \neq 0$) and $e < 0$. So the finite singularity is a focus and according to [36] we have a strong focus if $\mathcal{B}_1 \neq 0$ and we have a center if $\mathcal{B}_1 = 0$.

Thus considering Lemma 6 we arrive at the following two configurations:

- f; $\widehat{\binom{1}{2}}\,\widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H, \overline{\binom{2}{1}} S$: *Example* $\Rightarrow$ $(\varepsilon_2 = 1,\ e = -1,\ f = 1)$ (if $\kappa_1 \neq 0$, $\mathcal{B}_1 \neq 0$);
- c; $\widehat{\binom{1}{2}}\,\widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H, \overline{\binom{2}{1}} S$: *Example* $\Rightarrow$ $(\varepsilon_2 = 1,\ e = -1,\ f = 0)$ (if $\kappa_1 \neq 0$, $\mathcal{B}_1 = 0$).

β) *The case* $W_7 > 0$. Then we again have $\varepsilon_2 = 1, \varepsilon_1 = 0$ and hence $\kappa_1 \neq 0$. So the singular point is a generic node and by Lemma 6 we get the configuration

- n; $\widehat{\binom{1}{2}}\,\widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H, \overline{\binom{2}{1}} S$: *Example* $\Rightarrow$ $(\varepsilon_2 = 1,\ e = -1,\ f = 3)$.

γ) *The case* $W_7 = 0$. Then by (24) we have $\varepsilon_2 e(4\varepsilon_2 e + f^2) = 0$ and we consider two subcases: $\kappa_1 \neq 0$ and $\kappa_1 = 0$.

$\gamma_{\mathbf{1}}$) *The subcase* $\kappa_1 \neq 0$. Then we have $\varepsilon_2 = 1, \varepsilon_1 = 0$ and the condition $W_7 = 0$ gives $e = -f^2/4$. Considering the linearization matrix of the singularity $M_1(0,0)$ we conclude that systems (23) possess a node n^d. So by Lemma 6 we have the configuration

- n^d; $\widehat{\binom{1}{2}}\,\widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H, \overline{\binom{2}{1}} S$: *Example* $\Rightarrow$ $(\varepsilon_2 = 1,\ e = -1,\ f = 2)$.

$\gamma_{\mathbf{2}}$) *The subcase* $\kappa_1 = 0$. In this case we have $\varepsilon_2 = 0, \varepsilon_1 = 1$ and the linearization matrix of the singularity $M_1(0,0)$ is $\begin{pmatrix} 1 & 0 \\ e & f \end{pmatrix}$ with $f > 0$ due to $\mu_3 K_1 > 0$. So the systems (23) possess *i)* a generic node if $f \neq 1$; *ii)* an one-direction node if $f = 1$ and $e \neq 0$, and *iii)* a star node if $f = 1$ and $e = 0$. On the other hand for these systems in the considered case we have

$$U_7 = 12(f-1)x^4, \quad U_3 = -3x^4\big[efx + (1-f)y\big]$$

and clearly these invariant polynomials govern the possibilities mentioned above. So considering the condition $\kappa_1 = 0$ and Lemma 6 we get the following three configurations:

- n; $\binom{2}{2}\,\widehat{P}\,H - \widehat{P}\,H, \overline{\binom{1}{1}} SN$: *Example* $\Rightarrow$ $(\varepsilon_2 = 0,\ e = 1,\ f = 2)$ (if $U_7 \neq 0$);
- n^d; $\binom{2}{2}\,\widehat{P}\,H - \widehat{P}\,H, \overline{\binom{1}{1}} SN$: *Example* $\Rightarrow$ $(\varepsilon_2 = 0,\ e = 1,\ f = 1)$ (if $U_7 = 0$, $U_3 \neq 0$);
- n^*; $\binom{2}{2}\,\widehat{P}\,H - \widehat{P}\,H, \overline{\binom{1}{1}} SN$: *Example* $\Rightarrow$ $(\varepsilon_2 = 0,\ e = 0,\ f = 1)$ (if $U_7 = 0$, $U_3 = 0$).

2) *The possibility* $\widetilde{M} = 0$. In this case $m = 0$ and then the condition $\widetilde{M} = 0$ yields $dl = 0$. As $l \neq 0$ (due to $\mu_3 \neq 0$) we get $d = 0$ and then via the rescaling $(x, y) \mapsto (x/l,\ y/l)$ we may assume $l = 1$. Therefore we obtain the family of systems

$$\dot{x} = x, \quad \dot{y} = ex + fy + x^2, \tag{26}$$

for which calculations yield

$$\begin{gathered} \mu_0 = \mu_1 = \mu_2 = 0,\ \ \mu_3 = fx^3,\ \ \eta = \widetilde{M} = 0,\ \ C_2 = -x^3,\ \ K_1 = x^3, \\ \widetilde{K} = \kappa = \tilde{L} = \widetilde{N} = 0,\ \ G_3 = W_8 = 0,\ \ K_3 = 6(2-f)fx^6, \\ F_4F_5 = 6fx^6,\ \ \mathcal{T}_i = 0,\ i = 1,2,3,4,\ \ \sigma = 1 + f. \end{gathered} \tag{27}$$

a) *The case* $\mu_3 K_1 < 0$. Then we have $F_4F_5 < 0$ and according to [4] (see Table 1, line 179) the finite singularity is a saddle. Clearly this saddle is weak if and only if $f = -1$ and this is equivalent to $\sigma = 0$. However in the last case we get Hamiltonian systems and hence the weak saddle is an integrable one. So considering Lemma 6 we arrive at the configurations

- s; $\binom{3}{3}\,\widehat{P}_{\curlywedge} E E \widehat{P}_{\curlywedge} - \widehat{P}\,\widehat{P}$: *Example* $\Rightarrow$ $(e = 0,\ f = -2)$ (if $\sigma \neq 0$);

• $\$$; $\binom{3}{3}\,\widehat{P}_{\curlywedge}EE\,\widehat{P}_{\curlywedge}-\widehat{P}\,\widehat{P}$: *Example* $\Rightarrow$ $(e=0,\ f=-1)$ (if $\sigma=0$).

b) *The case* $\mu_3K_1>0$. In this case we have $F_4F_5>0$ (i.e. $f>0$) and considering the matrix of the linearization at the singular point, we conclude that the singular point $M_1(0,0)$ is a node. Moreover, this node is: *i)* generic if $f\neq 1$; *ii)* one-direction node if $f=1$ and $e\neq 0$, and *iii)* it is a star node if $f=1$ and $e=0$. On the other hand for these systems in the considered case we have

$$U_4=-6(f-1)x^3,\quad U_5\big|_{f=1}=-6ex^2.$$

On the other hand the behavior of the trajectories in the vicinity of the infinite singularity (which is of multiplicity six) according to Lemma 6 is governed by the invariant polynomial K_3. By (27) as $f>0$ we have $\operatorname{sign}(K_3)=\operatorname{sign}(2-f)$.

Thus we arrive at the following five geometrically distinct global configurations of singularities:

• n; $\binom{3}{3}\,H_{\curlywedge}\,\widehat{P}\,\widehat{P}\,H_{\curlywedge}-\widehat{P}\,\widehat{P}$: *Example* $\Rightarrow$ $(e=0,\ f=3)$ (if $K_3<0$);

• n; $\binom{3}{3}\,HH-\widehat{P}\,\widehat{P}$: *Example* $\Rightarrow$ $(e=0,\ f=2)$ (if $K_3=0$);

• n; $\binom{3}{3}\,\widehat{P}_{\curlywedge}\,\widehat{P}\,\widehat{P}\,\widehat{P}_{\curlywedge}-HH$: *Example* $\Rightarrow$ $(e=0,\ f=1/2)$ (if $K_3>0$, $U_4\neq 0$);

• n^d; $\binom{3}{3}\,\widehat{P}_{\curlywedge}\,\widehat{P}\,\widehat{P}\,\widehat{P}_{\curlywedge}-HH$: *Example* $\Rightarrow$ $(e=1,\ f=1)$ (if $K_3>0$, $U_4=0$, $U_5\neq 0$);

• n^*; $\binom{3}{3}\,\widehat{P}_{\curlywedge}\,\widehat{P}\,\widehat{P}\,\widehat{P}_{\curlywedge}-HH$: *Example* $\Rightarrow$ $(e=0,\ f=1)$ (if $K_3>0$, $U_4=0$, $U_5=0$).

As all the cases are considered we have got all 52 possible geometrically distinct global configurations of singularities of the family of quadratic systems with only one finite singularity which in addition is elemental.

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