

PERTURBED DAMPED PENDULUM: FINDING PERIODIC SOLUTIONS VIA AVERAGING METHOD

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ABSTRACT. Using the damped pendulum model we introduce the averaging method to study the periodic solutions of dynamical systems with small non-autonomous perturbation. We provide sufficient conditions for the existence of periodic solutions with small amplitude of the non-linear perturbed damped pendulum. The averaging method provides a useful means to study dynamical systems, accessible to Master and PhD students.

1. INTRODUCTION

Systems derived from the pendulum give to students important and practical examples of dynamical systems. For instance, we can see the *weight-driven pendulum clocks* which had its historical and dynamical aspects studied by Denny in a recent paper [1]. This system has been revisited by Llibre and Teixeira in [2], and using some simple techniques, from averaging theory, they got the same results. Usually, the systems involving pendulums have also been used to introduce mathematical concepts of classical mechanics, as we can see in [3].

In this paper we attempt to use a simple physical system, as the damped pendulum, to introduce some concepts and techniques of the important and useful averaging theory, which can be used to study the periodic solutions of dynamical systems. For instance, in [4], Llibre, Novaes and Teixeira have used the averaging theory to provide sufficient conditions for the existence of periodic solutions of the planar double pendulum with small oscillations perturbed by non-linear functions.

2. THE DAMPED PENDULUM

We consider a system composed of a point mass m moving in the plane, under gravity force, such that the distance between the point mass m and a given point P is fixed and equal to l . We also consider

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that the motion of the particle suffers a resistance proportional to its velocity. This system is called *Damped Pendulum*.

The position of the pendulum is determined by the angle θ shown in Figure 1. The equation of motion of this system is given by

$$(1) \quad \ddot{\theta} = -a \sin(\theta) - b\dot{\theta},$$

where $a > 0$ and $b > 0$ are real parameters, with $a = g/l$, g the acceleration of the gravity, l the length of the rod and b the damping coefficient. We shall also assume that the damping coefficient b is a small parameter.

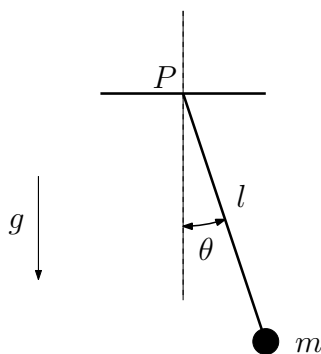


FIGURE 1. Pendulum.

There are many other kinds of resistance that the particle motion can suffer, providing many different dynamical behaviors (see Remark 1). For instance, the *Coulomb Friction* introduces a discontinuous term in the equation of motion (1). For more details about this last issue, see the book of Andronov, Vitt and Khaikin [5].

In the qualitative theory of dynamical systems, a singularity x^* of an autonomous differential system $\dot{x}(t) = F(x)$, i.e $F(x^*) = 0$, is called *Hyperbolic* if the eigenvalues of the linear transformation $DF(x^*)$ (derivative of F in x^*) has non-zero real components. In this case, applying the classical *Hartman-Grobman Theorem* (see Theorem 2.2.3 from [6]) we can study the local behavior of the system looking to the linearized system $\dot{y}(t) = D(x^*)y$.

Remark 1. Here, the behavior of a dynamical system can be informally understood as how the phase portrait looks like. For a general introduction to qualitative theory of dynamical systems see for instance the book of Arrowsmith and Place [6].

The linearized equation of motion of the damped pendulum is given by

$$(2) \quad \ddot{\theta} = -a\theta - b\dot{\theta}.$$

Considering the coordinates $(\theta, \dot{\theta})$, the system (2) has the following eigenvalues

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4a}}{2} \quad \text{and} \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4a}}{2},$$

which is the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 \\ -a & -b \end{pmatrix}.$$

Note that for $b^2 \geq 4a$ the eigenvalues λ_1 and λ_2 are both negative, then the singularity $(\theta, \dot{\theta}) = (0, 0)$ is a attractor node, represented in the Figure 2. Now, for $b^2 < 4a$ both eigenvalues λ_1 and λ_2 have the imaginary part different of zero and negative real part, then the singularity $(\theta, \dot{\theta}) = (0, 0)$ is a attractor focus, represented in the Figure 3. Both cases are topologically equivalent.

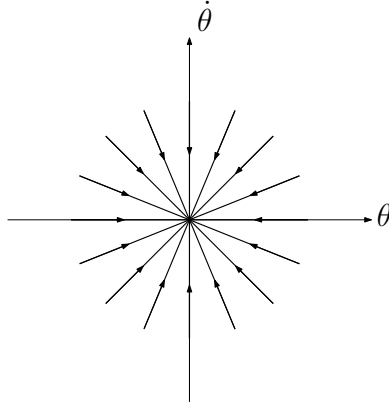


FIGURE 2. Attractor Node.

Observe that without the damping effect, i.e. $b = 0$, both eigenvalues λ_1 and λ_2 would be purely imaginary, then the singularity $(\theta, \dot{\theta}) = (0, 0)$ would be a center (which is a non-hyperbolic singularity), represented in the Figure 4. This last case is not topologically equivalent to the two cases above.

We emphasize here that, for $b \neq 0$, the linearized equation (2) can only be used to study the local behavior, at $(\theta, \dot{\theta}) = (0, 0)$, of the

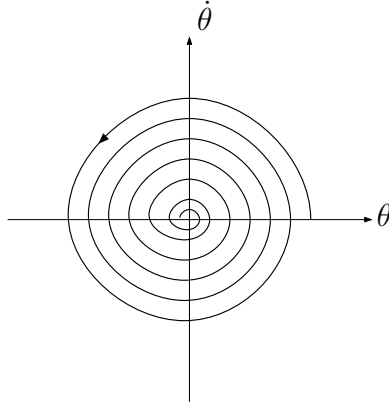


FIGURE 3. Attractor Focus.

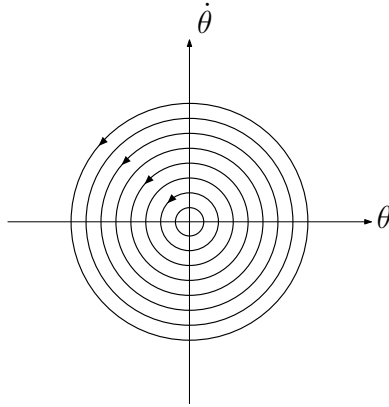


FIGURE 4. Center.

original (non-linear) system (1). However, a periodic solution of a differential system is a global element of the phase portrait. Thus to study these elements we have to consider the non-linear equation, which has the global information of the system.

As we have seen, when $b \neq 0$, the origin is an attractor singularity, thus the orbits of the system starting sufficiently closer to the origin tends to it. In other words, the damped pendulum always tends to stop. The following study provide conditions to drop this regime obtaining thus a periodic solution of the system which never reaches the origin.

3. PERTURBED SYSTEM

Let (f, g) be an ordered pair of functions, such that

$$f, g : \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R} \rightarrow \mathbb{R}.$$

We consider the non-autonomous perturbation of the system (1):

$$(3) \quad \ddot{\theta} = -a \sin(\theta) - b\dot{\theta} + \varepsilon f(t, \theta, \dot{\theta}) + \varepsilon^2 g(t, \theta, \dot{\theta}).$$

By a non-autonomous perturbation we understand that the function $(f, g)(t, \theta, \dot{\theta})$ is dependent of the variable t . We also assume that the ordered pair of functions (f, g) satisfies the following conditions:

- (C1) $f(t, \theta, \dot{\theta})$ and $g(t, \theta, \dot{\theta})$ are \mathcal{C}^2 functions;
- (C2) $f(t, \theta, \dot{\theta})$ and $g(t, \theta, \dot{\theta})$ are locally Lipschitz with respect to $(\theta, \dot{\theta})$;
- (C3) $f(t, \theta, \dot{\theta})$ and $g(t, \theta, \dot{\theta})$ are respectively T_f and T_g periodic in the variable t with

$$T_f = \frac{p_f}{q_f} \frac{2\pi}{\sqrt{a}} \quad \text{and} \quad T_g = \frac{p_g}{q_g} \frac{2\pi}{\sqrt{a}},$$

being p_i and q_i relatively prime positive integers for $i = f, g$;

- (C4) and $f(t, 0, 0) = 0$.

We say that the *Basic Conditions* are satisfied for an ordered pair of functions (f, g) (which define the non-autonomous perturbation on the system (3)) when (f, g) satisfies the conditions C_i , for $i = 1, 2, 3, 4$.

Remark 2. *For simplicity, instead condition C3 we can assume, without loss of generality, that the functions $f(t, \theta, \dot{\theta})$ and $g(t, \theta, \dot{\theta})$ are T -periodic with $T = 2p\pi/\sqrt{a}$ for some integer p . Indeed, taking p the least common multiple between p_f and p_g , we have that there exists integers n_f and n_g such that $p = n_f p_f = n_g p_g$. Hence*

$$p \frac{2\pi}{\sqrt{a}} = n_f q_f \frac{p_f}{q_f} \frac{2\pi}{\sqrt{a}} = n_g q_g \frac{p_g}{q_g} \frac{2\pi}{\sqrt{a}}.$$

4. CHANGE OF COORDINATES

As we shall see in the Section 6 the main result from the *Averaging Theory*, used in this present work, assumes that the perturbed system is given in an *Standard Form* (8). To get it, we have to introduce two changes of coordinates. The first one is done in the following.

If we take $\theta = \varepsilon\phi$ with $|\varepsilon| > 0$, then the system (3) becomes

$$(4) \quad \ddot{\phi} = -a \frac{\sin(\varepsilon\phi)}{\varepsilon} - \varepsilon \bar{b}\dot{\phi} + f(t, \varepsilon\phi, \varepsilon\dot{\phi}) + \varepsilon g(t, \varepsilon\phi, \varepsilon\dot{\phi}).$$

Note that, since b is a small parameter, we can assume that $b = \varepsilon \bar{b}$, with $\bar{b} > 0$ if we consider perturbations for $\varepsilon > 0$; and with $\bar{b} < 0$ if we consider perturbations for $\varepsilon < 0$. Henceforward we assume that $\varepsilon > 0$.

As a consequence of this coordinates change we have the following lemma.

Lemma 1. *There exists a continuous function $r(t, \phi, \dot{\phi}, \varepsilon)$, T -periodic in the variable t with $T = 2p\pi/\sqrt{a}$, such that the system (4) is written as*

$$(5) \quad \ddot{\phi} = -a\phi + \varepsilon \left(g_0(t) + f_1(t)\phi + (f_2(t) - \bar{b})\dot{\phi} \right) + \varepsilon^2 r(t, \phi, \dot{\phi}, \varepsilon),$$

where

$$g_0(t) = g(t, 0, 0), \quad f_1(t) = \frac{\partial f}{\partial \theta}(t, 0, 0), \quad \text{and} \quad f_2(t) = \frac{\partial f}{\partial \dot{\theta}}(t, 0, 0).$$

Proof. Set the variables t, ϕ and $\dot{\phi}$ as the parameters of the \mathcal{C}^2 function $s \mapsto F(s; t, \phi, \dot{\phi})$ defined as

$$F(s; t, \phi, \dot{\phi}) = -a \frac{\sin(s\phi)}{s} - s\bar{b}\dot{\phi} + f(t, s\phi, s\dot{\phi}) + sg(t, s\phi, s\dot{\phi}).$$

Applying the *Taylor's Formula with Lagrange Remainder* for the function $F(s; t, \phi, \dot{\phi})$ at $s = 0$ we conclude that there exists $0 < h < 1$ such that

$$\begin{aligned} F(\varepsilon; t, \phi, \dot{\phi}) &= F(0; t, \phi, \dot{\phi}) + \varepsilon F'(0; t, \phi, \dot{\phi}) + \varepsilon^2 \frac{F''(\varepsilon h; t, \phi, \dot{\phi})}{2}, \\ &= -ax + \varepsilon \left(g_0(t) + f_1(t)\phi + (f_2(t) - \bar{b})\dot{\phi} \right) \\ &\quad + \varepsilon^2 \frac{F''(\varepsilon h; t, \phi, \dot{\phi})}{2}. \end{aligned}$$

Since

$$\begin{aligned} F''(\varepsilon h; t, \phi, \dot{\phi}) &= \frac{\varepsilon 2ah \phi \cos(\varepsilon h\phi) + a(\varepsilon^2 h^2 \phi^2 - 2) \sin(\varepsilon h\phi)}{\varepsilon^3 h^3} \\ &\quad + \frac{d^2}{ds^2} \left(f(t, s\phi, s\dot{\phi}) + sg(t, s\phi, s\dot{\phi}) \right) \Big|_{s=\varepsilon h}, \end{aligned}$$

and

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon 2ah \phi \cos(\varepsilon h\phi) + a(\varepsilon^2 h^2 \phi^2 - 2) \sin(\varepsilon h\phi)}{\varepsilon^3 h^3} = \frac{ax^3}{3}.$$

It follows that the function

$$r : (t, \phi, \dot{\phi}, \varepsilon) \mapsto \frac{F''(\varepsilon h; t, \phi, \dot{\phi})}{2}$$

is continuous in the variable ε . By the other hand

$$r(t, \phi, \dot{\phi}, \varepsilon) = \frac{F(\varepsilon h; t, \phi, \dot{\phi}) + ax}{\varepsilon^2} - \frac{g_0(t) + f_1(t)\phi + (f_2(t) - \bar{b})\dot{\phi}}{\varepsilon}.$$

Thus we conclude that the function $r(t, \phi, \dot{\phi}, \varepsilon)$ is continuous in the variables t, ϕ and $\dot{\phi}$, and T -periodic in the variable t . \square

The *Taylor's Formula with Lagrange Remainder* has its statement and its proof done in the book of Lima [7].

To study the periodic solutions of the system (3) we shall study the periodic solutions of the system (5). Indeed, if $\psi(t, \varepsilon)$ is a solution of system (5), then $\varphi(t, \varepsilon) = \varepsilon\psi(t, \varepsilon)$ is a solution of the system (3). However, the change of coordinate, introduced above, restricts our study only for periodic solutions close to the origin, since $|\varphi(t, \varepsilon)| \rightarrow 0$ as $\varepsilon \rightarrow 0$ for $t \in I$ for every I compact interval of \mathbb{R} . In this case, we say that this periodic solution is bifurcating of the origin.

Now, denoting $(x, y) = (\phi, \dot{\phi})$, the second order differential equation (5) can be written as the first order differential system

$$(6) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -ax + \varepsilon (g_0(t) + f_1(t)x + (f_2(t) - \bar{b})y) + \varepsilon^2 r(t, x, y, \varepsilon). \end{aligned}$$

Observe that the unperturbed system, i.e. $\varepsilon = 0$, has the following eigenvalues

$$\lambda_1 = i\sqrt{a} \quad \text{and} \quad \lambda_2 = -i\sqrt{a},$$

thus it is a center at the origin, see Figure 4.

There are many works which deal with perturbation of centers, even in higher dimensions. For instance we can see the paper of Llibre and Teixeira [8].

The second change of coordinates (12) will be done in the Section 7.

5. STATEMENTS OF THE MAIN RESULTS

Our main goal, in this present work, is to find sufficient conditions on the ordered pair of functions (f, g) to assure the existence of periodic solutions of the system (3). For this, we shall provide a matrix M such that its non-singularity, i.e. $\det(M) \neq 0$, implies the existence of at least one periodic solution, for $\varepsilon > 0$ sufficiently small, of the system (3).

We define the matrix $M = (M_{ij})_{2 \times 2}$ as
(7)

$$M_{11} = -\frac{\bar{b}\pi}{\sqrt{a}} + \int_0^{\frac{2p\pi}{\sqrt{a}}} \sin(\sqrt{a}t) \left(-\frac{\cos(\sqrt{a}t)}{\sqrt{a}} f_1(t) + \sin(\sqrt{a}t) f_2(t) \right) dt,$$

$$M_{12} = \int_0^{\frac{2p\pi}{\sqrt{a}}} \frac{\sin(\sqrt{a}t)}{a} \left(-\sin(\sqrt{a}t) f_1(t) - \sqrt{a} \cos(\sqrt{a}t) f_2(t) \right) dt,$$

$$M_{21} = \int_0^{\frac{2p\pi}{\sqrt{a}}} \cos(\sqrt{a}t) \left(\cos(\sqrt{a}t) f_1(t) - \sqrt{a} \sin(\sqrt{a}t) f_2(t) \right) dt,$$

$$M_{22} = -\frac{\bar{b}\pi}{\sqrt{a}} + \int_0^{\frac{2p\pi}{\sqrt{a}}} \cos(\sqrt{a}t) \left(\frac{\sin(\sqrt{a}t)}{\sqrt{a}} f_1(t) + \cos(\sqrt{a}t) f_2(t) \right) dt.$$

Our main result on the periodic solutions of the damped pendulum with small non-autonomous perturbation (3) is the following.

Theorem 2. *Assume that the Basic Conditions are satisfied for the pair of functions (f, g) , which define the non-autonomous perturbation on the system (3). If $\det(M) \neq 0$, then for $\varepsilon > 0$ sufficiently small the perturbed damped pendulum (3) has a T -periodic solution $\theta(t, \varepsilon)$, such that*

$$(\theta(0, \varepsilon), \dot{\theta}(0, \varepsilon)) \rightarrow (0, 0),$$

when $\varepsilon \rightarrow 0$.

Theorem 2 is proved in section 7. Its proof is based in the averaging theory for computing periodic solutions, which will be introduced in section 6.

We provide an application of Theorem 2 in the following corollary.

Corollary 3. *Assume that the Basic Conditions are satisfied for the pair of functions (f, g) , which define the non-autonomous perturbation on the system (3). Moreover, suppose that*

$$\frac{\partial f}{\partial \theta}(t, 0, 0) = C_1 \quad \text{and} \quad \frac{\partial f}{\partial \dot{\theta}}(t, 0, 0) = C_2.$$

If $(C_1, C_2) \neq (0, \bar{b})$, then there exists a T -periodic solution $\theta(t, \varepsilon)$ of the perturbed damped pendulum (3), such that

$$(\theta(0, \varepsilon), \dot{\theta}(0, \varepsilon)) \rightarrow (0, 0),$$

when $\varepsilon \rightarrow 0$.

The Corollary 3 will be proved in section 7.

6. AVERAGING THEORY

We present in this section a basic result known as *First Order Averaging Theorem*. For a general introduction to averaging theory see for instance the book of Sanders and Verhulst [9] and the book of Verhulst [10].

We consider the differential equation

$$(8) \quad \dot{X} = \varepsilon F_1(s, X) + \varepsilon^2 R(s, X, \varepsilon),$$

where $F_1 : \mathbb{R} \times U \rightarrow \mathbb{R}^n$ is a smooth function and $R : \mathbb{R} \times U \times (-\varepsilon_f, \varepsilon_f) \rightarrow \mathbb{R}^n$ is a continuous function. These functions are both T -periodic in the first variable t and U is an open subset of \mathbb{R}^n .

Remark 3. *The equation (8) is the Standard Form, of a differential system, to apply the first order averaging theorem.*

We define the averaged system associated to system (8) as

$$(9) \quad \dot{Z}(t) = f_1(Z),$$

where $f_1 : U \rightarrow \mathbb{R}^n$ is given by

$$(10) \quad f_1(Z) = \int_0^T F_1(s, Z) ds.$$

In resume, the averaging theory gives a quantitative relation between the solutions of some non-autonomous differential system and the solutions of the averaged differential system, which is an autonomous one. In our case, as we are working with periodic systems, the averaging method also leads to the existence of periodic solutions.

The next theorem associates the singularities of the system (9) with the periodic solutions of the differential system (8).

Theorem 4. *Assume that*

- (i) F_1 and R are locally Lipschitz with respect to x ;
- (ii) for $a \in U$ with $f_1(a) = 0$, there exist a neighborhood V of a such that $f_1(Z) \neq 0$ for all $z \in \bar{V} \setminus \{a\}$ and $\det(df_1(a)) \neq 0$.

Then, for $|\varepsilon| > 0$ sufficiently small, there exist a T -periodic solution $X(t, \varepsilon)$ of the system (8) such that $X(0, \varepsilon) \rightarrow a$ as $\varepsilon \rightarrow 0$.

Using Brower degree theory, the hypotheses of Theorem 4 becomes weaker. For a proof of Theorem 4 see Buica and Llibre [11].

Remark 4. *Using the Averaging Theorem 4 we can study some solutions of (8) only studying the algebraic equation $f_1(Z) = 0$, instead solving the differential equation. This is one of the main characteristic of Averaging Theory.*

7. PROOFS OF THEOREM 2 AND COROLLARY 3

In order to use the Theorem 4 in the proof of Theorem 2, we have to modify the equation (6). If we denote

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix}, \quad F(t, \mathbf{x}) = \begin{pmatrix} 0 \\ g_0(t) + f_1(t)x + (f_2(t) - \bar{b})y \end{pmatrix},$$

and

$$R_0(t, \mathbf{x}) = \begin{pmatrix} 0 \\ r(t, \mathbf{x}, \varepsilon) \end{pmatrix},$$

then the equation (6) can be written in the matrix form

$$(11) \quad \dot{\mathbf{x}} = A\mathbf{x} + \varepsilon F(t, \mathbf{x}) + \varepsilon^2 R_0(t, \mathbf{x}).$$

Now, we introduce the main change of coordinates which makes us able to write the differential equation (11) in the *Standard Form* (8).

Take $\mathbf{y}(t) \in \mathbb{R}^2$ as

$$(12) \quad \mathbf{y}(t) = e^{-At}\mathbf{x}(t),$$

where

$$e^{At} = \begin{pmatrix} \cos(\sqrt{a}t) & \frac{\sin(\sqrt{a}t)}{\sqrt{a}} \\ -\sqrt{a}\sin(\sqrt{a}t) & \cos(\sqrt{a}t) \end{pmatrix}$$

is the matrix of the fundamental solution of the unperturbed differential system (11), i.e., $\varepsilon = 0$. This change of coordinates is done in the book of Sanders and Verhulst [12].

Observe that the application $t \mapsto e^{-At}$ is T -periodic function with $T = 2\pi/\sqrt{a}$, since the eigenvalues of A are purely imaginary. Moreover $y(0) = x(0)$.

Lemma 5. *The system (11) is written in the new variable \mathbf{y} as*

$$(13) \quad \dot{\mathbf{y}} = \varepsilon \tilde{F}(t, \mathbf{y}) + \varepsilon^2 \tilde{R}(t, \mathbf{y}, \varepsilon),$$

where $\tilde{F}(t, \mathbf{y}) = e^{-At}F(t, e^{At}\mathbf{y})$ and $\tilde{R}(t, \mathbf{y}, \varepsilon) = e^{-At}R_0(t, e^{At}\mathbf{y}, \varepsilon)$.

Proof.

$$\begin{aligned}
\dot{\mathbf{y}}(t) &= \frac{d}{dt} (e^{-At} \mathbf{x}(t)), \\
&= -Ae^{-At} \mathbf{x}(t) + e^{-At} \dot{\mathbf{x}}(t), \\
&= -Ae^{-At} \mathbf{x}(t) + e^{-At} (A\mathbf{x}(t) + \varepsilon F(t, \mathbf{x}(t)) + \varepsilon^2 R_0(t, \mathbf{x}(t))), \\
&= \varepsilon e^{-At} F(t, \mathbf{x}(t)) + \varepsilon^2 e^{-At} R_0(t, \mathbf{x}(t)), \\
&= \varepsilon e^{-At} F(t, e^{At} \mathbf{y}(t)) + \varepsilon^2 e^{-At} R_0(t, e^{At} \mathbf{y}(t)), \\
&= \varepsilon \tilde{F}(t, \mathbf{y}(t)) + \varepsilon^2 \tilde{R}(t, \mathbf{y}(t), \varepsilon).
\end{aligned}$$

□

Note that the system (13) is written in the standard form (8). Thus we are ready to prove the Theorem 2.

Proof of Theorem 2. The smooth functions $f(t, \mathbf{x})$ and $g(t, \mathbf{x})$ are T -periodic in the variable t , with $T = 2p\pi/\sqrt{a}$, which implies that the smooth functions $\tilde{F}(t, \mathbf{y})$ and $\tilde{R}(t, \mathbf{y}, \varepsilon)$ are also T -periodic in t .

We shall apply the Theorem 4 to the differential equation (13). Note that the equation (13) can be written as the equation (8) taking

$$X = \mathbf{y}, \quad F_1(t, X) = \tilde{F}(t, \mathbf{y}), \quad \text{and} \quad R(t, X, \varepsilon) = \tilde{R}(t, \mathbf{y}, \varepsilon).$$

Observe that, by Lemma 1, the function F_1 and R satisfy the assumptions of Theorem 4.

Given $\mathbf{z} \in \mathbb{R}^2$, we can compute the averaged function $f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined in (10), as

$$\begin{aligned}
f_1(\mathbf{z}) &= \int_0^{\frac{2p\pi}{\sqrt{a}}} e^{-At} F(t, e^{At} \mathbf{z}) dt, \\
&= M\mathbf{z} - v,
\end{aligned}$$

where M is defined in (7) and

$$v = \begin{pmatrix} \int_0^{\frac{2p\pi}{\sqrt{a}}} \frac{\sin(\sqrt{a}t)}{\sqrt{a}} g_0(t) dt \\ - \int_0^{\frac{2p\pi}{\sqrt{a}}} \cos(\sqrt{a}t) g_0(t) dt \end{pmatrix}.$$

Assuming that $\det(M) \neq 0$, we conclude that there exists a solution $\mathbf{z}_0 = (x_0, y_0)$ of the linear system $f_1(\mathbf{z}) = 0$ given by $\mathbf{z}_0 = M^{-1}v$ which satisfies the hypotheses of Theorem 4. Indeed

$$\det(df_1(\mathbf{z}_0)) = \det(M) \neq 0.$$

Moreover, $\det(M) \neq 0$ also implies the uniqueness of the solution \mathbf{z}_0 of the system $M\mathbf{z} = v$, thus $f_1(\mathbf{z}) \neq 0$ for all $\mathbf{z} \in \mathbb{R}^2 \setminus \{\mathbf{z}_0\}$.

Hence, applying Theorem 4, follows that there exists a T -periodic solution $\mathbf{y}(t, \varepsilon)$ of the system (13) such that

$$\mathbf{y}(0, \varepsilon) \rightarrow \mathbf{z}_0 \quad \text{when } \varepsilon \rightarrow 0,$$

which implies the existence of a periodic solution $(x(t, \varepsilon), y(t, \varepsilon))$ of the system (6) such that

$$(x(t, \varepsilon), y(t, \varepsilon)) = \mathbf{x}(t, \varepsilon) = e^{At}\mathbf{y}(t, \varepsilon).$$

Since $\mathbf{x}(0) = \mathbf{y}(0)$ it follows that

$$\begin{pmatrix} x(0, \varepsilon) \\ y(0, \varepsilon) \end{pmatrix} \rightarrow \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

when $\varepsilon \rightarrow 0$. Hence

$$(\theta(t, \varepsilon), \dot{\theta}(t, \varepsilon)) = \varepsilon(x(t, \varepsilon), y(t, \varepsilon))$$

is a T -periodic solution of the system (3) such that

$$(\theta(t, \varepsilon), \dot{\theta}(t, \varepsilon)) \rightarrow (0, 0),$$

when $\varepsilon \rightarrow 0$. □

Proof of Corollary 3. The hypotheses of Corollary 3 implies that

$$M = \begin{pmatrix} \frac{(C_2 - \bar{b})\pi}{\sqrt{a}} & -\frac{C_1\pi}{\sqrt{a^3}} \\ \frac{C_1\pi}{\sqrt{a}} & \frac{(C_2 - \bar{b})\pi}{\sqrt{a}} \end{pmatrix}.$$

Thus

$$\det(M) = \frac{(C_1^2 + a(\bar{b} - C_2)^2)\pi^2}{a^2}.$$

Hence $\det(M) = 0$ if and only if $(C_1, C_2) = (0, \bar{b})$. Since $(C_1, C_2) \neq (0, \bar{b})$, the result follows by applying Theorem 2. □

8. SIMULATION

Consider the following differential equation

$$(14) \quad \ddot{\theta} = -\sin(\theta) - \varepsilon\dot{\theta} + \varepsilon^2 \sin(t).$$

Observe that the equation (14) is a small perturbation of the damped pendulum system. Indeed, equation (3) becomes (14) by taking the parameters $a = 1$, and $b = \varepsilon$; and the functions $f(t, \theta, \dot{\theta}) = 0$, and

$g(t, \theta, \dot{\theta}) = \sin(t)$. Therefore, considering $C_1 = C_2 = 0$ we can apply the Corollary 3 to assure the existence of a periodic solution of the system (14).

Indeed, proceeding with the numerical simulation we find this periodic solution, represented by the blue line in Figure 5.

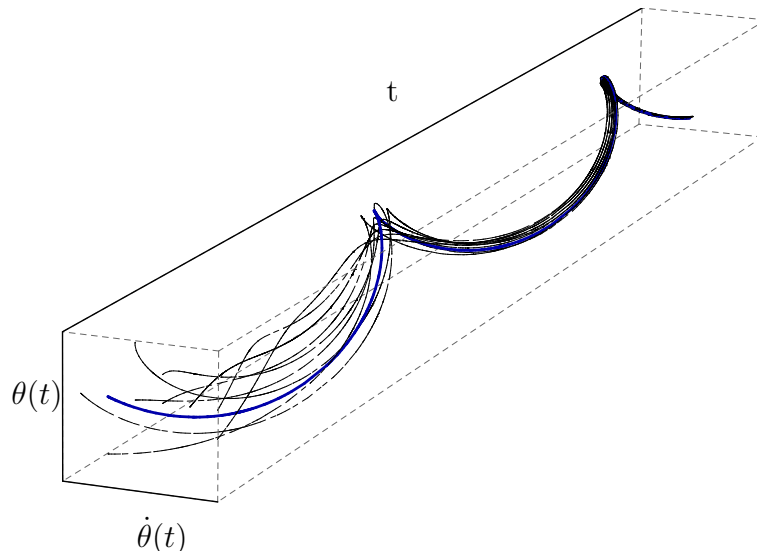


FIGURE 5. Solutions converging to the periodic solution.

This simulation has been done using the Wolfram Mathematica[®] 8 software.

9. CONCLUSIONS AND FUTURE DIRECTIONS

The averaging theory is a collection of techniques to study, via approximations, the behavior of the solutions of a dynamical system under small perturbations. As we have seen, it can also be used to find periodic solutions.

In this paper, we have presented one of these techniques and used it to find conditions that assure the existence of a periodic solution of the non-autonomous perturbed damped pendulum system. In resume, we have got an algebraic non-homogeneous linear system $M\mathbf{z} = v$ such that its solution, when $\det(M) \neq 0$, is associated with a periodic solution of such perturbed system.

Recently, Llibre, Novaes and Teixeira [13] have extended the averaging method for studying the periodic solutions of a class of differential

equations with discontinuous second member. Therefore we are able to consider for instance equations of kind

$$\ddot{\theta} = -a \sin(\theta) - b \operatorname{sign}(\dot{\theta}) + \varepsilon f(t, \theta, \dot{\theta}) + \varepsilon^2 g(t, \theta, \dot{\theta}).$$

Here, the term $b \operatorname{sign}(\dot{\theta})$ represents the Coulomb Friction, where the function $\operatorname{sgn}(z)$ denotes the sign function, i.e.

$$\operatorname{sgn}(z) = \begin{cases} -1 & \text{if } z < 0, \\ 0 & \text{if } z = 0, \\ 1 & \text{if } z > 0. \end{cases}$$

For instance, in [14], Llibre, Novaes and Teixeira have used the averaging theory to provide sufficient conditions for the existence of periodic solutions with small amplitude of the non-linear planar double pendulum perturbed by smooth or non-smooth functions.

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