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ALIEN LIMIT CYCLES IN LIÉNARD EQUATIONS

B. COLL, F. DUMORTIER, AND R. PROHENS

Abstract. This paper aims at providing an example of a family of polynomial Liénard equations exhibiting an alien limit cycle. This limit cycle is perturbed from a 2-saddle cycle in the boundary of an annulus of periodic orbits given by a Hamiltonian vector field. The Hamiltonian represents a truncated pendulum of degree 4. In comparison to a former polynomial example, not only the equations are simpler but a lot of tedious calculations can be avoided, making the example also interesting with respect to simplicity in treatment.

1. Introduction.

Periodic orbits in polynomial planar differential systems can be isolated or belong to an annulus of periodic orbits. In the isolated case they are called limit cycles.

For planar real polynomial vector fields, Hilbert's 16th problem (see [9]) is involved with the question of the existence of a finite upper bound, only depending on the degree of the vector field, of the number of limit cycles.

A way to get insight in Hilbert 16th problem is by using perturbative methods, i.e. to ask for the maximum number of limit cycles when perturbing from known situations like, for instance, from polynomial Hamiltonian systems, i.e. those described by a polynomial Hamiltonian function. The latter problem is known as the Infinitesimal Hilbert's 16th problem (see [1, 2]). It deals with differential systems of the form

$$\dot{x} = -\frac{\partial H}{\partial y} + \varepsilon p, \quad \dot{y} = \frac{\partial H}{\partial x} + \varepsilon q, \tag{1.1}$$

where $H = H(x, y)$ is the polynomial Hamiltonian function, $p, q \in \mathbb{R}[x, y]$ and ε is considered to take small positive values. From now on we will restrict to a single Hamiltonian function, but we will permit the polynomials p and q also to depend, in a polynomial way, on extra parameters $\lambda \in \Lambda$, with $\Lambda \subset \mathbb{R}^s$, for some $s \in \mathbb{N}$. We restrict our attention to a bounded period annulus of the Hamiltonian vector field X_H. We choose the Hamiltonian to be 0 on the outer boundary and positive on the side of the annulus. As usual, we denote by $\gamma_u \subset \{(x, y) \,|\, H(x, y) = u\}$ the part of the level curve of the Hamiltonian, representing one of the closed orbits in the period annulus.

The Poincaré return map, P, with respect to a regular section transverse to the orbits of the period annulus, for the perturbed Hamiltonian vector

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field (1.1), verifies

$$P(u,\varepsilon) = u + \varepsilon \oint_{\gamma_u} (p\,dy - q\,dx) + O(\varepsilon^2).$$

In perturbing from a polynomial Hamiltonian system, large part of the cyclicity problem can be reduced to studying the zeros of the integral

$$I(u) = \oint_{\gamma_u} (p\,dy - q\,dx)\,, \tag{1.2}$$

called Abelian integral related to system (1.1) and calculated along the ovals of the Hamiltonian function, see [13]. Nevertheless, this study does not finish the investigation near the Hamiltonian systems and attention has to be paid to the singular cycles situated at the boundary of the annuli of periodic orbits. Such a singular cycle contains both regular orbits and singularities, i.e. zeroes of the system. The limit cycles that can be perturbed from such a singular cycle cannot always been detected by the Abelian integral, see [5, 6]. Such limit cycles have been called “alien limit cycles”, see [3, 4].

The existence of alien limit cycles for concrete families of vector fields (1.1) is far from being trivial. It is e.g. not possible to find alien limit cycles when the boundary of the annulus is a hyperbolic loop. In [12], under certain genericity conditions on $I(u)$, it is proved that there is a bijective correspondence between limit cycles of the perturbed vector field (1.1) and the zero-set of I, for ε and u near zero.

As shown in [6] the simplest possibility to get alien limit cycles is the hyperbolic 2-saddle cycle. In that case there is a possibility of getting alien limit cycles merely by breaking one connection and keeping the other connection unbroken. As we will recall in Section 5, the asymptotic expansion of the equations governing the closed orbits, contains more terms than the asymptotic development of the Abelian integral does. This can be exploited, permitting alien limit cycles to appear and hence implying that the Abelian integral does not give the precise cyclicity of the perturbed vector field.

In [6] the theoretical elaboration was made on how to obtain alien limit cycles. The most generic conditions were described needed for proving the existence of an alien limit cycle in a polynomial family of planar vector fields. A first example was provided in [11]. Namely it was proven that the family

$$\left\{\begin{array}{rcl} \dot{x} &=& 1-\frac{1}{4}y^2-x^2+\varepsilon[\mu_3 xy+\mu_4 y^2 x+y(x^2+\frac{1}{12}y^2-1)(x-\frac{\sqrt{3}\pi}{8}xy)],\\ \dot{y} &=& 2xy+\varepsilon y(\mu_1+\mu_2 x). \end{array}\right. \tag{1.3}$$

contains an alien limit cycle, for small positive ε and small $\mu_i, i = 1,..,4$. The calculations, needed to check the conditions as described in [6], revealed to be very lengthy and complicated.

In this work we want to prove the existence of an example in the class of polynomial Liénard equations. We consider families of Liénard equations, written in the phase plane as

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x,\lambda)y, \tag{1.4}$$

where f and g are polynomials in x of respective degrees m and n and $\lambda \in \Lambda$ is a s-dimensional parameter. We remark that to confine Hilbert's

16th problem to Liénard equations is a problem of current interest, see Problem 13 in [14] for instance. Moreover we will see that a lot of tedious calculations can be avoided in the treatment of the example that we provide. We specifically consider the family of Liénard equations $X_{(\varepsilon,c,d)}$:

$$\left\{\begin{array}{rcl} \dot{x} &=& y, \\ \dot{y} &=& x^3 - x + \varepsilon d_0 + \varepsilon y \left(c_0 + c_1 x + (-13c_0 + d_2)x^2 + (27c_0 + d_4)x^4 \right. \\ && \left. \qquad +(-15c_0 + d_6)x^6\right), \end{array}\right. \tag{1.5}$$

where $c = (c_0, c_1) \in I\!R^2$, $d = (d_0, d_2, d_4, d_6) \in I\!R^4$, and $\varepsilon \in I\!R$ is a small positive value.

During the construction we will fix c_0 and c_1 at some non-zero values, while we will keep $d = (d_0, d_2, d_4, d_6) \in I\!R^4$ as parameters, close to zero. For a precise description of the main result we refer to Theorem 1.2.

We note that equation (1.5) is an ε-perturbation of a single Hamiltonian system X_H, with Hamiltonian function

$$H(x, y) = -\frac{1}{2}y^2 + \frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{1}{4}. \tag{1.6}$$

We observe that the flow of $X_{(0,c,d)} = X_H$ contains a period annulus bounded by a hyperbolic 2-saddle cycle, that we denote by $\mathcal{L}$, as in Figure 1. We re-

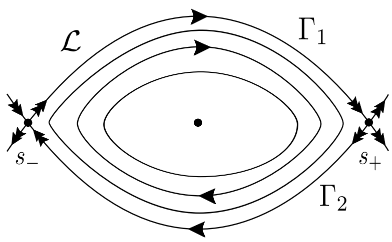

FIGURE 1. A 2-saddle cycle lying on the boundary of the period annulus around the origin of system (1.1).

call that a period annulus is a subset of the plane filled by closed orbits. The vector field X_H has an annulus of periodic orbits γ_u, contained in the level curves of $\{H = u\}$, for $u \in (0, 1/4)$. The hyperbolic 2-saddle cycle $\mathcal{L}$ is formed by two hyperbolic saddles, $s_- = (-1, 0)$ and $s_+ = (1, 0)$ plus their connections Γ_1 and Γ_2. In this way, we have that $s_- = \alpha(\Gamma_1) = \omega(\Gamma_2)$ while $s_+ = \alpha(\Gamma_2) = \omega(\Gamma_1)$. We choose H to be zero on $\mathcal{L}$, positive on the closed orbits and $H(0, 0) = 1/4$.

The existence of alien limit cycles in the unfolding $X_{(\varepsilon,c,d)}$, is obtained by assuming a "generic" unfolding of the vector field X_H, of codimension 4, leaving one connection of the 2-saddle cycle unbroken (see [6]). Roughly speaking, for an unfolding to be generic, besides a genericity condition on the related Abelian integral, some genericity property on the second order derivatives of the transition map along the saddle connections has to be satisfied as well.
Verifying the genericity conditions on the second order derivatives revealed

to be very complicated and tedious in the example (1.3). For equation (1.5) we can avoid making such calculations by not specifying which of the two connections we keep unbroken. It has to be proven, of course, that it is permitted working that way in equation (1.5). It is essentially due to some symmetries present in the Hamiltonian vector field X_H.

We strongly believe that equation (1.5) is the simplest polynomial example possible exhibiting an alien limit cycle as described in [6]. For Liénard equations this is definitely the case.

We know from [7] and [15] that the Abelian integrals related to equation (1.5) have at most 3 zeros, multiplicity taken into account, if we stay close to the 2-saddle cycle. Let us state this as a proposition to emphasize the contrast with the result that we will state in Theorem 1.2.

Proposition 1.1. *Consider system $X_{(\varepsilon,c,d)}$, representing a perturbation from a Hamiltonian vector field with (1.6) as Hamiltonian. Then the Abelian integrals associated to equation (1.5) have, for $c = (c_0, c_1)$ with $c_0 \neq 0$, d sufficiently close to 0, and $u > 0$ sufficiently small, at most 3 zeros, multiplicity taken into account.*

In Section 4 we will recall the proof of this proposition, since we will have to rely on these calculations in the rest of the elaboration.

The principal result in this paper is the following:

Theorem 1.2. *Let $X_{(\varepsilon,c,d)}$ be the unfolding given in (1.5), of the Hamiltonian vector field X_H with (1.6) as Hamiltonian. Let $\mathcal{L}$ be the 2-saddle cycle with saddle points $s_- = (-1, 0)$ and $s_+ = (1, 0)$, see Figure 2. Then, for any choice $c_0 = c_0^0$ with $c_0^0 \neq 0$, there exists (at least) an interval of choices $c_1 = c_1^0$, with $c_1^0 \neq 0$, such that the 4-parameter family $X_{\left(\varepsilon,(c_0^0,c_1^0),d\right)}$, with d sufficiently close to 0, contains, for $\varepsilon \sim 0$, with $\varepsilon > 0$, a swallowtail catastrophe of limit cycles. The limit cycles in this bifurcation are Hausdorff-close to $\mathcal{L}$ and the swallowtail catastrophe can be obtained by only breaking one of the connections, leaving the other unbroken. Moreover, for any value (c_0^0, c_1^0) with $c_0^0 \neq 0$ and d sufficiently close to 0, at most 4 limit cycles can perturb from $\mathcal{L}$ in the family $X_{\left(\varepsilon,(c_0^0,c_1^0),d,\right)}$, when breaking one connection and leaving the other unbroken.*

Remarks:

(1) In order to avoid lengthy and complicated calculations, as in [11], we provide a construction in which we cannot make precise for which c_1^0 the results hold. Most probably it holds for most c_1^0 but we restrict to proving the existence of an interval of choices for which the result holds. For the same reason we also do not specify which of the connections Γ_1 or Γ_2 gets broken and which remains unbroken. The way of proving the existence of the swallowtail catastrophe of

limit cycles will however require that one of the connections remains unbroken in order to be able to rely on the theoretical results from [6]. In Section 5 we will recall the essential ingredients from the theory developed in [6] since we will heavily rely on it.

(2) We recall that in a swallowtail catastrophe of limit cycles there is a limit cycle of multiplicity 4 that gets fully unfolded, inducing the occurrence of systems exhibiting 4 limit cycles.

In Section 2 we will study some important properties of the family (1.5), including a study of the normal form at the saddle points. In Section 3 we will investigate the conditions under which the heteroclinic connections break or remain unbroken. In section 4 we will calculate the Abelian integrals of the family (1.5), stressing their behaviour near the 2-saddle connection. In Section 5 we will recall the essential ingredients from [6], needed to prove Theorem 1.2 and we will check some conditions that need to be fulfilled for proving the theorem. In Section 6 we finalize the proof of Theorem 1.2. In Section 7 we provide some extra information on the Abelian integrals of the family (1.5), showing that no limit cycles will perturb from the interior of the annulus under the conditions of Theorem 1.2.

2. Basic properties of family (1.5), including normal forms at the saddles.

Important to observe is that the family of Liénard equations (1.5) is invariant under the change of variables and parameters

$$(x, y, \varepsilon, c_0, c_1, d_0, d_2, d_4, d_6, t) \mapsto (-x, y, \varepsilon, -c_0, c_1, -d_0, -d_2, -d_4, -d_6, -t), \tag{2.1}$$

as well as under the change of variables and parameters

$$(x, y, \varepsilon, c_0, c_1, d_0, d_2, d_4, d_6, t) \mapsto (-x, -y, \varepsilon, c_0, -c_1, -d_0, d_2, d_4, d_6, t). \tag{2.2}$$

Also the Hamiltonian H (see (1.6)) is unchanged under the transformations (2.1) and (2.2).

As mentioned in Section 1 we are interested in a study of family (1.5), more precisely of its limit cycles, near the 2-saddle cycle $\mathcal{L}$ (see Figure 1), under the conditions expressed in Theorem 1.2.

In studying limit cycles bifurcating from $\mathcal{L}$, instead of considering a first-return map or Poincaré-map with respect to some transverse section, we will, like in [6], consider a difference map Δ that we will introduce now. We therefore introduce the transverse sections Σ_0^i on $\{x = 0\}$, (see Figure 2), transverse to the according Γ_i. As a regular parameter on both Σ_0^1 and Σ_0^2 we take u, recalling that the values $u \in (0, 1/4)$, describe the periodic orbits γ_u in the period annulus bounded by $\mathcal{L}$. The intersection $\Gamma_i \cap \Sigma_0^i$ corresponds to $u = 0$. From section Σ_0^1 to Σ_0^2 we can now consider a passage map in forward time, that we denote by $\mathcal{H}_+$ and a passage map in backward time, that we denote by $\mathcal{H}_-$. The difference map that we want to consider is defined as $\Delta = \mathcal{H}_+ - \mathcal{H}_-$. Because of the symmetry (2.1) we see that the

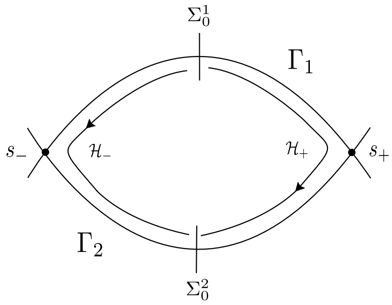

FIGURE 2. Transverse sections and passage maps for $\mathcal{L} = \Gamma_1 \cup \Gamma_2$.

passage map in backward time $\mathcal{H}_-$, called the passage map to the left, can be expressed as a passage map to the right, namely:

$$\mathcal{H}_-(u, \varepsilon, c_0, c_1, d_0, d_2, d_4, d_6) = \mathcal{H}_+(u, \varepsilon, -c_0, c_1, -d_0, -d_2, -d_4, -d_6).$$

To simplify notation, we simply write d for (d_0, d_2, d_4, d_6). We see that the difference map $\Delta : \Sigma_0^1 \mapsto \Sigma_0^2$ can be expressed as

$$\Delta(u, \varepsilon, c, d) = \mathcal{H}_+(u, \varepsilon, c_0, c_1, d) - \mathcal{H}_+(u, \varepsilon, -c_0, c_1, -d), \tag{2.3}$$

for $u > 0$ sufficiently small. We remark that, for $\varepsilon = 0$, Δ can be defined for $u \in (0, 1/4)$ and $\Delta(u, 0, c, d) = 0$.

For $\varepsilon \neq 0$, and in the region under consideration, the limit cycles of system (1.5) correspond to isolated solutions of $\Delta(u, \varepsilon, c, d) = 0$ and the cyclicity of $\mathcal{L}$ in the unfolding $X_{(\varepsilon,c,d)}$ can be expressed in terms of Δ by

$$\text{Cycl}(X_{(\varepsilon,c,d)}, \mathcal{L}) = \limsup_{\varepsilon\to 0, u\to 0} \{\text{number of isolated zeroes of } \Delta(u, \varepsilon, c, d)\}.$$

The cyclicity is the maximum number of limit cycles that can be generated by $\mathcal{L}$ (see e.g. [13]).

In order to describe the conditions that we will need for applying [6] we will need more information, of which some is related to the saddles s_+ and s_-.

A first thing we observe is that, for $\varepsilon \sim 0$, the position of the right saddle is given by $(S_+(\varepsilon d_0), 0)$, with $S_+(0) = 1$, and with S_+ solution of $x^3 - x + \varepsilon d_0 = 0$. As such $S_+(\varepsilon d_0) = 1 - (1/2)\varepsilon d_0 + O((\varepsilon d_0)^2)$. We will sometimes merely write s_+ for $(S_+(\varepsilon d_0), 0)$ and S_+ for $S_+(\varepsilon d_0)$.
The 1-jet of (1.5) at s_+ is given by:

$$\begin{pmatrix} 0 & 1 \\ 2 & A_+(\varepsilon, c, d) \end{pmatrix}, \tag{2.4}$$

with

$$A_+(\varepsilon, c, d) = \varepsilon(c_0 + c_1 S_+ + (-13c_0 + d_2)S_+^2 + (27c_0 + d_4)S_+^4 + (-15c_0 + d_6)S_+^6).$$

The eigenvalues of (2.4) at s_+ are given by $\lambda_+^- < 0 < \lambda_+^+$, with

$$\lambda_+^-(\varepsilon, c, d) = -\sqrt{2} + \frac{1}{2}\big(c_1 + d_2 + d_4 + d_6\big)\varepsilon + O(\varepsilon^2),$$
$$\lambda_+^+(\varepsilon, c, d) = \sqrt{2} + \frac{1}{2}\big(c_1 + d_2 + d_4 + d_6\big)\varepsilon + O(\varepsilon^2).$$

So the hyperbolicity ratio of s_+, defined by $r_+ = -\lambda_+^-/\lambda_+^+$ (see [13]), is given by

$$r_+(\varepsilon, c, d) = 1 - \frac{1}{\sqrt{2}}\big(c_1 + d_2 + d_4 + d_6\big)\varepsilon + O(\varepsilon^2). \tag{2.5}$$

For a similar study near $(-1, 0)$ we recall that, when passing near s_-, we work in backward time.

There is however no need to make new calculations because of the invariance of equation (1.5) under transformation (2.1).

We hence see that the left saddle of equation (1.5) is situated at $s_- = (S_-(\varepsilon d_0), 0)$, with $S_-(\varepsilon d_0) = -S_+(-\varepsilon d_0)$. The 1-jet at s_- is given by expression (2.4) changing A_+ by A_-, with

$$A_-(\varepsilon, c_0, c_1, d_0, d_2, d_4, d_6) = A_+(\varepsilon, -c_0, c_1, -d_0, -d_2, -d_4, -d_6).$$

We hence obtain eigenvalues $\lambda_-^- < 0 < \lambda_-^+$, that are, in a similar way, easily obtained from the expressions for λ_+^- and λ_+^+. We get a hyperbolicity ratio r_- given by

$$r_-(\varepsilon, c_0, c_1, d) = r_+(\varepsilon, -c_0, c_1, -d),$$

inducing that

$$r_-(\varepsilon, c, d) = 1 - \frac{1}{\sqrt{2}}\big(c_1 - d_2 - d_4 - d_6\big)\varepsilon + O(\varepsilon^2). \tag{2.6}$$

To enable expressing all the necessary conditions we need to work with normal forms near the saddles. Again it is only required to make calculations near the right saddle s_+, because of the symmetry (2.1).

Normal form theory induces the existence of $\mathcal{C}^\infty$ coordinates (m, n) near s_+, depending in a $\mathcal{C}^\infty$ way on the parameters, such that the Hamiltonian is given by mn and the family of vector fields (1.5) can be written as

$$\begin{cases} \dot{m} = & m\big(\lambda_+^-(\varepsilon, c, d) + M(mn, \varepsilon, c, d)\big), \\ \dot{n} = & n\big(\lambda_+^+(\varepsilon, c, d) + N(mn, \varepsilon, c, d)\big), \end{cases} \tag{2.7}$$

where both M and N are $\mathcal{C}^\infty$ functions that are $O(mn)$. Moreover, we have that $N(mn, 0, c, d) = -M(mn, 0, c, d)$, since X_H is Hamiltonian. For more information on this normal form we refer to [13] and to [11].

After a translation, a linear transformation and a near-identity mapping one easily obtains that the 3-jet of the normal form is, modulo $O(\epsilon^2)$, given by

$$\begin{cases} \dot{m} = & -m + \frac{3\sqrt{2}}{4}m^2 n + \frac{\sqrt{2}}{4}\varepsilon\big(c_1 + \frac{3}{\sqrt{2}}d_0 + d_2 + d_4 + d_6\big)m \\ & -\frac{\varepsilon}{32}\big(256c_0 + 2c_1 - 51\sqrt{2}d_0 + 4d_2 - 4d_4 - 28d_6\big)m^2 n, \\ \dot{n} = & n - \frac{3\sqrt{2}}{4}mn^2 + \frac{\sqrt{2}}{4}\varepsilon\big(c_1 - \frac{3}{\sqrt{2}}d_0 + d_2 + d_4 + d_6\big)n \\ & -\frac{\varepsilon}{32}\big(256c_0 - 24c_1 + 51\sqrt{2}d_0 - 22d_2 - 30d_4 - 54d_6\big)mn^2, \end{cases}$$

Like in [6] we now consider sections Σ_+^1 and Σ_+^2 near s_+ (see Figure 3) that

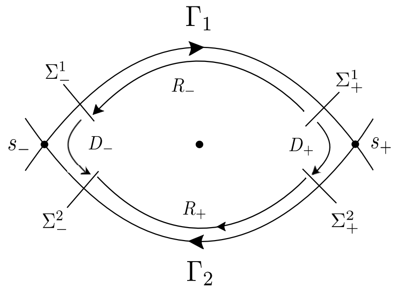

FIGURE 3. Dulac maps and regular transition maps.

are defined in the coordinates (m, n) as $\{m = 1\}$ and $\{n = 1\}$, respectively. In the coordinates (m, n), the quadrant $\{m \geq 0,\ n \geq 0\}$ corresponds to the period annulus under consideration.

Let us write $D_+ : \Sigma^1_+ \to \Sigma^2_+$ for the so-called Dulac map, describing the passage in forward time for the family of vector fields (2.7). We parametrize Σ^1_+ by n and Σ^2_+ by m. With calculations as in [13] we get

$$D_+(n) = n + \alpha_1[n\omega + o(n\omega)] + \alpha_2[n^2\omega + o(n^2\omega)] + O(n^3\omega), \tag{2.8}$$

where ω stands for $\omega(n, \alpha_1)$ and

$$\begin{cases} \omega(n, \alpha_1) & = \frac{n^{-\alpha_1} - 1}{\alpha_1}, \quad \text{if} \quad \alpha_1 \neq 0; \\ \omega(n, 0) & = -\ln n; \end{cases}$$

it is the so-called compensator of the hyperbolic saddle (see [13] for more explanation).

Near the saddle s_- we obtain, in a similar way, the sections Σ^1_- and Σ^2_- and the Dulac map $D_- : \Sigma^1_- \to \Sigma^2_-$.

To see the use that we will make of the sections $\Sigma^1_\pm$ and $\Sigma^2_\pm$ we refer Section 5.

For $\varepsilon = 0$, the vector field is Hamiltonian, and since we use the value of the Hamiltonian u as regular parameter on both the sections Σ^i_0 we see that $\mathcal{H}_+(u, \varepsilon, c, d) - \text{Id}$, $\mathcal{H}_-(u, \varepsilon, c, d) - \text{Id}$ and $\Delta(u, \varepsilon, c, d)$ are divisible by ε. Hence, we can write

$$\Delta = \varepsilon\overline{\Delta}, \tag{2.9}$$

for a $\mathcal{C}^\infty$ map $\overline{\Delta}$, that we will call reduced difference map. The limit of $\overline{\Delta}$, when $\varepsilon \to 0$, is the Abelian integral that we will study in Section 4,

$$\overline{\Delta}(u, 0, c, d) = I(u, c, d), \quad u \in (0, 1/4).$$

It is known that $I(u, c, d)$ admits, for $u \sim 0$, an asymptotic expansion in the logarithmic scale $\{1, u\ln u, u, u^2\ln u, u^2, \ldots, u^i\ln u, u^i, \ldots\}$. Hence, there exist smooth functions: p, q, r, s and t in c and d, such that

$$\begin{aligned} I(u, c, d) =& p(c, d) + q(c, d)u\ln u + r(c, d)u + s(c, d)u^2\ln u \\ & + t(c, d)u^2 + O(u^3\ln u), \quad u \downarrow 0. \end{aligned} \tag{2.10}$$

The coefficients in this expansion will be obtained in Section 4 by using Picard-Fuchs equations.

3. Heteroclinic connections in the family (1.5).

In this section we want to study the effect that the different parameters (c,d) have on the breaking of the heteroclinic connections Γ_1 and Γ_2 at first order in ε, emphasizing the conditions under which the connections remain unbroken.

Due to the symmetry (2.2) it is sufficient to make the study for Γ_1. Analogous conclusions for Γ_2 will then follow by applying (2.2).

From (1.6) we see that

$$\Gamma_1 = \{(x,y)\,|\, x \in (-1,1),\, y = \frac{1}{\sqrt{2}}(1-x^2)\}. \tag{3.1}$$

Γ_1 is the upper connection of X_H between the saddles $s_- = (-1,0)$ and $s_+ = (1,0)$. From Section 2 we know that, for $\varepsilon \sim 0$ the saddles are situated at $(S_-(\varepsilon d_0),0)$ and $(S_+(\varepsilon d_0),0)$ with $S_+(\varepsilon d_0) = 1 - 1/2\varepsilon d_0 + O((\varepsilon d_0)^2)$ and $S_-(\varepsilon d_0) = -S_+(-\varepsilon d_0)$.

We will now first adapt equation (1.5) in a way that saddles remain at fixed points $(-1,0)$ and $(1,0)$ for whatever value of (ε,c,d). We therefore change x by X in a way that

$$x = \frac{1}{2}(S_+ - S_-)X + \frac{1}{2}(S_+ + S_-), \tag{3.2}$$

where S_+ and S_- stand for respectively $S_+(\varepsilon d_0)$ and $S_-(\varepsilon d_0)$. The coordinates change (3.2) transforms the first equation of (1.5) into

$$\frac{1}{2}(S_+ - S_-)\dot{X} = y. \tag{3.3}$$

The second equation is like in (1.5), just replacing x by the value expressed in (3.2).

The expansions of S_+ and S_- show that

$$\begin{aligned} \frac{1}{2}(S_+ - S_-)(\varepsilon d_0) &= 1 + O((\varepsilon d_0)^2), \\ \frac{1}{2}(S_+ + S_-)(\varepsilon d_0) &= -\frac{1}{2}\varepsilon d_0 + O((\varepsilon d_0)^2). \end{aligned} \tag{3.4}$$

From (3.4) follows that (3.3) can be written as

$$\big(1 + O((\varepsilon d_0)^2)\big)\dot{X} = y.$$

Hence, at first order in ε (i.e. modulo terms of order ε^2) we will have $\dot{X} = y$ as the first equation of the transformed (1.5).

The second equation of (1.5), for $y = 0$, by the change (3.2), gets transformed into

$$(X^2 - 1)(X - \frac{3}{2}\varepsilon d_0 + O(\varepsilon^2 d_0^2)). \tag{3.5}$$

Modulo $O(\varepsilon^2)$, the rest of the second equation of (1.5) is unchanged under (3.2), as we see in (3.4).

For the problem under consideration we can continue working with (1.5), merely changing its second equation for $y = 0$ (writing x for X):

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= (x^2-1)(x - \frac{3}{2}\varepsilon d_0) + \varepsilon y\big(c_0 + c_1 x + (-13c_0 + d_2)x^2 \\ & \qquad + (27c_0 + d_4)x^4 + (-15c_0 + d_6)x^6\big). \end{cases} \tag{3.6}$$

To see, at first order in ε, the impact of the different parameters (c,d) on the breaking of Γ_1 we merely have to calculate the related Melnikov integral (see [10, §6.4.1]),

$$\int_{-\infty}^{\infty} \varphi(x(t),y(t))\,dt, \tag{3.7}$$

where $(x(t),y(t))$ describes the connecting orbit and φ is obtained by calculating the determinant

$$\begin{vmatrix} \dot{x} & \frac{\partial \dot{x}}{\partial \varepsilon}\Big|_{\varepsilon=0} \\ \dot{y} & \frac{\partial \dot{y}}{\partial \varepsilon}\Big|_{\varepsilon=0} \end{vmatrix} (x(t),y(t)). \tag{3.8}$$

From (3.8) it is clear that the integral (3.7) can be expressed as

$$\int_{-1}^{1} \left(\frac{\partial \dot{y}}{\partial \varepsilon}\Big|_{\varepsilon=0} \right)(x)\,dx. \tag{3.9}$$

We now introduce

$$J_i = \int_{-1}^{1} (1-x^2)x^i\,dx, \tag{3.10}$$

and, based on (3.10), we write the Melnikov integral (3.9) as

$$\begin{aligned}\big(\frac{3}{2}d_0+\frac{1}{\sqrt{2}}c_0\big)J_0 + \frac{1}{\sqrt{2}}c_1 J_1 + \frac{1}{\sqrt{2}}\big(-13c_0+d_2\big)J_2 + \frac{1}{\sqrt{2}}\big(27c_0+d_4\big)J_4 \\ +\frac{1}{\sqrt{2}}\big(-15c_0+d_6\big)J_6.\end{aligned} \tag{3.11}$$

We can easily calculate that

$$J_0 = 4/3, \quad J_1 = 0, \quad J_2 = 4/15, \quad J_4 = 4/35, \quad J_6 = 4/63. \tag{3.12}$$

Combining (3.11) and (3.12) and multiplying by $\sqrt{2}/4$, gives the Melnikov integral written as

$$\frac{1}{\sqrt{2}}d_0 + \frac{1}{15}d_2 + \frac{1}{35}d_4 + \frac{1}{63}d_6. \tag{3.13}$$

A zero value of (3.13) gives the first order condition to keep the connection Γ_1 unbroken. As already has been observed, to get a similar condition for the connection Γ_2 we apply the symmetry (2.2). As a conclusion we see that to keep the connection Γ_1 unbroken, we need the relation

$$d_0 = -\frac{\sqrt{2}}{15}d_2 - \frac{\sqrt{2}}{35}d_4 - \frac{\sqrt{2}}{63}d_6. \tag{3.14}$$

To keep the connection Γ_2 unbroken, we need the relation

$$d_0 = \frac{\sqrt{2}}{15}d_2 + \frac{\sqrt{2}}{35}d_4 + \frac{\sqrt{2}}{63}d_6. \tag{3.15}$$

To keep both connections unbroken we need

$$d_0 = 0, \quad \frac{d_2}{15} + \frac{d_4}{35} + \frac{d_6}{63} = 0. \tag{3.16}$$

Former calculations are up to first order in ε, but it is well known (due to the Implicit Function Theorem) that, tangent to the 3-spaces defined respectively by (3.14) and (3.15), it is possible to find manifolds of parameter values for which the connections under consideration remain unbroken. The same holds for $\mathcal{L}$ itself if we choose appropriate parameter values in a 2-dimensional manifold tangent to the plane defined by (3.16).

4. Abelian integrals of the family (1.5).

In this section we obtain a "closed" Picard-Fuchs equation for the Abelian integral I associated to the family (1.5). This equation will be useful to determine the coefficients of the expansion (2.10). The equations have been obtained in [7], in looking for the number of zeroes of Abelian integrals for a more general kind of quartic Hamiltonians. We repeat these calculations, restricting to the Hamiltonian (1.6) and correcting some mistakes that have been made in [7].

In this section we also prove Proposition 1.1 and we obtain the asymptotic behaviour of the Abelian integral close to $\mathcal{L}$ through its asymptotical development.

For the polynomial 1-form $\omega = p\,dy - q\,dx$ associated to equation (1.1), the Abelian integral I can be written as (1.2). Hence, in the particular case of the family of Liénard equations (1.5), we get

$$\begin{aligned} I(u,c,d) &= \oint_{\gamma_u} \big(d_0 + y\left(c_0 + c_1 x + (-13c_0 + d_2)x^2 + (27c_0 + d_4)x^4 \right. \\ &\qquad\qquad \left. +(-15c_0 + d_6)x^6\right)\big)\,dx \\ &= c_0\left(I_0(u) - 13I_2(u) + 27I_4(u) - 15I_6(u)\right) \\ &\qquad + d_2 I_2(u) + d_4 I_4(u) + d_6 I_6(u), \end{aligned} \tag{4.1}$$

where $I_i(u) = \oint_{\gamma_u} x^i y\,dx$. We note that in expression (4.1), the integrals corresponding to d_0 and c_1 are zero. For d_0 this is clear. Based on symmetry arguments on γ_u any integral $I_{2i+1}(u)$, for each natural number i, is zero too.

From now we will often represent $I(u,c,d)$ as $I(u,c_0,d_A)$, with $d_A = (d_2,d_4,d_6)$. We sometimes will simply write $I(u)$ if no confusion is possible.

Concerning $I_0(u)$ we remark that, by direct computation one verifies that $\lim_{u\to 0} I_0(u) = 4\sqrt{2}/3$ and that $I_0(u) \neq 0$ for all $u \in (0,1/4)$. This last fact follows easily from Green's Theorem, since $I_0(u) = \iint_{H(x,y)>u} dx\,dy$, i.e. $I_0(u)$ is a nonzero area.

Let ω_i be the monomial 1-form $x^i y\,dx$. Then, modulo closed 1-forms and writing the value of H as u, it is easy to prove (see [7]) that

$$(7+i)\omega_{i+4} = 2(i+4)\omega_{i+2} - 4(i+1)(1/4-u)\omega_i.$$

Consequently, by integrating the last expression for $i = 0$ and 2, we get

$$\begin{aligned} I_4(u) &= 8/7 I_2(u) - 4/7(1/4-u)I_0(u), \\ I_6(u) &= 4/3(25/28+u)I_2(u) - 16/21(1/4-u)I_0(u). \end{aligned}$$

The Abelian integral $I(u, c_0, d_A)$, can be written as:

$$\begin{aligned} I(u, c_0, d_A) = &\Big(4c_0 u - \frac{4}{7}(\frac{1}{4} - u)d_4 - \frac{16}{21}(\frac{1}{4} - u)d_6\Big) I_0(u) \\ &+ \Big(-20c_0 u + d_2 + \frac{8}{7}d_4 + \frac{4}{3}(\frac{25}{28} + u)d_6\Big) I_2(u). \end{aligned} \tag{4.2}$$

For $d_A = 0$, this gives

$$I(u, c_0, 0) = 4c_0 u \left(I_0(u) - 5I_2(u)\right). \tag{4.3}$$

To get extra information on $I_0(u)$ and $I_2(u)$, we introduce the column vector $\mathbf{I}(u) = \mathrm{col}\left(I_0(u), I_2(u)\right)$, having $I_0(u)$ and $I_2(u)$ as components.

Lemma 4.1 ([7])**.** *The vector* $\mathbf{I}(u)$ *satisfies the Picard-Fuchs equation*

$$\left(A\left(u - 1/4\right) + B\right)\mathbf{I}'(u) = \mathbf{I}(u), \tag{4.4}$$

where

$$A = \begin{pmatrix} \frac{4}{3} & 0 \\ \frac{4}{15} & \frac{4}{5} \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 0 & \frac{1}{3} \\ 0 & \frac{4}{15} \end{pmatrix}.$$

From equation (4.4) we get

$$\begin{aligned} 4u(1-4u)I_0'(u) &= -(1+12u)I_0(u) + 5I_2(u), \\ 4u(1-4u)I_2'(u) &= -(1-4u)I_0(u) + 5(1-4u)I_2(u). \end{aligned} \tag{4.5}$$

Since we are only interested in the behaviour of the Abelian integral for $u \sim 0$, we will calculate its development at $u = 0$, using following notation:

$$\begin{aligned} I_0(u) &= 4\sqrt{2}/3 + a_1 u \ln u + a_2 u + a_3 u^2 \ln u + O(u^2), \\ I_2(u) &= b_0 + b_1 u \ln u + b_2 u + b_3 u^2 \ln u + O(u^2). \end{aligned} \tag{4.6}$$

If we substitute this in the equations (4.5), we find the values

$$a_1 = \sqrt{2},\ a_3 = \frac{3}{8}\sqrt{2},\ b_0 = \frac{4}{15}\sqrt{2},\ b_1 = \sqrt{2},\ b_3 = -\frac{1}{8}\sqrt{2},$$

as well as the relation $b_2 = 4\sqrt{2} + a_2$. To finish the calculations of all relevant terms in the expansion (4.6) we will rely on following lemma:

Lemma 4.2. *The coefficient* a_2 *in the asymptotic scale expansion of function* I_0*, as given in* (4.6)*, is* $a_2 = -\sqrt{2}(1 + 4\ln 2)$*.*

Proof. From the first expression in (4.6) we have

$$a_2 = \lim_{u \to 0} \left(I_0'(u) - \sqrt{2}\left(\ln u + 1\right)\right). \tag{4.7}$$

From the definition of $I_0(u)$ we have

$$I_0(u) = 2\sqrt{2} \int_0^{x_+(u)} \sqrt{x^4 - 2x^2 + 1 - 4u}\, dx,$$

where $x_+(u)$ stands for the intersection of γ_u with the positive semi-axis. Hence, if we take $v^2 = 4u$, $v \in (0, 1)$, we get

$$I_0'(v) = -4\sqrt{2} \int_0^{\sqrt{1-v}} \frac{dx}{\sqrt{(1-x^2)^2 - v^2}} = -\frac{4\sqrt{2}}{\sqrt{1+v}}\, F\left(\frac{\pi}{2}, \sqrt{\frac{1-v}{1+v}}\right),$$

where F is the elliptic integral of the first kind. See [8, §3.131.6, §8.111.2], for instance. We observe that $F(\pi/2, k) = \mathbf{K}(k)$, where $\mathbf{K}$ is a complete elliptic integral of the first kind. By using the series representation of the $\mathbf{K}$ function, the result follows. □

If we substitute the expansions (4.6) in expression (4.2) then we get:

$$I(u, c_0, d_A) = \alpha_{00} + \alpha_{11} u \ln u + \alpha_{10} u + \alpha_{21} u^2 \ln u + \alpha_{20} u^2 + O(u^3 \ln u), \quad (4.8)$$

where

$$\begin{aligned}
\alpha_{00} &= \frac{4}{35} d_4 \sqrt{2} + \frac{4}{15} d_2 \sqrt{2} + \frac{4}{63} d_6 \sqrt{2}, \\
\alpha_{10} &= d_2 \left(4\sqrt{2} - \sqrt{2}\,(1 + 4 \ln 2)\right) + d_4 \left(\frac{16}{3}\sqrt{2} - \sqrt{2}\,(1 + 4 \ln 2)\right) \\
&\quad + d_6 \left(\frac{92}{15}\sqrt{2} - \sqrt{2}\,(1 + 4 \ln 2)\right), \\
\alpha_{11} &= d_2 \sqrt{2} + d_4 \sqrt{2} + d_6 \sqrt{2}, \\
\alpha_{20} &= c_0 \left(16\sqrt{2}\,(1 + 4 \ln 2) - 80\sqrt{2}\right) + d_6 \left(\frac{16}{3}\sqrt{2} - \frac{44}{21}\sqrt{2}\,(1 + 4 \ln 2)\right) \\
&\quad - \frac{4}{7} d_4 \sqrt{2}\,(1 + 4 \ln 2), \\
\alpha_{21} &= -\frac{1}{8} d_2 \sqrt{2} + \frac{3}{8} d_4 \sqrt{2} - 16 c_0 \sqrt{2} + \frac{15}{8} d_6 \sqrt{2}.
\end{aligned}$$

For $d_A = 0$, this gives

$$I(u, c_0, 0) = 4\sqrt{2} c_0 u^2 \big(-4 \ln u + 16(\ln 2 - 1) + O(u \ln u)\big). \quad (4.9)$$

Proof of Proposition 1.1. The asymptotic development of $I(u, c_0, d)$, for $u \sim 0$, is given in (2.10). From (4.9) we see that $s(c_0, 0) \neq 0$ when $c_0 \neq 0$. It is hence clear (see e.g. [13]) that $I(u, c_0, d)$, for $c_0 \neq 0$, $d \sim 0$ and $u \sim 0$, can have at most 3 zeroes, multiplicity taken into account. □

5. Generic unfolding of codimension 4 with an unbroken connection.

In this section we will recall the essential elements of the theory presented in [6], permitting to precisely describe the conditions on the parameters c and d, of system (1.5). We will not make the description as general as in [6], but we will restrict, as much as possible, to the family (1.5) and use the notations and properties that have been specified in Section 2.

In order to apply the results from [6] we first need to consider the hyperbolicity ratio at both s_- and s_+. Let us denote, as in [6], the hyperbolicity ratio at s_+ as

$$r_+(\varepsilon, c, d) = 1 + \alpha_+(c, d)\varepsilon + O(\varepsilon^2), \quad (5.1)$$

and the hyperbolicity ratio at s_- as

$$r_-(\varepsilon, c, d) = 1 + \alpha_-(c, d)\varepsilon + O(\varepsilon^2). \quad (5.2)$$

Let us call α_+ as well as α_- a "reduced hyperbolicity ratio". We remark that in [6] the indices are 1 and 2 instead of + and −, respectively. From (2.5) and (2.6) we know that

$$\alpha_+(c,0) = \alpha_-(c,0) = -\frac{c_1}{\sqrt{2}}. \tag{5.3}$$

In order to use the Theorems 11 and 12 from [6] we need the equality (5.3) as a first condition, combined with the fact that c_1 be different from zero.

We second need conditions on the Abelian integral (2.10). More precisely, the integral $I(u,c,d)$ needs, for a specific choice of c and for $d = 0$, to have a zero at $u = 0$ of codimension 3, i.e.

$$\begin{aligned} &p(c,0) = q(c,0) = r(c,0) = 0, \\ &s(c,0) \neq 0, \end{aligned} \tag{5.4}$$

and the mapping

$$(I\!R^4, 0) \mapsto (I\!R^3, 0) : d \mapsto (p(c,d), q(c,d), r(c,d)) \tag{5.5}$$

needs to be a submersion at $d = 0$.

In (4.9) we immediately see that the condition (5.4) holds for $I(u,c,0)$ if we keep $c_0 \neq 0$.

To check condition (5.5) we consider (4.8) and we calculate

$$\det\left(\frac{\partial(p,q,r)}{\partial(d_2,d_4,d_6)}\right)\bigg|_{d=0} = \frac{512\sqrt{2}}{4725} \neq 0. \tag{5.6}$$

In third place it must be possible to break one of the connections, keeping the other unbroken. As we have seen in Section 3 (3.14), Γ_1 remains unbroken if we take the parameter values d in an appropriate manifold tangent to the 3-space

$$\{d \,|\, d_0 = -\frac{\sqrt{2}}{15}d_2 - \frac{\sqrt{2}}{35}d_4 - \frac{\sqrt{2}}{63}d_6\}. \tag{5.7}$$

From relation (3.15) follows that we keep Γ_2 unbroken if we take the parameter values d in an appropriate manifold tangent to the 3-space

$$\{d \,|\, d_0 = \frac{\sqrt{2}}{15}d_2 + \frac{\sqrt{2}}{35}d_4 + \frac{\sqrt{2}}{63}d_6\}. \tag{5.8}$$

We keep both Γ_1 and Γ_2 unbroken (see (3.16)) if we choose d in an appropriate 2-dimensional manifold tangent to the 2-space

$$\{d \,|\, d_0 = 0, \quad \frac{d_2}{15} + \frac{d_4}{35} + \frac{d_6}{63} = 0\}. \tag{5.9}$$

Besides the three conditions stated above, there is a fourth one that we are allowed to check under the condition (5.9), as explained in [6].

We therefore recall that in Section 2 we have introduced the normal form coordinates (2.7) near the saddles s_+ and s_- and, in (2.8), we have introduced the so-called Dulac maps in these normalizing coordinates. These maps were defined as respectively $D_+ : \Sigma_+^1 \mapsto \Sigma_+^2$ and $D_- : \Sigma_-^1 \mapsto \Sigma_-^2$. Remind that we parametrized Σ_+^1 (and Σ_-^1) by n and Σ_+^2 (and Σ_-^2) by m. In all cases this is equal to the value u of the Hamiltonian.

We now also (see Figure 3) define a map $R_+ : \Sigma_+^2 \mapsto \Sigma_-^2$ that expresses the regular passage in forward time between the two given sections. In a similar way, we consider the passage map in backward time $R_- : \Sigma_+^1 \mapsto \Sigma_-^1$.

If both connections Γ_1 and Γ_2 remain unbroken we can write

$$\begin{aligned} R_-(u,c,d) =& u + \varepsilon\big(\gamma_-(\varepsilon,c,d)u + \eta_-(\varepsilon,c,d)u^2 + O(u^3)\big), \quad \text{and} \\ R_+(u,c,d) =& u + \varepsilon\left(\gamma_+(\varepsilon,c,d)u + \eta_+(\varepsilon,c,d)u^2 + O(u^3)\right). \end{aligned} \tag{5.10}$$

It is possible to reduce either $\gamma_-(\varepsilon,c,d)$ or $\gamma_+(\varepsilon,c,d)$ to zero by a linear change of variables in u, depending on the parameters (ε,c,d) and being the identity for $\varepsilon = 0$. Such changes are explicitly used in [6]. In any case such changes do not affect the values

$$\eta_-^0 = \eta_-(0,c,0) \quad \text{and} \quad \eta_+^0 = \eta_+(0,c,0).$$

In working with the inverses of these mappings only the sign of the expressions changes. If we want to apply Theorems 11 and 12 from [6] in a way that we keep Γ_1 unbroken while we break Γ_2, then we need the condition

$$\eta_-^0 \neq 2\eta_+^0. \tag{5.11}$$

At the same time we need that

$$\alpha_+(0,c) > 0, \tag{5.12}$$

implying a choice of $c_1 < 0$.

On the other hand, we could also try to apply Theorems 11 and 12 from [6] to a situation where we keep Γ_2 unbroken and break Γ_1.

We see that the conditions (5.4) and (5.5) remain unchanged, but (5.11) changes into

$$\eta_+^0 \neq 2\eta_-^0, \tag{5.13}$$

and (5.12) changes into

$$-\alpha_-(c,0) > 0, \tag{5.14}$$

implying a choice of $c_1 > 0$.

We can also remark that there exist relations between some quantities that we have just used. As it is shown in [6], and recalled in Section 2,

$$\overline{\Delta}(u,\varepsilon,c,d) = I(u,c_0,d_A) + O(\varepsilon).$$

Taking into account the expressions (2.8), (2.10) and (5.10) for the respective maps $D_\pm$ and $R_\pm$, we see that

$$p(c,d) = \alpha_+(c,d) - \alpha_-(c,d)$$

and

$$t(c,0) = \eta_+(0,c,0) - \eta_-(0,c,0) = \eta_+^0 - \eta_-^0. \tag{5.15}$$

From (5.15) and (4.9) follows that

$$\eta_+^0 - \eta_-^0 = 64\sqrt{2}(\ln 2 - 1)c_0. \tag{5.16}$$

We will sometimes also write $\eta_+^0(c_0,c_1)$ and $\eta_-^0(c_0,c_1)$.

Let us, for the sake of completeness, copy now the two results from [6] on which we will heavily rely in the proof of Theorem 1.2. We will state them, as propositions, using as much as possible the notations introduced before, recalling that we intend to fix $c = (c_0,c_1)$ keeping merely (ε,d), with $d = (d_0,d_2,d_4,d_6)$, as parameters.

In fact, we will formulate the propositions in terms of parameters p, q, r and s, as in expression (2.10) but chosen independently. We write $Y_{(\varepsilon,p,q,r)}$

as an abstract family of vector fields, recalling that we recover $X_{(\varepsilon,c,d)}$ for the chosen value of $c = (c_0, c_1)$ by expressing (p, q, r) as functions of (ε, c, d).

Proposition 5.1. [6, Th. 11, p.156] *Consider a $\mathcal{C}^\infty$ unfolding $Y_{(\varepsilon,p,q,r)}$ of a hyperbolic 2-saddle cycle $\mathcal{L}$. Suppose that, in the unfolding, the connection Γ_2 gets broken while Γ_1 remains unbroken. Suppose that, for the unfolding, the related Abelian integral is of codimension 3 at the 2-saddle cycle $\mathcal{L}$, i.e. $(p, q, r) \sim (0, 0, 0)$ and $s \neq 0$. Assume that the reduced hyperbolicity ratio at the saddle preceding Γ_2 is strictly positive and that $\eta_1^0 \neq 2\eta_2^0$.*

Then, there exists $v_0 > 0$ with the following property. For any $u_0 \in (0, v_0)$, there exists a unique parameter value, $\chi(u_0) = (\varepsilon(u_0), p(u_0), q(u_0), r(u_0))$ such that u_0 is a zero of multiplicity 4 of Δ,

$$\Delta(u_0) = \frac{\partial \Delta}{\partial u}(u_0) = \frac{\partial^2 \Delta}{\partial u^2}(u_0) = \frac{\partial \Delta^3}{\partial u^3}(u_0) = 0, \quad \textit{and} \quad \frac{\partial \Delta^4}{\partial u^4}(u_0) \neq 0.$$

The arc χ is smooth on $(0, v_0)$ and $\chi(u_0) \to 0$ if $u_0 \to 0$.

Remarks:

(1) Theorem 11 in [6] also contains some information on the asymptotics of χ for $u_0 \to 0$, but there is no need to repeat it here.

(2) With respect to Figure 3 the Proposition 5.1 can be applied to (Γ_1, Γ_2) as used in the statement with $\eta_1^0 = \eta_-^0$, $\eta_2^0 = \eta_+^0$ and with a positive reduced hyperbolicity ratio taken at s_+ implying that we take $c_1 < 0$. The proposition can however also be applied to cases were we keep Γ_2 unbroken and break Γ_1. We then choose $\eta_1^0 = \eta_+^0$, $\eta_2^0 = \eta_-^0$ and suppose to have a positive reduced hyperbolicity ratio at s_-; it implies that we take $c_1 > 0$.

Proposition 5.2. [6, Th. 12, p.160] *Under the conditions stated in Proposition 5.1 the curve χ, obtained in Proposition 5.1, is a curve of swallowtail bifurcations of limit cycles when varying (p, q, r). As a consequence, for every $u_0 \in (0, v_0)$, the value $(\varepsilon(u_0), p(u_0), q(u_0), r(u_0))$ is a limit of values $(\varepsilon_n, p_n, q_n, r_n)$ such that the vector field $Y_{(\varepsilon_n,p_n,q_n,r_n)}$ has four hyperbolic limit cycles converging, in the Hausdorff sense, towards the limit cycle of multiplicity four corresponding to u_0.*

Remark:
For $u_0 \to 0$ the limit cycles obtained in Proposition 5.2 tend, in a Hausdorff sense, to the 2-saddle cycle $\mathcal{L}$.

We do not give an idea of the proof of these propositions. The proof, which is quite involved, can be found in [6]. It requires the introduction of different compensators as well as the use of special asymptotic scales and so-called simple asymptotic scale deformations.

6. Proof of Theorem 1.2.

As explained in the previous section we only need to check, on the family (1.5), all the conditions that need to be fulfilled in order to apply the

Propositions 5.1 and 5.2. These conditions are stated in Proposition 5.1 and have been carefully explained and already partially checked in Section 5.

We hence continue the proof by essentially resuming the properties that have already been checked and adding the information that is missing.

We already know that the Abelian integral, related to the family (1.5) is of codimension 3 at the 2-saddle cycle $\mathcal{L}$, when taking $c_0 \neq 0$. This is shown in (4.9) and (5.6).

If we take $c_1 < 0$ then (see (5.3)) the reduced hyperbolicity ratio at s_+ is positive so that we have to break Γ_2 and keep Γ_1 unbroken in order to apply the Propositions 5.1 and 5.2. This is clearly possible as proven in Section 3. We also need that $\eta_-^0 \neq 2\eta_+^0$, i.e. we need to choose $c_1 < 0$ in a way that $\eta_-^0(c_0, c_1) \neq 2\eta_+^0(c_0, c_1)$, for the chosen c_0.

There hence will exist an interval of negative c_1-values where all necessary conditions hold or, for the chosen c_0, the identity

$$\eta_-^0(c_0, c_1) = 2\eta_+^0(c_0, c_1),$$

holds for all $c_1 < 0$. Together with (5.16) this implies that, for all $c_1 < 0$

$$\begin{cases} \eta_-^0(c_0, c_1) = & -2Ac_0, \\ \eta_+^0(c_0, c_1) = & -Ac_0, \end{cases} \tag{6.1}$$

with $A = 64\sqrt{2}(\ln 2 - 1)$.

In the latter case we also consider $c_1 > 0$. Then (see (5.3)) the restricted hyperbolicity ratio at s_- is positive (calculated in the direction of the flow of equation (1.5)). In order to be able using the Propositions 5.1 and 5.2 we have to break Γ_1 and keep Γ_2 unbroken, which is clearly possible as shown in Section 3.

We now need to choose $c_1 > 0$ in a way that $\eta_+^0(c_0, c_1) \neq 2\eta_-^0(c_0, c_1)$, for the chosen c_0.

There will exist an interval of positive c_1-values where all necessary conditions hold or, for the chosen c_0, the identity

$$\eta_+^0(c_0, c_1) = 2\eta_-^0(c_0, c_1)$$

holds for all $c_1 > 0$. Together with (5.16) this implies that, for all $c_1 > 0$

$$\begin{cases} \eta_-^0(c_0, c_1) = & Ac_0, \\ \eta_+^0(c_0, c_1) = & 2Ac_0. \end{cases} \tag{6.2}$$

However since, for fixed c_0, both functions η_-^0 and η_+^0 are smooth in c_1, for $c_1 \in \mathbb{R}$, we see that a simultaneous occurrence of (6.1) and (6.2) is not possible for $c_0 \neq 0$.

This finishes the proof of Theorem 1.2 with respect to the existence of a swallowtail catastrophe of limit cycles.

The last statement in Theorem 1.2 concerns the upperbound and the number of limit cycles that can perturb from $\mathcal{L}$ if one breaks one connection and leaves the other unbroken. This result is a direct consequence of Theorem 8 from [6], since the Abelian integral, related to the family (1.5), is of codimension 3 at $\mathcal{L}$, when taking $c_0 \neq 0$ (as shown in (4.9) and (5.6)).

7. Extra information on the Abelian integrals of the family (1.5).

In this Section we will add a study of the Abelian integral of the family (1.5) on the domain $u \in (0, 1/4]$, when $d = 0$ and $c_0 \neq 0$. We will show that this integral cannot have zeroes, implying that no limit cycles can perturb from the interior of the period annulus, under the conditions of Theorem 1.2.

From (4.3) we know that the Abelian integral under consideration is given by

$$I(u, c_0, 0) = 4c_0 u \left(I_0(u) - 5I_2(u)\right).$$

Zeroes of $I(u, c_0, 0)$ are hence given by solutions of

$$J(u) = 1/5, \tag{7.1}$$

with

$$J(u) = \frac{I_2(u)}{I_0(u)}. \tag{7.2}$$

We hence merely need to study $J(u)$, for $u \in (0, 1/4]$.
Since

$$\frac{d}{du} J(u) = \frac{I_2'(u)}{I_0(u)} - J(u)\frac{I_0'(u)}{I_0(u)},$$

and by using equations (4.5) we have that

$$u(1-4u)J'(u) = -(\frac{1}{4} - u) + (\frac{3}{2} - 2u)J(u) - \frac{5}{4}J^2(u).$$

This can better be written as a planar Riccati equation

$$\left\{ \begin{array}{rcl} \frac{dJ}{dt} & = & -\left(\frac{1}{4} - u\right) + \left(\frac{3}{2} - 2u\right) J - \frac{5}{4}J^2, \\ \frac{du}{dt} & = & u(1-4u), \end{array} \right. \tag{7.3}$$

for some new independent variable t.

The singularities of (7.3) are situated at $(u, J) = (0, 1/5)$, $(0, 1)$, $(1/4, 0)$ and $(1/4, 4/5)$. The lines $\{u = 0\}$ and $\{u = 1/4\}$ are invariant and we only need to consider the strip $(u, J) \in [0, 1/4] \times I\!R$.

We know from (4.9) that $\lim_{u \to 0} J(u) = 1/5$ and it is easy to check that $\lim_{u \to 1/4} J(u) = 0$.

Along $J = 0$ we see that $dJ/dt < 0$ for $u \in [0, 1/4]$, while along $J = 1/5$ we get $dJ/du = 3u/5 > 0$ for $u \in (0, 1/4]$.

The graph of J is hence contained in $[0, 1/4] \times [0, 1/5]$, as shown in Figure 4. Since (7.3) is a quadratic system the graph of J cannot have inflection points and J is strictly monotone. For the precise asymptotics of J near $u = 1/4$ it suffices to check that $(1/4, 0)$ is a hyperbolic saddle with J as stable manifold. Near $u = 0$ we can use (4.9). It shows that the graph of J is as represented in Figure 4.

We for sure see that equation (7.1) can have no solution for $u \in (0, 1/4]$, as claimed.

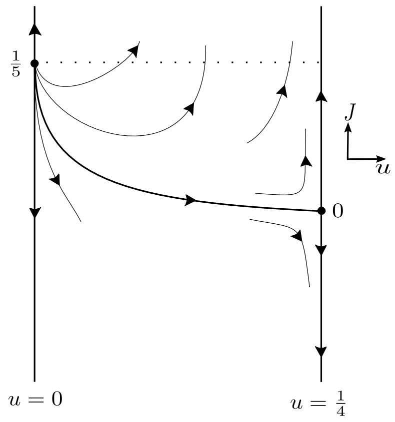

FIGURE 4. Graph of J.

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Dept. de Matemàtiques i Informàtica, Universitat de les Illes Balears, 07122 Palma de Mallorca, Illes Balears. Spain
E-mail address: tomeu.coll@uib.cat

Dept. Wiskunde, Universiteit Hasselt, Campus Diepenbeek, Agoralaan-Gebouw D, B-3590 Diepenbeek. Belgium
E-mail address: freddy.dumortier@uhasselt.be

Dept. de Matemàtiques i Informàtica, Universitat de les Illes Balears, 07122 Palma de Mallorca, Illes Balears. Spain
E-mail address: rafel.prohens@uib.cat