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# Orbital stability of solitary waves of moderate amplitude in shallow water

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## Abstract

We study the orbital stability of solitary traveling wave solutions of an equation for surface water waves of moderate amplitude in the shallow water regime. Our approach is based on a method proposed by Grillakis, Shatah and Strauss in 1987 [1], and relies on a reformulation of the evolution equation in Hamiltonian form. We deduce stability of solitary waves by proving the convexity of a scalar function, which is based on two nonlinear functionals that are preserved under the flow.

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## 1. Introduction

This paper is concerned with an equation for surface waves of moderate amplitude in the shallow water regime which arises as an approximation to the Euler equations. Since the exact governing equations for water waves have proven to be nearly intractable (Gerstner waves being the only known explicit solutions to the full equations [2, 3, 4, 5]), the quest for suitable simplified model equations was initiated at the earliest stages of the development of hydrodynamics. Evolving from an analysis confined to linear theory which dominated most studies until the early twentieth century [6], many competing nonlinear models are being proposed to this day to gain insight into phenomena like wave breaking or solitary waves. One of the most prominent examples is the Camassa–Holm equation [7], which is an integrable infinitedimensional Hamiltonian system [8, 9, 10] whose solitary waves are solitons [11, 12]. Some classical solutions of the Camassa–Holm equation develop singularities in finite time in the form of wave breaking, i.e. the solution remains bounded but its slope becomes unbounded [13]. After blow-up, the solutions recover in the sense of global weak solutions [14, 15]. With the objective of showing the relevance of this equation as a model for the propagation of shallow water waves, Johnson [16] demonstrates that the horizontal component of the velocity field at a certain depth within the fluid is described by a Camassa–Holm equation; see also the discussion in [17]. In search of a corresponding equation for the free surface, Constantin and Lannes follow up on earlier considerations by Johnson and derive an evolution equation for surface waves of moderate amplitude in the shallow water regime,

$$u_t + u_x + 6uu_x - 6u^2u_x + 12u^3u_x + u_{xxx} - u_{xxt} + 14uu_{xxx} + 28u_xu_{xx} = 0,$$
(1)

which approximates the governing equations to the same order as the Camassa-Holm equation [18]. Local well-posedness for the initial value problem associated to (1) was first established by Constantin and Lannes [18], and recently improved using semigroup theory for quasi-linear equations and an approach due to Kato [19]. Not much is known concerning the existence of global solutions of (1), unless one passes to a moving frame and studies waves traveling at constant speed c > 0 without changing their shape. For these so called traveling wave solutions, equation (1) takes the form

$$\varphi_x(1-c) + 6\varphi\varphi_x - 6\varphi^2\varphi_x + 12\varphi^3\varphi_x + (1+c)\varphi_{xxx} + 14\varphi\varphi_{xxx} + 28\varphi_x\varphi_{xx} = 0, \quad (2)$$

where x now denotes the independent variable in the moving frame. The existence of solitary wave solutions of (2), which have the additional property of decay at infinity, was proved using methods of dynamical systems [20]. Moreover, it was shown that solitary wave solutions of (2) are positive, symmetric with respect to the crest line and have a unique maximum which increases with the wave speed, i.e. faster waves are taller. In the present paper, we are concerned with the orbital stability of solitary waves of (2), cf. Definition 2.1 below. Our approach is inspired by a paper of Constantin and Strauss [11], which shows that solitary waves of the Camassa–Holm equation are solitons and that they are orbitally stable. In a similar vein, we employ an approach proposed by Grillakis, Shatah and Strauss [1] and prove the convexity of a scalar function, which is based on two nonlinear functionals preserved under the flow (1), to deduce stability of solitary traveling wave solutions.

#### 2. Preliminaries

We are interested in the nonlinear stability with respect to arbitrary perturbations of the initial data of solitary traveling wave solutions of the model equation (1). More precisely:

**Definition 2.1** (Orbital stability). A solitary traveling wave solution  $\varphi$  of (1) is called *stable* if for every  $\varepsilon > 0$  there exists  $\gamma > 0$  such that the following holds: If  $u \in C([0,T); H^2)$  is a solution of (1) for some  $T \in (0,\infty)$  with  $||u(0) - \varphi||_{H^2} \leq \gamma$ , then for every  $t \in [0,T)$  we have

$$\inf_{r \in \mathbb{R}} \|u(.,t) - \varphi(.-r)\|_{H^2} \le \varepsilon.$$

Otherwise, the solution is called *unstable*.

The verification of this type of stability relies strongly on a method introduced by Grillakis, Shatah and Strauss [1] and essentially follows from a theorem presented therein. To this end, we state the following assumptions:

- (A1) For every  $u_0 \in H^s(\mathbb{R})$ ,  $s > \frac{3}{2}$ , there exists a solution u of (1) in [0, T)such that  $u(0) = u_0$  where  $u \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); L^2(\mathbb{R}))$ . Furthermore, there exist functionals E(u) and F(u) which are conserved for solutions of (1).
- (A2) For every  $c \in (1, \infty)$ , there exists a traveling wave solution  $\varphi \in H^2$ of (1), where  $\varphi > 0$  and  $\varphi_x \not\equiv 0$ . The mapping  $c \mapsto \varphi(x - ct)$  is  $C^1((1, \infty); H^2)$ . Moreover  $cE'(\varphi) - F'(\varphi) = 0$ , where E' and F' are the variational derivatives of E and F, respectively.
- (A3) For every  $c \in (1, \infty)$ , the linearized Hamiltonian operator around  $\varphi$  defined by

$$H_c: H^1(\mathbb{R}) \to H^{-1}(\mathbb{R}), \quad H_c = cE''(\varphi) - F''(\varphi),$$
(3)

has exactly one negative simple eigenvalue, its kernel is spanned by  $\varphi_x$  and the rest of its spectrum is positive and bounded away from zero.

**Theorem 2.2** (Grillakis, Shatah & Strauss, 1987). Under the assumptions (A1), (A2) and (A3), a solitary wave solution  $\varphi(x - ct)$  of (1) is stable if and only if the scalar function

$$d(c) = cE(\varphi) - F(\varphi) \tag{4}$$

is convex in a neighborhood of c.

## 3. Hamiltonian formulation and conservation laws

In order to show that the assumptions (A1)-(A3) of Theorem 2.2 hold, we need to find conserved quantities of (1). To this end, we define the nonlinear functionals

$$E(u) = \frac{1}{2} \int_{\mathbb{R}} \left( u^2 + u_x^2 \right) dx, \tag{5}$$

which can be interpreted as the kinetic energy of the waves, and

$$F(u) = \int_{\mathbb{R}} \left( \frac{u^2}{2} + u^3 - \frac{u^4}{2} + \frac{3}{5}u^5 - \frac{u^2_x}{2} - 7uu^2_x \right) dx.$$
(6)

While proving the conservation of energy E(u) is straightforward, showing that the second quantity is conserved requires more attention. In the following, we prove that the evolution equation (1) has a Hamiltonian structure which will aid in showing that F(u) is also preserved by the flow (cf. 6 for some definitions).

**Theorem 3.1.** The evolution equation for surface waves of moderate amplitude in shallow water (1) can be expressed in Hamiltonian form, i.e.

$$m_t = -\partial_x (I - \partial_x^2) \delta F(m), \tag{7}$$

where m is the momentum defined as

$$m = u - u_{xx}.\tag{8}$$

*Proof.* Using (8) the evolution equation (1) can be rewritten as

$$m_t = -\partial_x \left( u + 3u^2 - 2u^3 + 3u^4 + u_{xx} + 14uu_{xx} + 7u_x^2 \right)$$

We compute the variational derivative of F(u) and find that

$$\delta F(u) = u + 3u^2 - 2u^3 + 3u^4 + u_{xx} + 14uu_{xx} + 7u_x^2.$$

Hence, the evolution equation takes the form

$$m_t = -\partial_x \delta F(u).$$

Using the fact that for any functional F we have  $\delta F(u) = (I - \partial_x^2) \delta F(m)$ , cf. Lemma 3.1 in [21], we obtain that

$$m_t = -\partial_x (I - \partial_x^2) \delta F(m).$$

Along the lines of a proof which shows that the Camassa-Holm equation has a bi-Hamiltonian structure [21], it is straightforward to verify that the operator

$$\mathcal{D} = -\partial_x (I - \partial_x^2) \tag{9}$$

is Hamiltonian, i.e. its Lie-Poisson bracket is skew-symmetric and satisfies the Jacobi identity [27]. Hence, the proof is complete.  $\Box$ 

**Lemma 3.2.** The nonlinear functionals E(u) and F(u) defined in (5) and (6) above are conserved quantities under the flow (1).

*Proof.* Multiplying (1) by u and integrating over  $\mathbb{R}$  we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}}(u^2+u_x^2)dx\right) = \frac{d}{dt}E(u) = 0.$$

To show that F(u) is conserved, we use the Hamiltonian structure of (1). For any solution u of (1), the momentum  $m = u - u_{xx}$  satisfies the corresponding evolution equation (7). It follows that

$$\frac{d}{dt}G(m(t)) = (\delta G(m), m_t) = (\delta G(m), -\partial_x (I - \partial_x^2) \,\delta F(m)) = [G, F](m),$$

in view of (7) and the fact that the operator  $\mathcal{D} = -\partial_x (I - \partial_x^2)$  is Hamiltonian, cf. 6.2. Using skew-symmetry, we conclude that

$$\frac{d}{dt}F(m(t)) = [F, F] = 0.$$

### 4. Stability

The main goal of this paper is to prove that solitary traveling wave solutions of (1) are orbitally stable. To this end, we will verify the assumptions of Theorem 2.2 and show that the function d(c) defined in (4) is convex. Assumption (A1) holds in view of Lemma 3.2 and the existence results presented in [19]. Regarding (A2), recall the existence of solitary wave solutions [20] and compute the first and second variational derivatives of the conserved quantities E and F, which are given by

$$E'(u)v = (u - \partial_x u_x, v), \qquad E''(u)v = (I - \partial_x^2, v),$$
$$F'(u)v = (u + 3u^2 - 2u^3 + 3u^4 + u_{xx} + 14uu_{xx} + 7u_x^2, v),$$

and

$$F''(u)v = (I + 6u - 6u^2 + 12u^3 + 14u_{xx} + 14u\partial_x^2 + 14u_x\partial_x + \partial_x^2, v)$$

It follows that equation (2) may be written as

$$cE'(\varphi) - F'(\varphi) = 0, \tag{10}$$

which ensures (A2). To show the validity of (A3), we study the spectral problem associated to the linearized Hamiltonian operator  $H_c$  defined in (3). We claim that for every c > 1, the operator  $H_c$  has exactly one negative simple eigenvalue while the rest of the spectrum is positive and bounded away from zero. Furthermore,  $H_c(\varphi_x) = 0$ . Indeed, observe that

$$H_{c} = -(1 + c + 14\varphi)\partial_{x}^{2} - 14\varphi_{xx} - 14\varphi_{x}\partial_{x} - I - 6\varphi + 6\varphi^{2} - 12\varphi^{3} + c$$
  
=  $-\partial_{x}((1 + c + 14\varphi)\partial_{x}) + c - I - 6\varphi + 6\varphi^{2} - 12\varphi^{3} - 14\varphi_{xx}.$ 

Therefore, we can write the spectral equation  $H_c v = \lambda v$  as the Sturm-Liouville problem

$$-(p v_x)_x + (q - \lambda) v = 0,$$

where

$$p(x) = 1 + c + 14\varphi, \quad q(x) = c - 1 - 6\varphi + 6\varphi^2 - 12\varphi^3 - 14\varphi_{xx}$$

Recall that any regular Sturm-Liouville system has an infinite sequence of real eigenvalues  $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$  with  $\lim_{n\to\infty} \lambda_n = \infty$  (cf. [22], Theorem 5, p. 320). The eigenfunction  $v_n(x)$  belonging to the eigenvalue  $\lambda_n$  is

uniquely determined up to a constant factor and has exactly n zeros. Furthermore, observe that  $H_c$  is a self-adjoint, second order differential operator. Therefore its eigenvalues  $\lambda$  are real and simple, and its essential spectrum is given by  $[c-1,\infty)$  in view of the fact that  $\liminf_{x\to\infty} q(x) = c-1$  (cf. [23], Theorem 18, p. 1450). It is straightforward to verify that (2) is equivalent to  $H_c(\varphi_x) = 0$ . Furthermore,  $\varphi_x$  has exactly one zero on  $\mathbb{R}$  in view of the fact that the solitary wave solutions of (2) have a unique maximum. We conclude that there is exactly one negative eigenvalue, and the rest of the spectrum is positive and bounded away from zero, which proves the claim.

Under the assumptions (A1)–(A3), Theorem 2.2 ensures that a solitary wave  $\varphi$  is stable if and only if the scalar function

$$d(c) = c E(\varphi) - F(\varphi),$$

is convex in a neighborhood of c. In view of (10) we find that

$$d'(c) = E(\varphi) + (cE'(\varphi) - F'(\varphi), \partial_c \varphi) = E(\varphi).$$

In the following theorem, we prove that  $d''(c) \ge 0$  to infer stability of  $\varphi$ .

**Theorem 4.1.** The kinetic energy E = d'(c) is an increasing function of the wave speed c. Therefore, all solitary traveling wave solutions of (1) are stable.

*Proof.* Let  $\varphi$  be a solitary traveling wave solution of (1). In [20] it was shown that  $\varphi$  satisfies

$$\varphi_x^2 = \varphi^2 \frac{c - 1 - 2\varphi + \varphi^2 - 6/5\varphi^3}{1 + c + 14\varphi},$$

and that it is symmetric with respect to the crest. Therefore,  $\varphi_x < 0$  on  $(0,\infty)$  and

$$\varphi = -\varphi_x \sqrt{\frac{S(\varphi,c)}{T(\varphi,c)}}$$

if we denote  $T(\varphi, c) := c - 1 - 2\varphi + \varphi^2 - 6/5\varphi^3$  and  $S(\varphi, c) := 1 + c + 14\varphi$ .

Thus, the kinetic energy can be rewritten as

$$d'(c) = \frac{1}{2} \int_{\mathbb{R}} (\varphi^2 + \varphi_x^2) dx = \int_0^\infty \varphi^2 \left(1 + \frac{T}{S}(\varphi, c)\right) dx$$
$$= -\int_0^\infty \varphi \varphi_x \left(\frac{S+T}{\sqrt{ST}}(\varphi, c)\right) dx$$
$$= \int_0^{h(c)} y \left(\frac{S+T}{\sqrt{ST}}(y, c)\right) dy.$$
(11)

In the last equality we have performed the change of variables  $\varphi(x) = y$  and used the fact that  $\varphi$  has a unique maximum h(c), which corresponds to the unique real root of  $T(\varphi, c)$ . To motivate the subsequent calculations showing that d(c) is convex in a neighborhood of c, we first perform numerical integration in MATLAB using the trapezoidal rule<sup>1</sup> to compute the integral in (11). If we plot the resulting approximation of d'(c), cf. Figure 1, it becomes apparent immediately that the kinetic energy is increasing with c, which implies that all solitary traveling wave solutions of (1) are stable.

We will now give a rigorous proof of this result. Observe that the integrand in (11) has a singularity in h(c), which precludes a straightforward differentiation of d'(c). To overcome this problem, we take advantage of the fact that the function  $h: [1, \infty) \to [0, \infty)$  is bijective, since it was shown in [20] that h(c) is strictly monotonically increasing, h(1) = 0 and  $\lim_{c\to\infty} h(c) = \infty$ . Denoting H = h(c), we find that  $c = h^{-1}(H) = 1 + 2H - H^2 + 6/5H^3$  which

$$Err_N = -\frac{h(c)^3}{12N^2}\partial_y^2 \left(y \,\frac{S+T}{\sqrt{ST}}(y,c)\right)(\xi),$$

where  $\xi \in [0, h(c)]$ , cf. [24]. For every c > 1 and N sufficiently large, this error can be made arbitrarily small.

<sup>&</sup>lt;sup>1</sup>The trapezoidal rule of integration is an example of the Newton-Cotes formula, cf. [24], based on the discretization of the integration domain and approximation of the integral by trapezoids. The error can be estimated in terms of the interval of integration and the number of discretization points N by the formula



Figure 1: The kinetic energy of solitary traveling waves of (1) is increasing with the wave speed, which implies that the waves are orbitally stable.

allows us to rewrite (11) as

$$\begin{split} e(H) &= \int_0^H y\left(\frac{S+T}{\sqrt{S\,T}}(y,h^{-1}(H))\right) dy \\ &= \int_0^1 H^2 z\left(\frac{S+T}{\sqrt{S\,T}}(Hz,h^{-1}(H))\right) dz, \end{split}$$

where we have performed the change of variables y = Hz. This improper integral is well defined on (0, 1) since the polynomials in the square root are positive in this interval, and the integrand becomes singular only when z = 1. Now, denote the integrand by f(z, H) and let  $I = [H_1, H_2]$  with  $H_1 > 0$ , so that we can view  $e(H) = \int_0^1 f(z, H)dz$  as a parameter integral. Observe that  $f(., H) \in \mathcal{L}^1(0, 1)$  for all  $H \in I$ , and  $f(z, .) \in C^1(I)$  for all  $z \in (0, 1)$ . Furthermore, we claim that there exists a function  $g \in \mathcal{L}^1(0, 1)$  such that  $|\partial_H f(z, H)| \leq g(z)$  for all  $(z, H) \in (0, 1) \times I$ . Indeed,

$$f(z,H) = H^2 z \left(\frac{S+T}{\sqrt{ST}}(Hz,h^{-1}(H))\right),$$

where

$$S(Hz, h^{-1}(H)) = 2 + 2H - H^2 + 6/5H^3 + 14Hz$$

and

$$T(Hz, h^{-1}(H)) = H(1-z)(2 - H(1+z) + 6/5H^2(1+z+z^2))$$

Differentiating f(z, H) with respect to H, we can find a positive constant K depending only on I such that

$$|\partial_H f(z,H)| \le K \frac{1}{\sqrt{1-z}},$$

for  $z \in (0, 1)$  and  $H \in I$ . We let  $g(z) := K(1-z)^{-1/2}$  and observe that  $g(z) \in \mathcal{L}^1(0, 1)$  which proves the claim. In view of the theorem on differentiation of parameter integrals (cf. [25], p.111), we obtain that

$$\partial_H e(H) = \int_0^1 \partial_H f(z, H) dz$$

for all  $H \in I$  and  $z \in (0, 1)$ . Since the choice of I was arbitrary, we can extend this result to all  $H \in (0, \infty)$ . Recall that H = h(c), so that

$$d''(c) = \partial_H e(H) h'(c).$$

Since h is an increasing function of c, it remains to show that  $\partial_H e(H) \ge 0$ . To this end, consider the integrand of this expression and notice that it may be rewritten as

$$\partial_H f(z, H) = \frac{H^2 z(z-1)N(z, H)}{(ST)^{\frac{3}{2}}},$$

where

$$\begin{split} N(z,H) = & \frac{1}{125} (-1500 - 5500H + 1700H^2 - 2700H^3 - \\ & 6600H^4 + 5540H^5 - 6120H^6 + 2160H^7 - 864H^8 - 21500Hz - \\ & 30300H^2z + 33300H^3z - 58800H^4z + 27140H^5z - \\ & 19080H^6z + 2160H^7z - 864H^8z - 78300H^2z^2 + \\ & 38550H^3z^2 - 46800H^4z^2 + 22790H^5z^2 - 17280H^6z^2 + \\ & 1080H^7z^2 - 864H^8z^2 + 19050H^3z^3 - 31750H^4z^3 + \\ & 3410H^5z^3 - 12060H^6z^3 - 1080H^7z^3 + 432H^8z^3 - \\ & 5975H^4z^4 - 11940H^5z^4 + 9315H^6z^4 - 1080H^7z^4 + \\ & 432H^8z^4 - 10290H^5z^5 + 8640H^6z^5 - 450H^7z^5 + \\ & 432H^8z^5 + 7560H^6z^6). \end{split}$$

In view of the fact that  $z \in (0,1)$  and  $H \in (0,\infty)$ , we have to prove that  $N(z,H) \leq 0$ . We will use a result for one-parameter families of polynomials in one variable (cf. Lemma 8.1 in [26]) which ensures that the polynomials do not change sign. To verify the assumptions of the Lemma, we first perform the change of variables  $z = x^2/(1 + x^2)$  and  $H = b^2$ , which maps the strip  $(0,1) \times (0,\infty)$  into the whole plane. Since the denominator of the resulting expression is everywhere positive, we only consider the numerator which is of the form

$$G_b(x) = g_{12}(b)x^{12} + g_{11}(b)x^{11} + \dots + g_1(b)x + g_0(b),$$

where the  $g_i$  depend polynomially on the real parameter b. Using the Sturm method one can check that  $G_1(x) < 0$  on  $\mathbb{R}$ . Next, we compute the discriminant of  $G_b(x)$  and find that it is non zero, since all coefficients are positive. Finally, we use the Sturm method once more to show that  $g_{12}(b) \neq 0$  for all  $b \in \mathbb{R}$ . Having verified the assumptions of the Lemma, we may conclude that  $G_b(x) < 0$  on  $\mathbb{R}$  for all  $b \in \mathbb{R}$ . This proves that  $N(z, H) \leq 0$ , and consequently  $d''(c) \geq 0$  which implies stability of  $\varphi$ .  $\Box$ 

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## 6. Appendix

Let X be a linear function space over  $\mathbb{R}$  and let F, G and H be functionals on X. The following definitions may be found in [27].

**Definition 6.1.** The variational derivative of F with respect to u is denoted by  $\delta F(u)$  and defined as

$$(\delta F(u), v) = \frac{d}{d\varepsilon} F(u + \varepsilon v)|_{\varepsilon = 0},$$

where (.,.) is the inner product on  $L^2(\mathbb{R})$ .

**Definition 6.2.** A linear operator  $\mathcal{D}$  on X is called *Hamiltonian* if its Lie-Poisson bracket defined by

$$[F, H](u) = (\delta F(u), \mathcal{D}\delta H(u))$$

satisfies the conditions of *skew-symmetry* 

$$[F,H] = -[H,F]$$

and the Jacobi identity

$$[[F,G],H] + [[G,H],F] + [[H,F],G] = 0.$$

**Definition 6.3.** Let  $A : X \to X$  be a (nonlinear) operator. We say that the evolution equation  $u_t = A(u)$  is expressible in *Hamiltonian form* if there exists a Hamiltonian operator  $\mathcal{D}$  and a functional H(u) such that the equation can be rewritten as

$$u_t = \mathcal{D}\delta H(u).$$

We say that F(u) is a *conserved functional* for this equation if the functional is independent of t for every solution u.

**Lemma 6.4.** The Hamiltonian operator  $\mathcal{D} = -\partial_x(I - \partial_x^2)$  defined in (9) is Hamiltonian.

*Proof.* We follow the proof presented by Constantin in [21]. To prove skewsymmetry, we use integration by parts and the symmetry of the  $L^2$ -inner product to show that

$$\begin{split} [F,H](m) &= (\delta F(m), \mathcal{D}\delta H(m)) \\ &= \int \delta F(m) \mathcal{D}\delta H(m) dx \\ &= \int \partial_x (\delta F(m)) (I - \partial_x^2) \delta H(m) dx \\ &= \int (\partial_x (\delta F(m)) \delta H(m) - \partial_x^3 (\delta F(m)) \delta H(m) dx \\ &= -\int \mathcal{D}\delta F(m) \delta H(m) dx = -(\delta H(m), \mathcal{D}\delta F(m)) \\ &= -[H,F](m). \end{split}$$

In order to prove the Jacobi identity, we define

$$M_H(m)[v] = \frac{d}{d\varepsilon} \delta H(m + \varepsilon v)|_{\varepsilon = 0}$$

and find that it is symmetric, i.e.  $(M_H(m)[v], w) = (M_H(m)[w], v)$ . Observe that

$$[[F,G],H](m) = \int \delta[F,G](m)\mathcal{D}\delta H(m)dx$$
$$= \int M_F(m)[\mathcal{D}\delta G(m)](\mathcal{D}\delta H(m))$$
$$- M_G(m)[\mathcal{D}\delta F(m)](\mathcal{D}\delta H(m))dx$$

Similar calculations for [[G, H], F] and [[H, F], G] show that

$$\begin{split} & [[F,G],H](m) + [[G,H],F](m) + [[H,F],G](m) \\ &= \int (M_F(m)[\mathcal{D}\delta G(m)](\mathcal{D}\delta H(m)) - M_G(m)[\mathcal{D}\delta F(m)](\mathcal{D}\delta H(m)))dx \\ &+ \int (M_G(m)[\mathcal{D}\delta H(m)](\mathcal{D}\delta F(m)) - M_H(m)[\mathcal{D}\delta G(m)](\mathcal{D}\delta F(m)))dx \\ &+ \int (M_H(m)[\mathcal{D}\delta F(m)](\mathcal{D}\delta G(m)) - M_F(m)[\mathcal{D}\delta H(m)](\mathcal{D}\delta G(m)))dx \\ &= 0 \end{split}$$

by the symmetry property. Hence, (9) is a Hamiltonian operator.

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