

ZERO–HOPF BIFURCATION IN A PREDATOR–PREY MODELS

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ABSTRACT. We study the competition between two species according the following modification of the Holling–Tanner II model

$$\begin{aligned}x' &= x \left[r \left(1 - \frac{x}{K} \right) - \frac{qy}{x^2 + a} \right], \\y' &= sy \left(1 - \frac{y}{nx + c} \right).\end{aligned}$$

Of course, $x \geq 0$, $y \geq 0$ and the parameters a , c , K , n , q , r and s are positive. We prove that its unique positive equilibrium point never exhibits a classical Hopf bifurcation, but for convenient values of the parameters from this equilibrium point bifurcates a periodic orbit, and during this local bifurcation the eigenvalues of such equilibrium remain purely imaginary.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The differential system

$$\begin{aligned}(1) \quad x' &= x \left[r \left(1 - \frac{x}{K} \right) - \frac{qy}{x + a} \right], \\y' &= sy \left(1 - \frac{y}{nx} \right),\end{aligned}$$

is a modification of the classical model of May (see [1, 3, 7, 11]) also known as the Holling–Tanner model (see [2, 5, 6]). Here the variables x , y and the parameters a , K , n , q , r and s are positive. As usual the prime denotes derivative with respect to the time t .

In the differential system (1) we have that $x(t)$ and $y(t)$ denote the prey and predator densities, respectively, as functions of the time t . Moreover, the parameters have the following meanings:

- (i) q is the maximal predator per capita consumption rate, in other words the maximum number of prey that can be eaten by a predator in each unit of time.
- (ii) a is the number of prey necessary to achieve one–half of the maximum rate q .
- (iii) K is the prey environment carrying capacity, see [3].
- (iv) n is a measure of the food quality that the prey provides for conversion into predator births.
- (v) r and s are the intrinsic growth rates, or biotic potential of the preys and predators, respectively.

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Here the effect of the predation is given by the function

$$\frac{qy}{x+a},$$

and it corresponds to a functional response of the predator Holling type II, see [10].

In [9] the authors describe the limit cycles of the differential systems (1) mainly studying the Hopf bifurcation that those systems can exhibit. Here we consider the following modified version of systems (1)

$$(2) \quad \begin{aligned} x' &= x \left[r \left(1 - \frac{x}{K} \right) - \frac{qy}{x^2 + a} \right], \\ y' &= sy \left(1 - \frac{y}{nx + c} \right). \end{aligned}$$

EXPLAIN WHY WE CONSIDER THIS MODIFIED SYSTEM.

Doing the rescaling of the time $t = \tau/s$ we get that the differential system (2) writes

$$\begin{aligned} \dot{x} &= x \left[\frac{r}{s} \left(1 - \frac{x}{K} \right) - \frac{(q/s)y}{x^2 + a} \right], \\ \dot{y} &= y \left(1 - \frac{y}{nx + c} \right), \end{aligned}$$

where now the dot denotes derivative with respect to the new independent variable τ . Renaming r/s and q/s by r and s we obtain the differential system (2) with $s = 1$. So in the rest of this paper without loss of generality we shall assume that $s = 1$.

As usual the first quadrant will be the region of the plane (x, y) where $x > 0$ and $y > 0$.

Proposition 1. *An equilibrium point (α, β) in the first quadrant of the differential system (2) must satisfy $\beta = c + n\alpha$, and it exists if and only if $q = (r(K - \alpha)(a + \alpha^2))/(K(c + n\alpha))$ and $K > \alpha$.*

Proposition 1 is proved in section 2.

Doing the change of variables $(x = \alpha X, y = \alpha Y)$ the differential system (2) can be written as

$$\begin{aligned} \dot{X} &= X \left[r \left(1 - \frac{X}{K/\alpha} \right) - \frac{(q/\alpha)Y}{X^2 + a/\alpha^2} \right], \\ \dot{Y} &= Y \left(1 - \frac{Y}{nX + c/\alpha} \right). \end{aligned}$$

So renaming $X, Y, K/\alpha, q/\alpha, a/\alpha^2$ and c/α as x, y, K, q, a and c , system (2) with the equilibrium point $(\alpha, c + n\alpha)$ becomes system (2) with the equilibrium point $(1, c + n)$. In short, without loss of generality we can assume that when an equilibrium point of the differential system (2) with $s = 1$ exists in the first quadrant, then it can be written as $(1, c + n)$, and we have that

$$(3) \quad q = \frac{r(K-1)(a+1)}{K(c+n)} \quad \text{and} \quad K > 1.$$

So, from now on we shall work with the differential system

$$(4) \quad \begin{aligned} \dot{x} &= rx \left[1 - \frac{x}{K} - \frac{(K-1)(a+1)}{K(c+n)} \frac{y}{x^2+a} \right], \\ \dot{y} &= y \left(1 - \frac{y}{nx+c} \right). \end{aligned}$$

Proposition 2. *The equilibrium point $(1, c+n)$ of the differential system (4) cannot be a strong focus with eigenvalues $\varepsilon \pm \omega i$ for any $\varepsilon \neq 0$ sufficiently small.*

Proposition 2 is proved in section 2.

We say that a Hopf bifurcation at the equilibrium point $(1, c+n)$ is *non-degenerate* or *classical* if it is created when a pair of complex eigenvalues of this equilibrium cross the imaginary axis. Then, from Proposition 2 it follows immediately the next result.

Corollary 3. *The differential system (4) cannot have a non-degenerate Hopf bifurcation in the quadrant $\{(x, y) : x > 0, y > 0\}$.*

We recall that also the differential systems (1) have non-degenerate Hopf bifurcations in the first quadrant, see [9].

Here we introduce the *zero-Hopf bifurcation in dimension 2* for a planar differential system depending on parameters, it is an equilibrium point having a pair of pure imaginary eigenvalues such that moving some parameters bifurcate from it some periodic orbit but during all this local bifurcation the equilibrium point always has a pair of pure imaginary eigenvalues. As far as we know this is a new kind of bifurcation for the differential systems here studied.

Our main result is the following one.

Theorem 4. *The differential systems (4) exhibits a zero-Hopf bifurcation in the quadrant $\{(x, y) : x > 0, y > 0\}$ for convenient values of the parameters.*

Theorem 4 is proved in section 2, computing the first terms of the analytic Poincaré map in a neighborhood of a weak focus. For a definition of Poincaré map see for instance [4].

2. PROOFS OF THE RESULTS

Taking a new time τ doing the rescaling

$$(5) \quad dt = K(c + nx)(a + x^2)d\tau,$$

the initial differential system (2) becomes the polynomial differential system

$$(6) \quad \begin{aligned} x' &= x(c + nx)(aKr - arx + Krx^2 - rx^3 - Kqy), \\ y' &= Ks(a + x^2)(c + nx - y)y, \end{aligned}$$

where the prime denotes derivative with respect to the new time τ . Note that $K(c + nx)(a + x^2) > 0$ in the first quadrant.

Proof of Proposition 1. From the expression of y' in (6) it follows that an equilibrium point (α, β) in the first quadrant must satisfy $\beta = c + n\alpha$. Then, from the expression of x' in (6) it follows that

$$-cKq + aKr - Knq\alpha - ar\alpha + Kr\alpha^2 - r\alpha^3 = 0.$$

Isolating q from this equation, and since q must be positive, the proof of the proposition is completed. \square

Proof of Proposition 2. The characteristic polynomial of the linear part of the differential system (6) satisfying (3) at the equilibrium point $(1, c + n)$ is

$$(7) \quad p(\lambda) = Krs(c+n)(a+1)(ac + aKn - 2cK + 3c - Kn + 2n) + (c+n)(aKs + ar - 2Kr + Ks + 3r)\lambda + \lambda^2.$$

Assume that $p(\lambda) = (\lambda - \varepsilon - \omega i)(\lambda - \varepsilon + \omega i)$ with $\varepsilon \neq 0$ sufficiently small, and we shall arrive to a contradiction. This implies that

$$\begin{aligned} c &= \frac{K(a - 2r + 1) + (a + 3)r)^2 \omega^2 + ((a + 1)^2 K^2 - 2(a + 1)(a(2K - 1) + 1)rK + (a - 2K + 3)^2 r^2) \varepsilon^2}{D}, \\ n &= \frac{-(K(a - 2r + 1) + (a + 3)r)^2 \omega^2 - ((a + 3)r - K(a + 2r + 1))^2 \varepsilon^2}{D}, \end{aligned}$$

where

$$D = 2(a + 1)^2(K - 1)Kr(K(a - 2r + 1) + (a + 3)r)\varepsilon.$$

Since the denominators of c and n are equal, the sign of cn only depends of their numerators. Then, the product of these two numerators are

$$-(K(a - 2r + 1) + (a + 3)r)^4 \omega^4 + O(\varepsilon^2) < 0.$$

So $cn < 0$, a contradiction with the fact that c and n are positives. Hence the proposition is proved. \square

Proof of Theorem 4. In order to simplify the huge computations that we should need to do for studying the Poincaré map in a neighborhood of the positive equilibrium point with arbitrary values of the parameters, we take $n = 1 - c$ and $K = 2$ (remember that from (3) $K > 1$), and we shall show for adequate values of the parameters that the equilibrium point $(1, c + n = 1)$ of the differential system (4) can be a weak focus, i.e. a focus with eigenvalues $\pm \omega i$.

We do the change of time given by (5) to the differential system (4) with $n = 1 - c$, $0 < c < 1$ and $K = 2$, and we obtain

$$(8) \quad \begin{aligned} \dot{x} &= rx(c + (1 - c)x) ((2 - x)(x^2 + a) - (a + 1)y), \\ \dot{y} &= 2y(x^2 + a)(c + (1 - c)x - y). \end{aligned}$$

Let

$$p(\lambda) = \lambda^2 + (ra + 2a - r + 2)\lambda - 2(a + 1)(ca - 2a + c)r$$

be the characteristic polynomial of the linear part of system (8) at the equilibrium point $(1, 1)$. Forcing that $p(\lambda) = (\lambda - 3i)(\lambda + 3i)$, we obtain that

$$(9) \quad \begin{aligned} c &= \frac{8a^3 + 16a^2 + 17a - 9}{4(a + 1)^3}, \\ r &= \frac{2(a + 1)}{1 - a}. \end{aligned}$$

Note that imposing that the eigenvalues of the equilibrium point $(1, 1)$ be $\pm 3i$, we have determined the parameters c and r of the differential system in function of the parameter a , which now is the unique free parameter of the system.

The unique real root of the polynomial $8a^3 + 16a^2 + 17a - 9$ is

$$\begin{aligned} a^* &= -\frac{2}{3} + \frac{\sqrt[3]{842 + 9\sqrt{8922}}}{6^{2/3}} - \frac{19}{6\sqrt[3]{2(842 + 9\sqrt{8922})}} \\ &\approx 0.3735505018762445... \end{aligned}$$

Since we must have $0 < c < 1$ and $r > 0$, it is easy to check that a must be in the interval $(a^*, 1)$.

Now we translate the equilibrium point $(1, 1)$ at the origin of coordinates doing the change of variables $x = X + 1$ and $y = Y + 1$. In these new variables the differential system (8) satisfying writes

$$\begin{aligned} (10) \quad \begin{aligned} X' &= \frac{1}{2(1-a)(a+1)^2} (X+1) (X^3 + X^2 + (a-1)X + (a+1)Y) \cdot \\ &\quad ((a-1)X(4a^2 + 8a + 13) - 4(a+1)^3), \\ Y' &= -\frac{1}{2(a+1)^3} (Y+1)(X^2 + 2X + a+1) \cdot \\ &\quad (4Y(a+1)^3 + (a-1)X(4a^2 + 8a + 13)). \end{aligned} \end{aligned}$$

Of course, the differential system (10) has the origin as an equilibrium point with eigenvalues $\pm 3i$.

We shall write the linear part of the differential system (10) in its real Jordan normal form doing the change of variables

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{2(a+1)}{3} & \frac{2(a+1)^2}{3(a-1)} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

In the new variables the differential system (10) becomes

$$\begin{aligned} (11) \quad \begin{aligned} u' &= \frac{1}{6(a-1)(a+1)^3} ((a-1)(3(v+1)^2 + a)(2(a+1)u + 3v) \cdot \\ &\quad (2(v-1)a^2 - (3u+4)a - 2v + 3u - 2) - \\ &\quad (a+1)(v+1)(2(a+1)v^3 + 2(a+1)v^2 + 3(a-1)u) \cdot \\ &\quad ((a-1)v(4a^2 + 8a + 13) - 4(a+1)^3)), \\ v' &= -\frac{1}{4(a-1)(a+1)^3} (2(a+1)v^3 + 2(a+1)v^2 + 3(a-1)u) \cdot \\ &\quad (v+1)((a-1)v(4a^2 + 8a + 13) - 4(a+1)^3). \end{aligned} \end{aligned}$$

Note that system (11) has its linear part at the origin in the real Jordan normal form.

Now we write the differential system (11) in polar coordinates (R, θ) where $u = R \cos \theta$ and $v = R \sin \theta$. Thus we have

$$\begin{aligned} (12) \quad \begin{aligned} r' &= a_2(\theta)R^2 + a_3(\theta)R^3 + a_4(\theta)R^4 + a_5(\theta)R^5, \\ \theta' &= \omega + b_1(\theta)R + b_2(\theta)R^2 + b_3(\theta)R^3 + b_4(\theta)R^4, \end{aligned} \end{aligned}$$

where

$$a_2(\theta) = \frac{1}{48(a-1)(a+1)^3} \left((16a^5 + 8a^4 + 160a^3 + 223a^2 + 242a - 137) \cdot \right. \\ \left. \cos \theta - (16a^5 + 152a^4 + 160a^3 - 65a^2 + 242a + 7) \cos(3\theta) + \right. \\ \left. 12(a+1)((2a^4 + 6a^3 + 5a^2 + 36a - 1) \sin \theta + \right. \\ \left. (2a^4 - 2a^3 - 19a^2 + 12a - 9) \sin(3\theta)) \right),$$

$$a_3(\theta) = \frac{\sin \theta}{12(a-1)(a+1)^3} \left(-72(a-1)^2(a+1) \cos^3 \theta - \right. \\ \left. 6(4a^5 + 9a^3 + 17a^2 - 49a + 19) \sin \theta \cos^2 \theta + \right. \\ \left. (16a^5 + 76a^4 + 288a^3 + 199a^2 + 370a + 75) \sin^2 \theta \cos \theta + \right. \\ \left. 6(4a^4 + 24a^3 + 39a^2 + 40a + 21) \sin^3 \theta \right),$$

$$a_4(\theta) = \frac{\sin^2 \theta}{24(a-1)(a+1)^3} \left(-8(a^5 + a^4 + 3a^3 - 14a^2 - 20a - 3) \cos \theta + \right. \\ \left. 8(a^5 + a^4 - 6a^3 - 5a^2 - 11a - 12) \cos(3\theta) + \right. \\ \left. 3((-8a^4 + a^2 + 114a + 85) \sin \theta + (8a^4 - 23a^2 - 14a - 35) \cdot \right. \\ \left. \sin(3\theta)) \right),$$

$$a_5(\theta) = -\frac{(4a^2 + 8a + 13) \sin^5 \theta (2(a+1) \cos \theta + 3 \sin \theta)}{6(a+1)^2},$$

$$b_1(\theta) = \frac{\sin \theta}{24(a-1)(a+1)^3} \left(-16a^5 - 80a^4 - 16a^3 - 97a^2 - 62a + \right. \\ \left. (16a^5 + 152a^4 + 160a^3 - 65a^2 + 242a + 7) \cos(2\theta) - \right. \\ \left. 12(2a^5 - 21a^3 - 7a^2 + 3a - 9) \sin(2\theta) - 241 \right),$$

$$b_2(\theta) = \frac{\sin^2 \theta}{24(a-1)(a+1)^3} \left(-16a^5 - 148a^4 - 216a^3 - 289a^2 - 118a + \right. \\ \left. (16a^5 + 76a^4 + 360a^3 + 127a^2 + 298a + 147) \cos(2\theta) + \right. \\ \left. 6(4a^5 + 4a^4 + 33a^3 + 56a^2 - 9a + 40) \sin(2\theta) - 237 \right),$$

$$b_3(\theta) = -\frac{\sin^3 \theta}{12(a-1)(a+1)^3} \left(-4(2a^3 - 2a^2 - a - 15)(a+1)^2 + \right. \\ \left. 8(a^5 + a^4 - 6a^3 - 5a^2 - 11a - 12) \cos(2\theta) + \right. \\ \left. 3(8a^4 - 23a^2 - 14a - 35) \sin(2\theta) \right),$$

$$b_4(\theta) = \frac{(4a^2 + 8a + 13) \sin^5 \theta (2(a+1) \sin \theta - 3 \cos \theta)}{6(a+1)^2}.$$

Taking θ as the new independent variable and developing R in a neighborhood of $R = 0$ up to order $O(R^6)$ we obtain that

$$(13) \quad \frac{dR}{d\theta} = c_2(\theta)R^2 + c_3(\theta)R^3 + c_4(\theta)R^4 + c_5(\theta)R^5 + O(R^6),$$

where

$$\begin{aligned} c_2(\theta) &= \frac{a_2}{\omega}, \\ c_3(\theta) &= \frac{a_3\omega - a_2b_1}{\omega^2}, \\ c_4(\theta) &= \frac{a_2b_1^2 - a_3\omega b_1 + a_4\omega^2 - a_2b_2\omega}{\omega^3}, \\ c_5(\theta) &= \frac{-a_2b_1^3 + a_3\omega b_1^2 - a_4\omega^2 b_1 + 2a_2b_2\omega b_1 + a_5\omega^3 - a_3b_2\omega^2 - a_2b_3\omega^2}{\omega^4}, \end{aligned}$$

where $a_i = a_i(\theta)$ and $b_i = b_i(\theta)$.

We know that the series (13) converges if R is small enough, and that the solution $R(\theta)$ of differential equation (13) satisfying the initial condition $R(0) = x$ can be expanded as

$$(14) \quad R(\theta, x) = \sum_{i=1}^{\infty} v_i(\theta)x^i,$$

where the $v_i(\theta)$'s satisfy the conditions

$$(15) \quad v_1(0) = 1 \quad \text{and} \quad v_i(0) = 0 \quad \text{for} \quad i = 2, 3, \dots$$

Substituting (14) in (13) and looking for the coefficients of the powers of x , we obtain the equations for determining all the v_k 's; i.e.

$$\begin{aligned} \frac{dv_1}{d\theta} &= 0, \\ \frac{dv_2}{d\theta} &= v_1^2 c_2, \\ \frac{dv_3}{d\theta} &= 2v_1 v_2 c_2 + v_1^3 c_3, \\ \frac{dv_4}{d\theta} &= (2v_1 v_3 + v_2^2) c_2 + 3v_1^2 v_2 c_3 + v_1^4 c_4, \\ \frac{dv_5}{d\theta} &= 2(v_1 v_4 + v_2 v_3) c_2 + 3(v_1 v_2^2 + v_1^2 v_3) c_3 + (4v_1^3 v_2 + v_2^2) c_4 + v_1^5 c_5, \\ &\dots \end{aligned}$$

Solving recursively this system of linear differential equations in the variables v_i 's with the initial conditions (15) we obtain the functions $v_i(\theta)$ for $i = 1, 2, 3, 4, 5$.

The Poincaré return map $h(x)$ is given by

$$h(x) = R(2\pi, x) = \sum_{i=1}^{\infty} v_i(2\pi)x^i.$$

We have that

$$\begin{aligned} v_1(2\pi) &= 1, \\ v_2(2\pi) &= 0, \\ v_3(2\pi) &= V_3, \\ v_4(2\pi) &= V_3 d_4, \\ v_5(2\pi) &= V_5 + d_5 V_3^2, \end{aligned}$$

where $V_3 = V_3(a)$ and $V_5 = V_5(a)$ are

$$V_3 = \frac{(16a^7 + 44a^6 + 280a^5 + 715a^4 + 1010a^3 - 80a^2 + 366a - 303)\pi}{216(a-1)^2(a+1)^3},$$

$$V_5 = \frac{\pi}{20155392(a-1)^4(a+1)^9} (480256a^{17} + 5994240a^{16} + 43031808a^{15} + 222493184a^{14} + 851334144a^{13} + 2476961184a^{12} + 5630286400a^{11} + 10064093904a^{10} + 14060220168a^9 + 15036900847a^8 + 12018681654a^7 + 6112974078a^6 + 366906086a^5 - 2104306572a^4 - 1487443758a^3 - 424804430a^2 + 58584906a + 15726333),$$

and d_4 and d_5 are polynomials in the variable a . The constants V_3 and V_5 are called the first two *Liapunov constants*, for more details see for instance [8].

The function $h(x) - x$ is called the *displacement function* and its zeros provides periodic orbits surrounding the equilibrium point $(0, 0)$ of the differential system (11), or equivalently surrounding the equilibrium point $(1, 1)$ of the differential system (4) with

$$K = 2,$$

$$r = \frac{2(a+1)}{1-a},$$

$$n = \frac{(1-a)(4a(a+2)+13)}{4(a+1)^3},$$

$$c = \frac{a(8a(a+2)+17)-9}{4(a+1)^3}.$$

We have that

$$(16) \quad h(x) - x = V_3(a)(x^3 + d_4x^4 + d_5V_3(a)x^5) + V_5(a)x^5 + O(x^6).$$

The polynomial of the numerator of $V_3(a)$ in the variable a has a unique real root $\tilde{a} = 0.47143970104460403\dots$. On this root the displacement function becomes

$$V_5(\tilde{a})x^5 + O(x^6) = -6.971890222792349\dots x^5 + O(x^6).$$

So, the weak focus at $(1, 1)$ of the differential system (4) is estable when $a = \tilde{a}$. Then, since $V_3(a) > 0$ if $a \gtrsim \tilde{a}$, at $a = \tilde{a}$ the weak focus change its stability when we increase the value of a . This change of stability forces the bifurcation of periodic orbit from the weak focus. Hence, the theorem is proved. \square

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