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Maximal rank for schemes of small multiplicity by Évain's differential Horace method

Joaquim Roé

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Abstract

The Hilbert function of the union of n general e-fold points in the plane is maximal if $n \geq 4e^2$ or n is a square. The Hilbert function of a union of A, D, E singularity schemes in general position is maximal in every degree > 28. The proofs use computation of limits of families of linear systems whose special members acquire base divisors, an interesting problem in itself.

Given a zero-dimensional scheme $Z \subset \mathbb{P}^2$ and a positive integer a, the curves of degree a containing Z form a linear system $|aL - Z|$ of dimension at least $v(Z, a) = a(a+3)/2 -$ length Z. $|aL - Z|$ is said to have *maximal rank* if it is empty or $\dim |aL - Z| = v(Z, a)$. Z itself has maximal rank (equivalently, its graded ideal has maximal Hilbert function) if $|aL - Z|$ has maximal rank for all $a > 0$.

Given positive integers n, e, denote $Z(e^n) = \bigcup_{i=1}^n p_i^e$ the scheme formed by n *general* points $p_1, \dots, p_n \in \mathbb{P}^2$ taken with multiplicity e. We prove three maximal rank statements:

Theorem 1. *If e, n are positive integers with $n \geq 4e^2$, $Z(e^n)$ has maximal rank.*

Theorem 2. *Assume that the characteristic of the base field is zero. Let $n = s^2$ be a square, and $e > s/2$. Then $Z(e^n)$ has maximal rank.*

Together with theorem 1, this says that every set of s^2 equimultiple general points has maximal rank, a result first proved by L. Évain [9].

Both these results are evidence for the homogeneous Segre–Harbourne–Gimigliano–Hirschowitz conjecture, which says that such Z have maximal rank for every e if $n \geq 9$. For comparison purposes, the only previous result showing maximal rank when $n \geq f(e)$ for some function f is due to J. Alexander and A. Hirschowitz [1], with $f \sim \exp(\exp(e))$. Note also that such Z have maximal rank if $e \leq 42$ by [6], so $Z = \bigcup_{i=1}^n p_i^e$ has maximal rank whenever $n \geq 9$ and $e \leq \max\{42, \sqrt{n}/2\}$.

Theorem 3. *Let $Z \subset \mathbb{P}^2$ be a scheme all whose components are singularity schemes of type A, D, or E. If Z is general among all schemes of the same type, then $|aL - Z|$ has maximal rank for all $a > 28$.*

This is evidence for the Greuel–Lossen–Shustin conjecture 6.3 in [10], which says that general singularity schemes have maximal rank in every degree larger than the sum of the three biggest multiplicities (which in the case of ADE singularities is 9). This reduces the proof of the Greuel–Lossen–Shustin conjecture for ADE singularities to checking a finite number of cases.

Except in theorem 2, we work over an algebraically closed field k of arbitrary characteristic. Theorems 1, 2 and 3 are proved by a generalization of the *differential Horace* method developed by J. Alexander–A. Hirschowitz [1] and L. Évain [9], applied to suitable specializations. This differential method consists in studying a family Z_t of zero-dimensional schemes whose special member Z_0 is tractable but gives dimension larger than Z_t, by taking $t^{p_1} = 0$, $t^{p_2} = 0$, ... for suitable exponents p_i. Like Évain's, our families Z_t are essentially monomial and can be described by combinatorial objects called staircases, but in our setting the reduction of algebraic computations to combinatorics is only valid under additional hypotheses on the p_i's, which we check for each Z_t involved.

1 General differential Horace & emptiness proofs

Our proof of theorem 2 relies on previous work by B. Harbourne–S. Holay–S. Fitchett [11], L. Évain [8] and A. Buckley–M. Zompatori [3]. By [8] and [3], $Z(e^n)$ has maximal rank for all e if n is a power of four or nine, and if n_1 and n_2 are such that $Z(e^{n_i})$ has maximal rank for all e then the same holds for $Z(e^{n_1 n_2})$. So we assume $n = s^2$ with $s \geq 5$ and odd.

Nonempty systems $|aL - Z(e^{s^2})|$ have been treated in [11], and in particular by its lemma 5.3, if $a \geq se + (s-3)/2$ then $\dim |aL - Z(e^{s^2})| = v(Z(e^{s^2}), a)$. Thus to prove theorem 2 it will be enough to show the following:

Proposition 1.1. *Assume* $\operatorname{char} k = 0$. *Let* $n = s^2$ *be an odd square,* $e > s/2$ *an integer and define* $a = se + (s-5)/2$. *Then* $|aL - Z(e^n)| = \emptyset$.

Such emptiness results can be proved by direct application of the differential Horace lemma of [9], which we recast in greater generality to be used later. Proposition 1.1 will be proved in section 1.2.

1.1 Higher order traces and residuals

Let R be an integral k-algebra. Consider $R_t = R \otimes k[[t]]$. Given $f_t \in R_t$, denote $f_0 \in R$ its image by the obvious morphism $t \mapsto 0$. Similarly, for an ideal I_t in R_t, $I_0 = (I_t + (t))/(t) \subset R_t/(t) \cong R$. Given $y \in R$, $I_t \subset R_t$, and an integer $p \geq 1$, consider the ideals:

$$\begin{aligned}
\operatorname{Tr}_p(I_t|y) =& \frac{\left((I_t + (y)) : t^{p-1}\right)_0}{(y)} \subset R/(y), \\
\mathbb{R}\mathrm{es}_p(I_t|y) =& (I_t + (t^p)) : y \subset R_t, \\
\operatorname{Res}_p(I_t|y) =& ((I_t + (t^p)) : y)_0 \subset R;
\end{aligned}$$

and inductively, given sequences $\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{Z}^m$ with $p_1 \geq p_2 \geq \cdots \geq p_m \geq 1$, $\mathbf{y} = (y_1, \ldots, y_m) \in R^m$, denote $\operatorname{Tr}^1_{\mathbf{p}}(I_t|\mathbf{y}) = \operatorname{Tr}_{p_1}(I_t|y_1)$, $\operatorname{Res}^1_{\mathbf{p}}(I_t|\mathbf{y}) = \operatorname{Res}_{p_1}(I_t|y_1)$ and $\mathbb{R}\mathrm{es}^1_{\mathbf{p}}(I_t|\mathbf{y}) = \mathbb{R}\mathrm{es}_{p_1}(I_t|y_1)$; and for every integer $1 < i \leq m$,

$$\begin{aligned}
\operatorname{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y}) =& \operatorname{Tr}_{p_i}(\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y})|y_i) \subset R/(y_i), \\
\mathbb{R}\mathrm{es}^i_{\mathbf{p}}(I_t|\mathbf{y}) =& \mathbb{R}\mathrm{es}_{p_i}(\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y})|y_i) \subset R_t, \\
\operatorname{Res}^i_{\mathbf{p}}(I_t|\mathbf{y}) =& \operatorname{Res}_{p_i}(\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y})|y_i) \subset R.
\end{aligned}$$

If $y_i = y \,\forall i$ we write $\mathrm{Tr}^i_{\mathbf{p}}(I_t|y) = \mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y})$, $\mathbb{R}\mathrm{es}^i_{\mathbf{p}}(I_t|y) = \mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y})$ and $\mathrm{Res}^i_{\mathbf{p}}(I_t|y) = \mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y})$, and if $p_i = p \,\forall i$ we write $\mathrm{Tr}^i_p(I_t|\mathbf{y}) = \mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y})$, etc. These ideals were introduced and used in [9] for vertical translations of monomial ideals (with different notation, for instance $\mathbb{R}\mathrm{es}^i_{\mathbf{p}}(I_t|y)$ is called J_{p_i}: there). Note that

$$\mathbb{R}\mathrm{es}^i_{\mathbf{p}}(I_t|\mathbf{y}) = (I_t + (t^{p_1}, y_1 t^{p_2}, \ \dots \ , y_1 y_2 \cdots y_{i-1} t^{p_i})) : (y_1 y_2 \cdots y_i). \qquad (1)$$

Sometimes we shall also write $\mathbb{R}\mathrm{es}^0_{\mathbf{p}}(I_t|y) = I_t$ and $\mathrm{Res}^0_{\mathbf{p}}(I_t|y) = I_0$. Proposition 1.2 and its immediate corollary 1.3 are a natural generalization of theorem 14 in [7], (proved in [9] for product ideals in product rings), which in turn refines proposition 8.1 of [1]. The proof of [9] works verbatim in this setting.

For a k-linear subspace $V \subset R$ and $y \in R$, let $\mathrm{Res}(V|y) = \{v \in R \mid vy \in V\}$.

Proposition 1.2. *Let $V \subset R$ be a k-linear subspace, and $I_t \subset R_t$ an ideal such that R_t/I_t is flat over $k[[t]]$. Let $m \geq 0$ and let $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$ with $p_1 \geq p_2 \geq \cdots \geq p_m \geq 1$, and $\mathbf{y} = (y_1, \dots, y_m) \in R^m$ be given sequences. Let $W = \{f \in R | \exists f_t \in V \otimes k[[t]] \cap I_t$ with $f_0 = f\}$. If for $1 \leq i \leq m$, the canonical map*

$$\frac{\mathrm{Res}(V|y_1 \cdots y_{i-1})}{\mathrm{Res}(V|y_1 \cdots y_{i-1}) \cap (y_i)} \longrightarrow \frac{R/(y_i)}{\mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y})}$$

is injective, then $W \subset y_1 \dots y_m \mathrm{Res}^m_{\mathbf{p}}(I_t|\mathbf{y})$.

Corollary 1.3. *In the same setting of the previous proposition, assume moreover that the canonical map*

$$\mathrm{Res}(V|y_1 \cdots y_m) \longrightarrow \frac{R}{\mathrm{Res}^m_{\mathbf{p}}(I_t|\mathbf{y})}.$$

is injective. Then the canonical map $\varphi_t : V \otimes k[[t]] \longrightarrow R_t/I_t$ is injective.

Let X be a projective variety and Z_t a family of subschemes of X. To be explicit, fix a quasi-projective smooth curve C, and Z a subscheme of $X \times C$ which is flat and finite over C; Z_t is the fiber of Z over $t \in C$.

Given an invertible sheaf L on X, denote $|L - Z_t|$ the linear system of effective divisors in $|L|$ containing Z_t. There is an open set $U \subset C$ such that $\forall t \in U$, $\dim |L - Z_t|$ is minimal and constant, say d. This gives a morphism $t \mapsto |L - Z_t|$ to the Grassmannian of d-dimensional linear subspaces of $|L|$, which can be extended to the whole of C because the Grassmannian is projective. For $t_0 \notin U$ we denote $\lim_{t \mapsto t_0} |L - Z_t|$ the image of t_0 by the extension of the morphism above to C. Clearly $\lim_{t \mapsto t_0} |L - Z_t| \subset |L - Z_{t_0}|$.

If $t_0 \notin U$, it is often because of a prime divisor $D \subset X$ with $\rho^D_{t_0} : H^0(D, \mathcal{O}_D(L)) \to H^0(D \cap Z_{t_0}, \mathcal{O}_{D \cap Z_{t_0}}(L))$ injective. In that case, D is a fixed part of $|L - Z_{t_0}|$; denoting $\mathcal{I}_{t_0}$ the ideal sheaf of Z_{t_0}, the residual linear system is $|(L - D) - \tilde{Z}_{t_0}|$, with $\tilde{Z}_{t_0}$ defined by the exact sequence

$$0 \to \mathcal{I}_{\tilde{Z}_{t_0}} \to \mathcal{I}_{Z_{t_0}} \to \mathcal{I}_{Z_{t_0}} \otimes \mathcal{O}_D \to 0.$$

If $\rho^D_{t_0}$ is not surjective, the expected dimension of $|(L - D) - \tilde{Z}_{t_0}|$ will be bigger than that of $|L - Z_t|$, so most likely it will be bigger than d.

We now describe how to use proposition 1.2 to construct a linear system $\mathcal{L}$ which contains $\lim_{t \mapsto t_0} |L - Z_t|$, and can be used to bound d. If $\mathcal{L}$ turns out to be empty or have codimension length Z_t in $|L|$, we can conclude that $d = \dim \mathcal{L}$ and $|L - Z_t|$ has maximal rank for general t. Assume $Z = (Z_{\mathrm{fix}} \times C) \cup Y \subset X \times C$, where $Z_{\mathrm{fix}} \subset X$ is a fixed zero-dimensional scheme and Y is finite and flat over C, with irreducible fibers. In other words, the moving part of Z is supported at a single (possibly moving) point of X. Let R be the local ring at the support of Y_{t_0} (or its completion), and $I_t \subset R_t$ the ideal of the family Y_t. Let $y_i = 0$ be a local equation for D_i, $i = 1, \dots, m$, and V the image of $H^0(\mathcal{O}_X(L))$ in R. Under the conditions of proposition 1.2, which are satisfied whenever $\rho_i : H^0(D_i, \mathcal{O}_{D_i}(L-D_1-\cdots-D_{i-1})) \to H^0(D_i \cap Z_{\mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y})}, \mathcal{O}_{D_i \cap Z_{\mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y})}}(L-D_1-\cdots-D_{i-1}))$ is injective for all i, $\lim_{t\mapsto t_0} |L - Z_t| \subset \mathcal{L}$, where $\mathcal{L}$ has $D_1 + \cdots + D_m$ as a fixed part and all divisors in $\mathcal{L} - D_1 - \cdots - D_m$ contain the scheme defined by $\mathrm{Res}^m_{\mathbf{p}}(I_t|\mathbf{y})$. If the aditional condition of corollary 1.3 is satisfied, then $\mathcal{L}$ is empty.

If the moving part Y_t of Z_t has more than one component, one takes R to be the product of the local rings at all points of the support of Y_{t_0} (see [9]).

1.2 A computation for squares

Let now R be the completion of the local ring at a point of $\mathbb{P}^2$ or a blowup of $\mathbb{P}^2$; thus $R \cong k[[x, y]]$. In particular, both R and R_t are regular local rings, with maximal ideals $\mathfrak{m} = (x, y)$ and $\mathfrak{m}_t = (x, y, t)$ respectively.

Proposition 1.4. *Let e, p, be positive integers with $e + 1 \geq p$, and let $I_t = (x, y - t) \subset R_t = k[[x, y, t]]$. Then*

1. *For every positive integer i, $\mathrm{tr}^i_p(I^e_t|y) \geq e + 2 - p - i$, and*
2. *Assume k has characteristic zero. Then $\mathrm{Res}^e_p(I^e_t|y) \subset (x, y)^{\lfloor \frac{p}{2} \rfloor}$.*

Proof. Let $f = y - t$. Given $g = \sum a_{ijk} x^i f^j t^k \in k[[x, f]][[t]] = R_t$, we set $\mathrm{ord}_t(g) = \min\{k | \exists i, j;\ a_{ijk} \neq 0\}$ and define the dominant part of g as $g^* = \sum a_{i,j,\mathrm{ord}_t(g)} x^i f^j t^{\mathrm{ord}_t(g)}$. An ideal $I_t \subset R_t$ determines its ideal of dominant terms $I^*_t = (g^*)_{g \in I_t} \subset R_t$. Observe that $(I^*_t)_0 = I_0$. The following elementary lemma is left to the reader:

Lemma 1.5. *Let $I_t \subset k[[x, y, t]]$ and $p \in \mathbb{Z}_{>0}$. Assume that $I^*_t \subset (x, f)^m + (t^p)$. Then $((I_t + (t^p)) : y)^* \subset (x, f)^{m-1} + (t^p)$*

By iteration, $\mathbb{R}\mathrm{es}^{i-1}_{p+1}(I^e_t|y) + (t) \subset I^{e-i+1}_t + (t)$. On the other hand, it is easy to see (and is proved as part of proposition 8.1 in [1]) that $\mathrm{tr}_p(I^{e-i+1}_t|y) = e + 2 - p - i$, whence the first claim. It remains to prove that

$$((I^e_t + (t^p)) : y^e)_0 \subset (x, y)^{\lfloor \frac{p}{2} \rfloor},$$

or equivalently (via the obvious $k[[t]]$-automorphism of R_t with $y \mapsto y - t$) that

$$(((x, y)^e + (t^p)) : (y + t)^e)_0 \subset (x, y)^{\lfloor \frac{p}{2} \rfloor}.$$

Let $g = \sum a_{ijk} x^i y^j t^k$ and assume that $h = g(y + t)^e = \sum b_{ijk} x^i y^j t^k \in (x, y)^e + (t^p)$, which means $b_{ijk} = 0$ for all $k < p$ and $i + j < e$. By definition,

$$h = \sum a_{ijk} \sum_{\ell=0}^{e} \binom{e}{\ell} x^i e^{\ell+j} t^{k+m-\ell}, \text{ so } b_{ijk} = \sum_{\ell=\ell_0}^{\ell_1} \binom{e}{\ell} a_{i,j-\ell,k+\ell-e},$$

where $\ell_0 = \max\{e - k, 0\}$ and $\ell_1 = \max\{j, e\}$. Therefore

$$\sum_{\ell=e-k}^{j} \binom{e}{\ell} a_{i,j-\ell,k+\ell-e} = 0, \qquad 0 \leq i, j, k;\ k < p;\ i + j \leq e - 1.$$

For each fixed i and $r = j + k - e$ satisfying $0 \le i \le e-1$ and $0 \le r < p - i - 1$ we have obtained a system of linear equations

$$\begin{pmatrix} \binom{e}{e-p+1} & \binom{e}{e-p+2} & \cdots & \binom{e}{r} \\ \binom{e}{e-p+2} & \binom{e}{e-p+3} & \cdots & \binom{e}{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{e}{e-1-i-r} & \binom{e}{e-i-r} & \cdots & \binom{e}{e-1-i} \end{pmatrix} \begin{pmatrix} a_{i,r,0} \\ a_{i,r-1,1} \\ \vdots \\ a_{i,0,r} \end{pmatrix} = 0,$$

which, if $e - i - 1 - r - (e - p + 1) \ge r$, admits only the trivial solution as its matrix has nonzero determinant (similar matrices are known in the litterature, e.g., in [2, p. 94], [14], [13], and this case is not difficult to treat by hand.)

In particular, for every (i, r) with $i + r < \lfloor \frac{p}{2} \rfloor$ (which trivially implies $e - i - 1 - r - (e - p + 1) \ge r$) we obtain $a_{i,r,0} = 0$. Thus $g_0 \in (x, y)^{\lfloor \frac{e}{2} \rfloor}$ as claimed. □

proof of proposition 1.1. Consider an irreducible smooth curve D of degree s, (and genus $g = (s-1)(s-2)/2$), let $p_1, \ldots, p_{s^2-1}$ be general points of D, and p_{s^2} a general point of $\mathbb{P}^2$. Denote by Z the union of these points taken with multiplicity e. The restriction of $\mathcal{I}_Z(a)$ to D is an invertible sheaf of degree $d = as - (s^2 - 1)e = e - s(s-5)/2$ which, by the genericity of the choice of the $s^2 - 1 > g$ points, is general among those of its degree.

If $d < g$ this invertible sheaf has no nonzero global sections, so D is a fixed part of $|aL - Z|$. The residual system is $|(a-s)L - Z'|$, where $a - s = s(e-1) + (s-5)/2$ and Z' consists of the points $p_1, \ldots, p_{s^2-1}$ with multiplicity $e-1$ and p_{s^2} with multiplicity e. The restriction of $\mathcal{I}_{Z'}(a-s)$ to D is a general invertible sheaf of degree $d' = d - 1 < g$, so D is again a fixed part of the linear system. Iterating, D is contained exactly e times in the curves of $|aL - Z|$, and the residual system consists of curves of degree $a - se = (s-5)/2$ with a point of multiplicity $e > s/2$, thus it is empty.

So assume $d \ge g$ and let $p = e + g - d = s + 2$ (so we trivially have $e + 1 \ge p \ge 1$). Now let p_{s^2} tend to D transversely, i.e.. choose a general point $q \in D$, let x, y be local parameters at q such that $y = 0$ is a local equation for D, and let $q_t = (0, -t)$. We claim that the limit of the linear systems formed by curves of degree a with multiplicity e at $p_1, \ldots, p_{s^2-1}$ and at q_t when $t \mapsto 0$ is empty. By the first claim of 1.4 and 1.2, the limit system consists of D counted e times plus a residual system contained in $|(a - se)L - Z''|$ where Z'' is the zeroscheme defined by $\mathrm{Res}^e_{p+1}(I^e_t|y)$. By the second claim of 1.4, Z'' contains the point q with multiplicity $p/2 > s/2$, and we are done. □

2 Preserving the number of conditions

Propositon 1.2 can be also used to prove maximal rank for nonempty systems, as explained at the end of section 1.1, but this requires some more work.

We recycle the notations of section 1.1. Z_t is a family of schemes defined by the ideal $I_t \subset R_t$, $D_1, \ldots, D_m$ are divisors through p defined locally by $y_i = 0$, $i = 1, \ldots, m$. Consider the quantities $\mathrm{tr}^i_{\mathbf{p}}(I_t|\mathbf{y}) = \dim_k(R/\mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y}))$, $\mathbb{res}^i_{\mathbf{p}}(I_t|\mathbf{y}) = \dim_k(R_t/\mathbb{Res}^i_{\mathbf{p}}(I_t|\mathbf{y}))$, and $\mathrm{res}^i_{\mathbf{p}}(I_t|\mathbf{y}) = \dim_k(R/\mathrm{Res}^i_{\mathbf{p}}(I_t|\mathbf{y}))$. As R_t/I_t is flat over $k[[t]]$, $\mathbb{res}^0_{\mathbf{p}}(I_t|y) = \infty$ and $\mathrm{res}^0_{\mathbf{p}}(I_t|y) = \dim_k R/I_0 = \dim_{k((t))}(R_t/I_t) \otimes k((t))$.

Divisors in the system $\mathcal{L}$ that bounds $\lim_{t \mapsto t_0} |L - Z_t|$ contain (a) the divisors D_1, $\ldots$, D_m (containing D_i accounts for at most $\mathrm{tr}^i_{\mathbf{p}}(I_t|\mathbf{y})$ linear conditions, and exactly this number if ρ_i is bijective) and (b) the residual zeroscheme (which accounts for at most $\mathrm{res}^m_{\mathbf{p}}(I_t|\mathbf{y})$ linear conditions). So assuming $\dim |L| \ge \mathrm{length}\, Z_t$, $\mathcal{L}$ will have the same expected dimension as $|L - Z_t|$ if and only if ρ_i is bijective for all i and

$$\mathrm{res}^m_{\mathbf{p}}(I_t|\mathbf{y}) + \sum_{i=1}^{m} \mathrm{tr}^i_{\mathbf{p}}(I_t|\mathbf{y}) = \mathrm{res}^0_{\mathbf{p}}(I_t|y).$$

The amount by which this equality fails is the number of conditions lost in the residuation process, $\Lambda^m_{\mathbf{p}}(I_t|\mathbf{y}) = \mathrm{res}^0_{\mathbf{p}}(I_t|y) - (\mathrm{res}^m_{\mathbf{p}}(I_t|\mathbf{y}) + \sum \mathrm{tr}^i_{\mathbf{p}}(I_t|\mathbf{y}))$. In this section we study $\Lambda^m_{\mathbf{p}}(I_t|\mathbf{y})$ and determine under what conditions it vanishes.

Given a nonincreasing sequence $\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{Z}^m$ and two integers j and q, with $1 \leq j < m$ and $q \leq p_j$, we denote $\mathbf{p}(q,j) = (p_1, p_2, \ldots, p_{j-1}, q)$ and, if $q \leq p_m$, $\mathbf{p} - q = (p_1 - q, \ldots, p_m - q)$.

Lemma 2.1. *Let $I_t \subset R_t$ be an ideal such that R_t/I_t is flat over $k[[t]]$. Let $m \geq 0$ and let $\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{Z}^m$ with $p_1 \geq p_2 \geq \cdots \geq p_m \geq 1$, and $\mathbf{y} = (y_1, \ldots, y_m) \in R^m$ be given sequences. Then for every integer $q \leq p_m$,*

$$\mathbb{R}\mathrm{es}^m_{\mathbf{p}}(I_t|\mathbf{y}) : t^q = \mathbb{R}\mathrm{es}^m_{\mathbf{p}-q}(I_t|\mathbf{y}).$$

Proof. Using (1), it is easy to check that

$$\begin{aligned}\mathbb{R}\mathrm{es}^m_{\mathbf{p}}(I_t|\mathbf{y}) : t^q &= (I_t + (t^{p_1}, y_1 t^{p_2}, \ \ldots\ , y_1 y_2 \cdots y_{m-1} t^{p_m})) : (y_1 y_2 \cdots y_m t^q) = \\ &= \big(I_t : t^q + (t^{p_1 - q}, y_1 t^{p_2 - q}, \ \ldots\ , y_1 y_2 \cdots y_{m-1} t^{p_m - q})\big) : (y_1 y_2 \cdots y_m).\end{aligned}$$

The claim follows noting that $I_t : t^q = I_t$ (by flatness) and using (1) again. □

Given $f \in R$ and $I \subset R$ an ideal, the residual exact sequence

$$0 \longrightarrow \frac{R}{I : f} \longrightarrow \frac{R}{I} \longrightarrow \frac{R}{I + (f)} \longrightarrow 0$$

shows that $\dim_k R/I = \dim_k R/(I : f) + \dim_k R/(I + (f))$ if these quantities are finite; this equality will be denoted $RES(I : f)$.

Proposition 2.2. *Let $I_t \subset R_t$ be an ideal such that R_t/I_t is flat over $k[[t]]$ and $\dim_k R/I_0 < \infty$. Let $m \geq 0$ and let $\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{Z}^m$ with $p_1 \geq p_2 \geq \cdots \geq p_m \geq 1$, and $\mathbf{y} = (y_1, \ldots, y_m) \in R^m$ be given sequences. Then*

$$\Lambda^i_{\mathbf{p}}(I_t|\mathbf{y}) = \sum_{j=1}^{i} \dim \frac{\mathbb{R}\mathrm{es}^{j-1}_{\mathbf{p}-1}(I_t|\mathbf{y}) + (t^{p_j - 1}, y_j)}{\mathbb{R}\mathrm{es}^{j-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_j - 1}, y_j)},$$

$$\dim \frac{\mathbb{R}\mathrm{es}^{j-1}_{\mathbf{p}-1}(I_t|\mathbf{y}) + (t^{p_j - 1}, y_j)}{\mathbb{R}\mathrm{es}^{j-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_j - 1}, y_j)} = \sum_{q=1}^{p_j - 1} \left(\mathrm{tr}^j_{\mathbf{p}(q,j)}(I_t|\mathbf{y}) - \mathrm{tr}^j_{(\mathbf{p}-1)(q,j)}(I_t|\mathbf{y})\right).$$

Proof. The residual exact sequence $RES((\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_i})) : t^{p_i - 1})$, applied recursively, and lemma 2.1, give

$$\dim \frac{R_t}{\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_i})} = \sum_{q=1}^{p_i} \mathrm{res}^{i-1}_{\mathbf{p}-q+1}(I_t|\mathbf{y}).$$

Then, the $RES((\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_i})) : y_i)$ and the $RES((\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_i}, y_i)) : t^{p_i - 1})$ give

$$\mathrm{res}^i_{\mathbf{p}}(I_t|\mathbf{y}) = \sum_{q=1}^{p_i} \mathrm{res}^{i-1}_{\mathbf{p}-q+1}(I_t|\mathbf{y}) - \mathrm{tr}^i_{\mathbf{p}}(I_t|\mathbf{y}) - \dim \frac{R_t}{\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_i - 1}, y_i)}.$$

Combining this with the $RES(\mathbb{R}\mathrm{es}^i_{\mathbf{p}}(I_t|\mathbf{y}) : t)$, it follows that

$$\begin{aligned}\mathrm{res}^i_{\mathbf{p}}(I_t|\mathbf{y}) = \mathrm{res}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y}) - \mathrm{tr}^i_{\mathbf{p}}(I_t|\mathbf{y}) - \\ - \left(\dim \frac{R_t}{\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_i-1}, y_i)} - \dim \frac{R_t}{\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}-1}(I_t|\mathbf{y}) + (t^{p_i-1}, y_i)} \right),\end{aligned}$$

which recursively applied yields the first claim. The second follows by applying $RES((\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_i}, y_i)) : t^{p_i-1})$ recursively. □

Corollary 2.3. *Assume that* $\Lambda^i_{\mathbf{p}}(I_t|\mathbf{y}) = 0$. *Then there exists an integer* q_i, $p_i \geq q_i \geq 0$, *such that* $\Lambda^{i+1}_{\mathbf{p}}(I_t|\mathbf{y}) = 0$ *if and only if* $p_{i+1} \leq q_i$.

Proof. We claim that $q_i = \min \left\{ q \,\middle|\, \mathrm{Tr}^i_{\mathbf{p}(q)}(I_t|\mathbf{y}) \neq \mathrm{Tr}^i_{(\mathbf{p}-1)(q)}(I_t|\mathbf{y}) \right\} - 1$ satisfies the claim. Indeed, the hypothesis and 2.2 tell us that

$$\begin{aligned}\mathrm{res}^{i+1}_{\mathbf{p}}(I_t|\mathbf{y}) + \sum_{j=1}^{i+1} \mathrm{tr}^j_{\mathbf{p}}(I_t|\mathbf{y}) = \mathrm{res}^0_{\mathbf{p}}(I_t|\mathbf{y}) - \dim \frac{\mathbb{R}\mathrm{es}^i_{\mathbf{p}-1}(I_t|\mathbf{y}) + (t^{p_{i+1}-1}, y_{i+1})}{\mathbb{R}\mathrm{es}^i_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_{i+1}-1}, y_{i+1})} = \\ = \mathrm{res}^0_{\mathbf{p}}(I_t|\mathbf{y}) - \sum_{q=1}^{p_i-1} \left(\mathrm{tr}^i_{\mathbf{p}(q)}(I_t|\mathbf{y}) - \mathrm{tr}^i_{(\mathbf{p}-1)(q)}(I_t|\mathbf{y}) \right).\end{aligned}$$

Since for all q, $\mathrm{Tr}^i_{\mathbf{p}(q)}(I_t|\mathbf{y}) \subseteq \mathrm{Tr}^i_{(\mathbf{p}-1)(q)}(I_t|\mathbf{y})$, the claim follows. □

Example 2.4. Non iterated traces and residuals (i.e., for $m = 1$, as in [1]) always preserve the number of conditions: $\Lambda^1_p(I_t|y) = 0$.

3 Oblique limit monomial ideals

A *staircase* is a finite subset $E \subset \mathbb{Z}^2_{\geq 0}$ whose complement satisfies $E^c + \mathbb{Z}^2_{\geq 0} \subset E^c$. The *length* of its ith stair is $\ell_E(i) = \min\{e \mid (e, i) \notin E\}$, the length of its ith *step* is $\hat{\ell}_E(i) = \ell_E(i) - \ell_E(i+1)$, and the *height* of its ith slice is $h_E(i) = \min\{e \mid (i, e) \notin E\}$. When there are no steps of height > 1, i.e., if $h_E(i) \leq h_E(i+1) + 1$ for all i, we say that E is *gentle*. We also define the total length and height of E as $\ell(E) = \ell_E(0)$ and $h(E) = h_E(0)$, and the minimal step length $\hat{\ell}_{\min}(E) = \min\{\hat{\ell}_E(i) \mid 0 \leq i < h(E) - 1\}$ (for technical reasons the latter does not take into account the length of the top step). Given a staircase E, and elements g, f of a ring, we denote

$$I_{E,g,f} = (g^{e_1} f^{e_2})_{(e_1,e_2) \notin E}\,.$$

The following elementary lemma is left to the reader:

Lemma 3.1. *For every staircase* E, *and every system of parameters* $g, f \in R \cong k[[x, y]]$,

1. $I_{E,g,f}$ *is* $\mathfrak{m}$*-primary, and has colength* $\#E$.
2. $I_{E,g,f}$ *depends only on finite jets of* g *and* f, *i.e., there exist integers* $a = a(E)$ *and* $b = b(E)$ *such that* $g_1 - g_2 \in \mathfrak{m}^a$, $f_1 - f_2 \in \mathfrak{m}^b$ *imply* $I_{E,g_1,f_1} = I_{E,g_2,f_2}$.
3. *If* E *is gentle then* $I_{E,g,f}$ *does not depend on* g, *i.e.,* $I_{E,g_1,f} = I_{E,g_2,f}$ *whenever* $(g_1, f) = (g_2, f) = \mathfrak{m}$. *In such a case we denote* $I_{E,f} = I_{E,g_1,f}$.

In this section we fix $I_t = I_{E,x,f} \subset k[[x,y,t]]$ where $f = x+y+t$. Given $\mathbf{m} = (m_1,\dots,m_\mu) \in \mathbb{Z}^\mu_{>0}$, let $\mathbf{y_m}$ be the concatenation of μ sequences of the form $(y,x,\dots,x)$ of lengths $m_1,\dots,m_\mu$. Let also $\sigma_{\mathbf{m}}(i)$ denote the number of x appearing in $\mathbf{y_m}$ up to the ith position, i.e., $\sigma_{\mathbf{m}}(i) = i-1-\max\left\{k \mid \sum_{j=1}^k m_j \leq i-1\right\}$, and $\tau(E,i)$ the staircase obtained from E by deleting the leftmost i slices and moving everything to the left, i.e., $\tau(E,i) = \{(e_1,e_2) \mid (e_1+i,e_2) \in E\}$.

Example 3.2. Similar families $I_t = I_{E,x,\hat{f}}$ with $\hat{f} = y+t$ have been studied by Évain [7], [9]. In that case traces can be computed from slices of the staircase, residuals are obtained by deleting these slices, and (in the notation of corollary 2.3) $q_i = p_i - 1$. By contrast, $\Lambda^3_{(8,7,6)}((x^2,f)^4|(y,y,x)) > 0$, because (as it is easy to check) in this example $q_2 = 5 < p_2 - 1$.

To a certain extent, Évain's computation of [7], [9] can be used in our setting, e.g., to prove propositions 3.3 and 3.4:

Proposition 3.3. *In the setting above,*

1. *R_t/I_t is flat over $k[[t]]$ and over $k[[y]]$,*
2. *$I_t : y_1 \dots y_i = I_{\tau(E,\sigma_{\mathbf{m}}(i))}$,*
3. *For every $q \geq p \geq 1$, $\mathbb{R}\mathrm{es}_p(I_t + (t^q)|x) = \mathbb{R}\mathrm{es}_p(I_t|x) = I_{\tau(E,1)} + (t^p)$, and $\mathrm{Tr}_p(I_t + (t^q)|x) = \mathrm{Tr}_p(I_t|x) = (y^{h(E)},x)/(x)$,*

Proposition 3.4. *If E is gentle, then for every couple of integers $q > p \geq 1$,*

1. $\mathrm{tr}_p(I_t + (t^q)|y) = \mathrm{tr}_p(I_t|y) = h_E(p-1)$,
2. *if $p = \ell_E(i)$ for some i then $\mathrm{Res}_p(I_t + (t^q)|y) = \mathrm{Res}_p(I_t|y) = (I_{E',x,f} + (t))/(t)$, where E' is the staircase obtained from E by deleting a slice of height $i+1 = h_E(p-1) = \mathrm{tr}_p(I_E|y)$ and moving everything to the left.*

Define for each $i = 1,\dots,m$, $\widetilde{\mathrm{tr}}^i_{\mathbf{p}}(I|\mathbf{y}) = \max_{1\leq q\leq p_i}\{\mathrm{tr}^i_{\mathbf{p}(q,i)}(I|\mathbf{y}) + q - p_i\}$, $\mathbf{y}_i = (y_i,\dots,y_m)$ and $\mathbf{p}_i = (p_i,\dots,p_m)$.

Corollary 3.5. *If $E \subset \mathbb{Z}^2_{\geq 0}$ is a gentle staircase, then $\widetilde{\mathrm{tr}}_p(I_E + (t^q)|x) = \widetilde{\mathrm{tr}}_p(I_E|x) = h(E)$ and $\widetilde{\mathrm{tr}}_p(I_E + (t^q)|y) = \widetilde{\mathrm{tr}}_p(I_E|y) = h_E(p-1)$ for all $q > p \geq 1$.*

Although $\mathrm{Res}^i_{\mathbf{p}}(I_t|\mathbf{y})$ in general is not computed by staircases, proposition 3.6 below shows that, up to a certain order, one can substitute $\mathrm{Res}^{i-j}_{\mathbf{p}_j}(I_t : (y_1 \dots y_j)|\mathbf{y}_j)$ instead. It will be useful to do so when the latter *can* be computed by 3.3 and 3.4.

Proposition 3.6. *If $p_1 > p_2 > \dots > p_m \geq 1$, then for $1 \leq j \leq i \leq m$,*

1. *$\mathbb{R}\mathrm{es}^i_{\mathbf{p}}(I_t|\mathbf{y}) + \mathfrak{m}_t^{p_j-i+j-1} = \mathbb{R}\mathrm{es}^{i-j}_{\mathbf{p}_j}(I_t : (y_1 \dots y_j)|\mathbf{y}_j) + \mathfrak{m}_t^{p_j-i+j-1}$,*
2. *if $j < i$, $\mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y}) + \mathfrak{m}^{p_j-p_i-i+j} = \mathrm{Tr}^{i-j}_{\mathbf{p}_j}(I_t : (y_1 \dots y_j)|\mathbf{y}_j) + \mathfrak{m}^{p_j-p_i-i+j}$.*

Proof. Observe that $I_{E,x,f}$ is a homogeneous ideal so, as x,y,t are homogeneous, all higher traces and residuals are homogeneous. Whenever I, J are homogeneous ideals and f is a homogeneous polynomial, for every integer e one has

$$I + \mathfrak{m}^e = J + \mathfrak{m}^e \;\Rightarrow I : f + \mathfrak{m}^{e-\deg(f)} = J : f + \mathfrak{m}^{e-\deg(f)}. \qquad (2)$$

The first claim, together with (2), implies the second. Also, using (2) $i-j$ times we see that the first claim will follow from

$$\mathbb{R}\mathrm{es}^j_{\mathbf{p}}(I_t|\mathbf{y}) + \mathfrak{m}_t^{p_j-1} = I_t : (y_1 \dots y_j) + \mathfrak{m}_t^{p_j-1},$$

which is proved by induction on j, taking into account that $p_j < p_{j-1}$. □

Corollary 3.7. *Assume that $p_1 > p_2 > \cdots > p_m \geq 1$ and for every $i > 1$ there is $j < i$ with $\Lambda^{i-j}_{\mathbf{p}_j}(I_t : (y_1 \dots y_j)|\mathbf{y}_j) = 0$ and*

$$p_j - p_i \geq i - j + \widetilde{\mathrm{tr}}^{i-j}_{\mathbf{p}_i(p_i-1)}(I_t : (y_1 \dots y_j)|\mathbf{y}_j) - 2.$$

Then $\Lambda^m_{\mathbf{p}}(I_t|\mathbf{y}) = 0$.

Proof. By proposition 2.2, we only need to show that for each $i = 1, \dots, m$ and $q = 1, \dots, p_i - 1$, $\mathrm{tr}^i_{\mathbf{p}(q)}(I_t|\mathbf{y}) = \mathrm{tr}^i_{(\mathbf{p}-1)(q)}(I_t|\mathbf{y})$. By hypothesis there is $j < i$ with $p_j - q - i + j + 1 \geq \mathrm{tr}^{i-j}_{\mathbf{p}_j(q)}(I_t : (y_1 \dots y_j)|\mathbf{y}_j)$ for all $1 \leq q \leq p_i - 1$, and therefore $\mathfrak{m}^{p_{i-1}-q-i+j+1}$ is contained in $\mathrm{Tr}^{i-j}_{\mathbf{p}_j(q)}(I_t : (y_1 \dots y_j)|\mathbf{y}_j)$. We also have

$$\begin{aligned}
\mathrm{Tr}^{i-j}_{\mathbf{p}_j(q)}(I_t : (y_1 \dots y_j)|\mathbf{y}_j) &\subset \mathrm{Tr}^{i-j}_{(\mathbf{p}_j-1)(q)}(I_t : (y_1 \dots y_j)|\mathbf{y}_j) \\
\mathrm{Tr}^{i-j}_{\mathbf{p}_j(q)}(I_t : (y_1 \dots y_j)|\mathbf{y}_i) &\subset \mathrm{Tr}^{i}_{\mathbf{p}(q)}(I_t|\mathbf{y}) \\
\mathrm{Tr}^{i-j}_{(\mathbf{p}_j-1)(q)}(I_t : (y_1 \dots y_j)|\mathbf{y}_i) &\subset \mathrm{Tr}^{i}_{(\mathbf{p}-1)(q)}(I_t|\mathbf{y}),
\end{aligned}$$

so in particular $\mathfrak{m}^{p_{i-1}-q-i+j+1}$ is contained in the four ideals involved. Using the second part of proposition 3.6, we get that $\mathrm{Tr}^i_{\mathbf{p}(q)}(I_t|\mathbf{y})$ equals

$$\mathrm{Tr}^i_{\mathbf{p}(q)}(I_t|\mathbf{y}) + \mathfrak{m}^{p_{i-1}-q-i+j+1} = \mathrm{Tr}^{i-j}_{\mathbf{p}_j(q)}(I_t : (y_1 \dots y_j)|\mathbf{y}_j) + \mathfrak{m}^{p_{i-1}-q-i+j+1}$$

which by proposition 2.2 equals

$$\mathrm{Tr}^{i-j}_{(\mathbf{p}_j-1)(q)}(I_t : (y_1 \dots y_j)|\mathbf{y}_j) + \mathfrak{m}^{p_{i-1}-q-i+j+1} = \mathrm{Tr}^{i}_{(\mathbf{p}-1)(q)}(I_t|\mathbf{y}) + \mathfrak{m}^{p_{i-1}-q-i+j+1},$$

and this is $\mathrm{Tr}^i_{(\mathbf{p}-1)(q)}(I_t|\mathbf{y})$. □

Remark 3.8. Proposition 3.6 and corollary 3.7 can be generalized to arbitrary dimension $R = k[[x_1, \dots, x_n]]$ whenever the ideal I_t is homogeneous and each y_i is homogeneous.

4 Proof of theorem 1

In this section we fix $\pi : S \to \mathbb{P}^2$ the blowup of $\mathbb{P}^2$ at a point p_1, $D = D_1$ the exceptional divisor, and $p_2 \in D$ a point. Let R be the completion of the local ring $\mathcal{O}_{S,p_2}$, and choose an isomorphism $R = \hat{\mathcal{O}}_{S,p_2} \cong k[[x,y]]$ such that $y = 0$ is a local equation for D. We shall abuse notation and denote both maximal ideals of $\mathcal{O}_{S,p_2}$ and $k[[x,y]]$ by $\mathfrak{m}$. By the second part of lemma 3.1, every ideal $I_{E,g,f} \subset k[[x,y]]$ can be defined by $f, g \in \mathcal{O}_{S,p_2}$. $I_{E,g,f} \cap \mathcal{O}_{S,p_2}$ is $\mathfrak{m}$-primary of the same colength $\#E$.

Let $C \subset S$ be a curve through p_2 and smooth at p_2, and f a local equation for C. If E is gentle, by 3.1 $I_{E,g,f}$ is independent of the choice of $f \in \mathcal{O}_{S,p_2}$ such that $(f,g) = \mathfrak{m}$; we denote $I_{E,g} = I_{E,f,g}$ and define $\mathcal{I}_{p_2,E,C}$ as the ideal sheaf with cosupport at p_2 and stalk $I_{E,g} \cap \mathcal{O}_{S,p_2}$, and $Z_{p_2,E,C} \subset S$ the scheme it defines.

Given positive integers n, e with $e \leq \sqrt{n}/2$, define $E_1(n,e) = \{(e_1,e_2) \in \mathbb{Z}^2_{\geq 0} \,|\, e\, e_1 + (n-1)e_2 \geq e(n-1)\}$, i.e., the staircase of height e with $\hat{\ell}_{E_1(n,e)}(i) = n-1$ for $0 \leq i < e$. Let $L = \pi^*(aH) - eD$, where H is the class of a line and d a positive integer. By well-known semicontinuity arguments [16], [12], theorem 1 will follow if we prove that $|L - Z_{p_2,E_1(n,e),C}|$ has maximal rank for some smooth curve C through p_2.

Remark 4.1. The fact that $|L - Z_{p_2,E_1(n,e),C}|$ has maximal rank implies a slightly stronger result: for every consistent weighted Enriques diagram $\mathbf{D}$ of n points, all of multiplicity e, general weighted clusters of type $\mathbf{D}$ on the projective plane have maximal rank in all degrees. See [4] for unexplained notions.

For the proof we specialize C so that $(C \cdot D)_{p_2}$ increases stepwise, and D becomes a fixed part of increasing multiplicity in the system. The successive limits are too complicated to give exact formulas, but they are defined by staircases with decreasing height, eventually reaching 2, in which case maximal rank has been established in [17].

4.1 Gentle staircases

Given an integer $r \geq 0$, we denote $h^r_E = \min\{i | \hat{\ell}_E(i) \leq r\}$. If the function $\hat{\ell}_E : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is non-increasing in the interval $[h^r_E, \infty)$ then we say that E is r-gentle. Observe that for $r > 0$ r-gentle implies $(r-1)$-gentle, and 0-gentle implies gentle.

Lemma 4.2. *Let E be an r-gentle staircase, and assume $(C \cdot D)_{p_2} = s \geq r$. Let $p_2, p_3, \dots, p_{r+2}$ be the first $r+1$ points on C infinitely near to p_2, and S' the surface obtained by blowing up $p_2, p_3, \dots, p_{r+1}$. Let $\tilde{C}$ be the strict transform of C on S' and $\tilde{E}$ the staircase with $\hat{\ell}_{\tilde{E}}(i) = \max\{\hat{\ell}_E(i) - r, 0\}$. Then $\mathcal{I}_{p_2,E,C} = \pi_*(\mathcal{I}_{p^{r+2},\tilde{E},\tilde{C}} \otimes \mathcal{O}_{S'}(-d_2D_2 - \cdots - d_{r+1}D_{r+1}))$, where $d_i = \max\{j | \hat{\ell}_E(j) \geq i-1\} + 1$ and D_i denotes the pullback in S' of the ith exceptional divisor.*

Remark 4.3. Observe that if $\hat{\ell}_{\min}(E) \geq r$ then $d_i = h_E(i-2)$.

Proof. We give the details for $r = 1$ only, as the general case follows iteratively. Both $\mathcal{I}_{p_2,E,C}$ and $\pi_*(\mathcal{I}_{p_3,\tilde{E},\tilde{C}} \otimes \mathcal{O}_{S'}(-h(E)D_2))$ have cosupport at p, and their stalk there is primary with respect to the maximal ideal. Therefore, it is enough to show that their extensions to the completion of $\mathcal{O}_{S,p_2}$ coincide.

Let $f = 0$, $y = 0$ be local equations of C, D respectively and fix an isomorphism $\hat{\mathcal{O}}_{S,p_2} \cong R = k[[x,y]]$ such that $(x,f) = \mathfrak{m}$. Then $\hat{\mathcal{O}}_{S,p_2} \cong k[[x,f]]$, and $\hat{\mathcal{O}}_{S',p_3} \cong R[[f/x]] = k[[x,f/x]]$. $f/x = 0$ is a local equation of $\tilde{C}$, $x = 0$ a local equation for D_2 and, if $\tilde{D}$ goes through p_3 (i.e., $(D \cdot C)_p > 1$), y/x is a local equation of $\tilde{D}$. The stalk at p of $\pi_*(\mathcal{I}_{\tilde{E},\tilde{C}} \otimes \mathcal{O}_{S_1}(-h(E)D_2))$ is $\mathcal{O}_{S,p_2} \cap (x^{h(E)} I_{\tilde{E},\tilde{C}})$, and its extension to R is

$$R \cap (x^{h(E)} I_{\tilde{E},\tilde{C}}) = k[[x,f]] \cap \left(\left(x^{h(E)+e_1} (f/x)^{e_2} \right)_{(e_1,e_2)\in\tilde{E}} \right) = $$
$$= \left(x^{h(E)+e_1-e_2} f^{e_2} \right)_{(e_1,e_2)\in\tilde{E}} \subset k[[x,f]] = R.$$

As E is 1-gentle, $(h(E) + e_1 - e_2, e_2) \in E$ if and only if $(e_1,e_2) \in \tilde{E}$. □

Corollary 4.4. *If E is r-gentle then* $\text{length}(Z_{p,E,C} \cap D) = \ell_E(h^r_E) + r h^r_E$. *If moreover $\hat{\ell}_{\min}(E) \geq r$ then* $\text{length}(Z_{p,E,C} \cap D) = \sum_{i=0}^{r-1} h_E(i)$.

Proof. Follows from the previous lemma and the projection formula. □

Remark 4.5. If $\hat{\ell}_{\min}(E) \geq r+1$, E is r-gentle and the last two results apply.

We say that a linear system Σ on S has type (L, E, r), where L is a divisor class on S and E an r-gentle staircase, if there is a curve C through p_2, smooth at p_2 and with $(C \cdot D)_{p_2} = r$, such that $\Sigma = |L - Z_{p_2,E,C}|$. If $L \cdot D \geq \ell_E(h_E^r) + rh_E^r$, then the type (L, E, r) is called *consistent*.

Given a family of curves C_t through p_2, the intersection number $(C_t \cdot D)_{p_2}$ may depend on the parameter t, i.e., one may have $(C_t \cdot D)_{p_2} = r$ and $(C_0 \cdot D)_{p_2} = r+1$, for instance. Then one obtains a family of linear systems Σ_t, $t \neq 0$ of type (L, E, r) whose limit when $t \mapsto 0$ is of different type.

Remark 4.6. Let (L, E, r) be a consistent type with $r > 1$. Every linear system of type (L, E, r) is the limit of a family of linear systems of type $(L, E, r-1)$.

Theorem 4.7. *Let (L, E, r) be a consistent type such that $\hat{\ell}_{\min}(E) \geq r + h_E^r + 1$ and $h_E^r \geq 2$. There exist an integer $\mu \geq 0$ and an $(r+1)$-gentle staircase E' such that $(L - \mu D, E', r+1)$ is consistent, for every linear system Σ of this type there is a family Σ_t of linear systems of type (L, E, r) with $\lim_{t \mapsto 0} \Sigma_t \subset \Sigma + \mu D$, and E' satisfies*

1. $\#E' + \mu(L \cdot D) + \binom{\mu+1}{2} = \#E$,
2. $\tau(E, \mu r) \subset E' \subset \tau(E, \mu(r+1))$,
3. $\ell(E') = \ell(E) - \mu(r+1)$,
4. $\hat{\ell}_{\min}(E') \geq \hat{\ell}_{\min}(E) - 1$,
5. *if* $\ell_E(h(E)-1) > \mu r + 1$ *and* $\mu \geq 1$ *then* $h(E') = h(E)$ *and* $\ell_{E'}(h(E')-1) = \ell_E(h(E)-1) - (\mu r + 1)$,

Proof. For every integer $i > 0$, let

$$s_i = \sum_{j=r(i-1)}^{ir-1} h_E(j)$$

and $tr_i = L \cdot D + i - s_i$. Clearly $s_1 \geq s_2 \geq \dots$ and $tr_1 < tr_2 < \dots$ Let $\mu = \max\{i \mid \ell_E(tr_i - 1) > ri\}$, $\mathbf{tr} = (tr_1, tr_2, \dots, tr_\mu)$. If $\mu = 0$ then either $tr_1 > h(E)$ or $tr_1 = h(E)$ and $\ell_E(h(E)-1) < r$, in which case $h_E^r = h(E) - 1$; in both cases $(L, E, r+1)$ is consistent and the claims follow from remark 4.6 setting $E' = E$.

So assume $\mu \geq 1$. The staircase E' obtained from E by deleting the leftmost μr slices, and further μ slices of heights $tr_1, tr_2, \dots, tr_\mu$, satisfies conditions 1–5 (the hypothesis on $\hat{\ell}_{\min}$ guarantees that such slices exist). Moreover, as $\hat{\ell}_{\min}(E') \geq \hat{\ell}_{\min}(E) - 1 \geq r + h_E^r > r+1$, E' is $(r+1)$-gentle and for every C through p_2 with $(C \cdot D)_{p_2} = r+1$, $\operatorname{length}(Z_{p_2,E',C} \cap D) = \sum_{i=0}^{r} h_{E'}(i)$, which by 2 is at most $\sum_{i=\mu r}^{(\mu+1)r} h_E(i)$ and by the definition of μ this is at most $\mathcal{L} \cdot D + \mu$. Therefore $(L - \mu D, E', r+1)$ is consistent.

Let now C be an arbitrary curve smooth at p_2 with $(C \cdot D)_{p_2} = r+1$, let $y = 0$, $f = 0$ be local equations for D and C respectively, and fix local coordinates (x, y) in $\mathcal{O}_{S,p_2}$. For every $t \neq 0$, $f + tx^r = 0$ is a local equation of a curve C_t with $(C_t \cdot D)_{p_2} = r$. Define $\Sigma_t = |L - Z_{p_2,E,C_t}|$. The first r points on C_t infinitely near to p_2 lie on D, so they do not depend on t; denote them $p_2, p_3, \dots, p_{r+1}$, and let $\pi : S' \to S$ be the blowup of these points. The $(r+1)$th point on C_t infinitely near to p depends on t; let it be $p_t^{r+2} \in S'$. By remark 4.5, $\mathcal{I}_{p,E,C_t} = \pi_*(\mathcal{I}_{p_t^{r+2},\tilde{E},\tilde{C}_t} \otimes \mathcal{O}_{S'}(-d_2 D_2 - \dots - d_{r+1} D_{r+1}))$, where $\tilde{E}$ is the staircase with $\hat{\ell}_{\tilde{E}}(i) = \max\{\hat{\ell}_E(i) - r, 0\}$ and $d_i = h_E(i-2)$. Setting $\tilde{\Sigma}_t = |(L - d_2 D_2 - \cdots - d_{r+1} D_{r+1}) - Z_{p_t^{r+2}, \tilde{E}, \tilde{C}_t}|$, it is clear that $\Sigma_t \overset{\pi_*}{\cong} \tilde{\Sigma}_t$, and

$$\lim_{t \mapsto 0} \Sigma_t = \pi_* \left(\lim_{t \mapsto 0} \tilde{\Sigma}_t \right). \tag{3}$$

On S', p_0^{r+2} belongs to the strict transforms $\tilde{C}$ and $\tilde{D}$, and to the exceptional divisor D_{r+1}; at p_0^{r+2}, $\tilde{D}$, D_{r+1} and $\tilde{C}$ are pairwise transverse. Thus, there exist $x_r, y_r \in \hat{\mathcal{O}}_{S', p_0^{r+2}}$ local parameters such that $y_r = 0$, $x_r = 0$, and $x_r + y_r = 0$ are local equations of $\tilde{D}$, D_{r+1} and $\tilde{C}$ respectively. Then $f_t = x_r + y_r + t = 0$, for t in a neighbourhood of 0, is an equation of C_t in a neighbourhood of p_0^{r+2}.

We now apply the results of section 3 to $I_t = I_{\tilde{E}, x_r, f_t}$, to show that

$$\lim_{t \mapsto 0} \tilde{\Sigma}_t = \mu D + (d_2' - d_2) D_2 + \ldots (d_{r+1}' - d_{r+1}) D_{r+1} + \\ + |(L - \mu D - d_2' D_2 - \cdots - d_{r+1}' D_{r+1}) - Z_{p_0^r, \tilde{E}', \tilde{f}}| \tag{4}$$

where $d_i' = h_{E'}(i)$ and $\tilde{E}'$ is the staircase obtained from $\tilde{E}$ by deleting the leftmost $m - \mu$ slices, and further μ slices of heights $tr_1, tr_2, \ldots, tr_\mu$.

Define, for $i = 1, \ldots, \mu$, $l_i = \ell_E(h_E(ri) - 1)$, $m_i = \min\{r, ri - l_{i-1}, l_i - ri\}$, $\mathbf{m} = (m_1, m_2, \ldots, m_\mu)$, $m = \sum m_i$, $n_i = \sum_{j<i}(m_j - 1)$, and $\mathbf{y} = \mathbf{y_m}$. $\mathbf{p}$ is defined as follows. $p_{n_j+j} = \ell_E(tr_j - 1) - n_j$ for $1 \le j \le \mu - 1$, $p_{n_\mu+\mu} = \ell_E(tr_\mu - 1) - n_\mu$ if $\ell_E(tr_\mu) > n_\mu$, $p_{n_\mu+\mu} = 1$ otherwise; $p_{n_j+j+1} = \ell_E(tr_j - 1) - n_j - h_E(n_j)$ for $1 \le j \le \mu - 1$ (so for instance $p_1 = \ell_E(tr_1 - 1)$ and $p_2 = \ell_E(tr_1 - 1) - h(E)$), $p_i = 1$ for $i > n_\mu + \mu$ and $p_i = p_{i-1} - 1$ for all other i. The bound on $\hat{\ell}_{\min}(E)$ guarantees that $p_1 > \cdots > p_{n_j+j} \ge p_{n_j+j+1} + h_E(n_j) \ge \cdots \ge p_m$.

Due to proposition 3.3, for all i and j with $j + n_j < i \le j + 1 + n_{j+1}$,

$$\widetilde{\mathrm{tr}}^{i-j-n_j}_{\mathbf{p}_{j+n_j}}(I_t : (y_1 \ldots y_{j+n_j}) | \mathbf{y}_{j+n_j}) = \widetilde{\mathrm{tr}}^{i-j-n_j}_{\mathbf{p}_{j+n_j}}(I_{\tau(\tilde{E}, \sigma_\mathbf{m}(j+n_j))} | \mathbf{y}_{j+n_j}) = \\ = \widetilde{\mathrm{tr}}^1_{\mathbf{p}_{i-1}}(I_{\tau(\tilde{E}, \sigma_\mathbf{m}(i-1))} | y_i). \tag{5}$$

If $i \le j + n_{j+1}$ then (5) can be evaluated using proposition 3.3, which gives

$$\widetilde{\mathrm{tr}}^1_{\mathbf{p}_{i-1}}(I_{\tau(\tilde{E}, \sigma_\mathbf{m}(i-1))} | y_i) = \mathrm{tr}_{p_i}(I_{\tau(\tilde{E}, i-j-1)} | y_i) = h(\tau(\tilde{E}, i-j-1)) = h_{\tilde{E}}(i-j-1).$$

On the other hand, if $i = j + 1 + n_{j+1}$, then using 3.4 we get

$$\widetilde{\mathrm{tr}}^1_{\mathbf{p}_{i-1}}(I_{\tau(\tilde{E}, \sigma_\mathbf{m}(i-1))} | y_i) = \mathrm{tr}_{p_i}(I_{\tau(\tilde{E}, i-j-1)} | y_i) = h_{\tau(\tilde{E}, n_{j+1})}(p_{j+1+n_{j+1}} - 1) = \\ = h_{\tilde{E}}(\ell_{\tilde{E}}(tr_{j+1} - 1) - n_{j+1} + n_{j+1} - 1) = tr_{j+1}.$$

In both cases the result is bounded above by $h_{\tilde{E}}(n_j)$ and therefore

$$\mathfrak{m}^{h_{\tilde{E}}(n_{j-1})} \subset \mathrm{Tr}^{i-j-n_j}_{\mathbf{p}_{j+n_j}}(I_t : (y_1 \ldots y_{j+n_j}) | \mathbf{y}_{j+n_j}) \subset \mathrm{Tr}^i_\mathbf{p}(I_t | \mathbf{y}).$$

But $p_{j+n_j} - p_i \ge h_{\tilde{E}}(n_j) + i - j - n_j - 1$, therefore $p_{j+n_j} - p_i + 1 - (i - j - n_j) \ge h_{\tilde{E}}(n_{j-1})$ and by 3.6, $\mathrm{tr}^i_\mathbf{p}(I_t | \mathbf{y})$ equals

$$\mathrm{tr}^{i-j-n_j}_{\mathbf{p}_{j+n_j}}(I_t : (y_1 \ldots y_{j+n_j}) | \mathbf{y}_{j+n_j}) = \begin{cases} tr_j & i = j + n_j, \\ h_{\tilde{E}}(i-j-1) & j + n_j < i \le j + n_{j+1}, \end{cases}$$

for all $j < \mu$; and $\mathrm{tr}_{\mathbf{p}}^{n_\mu+\mu}(I_t|\mathbf{y}) = tr_\mu$ if $\ell_{\tilde{E}}(tr_\mu) > n_\mu$.

Next we show inductively that $\mathrm{Res}_{\mathbf{p}}^i(I_t|\mathbf{y}) = (I_{E_i})_0$, where $E_0 = \tilde{E}$, E_{j+n_j} is obtained from E_{j+n_j-1} by deleting a slice of height tr_j and $E_i = \tau(E_{i-1}, 1)$ for $j+n_j < i \leq j+n_{j+1}$. Remark that $E_m = \tilde{E}'$. For $i = 0$ there is nothing to prove. For $i > 0$, proposition 3.6, part 1 implies that $\mathrm{Res}_{\mathbf{p}}^{i-1}(I_t|\mathbf{y}) + \mathfrak{m}^{p_i-1} = (I_t : (y_1 \dots y_{i-1}))_0 + \mathfrak{m}^{p_i-1} = (I_{E_{i-1}})_0 + \mathfrak{m}^{p_i-1}$, and $\mathrm{Res}_{\mathbf{p}}^i(I_t|\mathbf{y}) \subset \mathrm{Res}_{\mathbf{p}}^{i-1}(I_t|\mathbf{y}) : y_i$. Therefore $\mathrm{Res}_{\mathbf{p}}^i(I_t|\mathbf{y}) \subset (I_{E_{i-1}} : y_i)_0 + (\mathfrak{m}^{p_i-1} \cap (I_{\tau(E_{i-1},1)})_0) = (I_{E_i})_0$. On the other hand, from the first two claims and corollary 3.7 we get that $\mathrm{res}_{\mathbf{p}}^i(I_t|\mathbf{y}) = \#E_{i-1} - \mathrm{tr}_{\mathbf{p}}^i(I_t|\mathbf{y}) = \#E_i = \dim_k(R/(I_{E'})_0)$, and we are done.

Now, for $i > \mu + n_\mu$ and for $i = \mu + n_\mu$ if $\ell_E(tr_\mu) \leq n_\mu$, $p_i = 1$, hence by 3.3

$$\mathrm{Res}_{\mathbf{p}}^i(I_E|\mathbf{y}) = \mathbb{R}\mathrm{es}_{\mathbf{p}}^i(I_E|\mathbf{y})k[[t]] + (t) = I_{\tau(E_{\mu+n_\mu}, i-\mu+n_\mu)} + (t)$$

so $\mathrm{Res}_{\mathbf{p}}^i(I_E|\mathbf{y}) = (I_{E_i})_0$ in these cases as well and $\mathrm{tr}_{\mathbf{p}}^i(I_E|\mathbf{y}_m) = h_E(i - \mu - 1)$.

The staircase $\tilde{E}'$ satisfies $\hat{\ell}_{\tilde{E}'}(i) = \max\{\hat{\ell}_{E'}(i) - r, 0\}$. So by 4.2, $\mathcal{I}_{p,E',C} = \pi_*(\mathcal{I}_{p_0^{r+2},\tilde{E}',\tilde{C}} \otimes \mathcal{O}_{S'}(-d_2'D_2 - \cdots - d_{r+1}'D_{r+1}))$. In particular

$$|(L - \mu D) - Z_{p,E',C}| = |(L - \mu D - d_2'D_2 - \cdots - d_{r+1}'D_{r+1}) - Z_{p_0^{r+2},\tilde{E}',\tilde{C}}|. \tag{6}$$

Let $V \subset \hat{\mathcal{O}}_{S',p_0^{r+2}}$ be the image of the natural morphism $H^0(\mathcal{O}_{S'}(L - d_2D_2 - \cdots - d_{r+1}D_{r+1})) \to \hat{\mathcal{O}}_{S',p_0^{r+2}}$. Lemma 4.8 below shows that

$$\frac{\mathrm{Res}(V|y_1 \cdots y_{i-1})}{\mathrm{Res}(V|y_1 \cdots y_{i-1}) \cap (y_i)} \longrightarrow \frac{R/(y_i)}{\mathrm{Tr}_{\mathbf{p}}^i(I_t|\mathbf{y})} \tag{7}$$

is injective for $i = 1, \dots, \mu$, so theorem 1.2 applies as well, and therefore (4) holds. Now it suffices to put (3), (6) and (4) together to see that $\lim_{t\mapsto 0} \Sigma_t \subset \Sigma + \mu D$. □

Lemma 4.8. *For each i there is a divisor class F_i on S' such that*

1. $\mathrm{Res}(V|y_1 \cdots y_{i-1}) \subset \hat{\mathcal{O}}_{S_r,p_0^{r+2}}$ *is the image of the natural morphism*

$$H^0(\mathcal{O}_{S'}(F_i)) \xrightarrow{\rho_i} \hat{\mathcal{O}}_{S',p_0^{r+2}},$$

2. $F_i \cdot R_i = \mathrm{tr}_{\mathbf{p}}^i(I_t|\mathbf{y}) - 1$, *where E_i is the irreducible divisor defined locally by $y_i = 0$, that is, $R_i = \tilde{D}$ if $i = n_j + j$ for some j, and $R_i = D_{r+1}$ otherwise.*

3. $F_m = L - \mu D - d_2'D_2 - \cdots - d_{r+1}'D_{r+1}$.

Proof. We define F_i by recurrence. For $i = 1$, $F_1 = L - d_2D_2 - \cdots - d_{r+1}D_{r+1}$. Now assuming F_{i-1} satisfies the claims, $\mathrm{Res}(V|y_1 \cdots y_{i-1})$ is the image of the natural morphism

$$H^0(\mathcal{O}_{S'}(F_{i-1} - R_{i-1})) \longrightarrow \hat{\mathcal{O}}_{S',p_0^{r+2}}.$$

If $R_{i-1} = D_{r+1}$ and $F_{i-1} \cdot D_{r+1} = F_{i-1} \cdot D_j$ for $j \in [j_0, r]$, then the divisors D_j are in the fixed part of $|F_{i-1} - R_{i-1}|$. We define F_i to be the class obtained from $F_{i-1} - R_{i-1}$ by unloading, i.e., $F_i = F_{i-1} - \sum_{j=j_0}^{r+1} D_j$. It is then straightforward to check the claims. □

4.2 Termination of the specializations

Without loss of generality (see [12, 17]) we assume $e > 2$. For the staircase $E_1(n,e) := \{(e_1,e_2) \in \mathbb{Z}^2_{\geq 0} | ee_1 + (n-1)e_2 \geq e(n-1)\}$ defined above, $(L, E_1(n,e), 1)$ is a consistent type satisfying the requirement of theorem 4.7; let μ_1 and $E_2(n,e)$ be the integer and staircase given by 4.7. $(L - \mu_1 D, E_2(n,e), 2)$ is again a consistent type satisfying the requirement of theorem 4.7; let μ_2 and $E_3(n,e)$ be the corresponding integer and staircase. As long as the hypotheses of the theorem are satisfied, we keep using it to define integers $\mu_3, \mu_4, \dots$ and staircases $E_4(n,e), E_5(n,e), \dots$ For simplicity, we denote these staircases $E_1, E_2, \dots$ and also $L_2 = L - \mu_1 D$, $L_3 = L_2 - \mu_2 D$, ... Let $r_{\max}(n,e)$ be the last r such that E_r is defined, i.e., $(L_{r_{\max}(n,e)}, E_{r_{\max}(n,e)}(n,e), r_{\max}(n,e))$ is a consistent type and either $h^{r_{\max}(n,e)}_{E_{r_{\max}(n,e)}(n,e)} \leq 1$ or $\hat{\ell}_{\min}(E_{r_{\max}(n,e)}) \leq r_{\max}(n,e) + h^{r_{\max}(n,e)}_{E_{r_{\max}(n,e)}(n,e)}$.

Lemma 4.9. *Let r be a positive integer, and denote $M = \sum \mu_i$, with the summation running over all $i \leq \min\{r-1, r_{\max}(n,e)\}$.*

1. *if $r \leq r_{\max}(n,e)$, then*
   - *(a) $\hat{\ell}_{\min}(E_r) \geq n - r$,*
   - *(b) $(r-1)(r+2)e \geq 2(n-1)(e - h(E_r))$, and*
   - *(c) $\#(E_1 \setminus E_r) = eM + \binom{M+1}{2}$,*
2. *If $\binom{r}{2}e + r < n-1$, then $r \leq r_{\max}(n,e)$, and $\mu_1 = \mu_2 = \cdots = \mu_{r-1} = e$,*
3. *if $\binom{r}{2}e + r \geq n-1$ and $r \leq r_{\max}(n,e)$, then $\#(E_1 \setminus E_r) \geq e(n-1)/2$.*

Proof. Due to 4.7, claim 4, for every $1 < r \leq r_{\max}(n,e)$ one has $\hat{\ell}_{\min}(E_r) \geq \hat{\ell}_{\min}(E_{r-1}) - 1$, and so $\hat{\ell}_{\min}(E_r) \geq n - r$. Now because of 4.7, claim 2, $E_r \supset \tau(E_1, (e - h(E_r))(n-1))$. We also have $E_r \subset \tau(E_1, \sum_{i=1}^{r-1} \mu_i(i+1))$ and $\mu_i \leq e$, hence

$$\frac{(r-1)(r+2)}{2} e \geq (n-1)(e - h(E_r))$$

and claim 1c follows because of 4.7, claim 1. Now since claim 1a holds, $r + 1 \leq r_{\max}(n,e)$ whenever $2r \leq n - e - 1$, and in particular $r \leq r_{\max}(n,e)$ whenever $\binom{r}{2}e + r < n-1$; in such a case moreover, due to 4.7, claim 5, $\mu_1 = \mu_2 = \cdots = \mu_{r-1} = e$.

Finally for the third claim, let r_0 be the maximal integer with $\binom{r_0}{2}e + r_0 < n-1$. The hypothesis says $r \geq r_0$, and then $M = \mu_1 + \mu_2 + \cdots + \mu_{r-1} \geq e(r_0 - 1)$. Then by claim 1c, the definition of r_0, and the inequality $M \geq (r_0 - 1)e$,

$$\#(E_1 \setminus E_r) = eM + \binom{M+1}{2} \geq e(n-1)/2.$$ □

Lemma 4.10. *For every $r \leq \min\{r_{\max}(n,e), (n-1)/2\}$ such that $\binom{r}{2}e + r \geq n-1$, if $h(E_r) \geq 3$ then $4r^2 + 2r + e(2en - 3e - n) \leq \sqrt{3}e(n-1)(2e-1)$. Moreover, there exists an integer $r \leq r_{\max}(n,e)$ such that $h(E_r) \leq 2$.*

Proof. As $\binom{r}{2}e + r \geq n-1$, it follows from lemma 4.9, claim 3 that $\#(E_1 \setminus E_r) \geq e(n-1)/2$. On the other hand, we also have

$$\#(E_1 \setminus E_r) = \sum_i i \left(\hat{\ell}_{E_1}(i-1) - \hat{\ell}_{E_r}(i-1) \right),$$

$$\ell(E_1) - \ell(E_r) = \sum_i \left(\hat{\ell}_{E_1}(i-1) - \hat{\ell}_{E_r}(i-1) \right),$$

and by 4.7, claim 3, $\Delta_\ell := \ell(E_1)-\ell(E_r) = 2\mu_1+3\mu_2+\cdots+r\mu_{r-1}$. Write $\Delta_\ell = \kappa(n-1)+\rho$, with $0 \le \rho < n-1$. As $\hat{\ell}_{E_1}(i-1) - \hat{\ell}_{E_r}(i-1) \le n-1$ for all i,

$$\#(E_1 \setminus E_r) \le \sum_{i=e-\kappa+1}^{e} i(n-1) + (e-\kappa)\rho \le e\Delta_\ell - \frac{\Delta_\ell}{2}\left(\frac{\Delta_\ell}{n-1} - 1\right). \tag{8}$$

In particular $e\Delta_\ell \ge \#(E_1 \setminus E_r) \ge e(n-1)/2$. As $(x/2)(x/(n-1)-1)$ is an increasing function of x for $x \ge (n-1)/2$,

$$\frac{\Delta_\ell}{2}\left(\frac{\Delta_\ell}{n-1} - 1\right) \ge \frac{\#(E_1 \setminus E_r)/e}{2}\left(\frac{\#(E_1 \setminus E_r)/e}{n-1} - 1\right)$$

which combined with (8) gives, denoting as before $M = \mu_1 + \mu_2 + \cdots + \mu_{r_1-1}$,

$$\Delta_\ell \ge \left(M + \frac{\binom{M+1}{2}}{e}\right)\left(1 - \frac{1}{2e} + \frac{eM + \binom{M+1}{2}}{2e^2(n-1)}\right). \tag{9}$$

By claim 2 of theorem 4.7 applied recursively, $E_r \supset \tau(E_1, \Delta_\ell - M)$. Therefore

$$h_{E_r} \le h_{\tau(E_1,\Delta_\ell - M)} = h_{E_1}(\Delta_\ell - M) \le e - \frac{\Delta_\ell - M - (n-2)}{n-1},$$

which solving for Δ_ℓ and combining with (9), gives the bound

$$h_{E_r} \le e - \frac{eM + \binom{M+1}{2}}{e(n-1)}\left(1 - \frac{1}{2e} + \frac{eM + \binom{M+1}{2}}{2e^2(n-1)}\right) + \frac{M+n-2}{n-1}, \tag{10}$$

where the right hand side is a decreasing function of M. Since $(L - MD, E_r, r)$ is consistent and $\ell_{E_r}(h(E_r)-1) \ge r$, $M > r(h(E_r)-1) - e$. Plugging into (10) we deduce that $\left(r^2(h(E_r)-1)^2 + r(h(E_r)-1) + e(2en-3e-n)\right)^2$ is not larger than

$$e^2(n-1)^2\left(12e^2 + 4e + 1 - 8e\frac{e+r+1+(n-1-r)h(E_r)}{n-1}\right).$$

Now, if for some $r \le (n-1)/2$, $h(E_r) \ge 3$,then

$$4r^2 + 2r + e(2en-3e-n) \le \sqrt{3}e(n-1)(2e-1). \tag{11}$$

But from the proof of lemma 4.9, $\hat{\ell}_{\min}(E_r) \ge n-r$ implies $r_0 = \lfloor (n-e+1)/2 \rfloor \le r_{\max}(n,e)$. We claim that $h(E_{r_0}) \le 2$. Indeed, suppose $h(E_{r_0}) \ge 3$. Since $\binom{r_0}{2}e + r_0 \ge n-1$ and $2r_0 \le n-1$, r_0 satisfies (11). Then using $r_0 \ge (n-e)/2$ and $n \ge 4e^2$, we end up with the absurdity $e\,(2\,e-1)\left(4\,(3-\sqrt{3})\,e^2 + 1 + \sqrt{3}\right) \le 0$. □

The following result is a slight generalization of Theorem 1.1 in [17], whose proof, entirely analogous, follows by iteration from Theorem 4.3 in the same reference.

Proposition 4.11. *Let E be a staircase with $h(E) = 2$. Let c be a positive integer such that*

$$\hat{\ell}_E(0) + \frac{m}{2} > 1 + 3\sqrt{\ell_E(1) + \left(\frac{m}{2}\right)^2}.$$

Then for every divisor class L with $L \cdot D = m$ and every integer r such that the type (L, E, r) is consistent, general linear systems of type (L, E, r) have maximal rank.

We are now ready to prove our main theorem.

Proof. As remarked in the beginning of the section, we only need to prove that general systems on S of type $(L, E_1(n,e), 1)$ have maximal rank. By theorem 4.7, every system of type $(L_2, E_2(n,e), 2)$ contains a limit of linear systems of type $(L, E_1(n,e), 1)$ and their expected dimensions agree by claim 1 of theorem 4.7 and [5, 2.14], so it will be enough to show that general systems of type $(L_2, E_2(n,e), 2)$ have maximal rank. Iterating the process, it is enough to prove that general systems of type $(L_r, E_r(n,e), r)$ have maximal rank, for some $r \leq r_{\max}(n,e)$.

Let r be the minimal integer such that $h(E_r) \leq 2$. Such an r exists by lemma 4.10. Applying 4.11, it will be enough to show that

$$\hat{\ell}_{E_r}(0) + \frac{L_r \cdot D}{2} > 1 + 3\sqrt{\ell_{E_r}(1) + \left(\frac{L_r \cdot D}{2}\right)^2}. \tag{12}$$

Now, due to theorem 4.7, claim 1, $\ell_{E_r}(0) + \ell_{E_r}(1) + \binom{L_r \cdot D+1}{2} = n\binom{e+1}{2}$, so (12) is equivalent to

$$\hat{\ell}_{E_r}(0) + \frac{L_r \cdot D}{2} > 1 + 3\sqrt{\frac{n}{2}\binom{e+1}{2} - \frac{\hat{\ell}_{E_r}(0)}{2} - \frac{L_r \cdot D}{4}}.$$

Since $\ell_{E_r}(0) + \ell_{E_r}(1) \leq 3(n-1)$, $\binom{L_r \cdot D+1}{2} \geq n\binom{e+1}{2} - 3(n-1)$, which implies $L_r \cdot D \geq e\sqrt{n} - 1/2$ because $e > 2$, so it will be enough to prove

$$\hat{\ell}_{E_r}(0) + \frac{2e\sqrt{n}-1}{4} > 1 + 3\sqrt{\frac{n}{2}\binom{e+1}{2} - \frac{\hat{\ell}_{E_r}(0)}{2} - \frac{2e\sqrt{n}-1}{8}}. \tag{13}$$

But by claim 4 of theorem 4.7, $\hat{\ell}_{E_r}(0) \geq n - r$ and the minimality of r together with lemma 4.10 give

$$4(r-1)^2 + 2(r-1) + e(2en - 3e - n) \leq \sqrt{3}e(n-1)(2e-1). \tag{14}$$

It is now a simple calculus exercise to check that if e, n, r and $\hat{\ell}_{E_r}(0)$ are integers satisfying $e > 2$, $n \geq 4e^2$, $\hat{\ell}_{E_r}(0) \geq n - r$ and (14), then (13) holds. □

5 Proof of theorem 3

In this section, $E = E(a,b,c)$ will denote a gentle staircase of height at most three with stair lengths $\ell_E(2) = a$, $\hat{\ell}_E(1) = b$, $\hat{\ell}_E(0) = c$.

Singularity schemes of type A, D and E are defined by monomial ideals, whose corresponding staircase height at most three. Type A is characterized by $a = 0$, and satisfies $c \in \{b-1, b\}$. Type D is characterized by $a = 1$, and also satisfies $c \in \{b-1, b\}$. For type E, one has $b, c \in \{a-1, a\}$. A suitable "head-on collision" [12] can thus be used to specialize an arbitrary union of general schemes of types A, D, E to a monomial ideal whose staircase has height at most three and its stair lengths, being the sum of the stair lengths of the collided components, satisfy $2c \geq \max\{a, b\}$ and $2b \geq \max\{a, c\}$. By semicontinuity, theorem 3 will follow from a maximal rank statement for such monomial schemes.

Theorem 5.1. *Given stair lengths a, b, c, let s, Δ, m be the positive integers satisfying $a = s(3s-1)/2 + \Delta$, $0 \leq \Delta \leq 3s$, $m = 3s + \lceil \Delta/s \rceil$ (where $\lceil \cdot \rceil$ denotes the roundup to an integer). Assume that*

1. $2c \geq \max\{a, b\}$,
2. $2b \geq \max\{a, c\}$,
3. $2b \geq 5r + 8$ *and* $r \geq 5$,
4. $c \geq s + 3 - \frac{m}{2} + \frac{3\sqrt{4b-4s+m^2}}{2} \geq 0$.

Then a general monomial subscheme in $\mathbb{P}^2$ with staircase $E(a, b, c)$ has maximal rank.

Proof. Let $E_1 := \{(e_1, e_2) \in \mathbb{Z}^2_{\geq 0} \mid (e_1 + 3 - e_2, e_2) \in E\}$, and let $\pi : S \to \mathbb{P}^2$ be the blowup of a point, with exceptional divisor D. Let $L = \pi^*(aH) - 3D$, where H is the class of a line in the plane and d a positive integer. We have to show that general systems of type $(L, E_1, 1)$ have maximal rank.

$(L, E_1, 1)$ is a consistent type satisfying the requirement of theorem 4.7; as in the proof of theorem 1, apply 4.7 to iteratively define integers $\mu_1, \mu_2, \dots$ and staircases $E_2, E_3, \dots$ For simplicity, we denote $L_2 = L - \mu_1 D$, $L_3 = L_2 - \mu_2 D$, ...

Write $\Delta = ks - \delta$, where $0 \leq \delta < s$ (so $m = 3s + k$) and define $r = \lfloor (2s + k + \delta + 1)/2 \rfloor$. It is not hard to check that E_r is defined and has height 2, with $b' := \ell_{E_r}(1)$, $c' := \hat{\ell}_{E_r}(0)$ given by the following table, and $L_r = \pi^*(aH) - mD$.

	$\Delta \equiv k(s+1) \pmod 2$	$\Delta \not\equiv k(s+1) \pmod 2$
$k < 3$	$b' = b - \lfloor \frac{2s+3\delta+k+1}{2} \rfloor$ $c' = c - s - k + 1$	$b' = b - \lfloor \frac{2s+3\delta+k-1}{2} \rfloor$ $c' = c - s - k$
$k = 3$	$b' = b - \lfloor \frac{2s+3\delta+6}{2} \rfloor$ $c' = c - s - 1$	$b' = b - \lfloor \frac{2s+3\delta+4}{2} \rfloor$ $c' = c - s - 2$

The last hypothesis in the statement guarantees then that proposition 4.11 can be applied in each case, and so general linear systems of type (L_r, E_r, r) have maximal rank. By semicontinuity, general systems of type $(L, E_1, 1)$ have maximal rank. □

Proof of theorem 3. Observe first that if $a \geq 83$ (and therefore $r \geq 7$) then the last two hypotheses in theorem 5.1 are unnecessary, because they are implied by the first two. From this it follows that, if the length $3a + 2b + c$ is bigger than 455 (which implies that either $a \geq 83$ or b, c are large enough to automatically satisfy the last two hypotheses) then a general monomial subscheme in $\mathbb{P}^2$ satisfying the first two hypotheses in theorem 5.1 has maximal rank. So it only remains to see that for $\ell = 3a + 2b + c \leq 455$, a general monomial subscheme in $\mathbb{P}^2$ satisfying the first two hypotheses in theorem 5.1 impose ℓ independent conditions in degree $d = 29$. By suitably increasing c (and possibly b) it is always possible to create a larger monomial subscheme that contains the given zeroscheme and has length $\ell' = 3a' + 2b' + c' = 456$; this imposes 456 independent independent conditions in degree 29, so the original scheme imposes independent conditions as well. □

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