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Research Article

Some Properties of Distances and Best Proximity Points of Cyclic Proximal Contractions in Metric Spaces

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This paper presents some results concerning the properties of distances and existence and uniqueness of best proximity points of p-cyclic proximal, weak proximal contractions, and some of their generalizations for the non-self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ $(p \geq 2)$, where A_i and B_i, $\forall i \in \overline{p} = \{1, 2, \ldots, p\}$, are nonempty subsets of X which satisfy $T(A_i) \subseteq B_i, \forall i \in \overline{p}$, such that (X, d) is a metric space. The boundedness and the convergence of the sequences of distances in the domains and in their respective image sets of the cyclic proximal and weak cyclic proximal non-self-mapping, and of some of their generalizations are investigated. The existence and uniqueness of the best proximity points and the properties of convergence of the iterates to such points are also addressed.

1. Introduction

The characterization and study of existence and uniqueness of best proximity points is an important tool in fixed point theory concerning cyclic nonexpansive mappings including the problems of (strict) contractions, asymptotic contractions, contractive, weak-contractive mappings, and cyclic mappings and also in related problems of proximal contractions, weak proximal contractions, and approximation results and methods [1–15]. The application of the theory of fixed points in stability issues of dynamic systems, [16–21] has been proved to be a very useful tool. See, for instance, [22–26] and references therein. Some best approximation problems in semiconvex and locally convex structures and Hyers-Ulam type stability in multivalued functions and in additive-quadratic functional equations are investigated in [27–30] and some of the references therein. Recent trends concerning best proximity points and related problems are dealt with in [31–35] and some references therein. In particular, the problem of best proximity points of two mappings in a cyclic disposal is investigated in [31] under a nonlinear contractive condition. In [32], several results are obtained for proximal and weak proximal contractions of several types as well as for generalized proximal nonexpansive mappings. A modified Suzuki $\alpha - \psi$ proximal contraction is proposed and discussed in [33] and "ad hoc" best proximity and fixed point results are obtained. Generalizations of proximal contractions of first and second kinds are given in [34, 35] for non-self-mappings and related optimal approximate solution theorems are obtained.

This paper is devoted to formulating and proving some results being concerned with the boundedness and convergence properties of distances and the convergence of the built iterated sequences to unique existing best proximity points of p-cyclic proximal and weak proximal contractions of the form $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ $(p \geq 2)$ where A_i and B_i, for all $i \in \overline{p} = \{1, 2, \ldots, p\}$, are nonempty subsets of X which satisfy $T(A_i) \subseteq B_i$, for all $i \in \overline{p}$, with (X, d) being a metric space. In the most general case, all the A_i and B_i pairs of subsets, for all $i \in \overline{p}$, are assumed to be pairwise disjoint.

The results are also extended to a class of generalized p-cyclic proximal and weak proximal contractions in the sense that the contractiveness constraints are referred to finite sets of consecutive iterations rather than to each iteration. The boundedness and convergence of the sequences of distances in the domains and image sets of the cyclic proximal and weak cyclic proximal non-self-mappings are investigated. The existence and uniqueness of the best proximity points and their allocation as limit points, or limit cycles of best proximity points, are also addressed. These last properties are achieved if the metric space is complete under approximative compactness' assumptions of the image subsets of the cyclic mapping with respect to the domain subsets.

2. p-Cyclic Proximal Contractions, Extensions, Boundedness, and Convergence of Distances

Consider the metric space (X, d) and subsets A_i and B_i of X for $i \in \overline{p}$, where $\overline{p} = \{1, 2, \ldots, p\}$ with $p \geq 2$. Consider also a non-self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$, satisfying $T(A_i) \subseteq B_i$, for all $i \in \overline{p}$. Assume that $D_i = d(A_{i+1}, B_i)$, $D_A = d(A_i, A_{i+1})$, and $D_B = d(B_i, B_{i+1})$, for all $i \in \overline{p}$ by assuming also that $A_{np+i} = A_i$ and $B_{np+i} = B_i$, for all $i \in \overline{p}$, for all $n \in \mathbf{Z}_{0+}$. If the pair $(a_i, a_{i+1}) \in A_i \times A_{i+1}$ satisfies $d(a_{i+1}, Ta_i) = D_i$ for any $i \in \overline{p}$, then $a_{i+1} \in A_{i+1}$ and $Ta_i \in B_i$ are best proximity points in A_{i+1} and B_i with respect to $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$.

In the following, the fact that the best proximity points are best proximity points with respect to the mapping is not mentioned explicitly.

$A_{0i} \subseteq A_i$ is the set of best proximity points of A_i and $B_{0i} \subseteq B_i$ is the set of best proximity points of B_i. Through the paper, it is assumed that $A_{0i} \neq \varnothing$ and $B_{0i} \neq \varnothing$, for all $i \in \overline{p}$. An important remark is that the above statement can be considered for the particular case that $B_i \equiv A_{i+1}$ which is well known in the context of p-cyclic self-mappings $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ with $T(A_i) \subseteq B_i$ for all $i \in \overline{p}$. However, the proposed statement is more general in the sense of the following illustrative example.

Example 1. Consider a metric space (X, d) and $T : A_1 \cup A_2 \to B_1 \cup B_2$ such that $A_i, B_i \subset X$ are nonempty with $T(A_{0i}) \subseteq B_{0i}$, $D_i = d(A_{i+1}, B_i) = d(A_{0,i+1}, B_{0i})$, $D_A = d(A_i, A_{i+1}) = d(A_{0i}, A_{0,i+1})$, and $D_B = d(B_i, B_{i+1}) = d(B_{0i}, B_{0,i+1})$ for $i = 1, 2$. Assume also that $A_{01} = \{x_1, x_3\} \subset A_1$, $B_{01} = \{Tx_1, Tx_3\} \subset B_1$, $A_{02} = \{x_2\} \subset A_2$, and $B_{02} = \{Tx_2\} \subset B_2$. Then, we can formulate the following simple 2-cyclic proximal-type problem. Fix $x_1 \in A_{01}$ as a best proximity point of A_1 and then compute $x_2 \in A_{02}$ and $x_3 \in A_{01}$, best proximity points of A_2 and A_1, such that

$d(x_2, Tx_1) = d(A_2, B_1) = d(A_{02}, B_{01}) = D_1$ (2-cyclic proximal constraint, first step);

$d(x_3, Tx_2) = d(A_1, B_2) = d(A_{01}, B_{02}) = D_2$ (2-cyclic proximal constraint, second step);

$d(A_1, A_2) = d(A_{01}, A_{02}) = d(x_1, x_2) = d(x_2, x_3) = D_A$ (2-cyclic best proximity constraints);

$d(Tx_2, Tx_1) = d(Tx_3, Tx_2) = D_B = d(B_1, B_2) = d(B_{01}, B_{02})$ (2-cyclic associate best proximity constraints for the images).

Note that there are four potentially distinct constraints related to D_1, D_2, D_A, and D_B which can be distinct so that the problem is more general than the simple use of $D = d(A_i, A_{i+1})$ for $i = 1, 2$ for the 2-cyclic self-mapping $T : A_1 \cup A_2 \to A_1 \cup A_2$. A variant proximal-type problem arises if $A_1 = B_2$ and $A_2 = B_1$ and the best proximity points are taken as follows:

$x_1 \in A_{01}, Tx_1 \in A_{02}, x_2 \in A_{01}, Tx_2 \in A_{02}, x_3 \in A_{01}$, and then $d(x_2, Tx_1) = d(x_3, Tx_2) = D_A = d(A_1, A_2)$.

The following definitions will be then used through the paper.

Definition 2. $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is said to be a p-cyclic proximal contraction with respect to its domain (CPD$_p$) if there are real constants $\alpha_i \in [0, 1)$, for all $i \in \overline{p}$, such that any two sequences $\{x_{np+i}\} \subseteq A_{i+j}$ and $\{\overline{x}_{np+i}\} \subseteq A_{i+j}$, for all $i \in \overline{p}$, satisfy the constraints

$$d\left(x_{np+i+1}, \overline{x}_{np+i}\right) \leq \alpha_i d\left(x_{np+i}, \overline{x}_{np+i-1}\right) + (1 - \alpha_i) D_A, \quad \forall i \in \overline{p}, \quad \forall n \in \mathbf{Z}_{0+}, \tag{1}$$

$$d\left(x_{np+i+1}, \overline{x}_{np+i+1}\right) \leq \alpha_i d\left(x_{np+i}, \overline{x}_{np+i}\right), \quad \forall i \in \overline{p}, \quad \forall n \in \mathbf{Z}_{0+}, \tag{2}$$

provided that $x_0, \overline{x}_0 \in A_j$, for any given $j \in \overline{p}$ with $A_{i+j} = A_{i+j-p}$ if $i > p - j$ and that $d(x_{np+i+1}, Tx_{np+i}) = d(\overline{x}_{np+i+1}, T\overline{x}_{np+i}) = d(A_{i+1}, B_i) = D_i$, for all $i \in \overline{p}$.

Definition 3. $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is said to be a weak p-cyclic proximal contraction with respect to its domain (WCPD$_p$) if there are p real constants $\alpha_i \geq 0$, for all $i \in \overline{p}$, subject to $\alpha = \prod_{i=1}^{p} [\alpha_i] \in [0, 1)$, such that any two sequences $\{x_{np+i}\} \subseteq A_{i+j}$ and $\{\overline{x}_{np+i}\} \subseteq A_{i+j}$, for all $i \in \overline{p}$, satisfy the constraints (1) and (2) provided that $x_0, \overline{x}_0 \in A_j$ for any given $j \in \overline{p}$ and that $d(x_{np+i+1}, Tx_{np+i}) = d(\overline{x}_{np+i+1}, T\overline{x}_{np+i}) = D_i$, for all $i \in \overline{p}$.

Definition 4. $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is said to be a generalized p-cyclic proximal contraction with respect to its domain (GCPD$_p$) if there are p bounded real functions $\alpha_i : A_i \to \mathbf{R}_{0+}$, for all $i \in \overline{p}$, such that any sequences $\{x_{np+i}\} \subseteq A_{i+j}$ and $\{\overline{x}_{np+i}\} \subseteq A_{i+j}$, for all $i \in \overline{p}$, satisfy the constraints (1) and (2) with the replacements $\alpha_i \to \sup_{x\in A_i} \alpha_i(x)$, for all $i \in \overline{p}$, provided that $x_0, \overline{x}_0 \in A_j$ for any given $j \in \overline{p}$ and that $d(x_{np+i+1}, Tx_{np+i}) = d(\overline{x}_{np+i+1}, T\overline{x}_{np+i}) = D_i$, for all $i \in \overline{p}$.

Definition 5. $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is said to be a generalized weak p-cyclic proximal contraction with respect to its domain (GWCPD$_p$) if there are p bounded real functions $\alpha_i : A_i \to \mathbf{R}_{0+}$, for all $i \in \overline{p}$, and a strictly increasing sequence of integers $\{n_k\}$, subject to $n_0 \leq \overline{N}_0 <$

$+\infty$, $\limsup_{k\to\infty}(n_{k+1}-n_k) \le \overline{N} < +\infty$, and $\overline{\alpha} = \sup_{k\in\mathbf{Z}_{0+}}\alpha(n_k,n_{k+1}) \in [0,1)$, where

$$\alpha(n_k,n_{k+1}) = \prod_{j=n_k}^{n_{k+1}-1}\prod_{i=1}^{p}\left[\sup_{x_{j+i}\in A_i}\alpha_{jp+i}(x_{jp+i})\right], \quad \forall k\in\mathbf{Z}_{0+}, \tag{3}$$

such that any two sequences $\{x_{np+i}\} \subseteq A_{i+j}$ and $\{\overline{x}_{np+i}\} \subseteq A_{i+j}$, for all $i \in \overline{p}$, provided that $x_0, \overline{x}_0 \in A_j$ for any given $j \in \overline{p}$, satisfy the constraints

$$d\left(x_{n_{k+1}p}, \overline{x}_{n_{k+1}p-1}\right) \le \alpha(n_k,n_{k+1})\, d\left(x_{n_kp+1}, \overline{x}_{n_kp}\right) + (1-\alpha(n_k,n_{k+1}))D_A, \quad \forall k\in\mathbf{Z}_{0+}, \tag{4}$$

$$d\left(x_{n_{k+1}p+i}, \overline{x}_{n_{k+1}p+i-1}\right) \le \left(\prod_{j=1}^{i-1}\left[\sup_{x\in A_i}\alpha_i(x)\right]\right) d\left(x_{n_{k+1}p}, \overline{x}_{n_{k+1}p-1}\right) + \sum_{j=1}^{i-1}\left(\prod_{k=j+1}^{i-1}\left[\sup_{x\in A_k}\alpha_k(x)\right]\right)\left(1-\sup_{x\in A_j}\alpha_j(x)\right)D_A, \quad \forall k\in\mathbf{Z}_{0+}, \tag{5}$$

$$d\left(x_{n_{k+1}p+i}, \overline{x}_{n_{k+1}p+i}\right) \le \left(\prod_{j=1}^{i-1}\left[\sup_{x\in A_i}\alpha_i(x)\right]\right)\alpha(n_k,n_{k+1})\, d\left(x_{n_kp}, \overline{x}_{n_kp}\right), \quad \forall k\in\mathbf{Z}_{0+}, \tag{6}$$

and the constraints (1) and (2) provided that $x_0, \overline{x}_0 \in A_j$ for any given $j \in \overline{p}$ and that $d(x_{np+i+1}, Tx_{np+i}) = d(\overline{x}_{np+i+1}, T\overline{x}_{np+i}) = D_i$, for all $i \in \overline{p}$.

The following assertions are obvious without proof from Definitions 2–5.

Assertions 1. If $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is CPD_p, then it is WCPD_p.

If $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is WCPD_p, then it is GCPD_p.

If $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is GCPD_p, then it is GWCPD_p.

Note that the converse implications of those in Assertions 1 are not true in general. The relevant distances satisfy the following convergence and boundedness result.

Lemma 6. *Consider a metric space (X,d) with subsets $A_i, B_i \subset X$ and a p-cyclic mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ which is GWCPD_p, subject to $D_i = d(A_{i+1}, B_i)$, $D_A = d(A_i, A_{i+1})$, and $D_B = d(B_i, B_{i+1})$, with $T(A_{0i}) \subseteq B_{0i}$, for all $i \in \overline{p}$ such that $A_{0i} \subseteq A_i$ and $B_{0i} \subseteq B_i$ are nonempty, for all $i \in \overline{p}$. Consider also any sequences $\{x_{np+j}\}, \{\overline{x}_{np+j}\} \subseteq \bigcup_{i\in\overline{p}} A_i$ which satisfy $d(x_{np+i+1}, Tx_{np+i}) = d(\overline{x}_{np+i+1}, T\overline{x}_{np+i}) = D_i$, for all $i \in \overline{p}$. Then, the following properties hold.*

(i) The sequences of distances $\{d(x_n, \overline{x}_{n+1})\} \to D_A$, $\{d(x_n, \overline{x}_n)\} \to 0$ and they are bounded for any given initial points $x_0, \overline{x}_0 \in A_j \subset \bigcup_{i\in\overline{p}} A_i$, for any given $j \in \overline{p}$.

If, furthermore, $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is continuous in $\mathrm{cl}\,T(A_{0i})$, for all $i \in \overline{p}$, then $\{d(Tx_n, T\overline{x}_n)\} \to 0$ and $\{d(Tx_n, T\overline{x}_{n+1})\} \to D_B$ and both sequences of distances are bounded for any given initial points $x_0, \overline{x}_0 \in A_j \subset \bigcup_{i\in\overline{p}} A_i$, for any given $j \in \overline{p}$.

If the sets of best proximity points A_{0i} and B_{0i}, for all $i \in \overline{p}$, are bounded, then the sequences $\{d(x_n, \overline{x}_{n+1})\}$, $\{d(x_n, \overline{x}_n)\}$, $\{d(Tx_n, T\overline{x}_n)\}$, and $\{d(Tx_n, T\overline{x}_{n+1})\}$ are uniformly bounded for any initial points $x_0, \overline{x}_0 \in A_j \subset \bigcup_{i\in\overline{p}} A_i$ for some $j \in \overline{p}$.

(ii) The sequences $\{x_{np+i}\} \subseteq \mathrm{cl}A_{i+j}$, for all $i \in \overline{p}$ (note that $A_{i+j} = A_{i+j-p}$ and $B_{i+j} = B_{i+j-p}$ for $i > p-j$) are Cauchy sequences for any initial points any given initial point $x_0 \in \bigcup_{i\in\overline{p}} A_i$ for any arbitrary given $j \in \overline{p}$. The corresponding image sequences $\{Tx_{np+i}\} \subseteq \mathrm{cl}B_{i+j}$, for all $i \in \overline{p}$, are also Cauchy sequences if $T(A_{0i}) \subseteq B_{0i}$, for all $i \in \overline{p}$, and $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is continuous in $\mathrm{cl}T(A_{0i})$, for all $i \in \overline{p}$.

Proof. Note that, for any $i \in \overline{p}$, $\varnothing \ne A_{0i} \subseteq A_i$ implies $A_i \ne \varnothing$ and $\varnothing \ne B_{0i} \subseteq B_i$ implies $B_i \ne \varnothing$. Take sequences with initial points $x_0, \overline{x}_0 \subset \bigcup_{i\in\overline{p}} A_i$ such that $x_0 \in A_j$ and $\overline{x}_0 \in A_i$ for some $i, j \in \overline{p}$. The problem of boundedness and convergence of distances can be described equivalently from initial points $x_0, \overline{x}'_0 \in A_j$ (i.e., both initial conditions at the same set), with $\overline{x}'_0 = T^{\ell}x_0$ and denoting $\overline{x}'_0 \to \overline{x}_0$ for some $\ell \in \overline{p-1} \cup \{0\}$ (in particular, $\ell = 0$ if $i = j$ for the set of both initial points) since $\ell \le p-1$. One has from (1)-(2) and (3)–(6) that, for any sequences $\{x_{np+j+i}\} \subseteq A_{j+i}$ and $\{\overline{x}_{np+j+i}\} \subseteq A_{j+i}$ fulfilling $\{x_0\} \in A_j, \{\overline{x}_0\} \in A_j$ and $d(x_{np+i+1}, Tx_{np+i}) = d(\overline{x}_{np+i+1}, T\overline{x}_{np+i}) = D_i$, for all $j, i \in \overline{p}$, such that $A_{j+i} = A_{j+i-p}$ if $i > p-j$ and

$$D_A \le d\left(x_{np+j+1}, \overline{x}_{np+j}\right) \le \left(\sup_{x\in A_j}\alpha_j(x)\right) d\left(x_{np+j}, \overline{x}_{np+j-1}\right) + \left(1-\sup_{x\in A_j}\alpha_j(x)\right)D_A, \quad \forall j\in\overline{p},\ \forall n\in\mathbf{Z}_{0+},$$

$$D_A \le d\left(x_{np+j+2}, \overline{x}_{np+j+1}\right) \le \left(\sup_{x\in A_{j+1}}\alpha_{j+1}(x)\right) \times \left(\left(\sup_{x\in A_j}\alpha_j(x)\right) d\left(x_{np+j}, \overline{x}_{np+j-1}\right) + \left(1-\sup_{x\in A_j}\alpha_j(x)\right)D_A\right) + \left(1-\sup_{x\in A_j}\alpha_{j+1}(x)\right)D_A$$

$$= D_A + \left(\sup_{x \in A_{j+1}} \alpha_{j+1}(x) \right) \left(\sup_{x \in A_j} \alpha_j(x) \right) \times \left(d\left(x_{np+j}, \overline{x}_{np+j-1} \right) - D_A \right),$$

$$D_A \le d\left(x_{np+j+3}, \overline{x}_{np+j+2} \right) \le D_A + \left(\sup_{x \in A_{j+2}} \alpha_{j+1}(x) \right) \times \left(\sup_{x \in A_{j+1}} \alpha_{j+1}(x) \right) \left(\sup_{x \in A_j} \alpha_j(x) \right) \times \left(d\left(x_{np+j}, \overline{x}_{np+j-1} \right) - D_A \right), \tag{7}$$

$$D_A \le d\left(x_{n_{k+1}p}, \overline{x}_{n_{k+1}p-1} \right) \le \alpha(n_k, n_{k+1})\, d\left(x_{n_k p+1}, \overline{x}_{n_k p} \right) + \left(1 - \alpha(n_k, n_{k+1}) \right) D_A$$
$$\le D_A + \alpha(n_k, n_{k+1})\, \alpha(n_{k-1}, n_{k-2}) \times \left(d\left(x_{n_{k-1}p+1}, \overline{x}_{n_{k-1}p} \right) - D_A \right)$$
$$\le D_A + \left(\prod_{j=0}^{k} \left[\alpha(n_j, n_{j+1}) \right] \right) \left(d\left(x_{n_0 p+1}, \overline{x}_{n_0 p} \right) - D_A \right)$$
$$\le D_A + \overline{\alpha}^{k+1} \left(d\left(x_{n_0 p+1}, \overline{x}_{n_0 p} \right) - D_A \right) \tag{8a}$$

$$< d\left(x_{n_0 p+1}, \overline{x}_{n_0 p} \right) < +\infty, \quad \forall k \in \mathbf{Z}_{0+}, \tag{8b}$$

$$D_A \le d\left(x_{n_{k+1}p+i}, \overline{x}_{n_{k+1}p+i-1} \right) \le D_A + \left(\prod_{j=1}^{i-1} \left[\sup_{x \in \bigcup_{j \in \overline{p}} A_j} \alpha_j(x) \right] \right) \times \left(\prod_{j=0}^{k} \left[\alpha(n_j, n_{j+1}) \right] \right) \left(d\left(x_{n_0 p+1}, \overline{x}_{n_0 p} \right) - D_A \right)$$
$$\le D_A + \left(\prod_{j=1}^{i-1} \left[\sup_{x \in \bigcup_{j \in \overline{p}} A_j} \alpha_j(x) \right] \right) \times \overline{\alpha}^{k+1} \left(d\left(x_{n_0 p+1}, \overline{x}_{n_0 p} \right) - D_A \right)$$
$$\le D_A + \overline{\alpha}^{k+1} \left(1 + \overline{N} + \overline{N}' \right) \widehat{\alpha}_M \left(d\left(x_{n_0 p+1}, \overline{x}_{n_0 p} \right) - D_A \right) \tag{9a}$$

$$< D_A + \left(\prod_{j=1}^{i-1} \left[\sup_{x \in \bigcup_{j \in \overline{p}} A_j} \alpha_j(x) \right] \right) \times \left(d\left(x_{n_0 p+1}, \overline{x}_{n_0 p} \right) - D_A \right)$$
$$\le \max\left(1, \prod_{j=1}^{i-1} \left[\sup_{x \in \bigcup_{j \in \overline{p}} A_j} \alpha_j(x) \right] \right) d\left(x_{n_0 p+1}, \overline{x}_{n_0 p} \right) \tag{9b}$$
$$\le \left(1 + \overline{N} + \overline{N}' \right) \widehat{\alpha}_M d\left(x_{n_0 p+1}, \overline{x}_{n_0 p} \right), \quad \forall k \in \mathbf{Z}_{0+},$$

for all $i \in (1, \max(n_{k+1} - n_k)) \cap \mathbf{Z}_{0+}$ since $\overline{\alpha} \in [0,1)$, $\limsup_{k \to \infty}(n_{k+1} - n_k) \le \overline{N} < +\infty$ and $\alpha_i \in B(A_i; \mathbf{R}_{0+})$ with $\max_{i \in \overline{p}}(\sup_{x \in A_i} \alpha_i(x)) \le \widehat{\alpha}_M < +\infty$.

Thus, $\exists \lim_{k \to \infty} d(x_{n_{k+1}p+i}, \overline{x}_{n_{k+1}p+i-1}) = \lim_{k \to \infty} d(x_{n_k p+i}, \overline{x}_{n_{k+1}p+i-1}) = \lim_{k \to \infty} d(x_{n_{k+1}p+i}, \overline{x}_{n_k p+i-1}) = D_A$, for all $i \in (1, \max(n_{k+1} - n_k)) \cap \mathbf{Z}_{0+}$ from (9a) and the distance subsequence $\{d(x_{n_{k+1}p+i}, \overline{x}_{n_{k+1}p+i-1})\}$, for all $i \in (1, \max(n_{k+1} - n_k)) \cap \mathbf{Z}_{0+}$ is bounded from (9b) for any given initial points $x_0, \overline{x}_0 \in \bigcup_{i \in \overline{p}} A_i$. Also, one gets from (6), subject to (3), that

$$\exists \lim_{k \to \infty} d\left(x_{n_{k+1}p+i}, \overline{x}_{n_{k+1}p+i} \right) = \lim_{k \to \infty} d\left(x_{n_k p+i}, \overline{x}_{n_{k+1}p+i} \right) = \lim_{k \to \infty} d\left(\overline{x}_{n_k p+i}, x_{n_{k+1}p+i} \right) = 0, \tag{10}$$

for all $i \in (1, \max(n_{k+1} - n_k)) \cap \mathbf{Z}_{0+}$. Those results also imply that the sequences of distances $\{d(x_n, \overline{x}_{n+1})\} \to D_A$, $\{d(x_n, \overline{x}_n)\} \to 0$. It is now proved by contradiction that $\{d(Tx_n, T\overline{x}_{n+1})\} \to D_B$ and $\{d(Tx_n, T\overline{x}_n)\} \to 0$. Assume that, for each given $x_0 \in A_{0j} \subset \bigcup_{i \in \overline{p}} A_i$ for some $j \in \overline{p}$ and any $\varepsilon \in \mathbf{R}_+$, there are some $\delta \in \mathbf{R}_+$, some $i \in \overline{p}$, some $k_0 = k_0(\varepsilon, x_0) \in \mathbf{Z}_{0+}$, sequences of integers $\{n_{k_\ell}\} \subseteq \{n_k\} \subseteq \mathbf{Z}_{0+}$, and sequences of best proximity points $\{x^*_{n_k p+i}\} \subseteq \mathrm{cl}(A_{0,i+j})$ and $\{Tx^*_{n_k p+i}\} \subseteq \mathrm{cl}(B_{0,i+j})$ for $k > k_0$, such that $\{d(Tx_n, T\overline{x}_n)\}$ does not converge to zero so that it has some subsequence which does not converge either:

$$d(A_i, A_{i+1}) = d(A_{0i}, A_{0,i+1}) = D_A,$$
$$d(T(A_{0i}), T(A_{0,i+1})) = d(B_i, B_{i+1}) = d(B_{0i}, B_{0,i+1}) = D_B, \quad \forall i \in \overline{p},$$
$$d\left(x_{n_k p+i}, x^*_{n_k p+i-1} \right) < D_A + \varepsilon,$$
$$d\left(Tx_{n_{k_\ell} p+i}, Tx^*_{n_{k_\ell} p+i-1} \right) \ge \delta + D_B, \quad \forall k > k_0,$$
$$d\left(x_{n_k p+i}, x^*_{n_k p+i} \right) < \varepsilon, \qquad d\left(Tx_{n_{k_\ell} p+i}, Tx^*_{n_{k_\ell} p+i} \right) \ge \delta, \quad \forall k > k_0,$$

$$Tx^*_{n_kp+i} \in \operatorname{cl} B_{0,i+j} \subseteq \operatorname{cl} B_{i+j},$$
$$Tx_{np+i-1} \in \operatorname{cl} B_{0,i+j-1} \subseteq \operatorname{cl} B_{i+j-1}, \quad \forall j \in \overline{p} \tag{11}$$

since $T(A_{0i}) \subseteq B_{0i}$, for all $i \in \overline{p}$. This implies that $\{x_{n_{k_\ell}p+i} - x^*_{n_{k_\ell}p+i}\} \to 0$ while $\{Tx_{n_{k_\ell}p+i} - Tx^*_{n_{k_\ell}p+i}\}$ does not converge to zero.

Since $\{x_{n_{k_\ell}p+i} - x^*_{n_{k_\ell}p+i}\} \to 0$, $\{x^*_{n_kp+i}\} \subseteq \operatorname{cl}(A_{0,i+j})$, $\{Tx^*_{n_kp+i}\} \subseteq \operatorname{cl}(T(A_{0,i+j})) \subseteq \operatorname{cl}(B_{0,i+j})$, and $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is continuous in $\operatorname{cl}(A_{0,i+j})$, for all $j \in \overline{p}$ and any given $j \in \overline{p}$, then $\{Tx_{n_{k_\ell}p+i} - Tx^*_{n_{k_\ell}p+i}\} \to 0$. Also, $\{x^*_{n_kp+i-1}\} \subseteq \operatorname{cl}(A_{0,i+j-1})$, $\{Tx^*_{n_kp+i-1}\} \subseteq \operatorname{cl}(T(A_{0,i+j-1})) \subseteq \operatorname{cl}(B_{0,i+j-1})$, for all $i \in \overline{p}$ and any given $j \in \overline{p}$; then $\{d(Tx_{n_{k_\ell}p+i}, Tx^*_{n_{k_\ell}p+i-1})\} \to D_B$, $\{d(Tx_{n_{k_\ell}p+i-1}, Tx^*_{n_{k_\ell}p+i})\} \to D_B$. Thus, $\{d(Tx_n, T\overline{x}_n)\} \to 0$ and $\{d(Tx_n, T\overline{x}_{n+1})\} \to 0$ if $x_0, \overline{x}_0 \in A_j \subset \bigcup_{i\in\overline{p}} A_i$, for any given $j \in \overline{p}$.

On the other hand, $\{d(Tx_n, T\overline{x}_n)\}$ and $\{d(Tx_n, T\overline{x}_{n+1})\}$ are bounded, since $\{d(x_n, \overline{x}_n)\}$ is bounded from (6) because $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is GWCPD$_p$ and since one has for some positive real constant $M = M(d(x_0, \overline{x}_0))$ that

$$d(Tx_n, T\overline{x}_n) \leq d(Tx_n, x_n) + d(\overline{x}_n, T\overline{x}_n) + d(x_n, \overline{x}_n) \leq 2\max_{1\leq i\leq p} D_i + M, \quad \forall n \in \mathbf{Z}_{0+},$$
$$d(Tx_n, T\overline{x}_{n+1}) \leq d(Tx_n, x_{n+1}) + d(\overline{x}_{n+1}, T\overline{x}_{n+1}) + d(x_{n+1}, \overline{x}_{n+1}) \leq 2\max_{1\leq i\leq p} D_i + M, \quad \forall n \in \mathbf{Z}_{0+}. \tag{12}$$

If the sets of best proximity points A_{0i} for all $i \in \overline{p}$ are bounded, then the sequences of distances $\{d(x_n, \overline{x}_{n+1})\}$, $\{d(x_n, \overline{x}_n)\}$, $\{d(Tx_n, T\overline{x}_n)\}$, and $\{d(Tx_n, T\overline{x}_{n+1})\}$ are uniformly bounded for any initial best proximity points $x_0, \overline{x}_0 \in A_{0j} \subset \bigcup_{i\in\overline{p}} A_i$ for some $j \in \overline{p}$ which follows by taking $M = M \sup_{x_0,\overline{x}_0 \in \bigcup_{i\in\overline{p}} A_i}(d(x_0, \overline{x}_0))$. Property (i) has been fully proved.

To prove Property (ii), take any sequences $\{x_{np+i}\} \subseteq A_{0,i+j}$ and $\{\overline{x}_{np+i}\} \subseteq A_{0,i+j}$, for all $i \in \overline{p}$, for given initial points $x_0, \overline{x}_0 \in A_j \subset \bigcup_{i\in\overline{p}} A_i$ for some $j \in \overline{p}$. Note, from (6) for $\{\overline{x}_{n_{k+1}p+i}\} = \{x_{n_kp+i}\}$, for all $i \in \overline{p}$, that

$$d(x_{n_{k+1}p+i}, x_{n_kp+i}) \leq Hd(x_{n_kp}, x_{n_{k-1}p}) \leq h_0\overline{\alpha} d(x_{n_kp}, x_{n_{k-1}p}), \quad \forall k \in \mathbf{Z}_{0+}, \quad \forall i \in \overline{p}, \tag{13}$$

$$\begin{aligned} d(x_{n_{k+1}p+i+\ell h}, x_{n_kp+i}) &\leq h_0\overline{\alpha}(1 + \overline{\alpha} + \cdots + \overline{\alpha}^\ell) d(x_{n_kp}, x_{n_{k-1}p}) \\ d(x_{n_{k+1}p+i+\ell h}, x_{n_kp+i}) &\leq h_0 \frac{\overline{\alpha}(1 - \overline{\alpha}^{\ell+1})}{1 - \overline{\alpha}} d(x_{n_kp}, x_{n_{k-1}p}) \\ &\leq h_0 \frac{\overline{\alpha}}{1 - \overline{\alpha}} d(x_{n_kp}, x_{n_{k-1}p}), \quad \forall k, \ell \in \mathbf{Z}_{0+}, \ \forall i \in \overline{p}, \end{aligned} \tag{14}$$

with the given upper bound being independent of the integers ℓ and i. Thus, one has for any $\mathbf{Z}_{0+}$ that

$$d(x_{n_{k+m+1}p+i+\ell j}, x_{n_{k+m}p+i}) \leq \overline{\alpha}^m h_0 \frac{\overline{\alpha}}{1 - \overline{\alpha}} d(x_{n_kp}, x_{n_{k-1}p}), \quad \forall k \in \mathbf{Z}_{0+}, \quad \forall i \in \overline{p}, \quad \forall j (\leq h), \ell \in \mathbf{Z}_{0+}, \tag{15}$$

where $H = \sup_{k\in\mathbf{Z}_{0+}}(\prod_{j=1}^p [\sup_{x\in A_i} \alpha_i(x)]\alpha(n_k, n_{k+1})) \leq h_0\overline{\alpha}$, with $h_0 \geq \sup_{k\in\mathbf{Z}_{0+}}(\prod_{j=1}^p [\sup_{x\in A_i} \alpha_i(x)])$ and $1 \leq j \leq h = \max(1, \sup_{k\in\mathbf{Z}_{0+}}(n_{k+1} - n_k))$, since Definition 5 holds for $\overline{\alpha} = \sup_{k\in\mathbf{Z}_{0+}} \alpha(n_k, n_{k+1}) \in [0, 1)$. Thus, one gets from (15) that, for any given real $\varepsilon \in \mathbf{R}_+$, $d(x_{n_{k+m+1}p+i+\ell j}, x_{n_{k+m}p+i}) < \varepsilon$, for all $k, \ell, j(\leq h) \in \mathbf{Z}_{0+}$ for any $i \in \overline{p}$ and any given integers $n_{k-1}, n_k \in \mathbf{Z}_{0+}$ if $m(\in \mathbf{Z}_{0+}) > \ln(h_0 d(x_{n_kp}, x_{n_{k-1}p})/(1 - \overline{\alpha})\varepsilon)/|\ln\alpha| - 1$. Thus, the sequences $\{x_{np+i}\} \subseteq \operatorname{cl} A_{0i}$, for all $i \in \overline{p}$, are Cauchy sequences for any given initial point $x_0 \in A_{0j} \subset \bigcup_{i\in\overline{p}} A_i$ and any $j \in \overline{p}$. This implies also that the sequences of images of the above points are also Cauchy sequences since $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is contractive and then continuous. □

From Assertions 1, we also have the subsequent parallel result to Lemma 6.

Lemma 7. *Assume that $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is either GCPD$_p$ or WCPD$_p$ or CPD$_p$ with the assumptions of Lemma 6, and consider any sequences $\{x_{np+j}\}$, $\{\overline{x}_{np+j}\} \subseteq \bigcup_{i\in\overline{p}} A_i$ which satisfy $d(x_{np+i+1}, Tx_{np+i}) = d(\overline{x}_{np+i+1}, T\overline{x}_{np+i}) = D_i$, for all $i \in \overline{p}$. Then, the following properties hold.*

(i) The sequences of distances $\{d(x_n, \overline{x}_{n+1})\} \to D_A$, $\{d(x_n, \overline{x}_n)\} \to 0$ and they are bounded for any given initial points $x_0, \overline{x}_0 \in A_j \subset \bigcup_{i\in\overline{p}} A_i$ for any given $j \in \overline{p}$.

If, furthermore, $T(A_{0i}) \subseteq B_{0i}$, for all $i \in \overline{p}$, and $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is continuous in $\operatorname{cl} T(A_{0i})$, for all $i \in \overline{p}$, then $\{d(Tx_n, T\overline{x}_n)\} \to 0$ and $\{d(Tx_n, T\overline{x}_{n+1})\} \to D_B$ and it is bounded for any initial points for any given initial points $x_0, \overline{x}_0 \in A_{0j} \subset \bigcup_{i\in\overline{p}} A_i$, for some $j \in \overline{p}$.

If the sets of best proximity points A_{0i} and B_{0i}, for all $i \in \overline{p}$, are bounded, then the sequences $\{d(x_n, \overline{x}_{n+1})\}$, $\{d(x_n, \overline{x}_n)\}$, $\{d(Tx_n, T\overline{x}_n)\}$, and $\{d(Tx_n, T\overline{x}_{n+1})\}$ are uniformly bounded for any initial best proximity points $x_0, \overline{x}_0 \in A_{0j} \subset \bigcup_{i\in\overline{p}} A_i$ for some $j \in \overline{p}$.

(ii) The sequences $\{x_{np+i}\} \subseteq \operatorname{cl} A_{0,i+j}$, for all $i \in \overline{p}$, are Cauchy sequences for any given initial point $x_0 \in A_{0j} \subset$

$\bigcup_{i\in\overline{p}} A_i$ for any given $j \in \overline{p}$. The corresponding image sequences $\{Tx_{np+i}\} \subseteq \operatorname{cl}B_{0,i+j}$, for all $i \in \overline{p}$, are also convergent, then Cauchy sequences if $T(A_{0i}) \subseteq B_{0i}$, for all $i \in \overline{p}$, and $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is continuous in $\operatorname{cl}T(A_{0i})$, for all $i \in \overline{p}$.

Definition 8. $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is said to be a p-cyclic proximal contraction with respect to its image (CPI_p) if there are real constants $\beta_i \in [0,1)$, for all $i \in \overline{p}$, such that any two sequences $\{Tx_{np+i}\} \subseteq T(A_{i+j}) \subseteq B_{i+j}$ and $\{\overline{x}_{np+i}\} \subseteq T(A_{i+j}) \subseteq B_{i+j}$, for all $i \in \overline{p}$, being point-to-point images of sequences $\{x_{np+i}\} \subseteq A_{i+j}$ and $\{\overline{x}_{np+i}\} \subseteq A_{i+j}$ for any given $j \in \overline{p}$ which satisfy $d(x_{np+i+1}, Tx_{np+i}) = d(\overline{x}_{np+i+1}, T\overline{x}_{np+i}) = D_i$, for all $i \in \overline{p}$, where $B_{i+j} = B_{i+j-p}$ if $i > p-j$ such that the initial points $Tx_0, T\overline{x}_0 \in T(A_j) \subseteq B_j$ are the images of points $x_0, \overline{x}_0 \in A_j$, for any given $j \in \overline{p}$, satisfy the constraints

$$d\left(Tx_{np+i+1}, T\overline{x}_{np+i}\right) \leq \beta_i d\left(Tx_{np+i}, T\overline{x}_{np+i-1}\right) + \left(1-\beta_i\right) D_B, \quad \forall i \in \overline{p}, \quad \forall n \in \mathbf{Z}_{0+}, \tag{16}$$

$$d\left(Tx_{np+i+1}, T\overline{x}_{np+i+1}\right) \leq \beta_i d\left(Tx_{np+i}, T\overline{x}_{np+i}\right), \quad \forall i \in \overline{p}, \quad \forall n \in \mathbf{Z}_{0+}. \tag{17}$$

Definition 9. $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is said to be a weak p-cyclic proximal contraction with respect to its image (WCPI_p) if there are p real constants $\beta_i \geq 0$, for all $i \in \overline{p}$, subject to $\beta = \prod_{i=1}^{p}[\beta_i] \in [0,1)$, such that any two sequences $\{Tx_{np+i}\} \subseteq T(A_{i+j}) \subseteq B_{i+j}$ and $\{T\overline{x}_{np+i}\} \subseteq T(A_{i+j}) \subseteq B_{i+j}$, for all $i \in \overline{p}$, being point-to-point images of sequences $\{x_{np+i}\} \subseteq A_{i+j}$ and $\{\overline{x}_{np+i}\} \subseteq A_{i+j}$ for any given $j \in \overline{p}$, where $B_{i+j} = B_{i+j-p}$ for $i > p-j$, for all $i \in \overline{p}$, such that the initial points $Tx_0, T\overline{x}_0 \in T(A_j) \subseteq B_j$ are the images of points $x_0, \overline{x}_0 \in A_j$, for any given $j \in \overline{p}$, satisfy constraints (16) and (17).

Definition 10. $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is said to be a generalized p-cyclic proximal contraction with respect to its image (GCPI_p) if there are p bounded real functions $\beta_i : A_i \to \mathbf{R}_{0+}$, for all $i \in \overline{p}$, such that any sequences $\{Tx_{np+i}\} \subseteq T(A_{i+j}) \subseteq B_{i+j}$ and $\{T\overline{x}_{np+i}\} \subseteq T(A_{i+j}) \subseteq B_{i+j}$, for all $i \in \overline{p}$, being point-to-point images of sequences $\{x_{np+i}\} \subseteq A_{i+j}$ and $\{\overline{x}_{np+i}\} \subseteq A_{i+j}$ for any given $j \in \overline{p}$, where $B_{i+j} = B_{i+j-p}$ for $i > p-j$, for all $i \in \overline{p}$, such that the initial points $Tx_0, T\overline{x}_0 \in T(A_j) \subseteq B_j$ are the images of points $x_0, \overline{x}_0 \in A_j$, satisfy the constraints (16) and (17) with the replacements $\beta_i \to \sup_{x\in A_i}\beta_i(Tx)$, for all $i \in \overline{p}$.

Definition 11. $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is said to be a generalized weak p-cyclic proximal contraction with respect to its image (GWCPI_p) if there are p bounded real functions $\beta_i : A_i \to \mathbf{R}_{0+}$, for all $i \in \overline{p}$, and a strictly increasing sequence of integers $\{n_k\}$, subject to $n_0 \leq \overline{N}_0 < +\infty$, $\limsup_{k\to\infty}(n_{k+1} - n_k) \leq \overline{N} < +\infty$, and $\overline{\alpha} = \sup_{k\in\mathbf{Z}_{0+}}\beta(n_k, n_{k+1}) \in [0,1)$, where

$$\beta\left(n_k, n_{k+1}\right) = \prod_{j=n_k}^{n_{k+1}-1}\prod_{i=1}^{p}\left[\sup_{x_{j+i}\in A_i}\beta_{jp+i}\left(Tx_{jp+i}\right)\right], \quad \forall k \in \mathbf{Z}_{0+}, \tag{18}$$

such that any two sequences $\{Tx_{np+i}\} \subseteq T(A_{i+j}) \subseteq B_{i+j}$ and $\{\overline{x}_{np+i}\} \subseteq T(A_{i+j}) \subseteq B_{i+j}$, for all $i \in \overline{p}$, being point-to-point images of sequences $\{x_{np+i}\} \subseteq A_{i+j}$ and $\{\overline{x}_{np+i}\} \subseteq A_{i+j}$ for any given $j \in \overline{p}$, where $B_{i+j} = B_{i+j-p}$ for $i > p-j$, for all $i \in \overline{p}$, such that the initial points $Tx_0, T\overline{x}_0 \in T(A_j) \subseteq B_j$ are the images of points $x_0, \overline{x}_0 \in A_j$, for any given $j \in \overline{p}$, satisfy the following constraints:

$$d\left(Tx_{n_{k+1}p}, T\overline{x}_{n_{k+1}p-1}\right) \leq \beta\left(n_k, n_{k+1}\right) d\left(Tx_{n_k p+1}, T\overline{x}_{n_k p}\right) + \left(1-\beta\left(n_k, n_{k+1}\right)\right) D_B, \quad \forall k \in \mathbf{Z}_{0+}, \tag{19}$$

$$\begin{aligned} d\left(Tx_{n_{k+1}p+i}, T\overline{x}_{n_{k+1}p+i-1}\right) &\leq \left(\prod_{j=1}^{i-1}\left[\sup_{x\in A_i}\beta_i(Tx)\right]\right) d\left(Tx_{n_{k+1}p}, T\overline{x}_{n_{k+1}p-1}\right) \\ &\quad + \sum_{j=1}^{i-1}\left(\prod_{k=j+1}^{i-1}\left[\sup_{x\in A_k}\beta_k(Tx)\right]\right)\left(1-\sup_{x\in A_j}\beta_j(x)\right) D_B, \quad \forall k \in \mathbf{Z}_{0+}, \end{aligned} \tag{20}$$

$$d\left(Tx_{n_{k+1}p+i}, T\overline{x}_{n_{k+1}p+i}\right) \leq \left(\prod_{j=1}^{i-1}\left[\sup_{x\in A_i}\beta_i(Tx)\right]\right)\beta\left(n_k, n_{k+1}\right) d\left(Tx_{n_k p}, T\overline{x}_{n_k p}\right), \quad \forall k \in \mathbf{Z}_{0+}, \tag{21}$$

and constraints (16) and (17) with the replacements $\beta_i \to \sup_{x\in A_i}\beta_i(Tx)$, for all $i \in \overline{p}$.

The following assertions are obvious without proof from Definitions 8–11 and are a parallel result to Assertions 1.

Assertions 2. If $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is CPI_p, then it is WCPI_p.

If $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is WCPI_p, then it is GCPI_p.

If $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is GCPI_p, then it is GWCPI_p.

Note that the converse implications of those in Assertions 1 are not true in general.

The relevant distances satisfy the following convergence and boundedness result which is a counterpart of Lemmas 6 and 7. Its proof is close to that of Lemma 6 and Assertions 2 by using (16) and (17) for Definition 11 and their variants for Definitions 8–10.

Lemma 12. *Assume that $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is either $GWCPI_p$ or $GCPI_p$ or $WCPI_p$ or CPI_p under the assumptions of Lemma 6, and consider any sequences $\{x_{np+j}\}, \{\overline{x}_{np+j}\} \subseteq \bigcup_{i\in\overline{p}} A_i$ which satisfy $d(x_{np+i+1}, Tx_{np+i}) = d(\overline{x}_{np+i+1}, T\overline{x}_{np+i}) = D_i$, for all $i \in \overline{p}$. Then, the following properties hold.*

(i) The sequences of distances $\{d(x_n, \overline{x}_{n+1})\} \to D_A$, $\{d(x_n, \overline{x}_n)\} \to 0$ and they are bounded for any given initial points $x_0, \overline{x}_0 \in A_j \subset \bigcup_{i\in\overline{p}} A_i$, for some $j \in \overline{p}$.

If, furthermore, $T(A_{0i}) \subseteq B_{0i}$, for all $i \in \overline{p}$, and $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is continuous in $\mathrm{cl}T(A_{0i})$, for all $i \in \overline{p}$, then $\{d(Tx_n, T\overline{x}_n)\} \to 0$ and $\{d(Tx_n, T\overline{x}_{n+1})\} \to D_B$ and it is bounded any given initial points $x_0, \overline{x}_0 \in A_{0j} \subset \bigcup_{i\in\overline{p}} A_i$ for some $j \in \overline{p}$.

If the sets of best proximity points A_{0i} and B_{0i}, for all $i \in \overline{p}$, are bounded, then the sequences $\{d(x_n, \overline{x}_{n+1})\}$, $\{d(x_n, \overline{x}_n)\}$, $\{d(Tx_n, T\overline{x}_n)\}$, and $\{d(Tx_n, T\overline{x}_{n+1})\}$ are uniformly bounded for any initial best proximity points $x_0, \overline{x}_0 \in A_{0j} \subset \bigcup_{i\in\overline{p}} A_i$ for some $j \in \overline{p}$.

(ii) The sequences $\{x_{np+i}\} \subseteq \mathrm{cl}A_{0,i+j}$, for all $i \in \overline{p}$, are Cauchy sequences for any given initial point $x_0 \in A_{0j} \subset \bigcup_{i\in\overline{p}} A_i$ for any given $j \in \overline{p}$. The corresponding image sequences $\{Tx_{np+i}\} \subseteq \mathrm{cl}B_{0,i+j}$, for all $i \in \overline{p}$, are also convergent; then Cauchy sequences if $T(A_{0i}) \subseteq B_{0i}$, for all $i \in \overline{p}$, and $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ are continuous in $\mathrm{cl}T(A_{0i})$, for all $i \in \overline{p}$.

Remark 13. The result $\{d(Tx_n, T\overline{x}_{n+1})\} \to D_B$ of Lemma 12, as well as Lemma 12(ii), obtained under the assumption that $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is continuous in $\mathrm{cl}\, T(A_{0i})$ and also holds without such a continuity assumption if the contractive conditions (16) and (17) become modified to the right limits as follows:

$$d\left(Tx^{+}_{np+i+1}, T\overline{x}^{+}_{np+i}\right) \leq \beta_i d\left(Tx^{+}_{np+i}, T\overline{x}^{+}_{np+i-1}\right) + (1-\beta_i) D_B, \quad \forall i \in \overline{p}, \quad \forall n \in \mathbf{Z}_{0+}, \tag{22}$$

$$d\left(Tx^{+}_{np+i+1}, T\overline{x}^{+}_{np+i+1}\right) \leq \beta_i d\left(Tx^{+}_{np+i}, T\overline{x}^{+}_{np+i}\right), \quad \forall i \in \overline{p}, \quad \forall n \in \mathbf{Z}_{0+}, \tag{23}$$

provided that any discontinuity points in $\bigcup_{i\in p} \mathrm{clT}(A_{0i})$, if any, are of first-class finite-jump type under right best proximity constraints

$$d\left(x_{np+i+1}, Tx^{+}_{np+i}\right) = d\left(\overline{x}_{np+i+1}, T\overline{x}^{+}_{np+i}\right) = D_i, \quad \forall i \in \overline{p}. \tag{24}$$

In the same way, the result $\{d(Tx_n, T\overline{x}_{n+1})\} \to D_B$ of Lemmas 6 and 7, as well as their properties (ii) obtained under the assumption that $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$, is continuous in $\mathrm{cl}\, T(A_{0i})$ and also holds under finite-jump discontinuities in $\mathrm{cl}\, T(A_{0i})$ for sequences $\{x_n\}$, $\{\overline{x}_n\}$, $\{Tx_n\}$, and $\{T\overline{x}_n\}$ satisfying the contractive proximal conditions (1) and (2) if Definition 2, or their counterparts of Definitions 3–5 for right values Tx^{+}_n and $T\overline{x}^{+}_n$ under right best proximity constraints (24).

3. Best Proximity Points and Related Convergence Results

We first recall the subsequent useful definition [2–4, 7] as follows.

Definition 14. Let A and B be two nonempty subsets of a metric space (X, d) and let $d(y, A) = \inf\{d(y, x) : x \in A\}$ for $y \in X$. A is said to be approximately compact with respect to B if each sequence $\{x_n\} \subset A$ satisfying $\{d(y, x_n)\} \to d(y, A)$ for some $y \in B$ has a convergent subsequence.

Note that if the sets of best proximity points $A_0 \subseteq A$ and $B_0 \subseteq B$ are nonempty if Definition 14 holds, then A is approximately compact with respect to B if every sequence $\{x_n\} \subset A$ such that $\{d(y, x_n)\} \to D$ for some $y \in B_0$ has a convergent subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ since $D = d(y, A) = d(B_0, A) = d(B_0, A_0)$. Note that every set is approximately compact with respect to itself and that every compact set is approximately compact with respect to any nonempty subset of a metric space. Also, if B is compact and A is approximately compact with respect to B, each sequence $\{x_n\} \subset A$ has a convergent sequence. If A and B are nonempty and closed and A is approximately compact with respect to B, then B_0 is closed. See, for instance, [2–4, 7]. A result on existence and uniqueness of best proximity points follows for p-cyclic proximal contraction fulfilling Definitions 2–5 under Lemmas 6 and 7 follows.

Theorem 15. *Consider a complete metric space (X, d) with nonempty closed subsets $A_i, B_i \subset X$ and a p-cyclic mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ being either $GWCPD_p$ or $GCPD_p$ or $WCPD_p$ or CPD_p, subject to set distances $D_i = d(A_{i+1}, B_i)$, $D_A = d(A_i, A_{i+1})$, and $D_B = d(B_i, B_{i+1})$, for all $i \in \overline{p}$ such that A_{0i} is nonempty and B_i is approximately compact with respect to A_i and $T(A_{0i}) \subseteq B_{0i}$, for all $i \in \overline{p}$. The following properties hold.*

(i) *$T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ has a unique best proximity point x^*_i at A_i such that $d(x^*_i, x^*_{i+1}) = D_A$, for all $i \in \overline{p}$, and all the sequences $\{x_n\} \subset \bigcup_{i\in\overline{p}} A_i$ converge to a unique limit cycle $\{x^*_1, x^*_2, \ldots, x^*_p\}$.*

(ii) *Furthermore, if $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ either is continuous, respectively, has eventual finite-jump discontinuity points, then $z^*_i = Tx^*_i$, for all $i \in \overline{p}$, respectively, $z^{*^+}_i = (Tx^*_i)^+$, for all $i \in \overline{p}$, are unique best proximity points such that $d(Tx^*_i, Tx^*_{i+1}) = D_B$, for all $i \in \overline{p}$, respectively, $d(Tx^{*^+}_i, Tx^{*^+}_{i+1}) = D_B$, for all $i \in \overline{p}$, and all the sequences $\{x_n\} \subset \bigcup_{i\in\overline{p}} A_i$ converge to a unique limit cycle $\{Tx^{*^+}_1, Tx^{*^+}_2, \ldots, Tx^{*^+}_p\}$.*

Proof. Since A_{0i} is nonempty and $T(A_{0i}) \subseteq B_{0i}$, for all $i \in \overline{p}$, then $T(A_{0i})$ and B_{0i} are nonempty, for all $i \in \overline{p}$. Also, A_{0i} is closed since B_i is approximately compact with respect to

A_i. Consider any sequences $\{x_{np+j}\}, \{\overline{x}_{np+j}\} \subseteq \bigcup_{i\in\overline{p}} A_i$ which satisfy

$$d\left(x_{np+i+1}, Tx_{np+i}\right) = d\left(\overline{x}_{np+i+1}, T\overline{x}_{np+i}\right) = D_i, \quad \forall i \in \overline{p}. \tag{25}$$

One gets, from Lemma 6(i), if the mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is GWCPD_p and, from Lemma 7(i), if the mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is either GCPD_p or WCPD_p or CPD_p that, since $T(A_{0i}) \subseteq B_{0i}$, for all $i \in \overline{p}$,

$$\begin{gathered}\left\{d\left(\overline{x}_{p(n+1)+i+j}, x_{pn+i+j}\right)\right\} \longrightarrow 0, \\ \left\{d\left(\overline{x}_{p(n+1)+i+j+1}, x_{pn+i+j+1}\right)\right\} \longrightarrow 0,\end{gathered} \tag{26}$$

$$\begin{aligned}&\left\{d\left(\overline{x}_{pn+i+j}, x_{pn+i+j+1}\right)\right\} \\ &\quad\longrightarrow D_A = d\left(y_{i+j}, A_{i+1}\right) = d\left(y_{i+j}, A_{0,i+1}\right) \\ &\quad= d\left(y_{i+j}, x_{pn_k+i+j+1}\right),\end{aligned} \tag{27}$$

$$\begin{aligned}&\left\{d\left(\overline{x}_{pn+i+j+1}, Tx_{pn+i+j}\right)\right\} \\ &\quad\longrightarrow D_i = d\left(y_{i+j+1}, B_i\right) = d\left(y_{i+j+1}, T\left(A_i\right)\right) \\ &\quad= d\left(y_{i+j+1}, T\left(A_{0i}\right)\right) = d\left(y_{i+j+1}, B_{0i}\right) \\ &\quad= d\left(y_{i+j+1}, Tx_{pn_k+i+j}\right), \quad \forall i \in \overline{p},\end{aligned} \tag{28}$$

for some $y_{i+j} \in A_{0,i+j}$ and $y_{i+j+1} \in A_{0,i+j+1}$ since $D_A = d(A_i, A_{i+1}) = d(A_{0i}, A_{0,i+1})$ and $D_i = d(A_{i+1}, B_i) = d(A_{0i+1}, B_{0i})$, for all $i \in \overline{p}$, and some subsequences

$$\begin{gathered}\left\{\overline{x}_{pn_k+i+j+1}\right\}, \left\{x_{pn_k+i+j+1}\right\} \subseteq A_{0,i+j+1}, \\ \left\{\overline{x}_{pn_k+i+j}\right\}, \left\{x_{pn_k+i+j}\right\} \subseteq A_{0,i+j}\end{gathered} \tag{29}$$

of the sequences $\{\overline{x}_{pn+i+j+1}\}, \{x_{pn+i+j+1}\} \subseteq A_{i+j+1}$ and $\{\overline{x}_{pn+i+j}\}, \{x_{pn+i+j}\} \subseteq A_{i+j}$, for all $i \in \overline{p}$, respectively, for any given initial points $x_0, \overline{x}_0 \in A_j$ for any given $j \in \overline{p}$. The following results hold.

(1) From (26) and by taking $\{\overline{x}\}_n \equiv \{x_n\}$ and $\{d(x_{p(n+1)+i+j}, x_{pn+i+j})\} \to 0$, for all $i \in \overline{p}$ for $x_0 \in A_j$ for any given $j \in \overline{p}$, one gets $\{x_{pn+i+j}\} \to x_i^*$, since A_{0i} is closed, for all $i \in \overline{p}$ and, from (27), $d(x_i^*, x_{i+1}^*) = D_A$, for all $i \in \overline{p}$.

(2) Again, from (26) $\{d(\overline{x}_{p(n+1)+i+j}, \overline{x}_{pn+i+j})\} \to 0$, for all $i \in \overline{p}$ for $x_0 \in A_j$ for any given $j \in \overline{p}$, $\{\overline{x}_{pn+i+j}\} \to \overline{x}_i^*$, for all $i \in \overline{p}$.

(3) Combining results (1) and (2) with (26), it follows that $\overline{x}_i^* = x_i^*$, for all $i \in \overline{p}$.

(4) Results (1)–(3) hold irrespective of the subset A_j for $j \in \overline{p}$ where the initial conditions of the sequences belong to, so for any $x_0, \overline{x}_0 \subset \bigcup_{i\in\overline{p}} A_i$ (see the beginning of the proof of Lemma 6). Thus, from result (3), there are unique limit points x_i^* at each subset A_i of all the sequences $\{x_n\} \subset \bigcup_{i\in\overline{p}} A_i$ such that any such sequence converges to a unique limit cycle $\{x_1^*, x_2^*, \ldots, x_p^*\}$ consisting of best proximity points of adjacent subsets A_i, for all $i \in \overline{p}$.

(5) Since B_i is closed and approximately compact with respect to A_i, for all $i \in \overline{p}$, one gets from (28) that a subsequence of $\{Tx_{pn+i+j}\}$ is convergent for each $i \in \overline{p}$; say $\{Tx_{pn_k+i+j}\} \to z_i^* \in T(A_i) \subseteq B_{0i} \subset B_i$, for all $i \in \overline{p}$. Since $\{x_{pn+i+j}\} \to x_i^*$, all its subsequences converge to the same limit so that $\{x_{pn_k+i+j}\} \to x_i^*$ and then $z_i^* = Tx_i^*$ is unique, since each x_i^* is unique, within each B_i, for all $i \in \overline{p}$ and, again, from (28), $D_i = d(x_{i+1}^*, Tx_i^*)$, for all $i \in \overline{p}$ if $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is continuous at x_i^* and $D_i = d(x_{i+1}^*, Tx_i^{*^+})$ if $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ has a finite-jump discontinuity at x_i^*, then $z_i^{*^+} = Tx_i^{*^+}$ (see Remark 13). The result has been proved. □

A further result on the existence and uniqueness of best proximity points follows for p-cyclic proximal contractions subject to Definitions 8–11 under Lemma 12 and whose proof is very close to that of Theorem 15.

Theorem 16. *Consider a complete metric space (X, d) with nonempty subsets $A_i, B_i \subset X$ and a p-cyclic mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ being either GWCPI_p or GCPI_p or WCPI_p or CPI_p, subject to set distances $D_i = d(A_{i+1}, B_i)$, $D_A = d(A_i, A_{i+1})$, and $D_B = d(B_i, B_{i+1})$, for all $i \in \overline{p}$, such that A_{0i} is nonempty and closed and B_{0i} is nonempty and A_i is approximately compact with respect to B_i and $T(A_{0i}) \subseteq B_{0i}$, for all $i \in \overline{p}$. The following properties hold.*

(i) *$T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ has a unique best proximity point x_i^* at A_i such that $d(x_i^*, x_{i+1}^*) = D_A$, for all $i \in \overline{p}$, and all the sequences $\{x_n\} \subset \bigcup_{i\in\overline{p}} A_i$ converge to a unique limit cycle $\{x_1^*, x_2^*, \ldots, x_p^*\}$.*

(ii) *Furthermore, if $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ either is continuous, respectively, has eventual finite-jump discontinuity points, then $z_i^* = Tx_i^*$, for all $i \in \overline{p}$, respectively, $z_i^{*^+} = (Tx_i^*)^+$, for all $i \in \overline{p}$, are unique best proximity points such that $d(Tx_i^*, Tx_{i+1}^*) = D_B$, for all $i \in \overline{p}$, respectively, $d(Tx_i^{*^+}, Tx_{i+1}^{*^+}) = D_B$, for all $i \in \overline{p}$, and all the sequences $\{x_n\} \subset \bigcup_{i\in\overline{p}} A_i$ converge to a unique limit cycle $\{Tx_1^{*^+}, Tx_2^{*^+}, \ldots, Tx_p^{*^+}\}$.*

Proof. Since $T(A_{0i}) \subseteq B_{0i}$, for all $i \in \overline{p}$, then $T(A_{0i})$ and B_{0i} are nonempty, since A_{0i} is nonempty, for all $i \in \overline{p}$. B_{0i} is, furthermore, closed since A_i is approximately compact with respect to B_i. Thus, A_{0i}, $T(A_{0i})$, and B_{0i} are nonempty and closed, for all $i \in \overline{p}$. Consider any sequences $\{x_{np+j}\}, \{\overline{x}_{np+j}\} \subseteq \bigcup_{i\in\overline{p}} A_i$ which satisfy

$$d\left(x_{np+i+1}, Tx_{np+i}\right) = d\left(\overline{x}_{np+i+1}, T\overline{x}_{np+i}\right) = D_i, \quad \forall i \in \overline{p}. \tag{30}$$

One gets from Lemma 12 that, since $T(A_{0i}) \subseteq B_{0i}$, for all $i \in \overline{p}$,

$$\left\{d\left(T\overline{x}_{p(n+1)+i+j}, Tx_{pn+i+j}\right)\right\} \longrightarrow 0,$$
$$\left\{d\left(T\overline{x}_{p(n+1)+i+j+1}, Tx_{pn+i+j+1}\right)\right\} \longrightarrow 0, \tag{31}$$

$$\begin{aligned} &\left\{d\left(T\overline{x}_{pn+i+j}, Tx_{pn+i+j+1}\right)\right\} \\ &\quad \longrightarrow D_B = d\left(y_{i+j}, B_{i+1}\right) = d\left(y_{i+j}, B_{0,i+1}\right) \\ &\quad = d\left(y_{i+j}, x_{pn_k+i+j+1}\right), \end{aligned} \tag{32}$$

$$\begin{aligned} &\left\{d\left(\overline{x}_{pn+i+j+1}, Tx_{pn+i+j}\right)\right\} \\ &\quad \longrightarrow D_i = d\left(y_{i+j+1}, B_i\right) = d\left(y_{i+j+1}, T\left(A_i\right)\right) \\ &\quad = d\left(y_{i+j+1}, T\left(A_{0i}\right)\right) = d\left(y_{i+j+1}, B_{0i}\right) \\ &\quad = d\left(y_{i+j+1}, Tx_{pn_k+i+j}\right), \quad \forall i \in \overline{p}, \end{aligned} \tag{33}$$

for some $y_{i+j} \in B_{0,i+j}$, $y_{i+j+1} \in B_{0,i+j+1}$ since $D_B = d(B_i, B_{i+1}) = d(B_{0i}, B_{0,i+1})$ and $D_i = d(A_{i+1}, B_i) = d(A_{0i+1}, B_{0i})$, for all $i \in \overline{p}$, and some subsequences

$$\begin{aligned} &\left\{T\overline{x}_{pn_k+i+j+1}\right\}, \left\{Tx_{pn_k+i+j+1}\right\} \subseteq T\left(A_{0,i+j+1}\right) \subseteq B_{0,i+j+1}, \\ &\left\{T\overline{x}_{pn_k+i+j}\right\}, \left\{Tx_{pn_k+i+j}\right\} \subseteq T\left(A_{0,i+j+1}\right) \subseteq B_{0,i+j} \end{aligned} \tag{34}$$

of the sequences $\{T\overline{x}_{pn+i+j+1}\}, \{Tx_{pn+i+j+1}\} \subseteq B_{i+j+1}$ and $\{T\overline{x}_{pn+i+j}\}, \{Tx_{pn+i+j}\} \subseteq B_{i+j}$, for all $i \in \overline{p}$, respectively, for any given initial points $x_0, \overline{x}_0 \in A_j$ for any given $j \in \overline{p}$. The following results hold.

(6) From (31) and by taking $\{\overline{x}\}_n \equiv \{x_n\}$, $\{d(Tx_{p(n+1)+i+j}, Tx_{pn+i+j})\} \to 0$, for all $i \in \overline{p}$ for $x_0 \in A_j$ for any given $j \in \overline{p}$, one gets $\{Tx_{pn+i+j}\} \to z_i^*$, since B_{0i} is closed, for all $i \in \overline{p}$ and, from (32), $d(z_i^*, z_{i+1}^*) = D_B$, for all $i \in \overline{p}$.

(7) Again from (31), $\{d(T\overline{x}_{p(n+1)+i+j}, T\overline{x}_{pn+i+j})\} \to 0$, for all $i \in \overline{p}$ for $x_0 \in A_j$ for any given $j \in \overline{p}$, so that $\{T\overline{x}_{pn+i+j}\} \to \overline{z}_i^*$, for all $i \in \overline{p}$.

(8) Combining results (6) and (7) with (31), it follows that $\overline{z}_i^* = z_i^*$, for all $i \in \overline{p}$.

(9) Results (6)–(9) hold irrespective of the subset A_j for $j \in \overline{p}$ where the initial conditions of the sequences belong to then for any $x_0, \overline{x}_0 \subset \bigcup_{i\in\overline{p}} A_i$. Thus, considering result (8), there are unique limit points z_i^* at each subset B_i of all the sequences $\{Tx_n\} \subset \bigcup_{i\in\overline{p}} T(A_i)$ such that any such sequence converges to a unique limit cycle $\{z_1^*, z_2^*, \ldots, z_p^*\}$ consisting of best proximity points of adjacent subsets A_i, for all $i \in \overline{p}$.

Since B_i is closed and approximately compact with respect to A_i, for all $i \in \overline{p}$, one gets from (33) that a subsequence of $\{x_{pn+i+j}\}$ is convergent for each $i \in \overline{p}$; say $\{x_{pn_k+i+j}\} \to x_i^* \in A_{0i}$, for all $i \in \overline{p}$. Since $\{Tx_{pn+i+j}\} \to z_i^*$, all its subsequences converge to the same limit so that $\{Tx_{pn_k+i+j}\} \to z_i^*$ and $x_i^* \in A_{0i}$ fulfilling $z_i^* = Tx_i^*$ which is unique. Assume not so that there are $x_i^*, \overline{x}_i^* (\neq x_i^*) \in A_{0i}$ for some $i \in \overline{p}$ such that $z_i^* = Tx_i^* = T\overline{x}_i^*$. Assume a sequence $\{x_{pn_k+i+j}\} \to x_i^* \in A_{0i}$ with $x_0 \in A_j$ and a sequence $\{x_{pn_k+i+j}\} \to \overline{x}_i^* \in A_{0i}$ with initial point $\overline{x}_{00} \in A_\ell$ and some $\overline{x}_0 = T^\delta \overline{x}_{00} \in A_j$ for some $j, \ell \in \overline{p}$ and some nonnegative integer $\delta < p$. But then $\{d(T\overline{x}_{p(n+1)+i+j}, T\overline{x}_{pn+i+j})\}$ does not converge to zero so that $\overline{x}_i^* = x_i^* \in A_{0i}$ is unique, for all $i \in \overline{p}$. The distance convergence properties are independent of the fact that for the initial condition ℓ is as equal or distinct as j, as discussed in Lemma 6. If and another sequence. Since each z_i^* is unique, within each A_i, for all $i \in \overline{p}$ and, again, from (33), $D_i = d(x_{i+1}^*, Tx_i^*)$, for all $i \in \overline{p}$ if $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ is continuous at x_i^* and $D_i = d(x_{i+1}^*, Tx_i^{*^+})$ if $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} B_i$ has a finite-jump discontinuity at x_i^*, then $z_i^{*^+} = Tx_i^{*^+}$ (see Remark 13). The result has been proved. □

Example 17. Consider a 2-cyclic proximal contraction with respect to its domain: $T : A_1 \cup A_2 \to B_1 \cup B_2$, where A_i and B_i for $i = 1, 2$ are nonempty closed subsets of $\mathbf{R}$. Take any sequences $\{x_n\} \subset A_1$, $\{y_n\} \subset A_2$ being subsequences of $\{z_n\} \subset A_1 \cup A_2$ defined either by $z_{2n} = x_n$, $z_{2n+1} = y_n$ or by $z_{2n} = y_n$, $z_{2n+1} = x_n$, for all $n \in \mathbf{Z}_{0+}$, and subject to the constraints below under the Euclidean metric $d(x, y) = |x - y|$, for all $x, y \in \mathbf{R}$ for some contractive real constant $\alpha \in [0, 1)$ such that $\alpha_1 = \alpha_2 = \alpha$ and $(\mathbf{R}, d)$ is a complete metric space and also a Banach space. Assume that $A_1 = [\underline{a}_1, \overline{a}_1]$ and $A_2 = [\underline{a}_2, \overline{a}_2]$ with $\overline{a}_1 < 0$ and $\underline{a}_2 > 0$ and $B_1 = [\underline{b}_1, \overline{b}_1]$ and $B_2 = [\underline{b}_2, \overline{b}_2]$ with $\overline{a}_1 \leq 0$ and $\underline{a}_2 \geq 0$, $\underline{b}_1 \leq \underline{a}_1$, $0 \geq \overline{b}_1 \geq \overline{a}_1$, $0 \leq \underline{b}_2 \leq \underline{a}_2$, and $\overline{b}_2 \geq \overline{a}_2$, so that $T(A_1) \subseteq B_1$, $T(A_2) \subseteq B_2$ with

$$\begin{aligned} D_A &= d\left(A_1, A_2\right) = \left|\overline{a}_1\right| + \underline{a}_2, \\ D_B &= d\left(B_1, B_2\right) = \left|\overline{b}_1\right| + \underline{b}_2, \end{aligned} \tag{35}$$

$$\begin{aligned} D_1 &= d\left(A_2, B_1\right) = \underline{a}_2 + \left|\overline{b}_1\right|, \\ D_2 &= d\left(A_1, B_2\right) = \left|\overline{a}_1\right| + \underline{b}_2, \end{aligned} \tag{36}$$

and $T : A_1 \cup A_2 \to B_1 \cup B_2$ is a CPD_2 (Definition 2 with $p = 2$) if the subsequent constraints hold for all $n \in \mathbf{Z}_{0+}$:

$$y_{n+1} + \left|Tx_n\right| = D_1, \qquad Ty_n + \left|x_{n+1}\right| = D_2, \tag{37}$$

$$\begin{aligned} &\left|x_{n+2} - x_{n+1}\right| \\ &\quad = \left|\left|x_{n+1}\right| - \left|x_{n+2}\right|\right| = \left|Ty_n - Ty_{n+1}\right| \leq \alpha \left|x_{n+1} - x_n\right|, \end{aligned} \tag{38}$$

$$\left|y_{n+2} - y_{n+1}\right| = \left|\left|Tx_n\right| - \left|Tx_{n+1}\right|\right| \leq \alpha \left|y_{n+1} - y_n\right|, \tag{39}$$

$$\begin{aligned} &\left|y_{n+1} - x_{n+1}\right| \\ &\quad = y_{n+1} + \left|x_{n+1}\right| = D_1 + D_2 - \left|Tx_n\right| - Ty_n && \text{(40a)} \\ &\quad \leq \alpha \left|y_n - x_n\right| + (1-\alpha) D_A \\ &\quad \leq \alpha \left|y_n - x_n\right| + (1-\alpha) D_A \\ &\quad = \alpha \left(y_n + \left|x_n\right|\right) + (1-\alpha) D_A. && \text{(40b)} \end{aligned}$$

In particular, (37)–(40a) and (40b) are satisfied if, for all $n \in \mathbf{Z}_{0+}$,

$$Ty_n \geq D_2 - \alpha y_n - \frac{1-\alpha}{2} D_A, \quad (41)$$

$$Tx_n \leq -D_1 - \alpha x_n + \frac{1-\alpha}{2} D_A$$
$$\left(\text{equivalently}, D_1 - \alpha \left|x_n\right| - \frac{1-\alpha}{2} D_A \leq \left|Tx_n\right|\right), \quad (42)$$

$$y_{n+1} \leq \alpha \left|x_n\right| - \frac{1-\alpha}{2} D_A,$$
$$\left|x_{n+1}\right| \leq \alpha y_n + \frac{1-\alpha}{2} D_A. \quad (43)$$

Parallel results for the case when $T : A_1 \cup A_2 \rightarrow B_1 \cup B_2$ is WCPD_2, GCPD_2, or GWCPD_2 (Definitions 3–5 with $p = 2$) can be discussed in the same way with the appropriate extensions for the contractive constant or function. It follows that $\{d(y_n, y_{n+1})\} \rightarrow 0$, $\{d(x_n, x_{n+1})\} \rightarrow 0$, $\{d(z_n, z_{n+1})\} \rightarrow D_A$, $\{y_n\} \rightarrow \underline{a}_2$, $\{x_n\} \rightarrow \overline{a}_1$, $\{d(Ty_n, Ty_{n+1})\} \rightarrow 0$, $\{d(Tx_n, Tx_{n+1})\} \rightarrow 0$ and, according to (40a), (40b), and (35)-(36), $\{d(Tz_n, Tz_{n+1})\} \rightarrow D_B$, since $\{|T(y_n - |x_n|)|\} \rightarrow D_B = D_1 + D_2 - D_A$, $\{Ty_n\} \rightarrow T\underline{a}_2 = \underline{b}_2$, and $\{Tx_n\} \rightarrow T\overline{a}_1 = \overline{b}_1$.

Example 18. Consider Example 17 in the case that $T : A_1 \cup A_2 \rightarrow B_1 \cup B_2$ is CPI_2, WCPI_2, GCPI_2, or GWCPI_2 (Definitions 8–11 with $p = 2$); (40a) using (37) can be reformulated accordingly. In particular, if it is CPI_2, then one gets for some real constant $\beta \in [0, 1)$

$$\begin{aligned} &\left|Tx_{n+1}\right| + Ty_{n+1} \\ &\quad = Ty_{n+1} - Tx_{n+1} = D_1 + D_2 - \left|x_{n+2}\right| - y_{n+2} \\ &\quad = D_A + D_B - \left|x_{n+2}\right| - y_{n+2} \\ &\quad \leq \beta\left(\left|x_n\right| + y_n\right) + (1-\beta) D_B, \quad \forall n \in \mathbf{Z}_{0+}. \end{aligned} \quad (44)$$

Then, $\{|T(y_n - |x_n|)|\} \equiv \{d(Tz_n, Tz_{n+1})\} \rightarrow D_B = D_1 + D_2 - D_A$, $\{|x_n| + y_n\} \equiv \{d(z_n, z_{n+1})\} \rightarrow D_A$, $\{d(y_n, y_{n+1})\} \rightarrow 0$, $\{d(x_n, x_{n+1})\} \rightarrow 0$, $\{Ty_n\} \rightarrow T\underline{a}_2 = \underline{b}_2$, $\{Tx_n\} \rightarrow T\overline{a}_1 = \overline{b}_1$, $\{y_n\} \rightarrow \underline{a}_2$, and $\{x_n\} \rightarrow \overline{a}_1$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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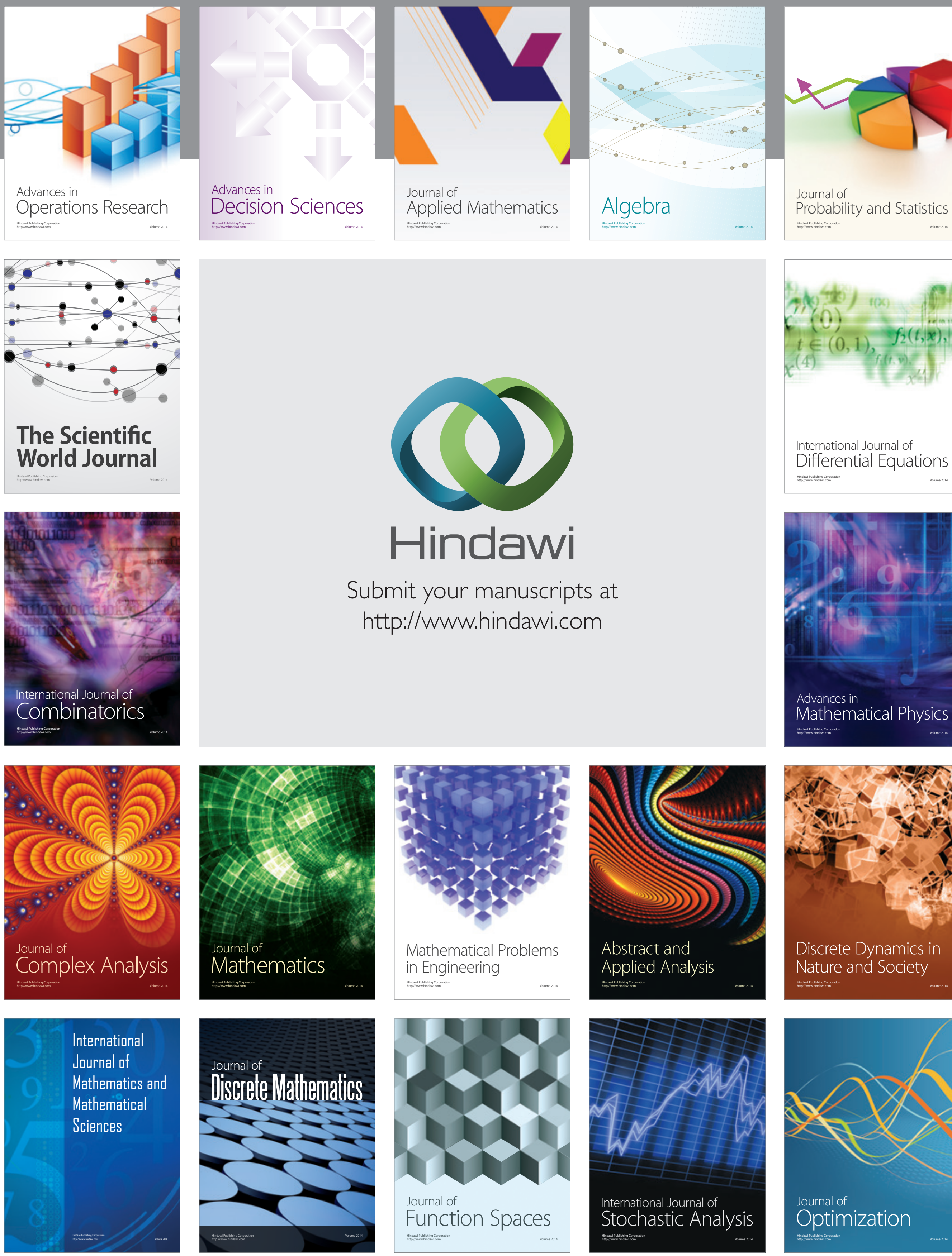

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