A NOTE ON COULHON TYPE INEQUALITIES

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ABSTRACT. T. Coulhon introduced an interesting reformulation of the usual Sobolev inequalities. We characterize Coulhon type inequalities in terms of rearrangement inequalities.

1. INTRODUCTION

Let (X, d, μ) be a connected Borel metric measure space. The perimeter or Minkowski content of a Borel set $A \subset X$, is defined by

$$\mu^+(A) = \liminf_{h \to 0} \frac{\mu(A_h) - \mu(A)}{h},$$

where $A_h = \{x \in \Omega : d(x, A) < h\}$, and the isoperimetric profile $I = I_{(\Omega, d, \mu)}$ is defined by

$$I_{(\Omega, d, \mu)}(t) = \inf_A \{\mu^+(A) : \mu(A) = t\}.$$

We assume throughout that (X, d, μ) is such that $I_{(\Omega, d, \mu)}$ is concave, continuous with $I(0) = 0$. Moreover, we also assume that (X, d, μ) is such that for each $c \in R$, and each $f \in Lip_0(X), |\nabla f(x)| = 0, a.e.$ in the set $\{x : f(x) = c\}$. Under these conditions[1] we showed in [18] that the Gagliardo-Nirenberg-Ledoux inequality

$$(1.1) \qquad \int_0^\infty I(\mu_f(t)) dt \leq \||\nabla f|\|_{L^1(X)}, \text{ for all } f \in Lip_0(X)$$

is equivalent to

$$(1.2) \qquad f^{**}(t) - f^*(t) \leq \frac{t}{I(t)} |\nabla f|^{**}(t),$$

where $Lip_0(X)$ are the functions in $Lip(X)$ of compact support,

$$|\nabla f(x)| = \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)},$$

[1]In [18] the result is shown for metric probability spaces such that $I(t)$ is symmetric about $1/2$, in which case we can replace $Lip_0(X)$ by $Lip(X)$ in the statement. With minor modifications one can also show its validity for infinite measure spaces (cf. also [22], [19]).

$\mu_f(t) = \mu\{|f| > t\}$, f^* is the non increasing rearrangement[2] of f with respect to the measure μ and $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$.

Conversely, if an inequality of the form (1.2) holds for some continuous concave function $I_1(t)$, it was shown in [18] that $I_1(t)$ satisfies the isoperimetric inequality $I_1(\mu(A)) \leq \mu^+(A)$ for any Borel set[3] $A \subset\subset X$. In particular, for $\mathbb{R}^n$ it is well known that (cf. [23, Chapter 1]) $I(t) = c_n t^{1-1/n}$, and therefore (1.2) becomes (cf. [4] and the references therein)

$$(1.3) \qquad f^{**}(t) - f^*(t) \leq c_n^{-1} t^{1/n} |\nabla f|^{**}(t).$$

It follows that (1.1) gives

$$c_n \int_0^\infty \mu_f(t)^{1-1/n}dt = c_n \frac{1}{n'} \int_0^\infty t^{1/n'} f^*(t)\frac{dt}{t} \leq \||\nabla f|\|_{L^1(\mathbb{R}^n)}$$

i.e.

$$\|f\|_{L^{\frac{n}{n-1},1}(\mathbb{R}^n)} \leq c \||\nabla f|\|_{L^1(\mathbb{R}^n)}, \text{ for all } f \in Lip_0(\mathbb{R}^n).$$

In other words, (1.1) represents a generalization of the sharp form of the Euclidean Gagliardo-Nirenberg inequality that uses Lorentz spaces (cf. [24] and [22] (for Euclidean spaces), [16] (Gaussian spaces), and [6], [18] (for metric spaces); for the corresponding rearrangement inequalities we refer to [4], [20], [18], as well as the references therein).

The corresponding Sobolev inequalities when $|\nabla f| \in L^p$, $p > 1$ are also known to self improve (cf. [23], [2], [19], and the references therein) but an analogous rearrangement inequality characterization in this case has remained an open problem. On the other hand, Coulhon (cf. [9], [8], [7]) and Bakry-Coulhon-Ledoux [2] introduced and studied a different scale of Sobolev inequalities. For $p \in [1, \infty]$, and ϕ an increasing function on the positive half line, these authors studied the validity of inequalities of the form

$$(S_\phi^p) \qquad \|f\|_p \leq \phi(\|f\|_0) \||\nabla f|\|_p, \ f \in Lip_0(X),$$

where

$$\|f\|_0 = \mu\{support(f)\}, \ \|f\|_p = \|f\|_{L^p(X)}.$$

In particular, it was shown by Coulhon et al. that the (S_ϕ^p) inequalities encapsulate the classical Sobolev inequalities, as well as the Faber-Krahn inequalities. For $p = 1$, (S_ϕ^1) is equivalent to the isoperimetric inequality in the sense that[4]

$$\frac{t}{I(t)} \leq \phi(t).$$

Moreover, for $p = \infty$, the (S_ϕ^∞) conditions are explicitly connected with volume growth. For a detailed discussion of the different geometric interpretations for different $p's$ we refer to [9], [12], [23], and the references quoted therein.

It follows from this discussion that, for a suitable class of metric measure spaces, the (S_ϕ^1) condition can be characterized by means of the symmetrization inequality (1.2):

$$(S_\phi^1) \text{ holds } \Leftrightarrow (1.2) \text{ holds.}$$

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[2]For background we refer to [5] (on rearrangements), and [17], [23] (on Sobolev spaces) .

[3]Therefore, $I_1(t) \leq \inf\{\mu^+(A) : \mu(A) = t\} = I(t)$, and consequently $\frac{t}{I(t)} \leq \frac{t}{I_1(t)}$.

[4]See Section 2 below.

The purpose of this paper is to provide an analogous rearrangement characterization of the (S_ϕ^p) conditions, for $1 \le p < \infty$. Our main result extends (1.2) as follows

Theorem 1. *Let (X, d, μ) be a connected Borel metric measure space as described above, and let $p \in [1, \infty)$. The following statements are equivalent*

(1) (S_ϕ^p) *holds, i.e.*

(1.4) $$\|f\|_p \le \phi(\|f\|_0) \, \||\nabla f|\|_p \,, \ \text{for all } f \in Lip_0(X).$$

(2) *Let $k \in \mathbb{N}$ be such that $k < p \le k+1$, then for all $f \in Lip_0(X)$*

(1.5) $$\left(\frac{f_{(p)}^{**}(t)}{\phi_{(p)}(t)} \right)^{1/p} - \left(\frac{f_{(p)}^{*}(t)}{\phi_{(p)}(t)} \right)^{1/p} \le 2^{\frac{k+1}{p}-1} \left(|\nabla f|_{(p)}^{**}(t) \right)^{1/p},$$

where

$$f_{(p)}^{*}(t) = (f^*(t))^p \,, \ f_{(p)}^{**}(t) = \frac{1}{t} \int_0^t f_{(p)}^{*}(s)ds, \ \phi_{(p)}(t) = (\phi(t))^p \,.$$

(3) *Let $k \in \mathbb{N}$ be such that $k < p \le k+1$, then for all $f \in Lip_0(X)$, $f_{(p)}^{*}$ is absolutely continuous (cf. [17]) and*

(1.6) $$-\frac{\partial}{\partial t} \left(f_{(p)}^{**}(t) \right)^{1/p} = -\frac{\partial}{\partial t} \left(\frac{1}{t} \int_0^t f_{(p)}^{*}(s)ds \right)^{1/p} \le 2^{\frac{k+1}{p}} \frac{\phi(t)}{t} \left(|\nabla f|_{(p)}^{**}(t) \right)^{\frac{1}{p}}.$$

Note that for $p = 1$ the inequality (1.5) of Theorem 1 coincides with (1.2). This new characterization for $p \ge 1$ is independent of [18], and, in fact, it provides a new approach to (1.2) as well. On the other hand, as it is well known (cf. [9]), the (S_ϕ^p) conditions get progressively weaker as p increases. Indeed, below we will also show that (1.2) implies (1.5) via an extended form of the chain rule, that is valid for metric spaces.

The note is organized as follows. In section 2 we give a somewhat more detailed discussion of the (S_ϕ^p) conditions and, in particular, we develop a connection with [18]. In section 3 we provide a proof of Theorem 1 and, finally, in section 4, we discuss, rather briefly, connections with Nash type inequalities, Sobolev and Faber-Krahn inequalities and interpolation/extrapolation theory.

As usual, the symbol $f \simeq g$ will indicate the existence of a universal constant $c > 0$ (independent of all parameters involved) so that $(1/c)f \le g \le c\,f$, while the symbol $f \preceq g$ means that $f \le c\,g$.

2. The (S_ϕ^p) conditions

From now on (X, d, μ) will be a connected metric measure space with a continuous isoperimetric profile I such that $\frac{t}{I(t)}$ increases and such that $I(0) = 0$. Moreover, we also assume that (X, d, μ) is such that for each $c \in \mathbb{R}$, and each $f \in Lip_0(X), |\nabla f(x)| = 0, a.e.$ in the set $\{x : f(x) = c\}$. The isoperimetric profile $I = I_{(\Omega, d, \mu)}$ is defined by

$$I_{(\Omega, d, \mu)}(t) = \inf_A \{ \mu^+(A) : \mu(A) = t \},$$

where $\mu^+(A)$ is the perimeter or Minkowski content of the Borel set $A \subset X$, defined by

$$\mu^+(A) = \liminf_{h \to 0} \frac{\mu(A_h) - \mu(A)}{h},$$

where $A_h = \{x \in \Omega : d(x, A) < h\}$.

2.1. The (S_ϕ^1) condition. From [18] (cf. also [19]) we know that

$$(2.1) \qquad f^{**}(t) - f^*(t) \leq \frac{t}{I(t)} |\nabla f|^{**}(t), \ f \in Lip_0(X),$$

is equivalent to the isoperimetric inequality. If we combine these results with the characterization of (S_ϕ^1) given in [7] we can see the equivalence between (2.1) and the (S_ϕ^1) condition. To understand the discussion of the next section it is instructive to provide an elementary direct approach. So we shall now show that (2.1) implies (S_ϕ^1) with $\phi(t) = t/I(t)$, and that this choice is in some sense the best possible (S_ϕ^1) condition.

Suppose that (2.1) holds. Multiplying both sides of (2.1) by $t > 0$ we obtain

$$t \left(f^{**}(t) - f^*(t) \right) \leq \frac{t}{I(t)} \int_0^t |\nabla f|^* (s) ds.$$

Since formally $f^*(t) = \mu_f^{-1}(t)$, drawing a diagram it is easy to convince oneself that

$$t \left(f^{**}(t) - f^*(t) \right) = \int_0^t f^*(s) ds - t f^*(t)$$

$$= \int_{f^*(t)}^\infty \mu_f(s) ds.$$

Consequently, if we let $t = \|f\|_0$, we see that $f^*(\|f\|_0) = 0$, $\int_{f^*(\|f\|_0)}^\infty \mu_f(s) ds = \|f\|_1$, and $\int_0^{\|f\|_0} |\nabla f|^* (s) ds = \||\nabla f|\|_1$. Thus,

$$\|f\|_1 \leq \frac{\|f\|_0}{I(\|f\|_0)} \||\nabla f|\|_1.$$

In other words, the (S_ϕ^1) condition holds with $\phi(t) = \frac{t}{I(t)}$, and consequently the $(S_{\tilde{\phi}}^1)$ condition holds for any $\tilde{\phi}(t) \geq \frac{t}{I(t)}$. On the other hand, consider an $(S_{\tilde{\phi}}^1)$ condition for a continuous, increasing but arbitrary function $\tilde{\phi}$. Let A be a Borel set, $A \subset\subset X$, with $\mu(A) = t$. Formally inserting $f = \chi_A$ in the corresponding $(S_{\tilde{\phi}}^1)$ inequality (this is done rigorously by approximation), yields

$$\|\chi_A\|_1 = t = \mu(A) \leq \tilde{\phi}(t) \mu^+(A).$$

Consequently,

$$\frac{t}{\tilde{\phi}(t)} \leq \inf\{\mu^+(B) : \mu(B) = t\}$$

$$= I(t),$$

and therefore

$$\frac{t}{I(t)} \leq \tilde{\phi}(t).$$

2.2. $(S_\phi^1) \Rightarrow (S_\phi^p)$, $p > 1$. In the Euclidean space $\mathbb{R}^n$, $I(t) = d_n t^{1-1/n}$, $\phi(t) \simeq t^{1/n}$ and the best possible (S_ϕ^1) inequality can be written as

$$\|f\|_1 \le c_n \|f\|_0^{1/n} \||\nabla f|\|_1 .$$

As was shown in [9] the corresponding inequalities for $p > 1$ then follow by the (classical) chain rule, the fact that $\||f|^p\|_0 = \||f|\|_0 = \|f\|_0$, and Hölder's inequality. In detail,

$$\|f\|_p^p = \||f|^p\|_1$$
$$\le c_n p \|f\|_0^{1/n} \||f|^{p-1} |\nabla |f||\|_1$$
$$\le c_n p \|f\|_0^{1/n} \|f\|_p^{p-1} \||\nabla |f|\|_p .$$

Consequently,

$$\|f\|_p \le c_n p \|f\|_0^{1/n} \||\nabla f|\|_p ,$$

and therefore, modulo constants, we have that $(S_\phi^1) \Rightarrow (S_\phi^p)$, for $p > 1$. More generally, this argument, taken from [9], shows that the (S_ϕ^p) conditions become weaker as p increases. In the general setting of metric spaces, the classical chain rule needs to be replaced by an inequality[5]: for $r > 1$,

$$(2.2) \qquad\qquad |\nabla f^r(x)| \le 2r |f^{r-1}(x)| |\nabla f(x)| .$$

Next, we use the generalized chain rule to explain the origin of the awkward looking condition (1.5). Informally, we shall now show that[6] $(S_\phi^1) \Rightarrow (S_\phi^p)$ at the level of rearrangements, i.e. (1.2)$\Rightarrow$(1.5).

Assume the validity of (S_ϕ^1). Let $f \in Lip_0(X)$; we may assume without loss that f is positive. Apply the (S_ϕ^1) inequality to $f_{(p)} = f^p$, where $p > 1$ is fixed. Then, by the chain rule (2.2)

$$f_{(p)}^{**}(t) - f_{(p)}^*(t) \preceq \phi(t) |\nabla f|_{(p)}^{**}(t)$$
$$\preceq \phi(t)(f^{p-1} |\nabla f|)^{**}(t).$$

By a result due to O'Neil (cf. [5, page 88, Exercise 10]) and Hölder's inequality

$$(f^{p-1} |\nabla f|)^{**}(t) \le \frac{1}{t} \int_0^t (f^*(s))^{p-1} |\nabla f|^*(s)ds$$
$$\le \frac{1}{t} \left(\int_0^t f_{(p)}^*(s)ds \right)^{1/p'} \left(\int_0^t |\nabla f|_{(p)}^*(s)ds \right)^{1/p}$$
$$= \left(f_{(p)}^{**}(t) \right)^{1-1/p} \left(|\nabla f|_{(p)}^{**}(t) \right)^{1/p} .$$

Combining inequalities we obtain,

$$f_{(p)}^{**}(t) - f_{(p)}^*(t) \preceq \phi(t) \left(f_{(p)}^{**}(t) \right)^{1-1/p} \left(|\nabla f|_{(p)}^{**}(t) \right)^{1/p} .$$

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[5]The underlying elementary inequality is

$$|a^r - b^r| \le r |a^{r-1} + b^{r-1}| |a - b| .$$

[6]With slightly more labor the same method will similarly show that, more generally, $(S_\phi^p) \Rightarrow (S_\phi^q)$, for $q > p$.

Hence,

$$\left(f_{(p)}^{**}(t)\right)^{1/p} - \frac{f_{(p)}^{*}(t)}{\left(|f|_{(p)}^{**}(t)\right)^{1/p'}} \preceq \phi(t)\left(|\nabla f|_{(p)}^{**}(t)\right)^{1/p}.$$

But, since

$$\left(|f|_{(p)}^{**}(t)\right)^{1/p'} \geq \left(|f|_{(p)}^{*}(t)\right)^{1/p'} = \left(|f|_{(p)}^{*}(t)\right)^{1-1/p},$$

we have

$$\left(f_{(p)}^{*}(t)\right)^{1/p} \geq \frac{f_{(p)}^{*}(t)}{\left(|f|_{(p)}^{**}(t)\right)^{1/p'}},$$

and we conclude that

$$\left(f_{(p)}^{**}(t)\right)^{1/p} - \left(f_{(p)}^{*}(t)\right)^{1/p} \preceq \left(f_{(p)}^{**}(t)\right)^{1/p} - \frac{f_{(p)}^{*}(t)}{\left(|f|_{(p)}^{**}(t)\right)^{1/p'}}$$

$$\preceq \phi(t)\left(|\nabla f|_{(p)}^{**}(t)\right)^{1/p}.$$

Therefore,

$$\left(\frac{f_{(p)}^{**}(t)}{\phi_{(p)}(t)}\right)^{1/p} - \left(\frac{f_{(p)}^{*}(t)}{\phi_{(p)}(t)}\right)^{1/p} \preceq \left(|\nabla f|_{(p)}^{**}(t)\right)^{1/p},$$

and (1.5) holds.

3. Proof of Theorem 1

Before going through the proof let us make a few useful remarks. Let $[x]_+ = \max(x, 0)$, and let $f \geq 0$, then, for all $\lambda > 0$, we have

(3.1)

$$\int_{\{f>\lambda\}} (f(s) - \lambda)\, d\mu(s) = \int [f(s) - \lambda]_+\, d\mu(s) = \int_0^\infty [f^*(s) - \lambda]^+\, ds$$

$$= \int_0^\infty \mu_{[f^*-\lambda]_+}(s)\, ds = \int_\lambda^\infty \mu_{f^*}(s)\, ds = \int_\lambda^{\|f\|_\infty} \mu_f(s)\, ds.$$

Thus, inserting $\lambda = f^*(t)$ in (3.1), and taking into account that f^* is decreasing, we obtain

$$t(f^{**}(t) - f^*(t)) = \int_0^t (f^*(x) - f^*(t))\, dx = \int_0^\infty [f^*(x) - f^*(t)]_+\, dx$$

$$= \int_{\{f>f^*(t)\}} [f(s) - f^*(t)]_+\, d\mu(s).$$

In order to deal with L^p norms, $p > 1$, we need to extended the formulae above. This will be achieved through the following variant of the binomial formula, whose proof will be provided at the end of this section.

Lemma 1. *Let $p > 1$, and let $k \in \mathbb{N}$ be such that $k < p \leq k + 1$. Then, for $a \geq b \geq 0$,*

(3.2)
$$(a - b)^p \geq a^p - b^p - \sum_{j=1}^{k} \binom{p}{j} b^{p-j}(a-b)^j,$$

and

$$(3.3) \qquad a^p + b^p + \sum_{j=1}^{k} \binom{p}{j} b^{p-j}(a-b)^j \leq (c(p)a+b)^p,$$

where $c(p) = 2^{\frac{k+1}{p}-1}$.

We are now ready to give the proof of Theorem 1.

Proof. $1 \to 2$. Suppose that (S_ϕ^p) holds. We may assume without loss that f is positive. Let $t > 0$; we will apply (1.4) to $[f - f^*(t)]_+$. Observe that

$$\left\| [f - f^*(t)]_+ \right\|_0 = \mu\{f > f^*(t)\} \leq t,$$

and, moreover, since $\int_{\{f=f^*(t)\}} |\nabla [f(x) - f^*(t)]| \, dx = 0$,

$$\left\| \nabla [f - f^*(t)]_+ \right\|_{L^p}^p = \int_{\{f>f^*(t)\}} \left(|\nabla f|^*(s) \right)^p ds.$$

Therefore,

$$\left\| [f - f^*(t)]_+ \right\|_p^p \leq \left\{ \phi(\left\| [f - f^*(t)]_+ \right\|_0) \right\}^p \left\| \nabla [f - f^*(t)]_+ \right\|_{L^p}^p$$

$$\leq \phi(t)^p \int_{\{f>f^*(t)\}} \left(|\nabla f|^*(s) \right)^p ds$$

$$\leq t\phi(t)^p \left(\frac{1}{t} \int_0^t \left(|\nabla f|^*(s) \right)^p ds \right)$$

$$(3.4) \qquad = t\phi(t)^p |\nabla f|_{(p)}^{**}(t).$$

Now,

$$\left\| [f - f^*(t)]_+ \right\|_p^p = \int_{\{f>f^*(t)\}} (f(s) - f^*(t))^p \, d\mu(s)$$

$$\geq \int_{\{f>f^*(t)\}} (f^p(s) - f^*(t)^p) \, d\mu(s)$$

$$- \sum_{j=1}^{k} \binom{p}{j} f^*(t)^{p-j} \int_{\{f>f^*(t)\}} (f(s) - f^*(t))^j d\mu(s) \quad \text{(by (3.2))}$$

$$= \int_{\{f_{(p)}>f_{(p)}^*(t)\}} \left(f_{(p)}(s) - f_{(p)}^*(t) \right) d\mu(s)$$

$$- \sum_{j=1}^{k} \binom{p}{j} f^*(t)^{p-j} \int_{\{f>f^*(t)\}} (f(s) - f^*(t))^j d\mu(s)$$

$$(3.5)$$

$$= t \left(f_{(p)}^{**}(t) - f_{(p)}^*(t) \right) - \sum_{j=1}^{k} \binom{p}{j} f^*(t)^{p-j} \int_{\{f>f^*(t)\}} (f(s) - f^*(t))^j d\mu(s).$$

We estimate each of the integrals in the sum using Hölder's inequality as follows,

$$
\int_{\{f>f^*(t)\}} (f(s) - f^*(t))^j d\mu(s) \leq \left(\int_{\{f>f^*(t)\}} (f(s) - f^*(t))^p d\mu(s) \right)^{\frac{j}{p}} \left(\int_{\{f>f^*(t)\}} d\mu(s) \right)^{\frac{p-j}{p}}
$$

$$
= \left(\int_{\{f>f^*(t)\}} (f(s) - f^*(t))^p d\mu(s) \right)^{\frac{j}{p}} (\mu_f(f^*(t)))^{\frac{p-j}{p}}
$$

$$
\leq \left(\int_{\{f>f^*(t)\}} (f(s) - f^*(t))^p d\mu(s) \right)^{\frac{j}{p}} t^{\frac{p-j}{p}}
$$

$$
= \left\| [f - f^*(t)]_+ \right\|_p^j t^{\frac{p-j}{p}}
$$

$$
\leq \phi(t)^j \left(|\nabla f|_{(p)}^{**}(t) \right)^{\frac{j}{p}} t^{\frac{j}{p}} t^{\frac{p-j}{p}} \quad \text{(by (3.4))}
$$

$$
(3.6) \qquad = t\phi(t)^j \left(|\nabla f|_{(p)}^{**}(t) \right)^{\frac{j}{p}}.
$$

Combining (3.5) and (3.6) we get

$$
\left\| [f - f^*(t)]_+ \right\|_p^p = t \left(f_{(p)}^{**}(t) - f_{(p)}^*(t) \right) - \sum_{j=1}^{p-1} \binom{p}{j} f^*(t)^{p-j} \int_{\{f>f^*(t)\}} (f^*(t) - f(s))^j d\mu(s)
$$

$$
\geq t \left(f_{(p)}^{**}(t) - f_{(p)}^*(t) \right) - t \left(\sum_{j=1}^{p-1} \binom{p}{j} f^*(t)^{p-j} \phi(t)^j \left(|\nabla f|_{(p)}^{**}(t) \right)^{\frac{j}{p}} \right).
$$

Therefore, we see that

$$
t \left(f_{(p)}^{**}(t) - f_{(p)}^*(t) \right) \leq \left\| [f - f^*(t)]_+ \right\|_p^p + t \left(\sum_{j=1}^{k} \binom{p}{j} f^*(t)^{p-j} \phi(t)^j \left(|\nabla f|_{(p)}^{**}(t) \right)^{\frac{j}{p}} \right)
$$

$$
\leq t\phi(t)^p |\nabla f|_{(p)}^{**}(t) + t \left(\sum_{j=1}^{k} \binom{p}{j} f^*(t)^{p-j} \phi(t)^j \left(|\nabla f|_{(p)}^{**}(t) \right)^{\frac{j}{p}} \right) \quad \text{(by (3.4))}
$$

$$
= t\phi_{(p)}(t) \left(|\nabla f|_{(p)}^{**}(t) + \sum_{j=1}^{k} \binom{p}{j} \left(\frac{f_{(p)}^*(t)}{\phi_{(p)}(t)} \right)^{\frac{p-j}{p}} \left(|\nabla f|_{(p)}^{**}(t) \right)^{\frac{j}{p}} \right).
$$

Consequently,

$$
(3.7) \qquad \frac{f_{(p)}^{**}(t) - f_{(p)}^*(t)}{\phi_{(p)}(t)} \leq |\nabla f|_{(p)}^{**}(t) + \sum_{j=1}^{k} \binom{p}{j} \left(\frac{f_{(p)}^*(t)}{\phi_{(p)}(t)} \right)^{\frac{p-j}{p}} \left(|\nabla f|_{(p)}^{**}(t) \right)^{\frac{j}{p}}.
$$

We can rewrite (3.7) as

$$
\frac{f_{(p)}^{**}(t) - f_{(p)}^*(t)}{\phi_{(p)}(t)} \leq |\nabla f|_{(p)}^{**}(t) + \sum_{j=1}^{k} \binom{p}{j} \left(\frac{f_{(p)}^*(t)}{\phi_{(p)}(t)} \right)^{\frac{p-j}{p}} \left(|\nabla f|_{(p)}^{**}(t) \right)^{\frac{j}{p}} + \frac{f_{(p)}^*(t)}{\phi_{(p)}(t)} - \frac{f_{(p)}^*(t)}{\phi_{(p)}(t)}
$$

$$
= \left(2^{\frac{k+1}{p}-1} \left(|\nabla f|_{(p)}^{**}(t) \right)^{1/p} + \left(\frac{f_{(p)}^*(t)}{\phi_{(p)}(t)} \right)^{1/p} \right)^p - \frac{f_{(p)}^*(t)}{\phi_{(p)}(t)} \quad \text{(by (3.3))}
$$

Hence

$$\frac{f_{(p)}^{**}(t)}{\phi_{(p)}(t)} \leq \left(2^{\frac{k+1}{p}-1}\left(|\nabla f|_{(p)}^{**}(t)\right)^{1/p} + \left(\frac{f_{(p)}^{*}(t)}{\phi_{(p)}(t)}\right)^{1/p}\right)^{p},$$

yielding

$$\left(\frac{f_{(p)}^{**}(t)}{\phi_{(p)}(t)}\right)^{1/p} \leq 2^{\frac{k+1}{p}-1}\left(|\nabla f|_{(p)}^{**}(t)\right)^{1/p} + \left(\frac{f_{(p)}^{*}(t)}{\phi_{(p)}(t)}\right)^{1/p}.$$

Summarizing, we have obtained

$$\left(\frac{f_{(p)}^{**}(t)}{\phi_{(p)}(t)}\right)^{1/p} - \left(\frac{f_{(p)}^{*}(t)}{\phi_{(p)}(t)}\right)^{1/p} \leq 2^{\frac{k+1}{p}-1}\left(|\nabla f|_{(p)}^{**}(t)\right)^{1/p}.$$

$2 \to 3$. Once again we use the elementary inequality

$$(x^{p} - y^{p}) \leq p(x - y)\left(x^{p-1} + y^{p-1}\right), \quad (x \geq y \geq 0),$$

with $x = \left(f_{(p)}^{**}(t)\right)^{1/p}$ and $y = \left(f_{(p)}^{*}(t)\right)^{1/p}$. We obtain,

$$f_{(p)}^{**}(t) - f_{(p)}^{*}(t) \leq p\left(\left(f_{(p)}^{**}(t)\right)^{1/p} - \left(f_{(p)}^{*}(t)\right)^{1/p}\right)\left(\left(f_{(p)}^{**}(t)\right)^{\frac{p-1}{p}} + \left(f_{(p)}^{*}(t)\right)^{\frac{p-1}{p}}\right)$$

$$\leq p2^{\frac{k+1}{p}-1}\phi(t)\left(|\nabla f|_{(p)}^{**}(t)\right)^{1/p}\left(\left(f_{(p)}^{**}(t)\right)^{\frac{p-1}{p}} + \left(f_{(p)}^{*}(t)\right)^{\frac{p-1}{p}}\right) \quad \text{by (1.5)}$$

$$\leq p2^{\frac{k+1}{p}-1}\phi(t)\left(|\nabla f|_{(p)}^{**}(t)\right)^{1/p}\left(2\left(f_{(p)}^{**}(t)\right)^{\frac{p-1}{p}}\right).$$

Consequently,

$$\frac{1}{p}\left(f_{(p)}^{**}(t)\right)^{\frac{1}{p}-1}\left(f_{(p)}^{**}(t) - f_{(p)}^{*}(t)\right) \leq p2^{\frac{k+1}{p}}\phi(t)\left(|\nabla f|_{(p)}^{**}(t)\right)^{\frac{1}{p}}.$$

Now observe that

$$\frac{1}{p}\left(f_{(p)}^{**}(t)\right)^{\frac{1}{p}-1}\left(\frac{f_{(p)}^{**}(t) - f_{(p)}^{*}(t)}{t}\right) = -\frac{\partial}{\partial t}\left(\frac{1}{t}\int_{0}^{t}(f^{*}(s))^{p}\,ds\right)^{1/p}.$$

$3 \to 1$.

Let $\Omega \subset\subset X$, and let $f \in Lip_{0}(\Omega)$, then, for $t = \mu(\Omega)$, we have

$$f_{(p)}^{**}(t) = \frac{1}{t}\int_{0}^{t}(f^{*}(t))^{p}\,dt = \frac{1}{t}\|f\|_{p}^{p}$$

and, similarly,

$$|\nabla f|_{(p)}^{**}(t) = \frac{1}{t}\||\nabla f|\|_{p}^{p}.$$

Since

$$f_{(p)}^{*}(\mu(\Omega)) = \inf_{x \in \Omega}|f(x)|^{p} = 0,$$

the inequality (1.6) becomes

$$\frac{1}{t}\|f\|_{p}^{p} \leq p2^{\frac{k+1}{p}}\phi(\mu(\Omega))\frac{1}{t}\||\nabla f|\|_{p}\|f\|_{p}^{p-1},$$

which is (1.4), up to constants. □

To complete the proof it remains to prove Lemma 1.

Proof. (of Lemma 1) We prove (3.2). Towards this end let us define

$$f(x) = (x-b)^p - x^p + b^p + \sum_{j=1}^{k} \binom{p}{j} b^{p-j}(x-b)^j, \quad (x \geq b).$$

An elementary computation shows that $f(b) = \frac{\partial}{\partial x} f(b) = \frac{\partial^{k-1}}{\partial x} f(b) = 0$. Moreover, since

$$\frac{\partial^k}{\partial x} f(x) = p(p-1)\ldots(p-k+1)\left((x-b)^{p-k} - x^{p-k} + b^{p-k}\right),$$

and $0 < p - k \leq 1$, we see that

$$(x-b)^{p-k} - x^{p-k} + b^{p-k} \geq 0,$$

consequently,

$$f(x) \geq f(b) = 0.$$

To see (3.3) let us write $a = xb$ $(x \geq 1)$. We would like to show that

$$g(x) = (c(p)x+1)^p - x^p - 1 - \sum_{j=1}^{k} \binom{p}{j}(x-1)^j \geq 0.$$

An easy computation shows that $g(1) \geq 0, \frac{\partial}{\partial x} g(1) \geq 0, \cdots \frac{\partial^{k-1}}{\partial x} g(1) \geq 0$, and $\frac{\partial^k}{\partial x} g(1) \geq 0$. Therefore, it will be enough to prove that $\frac{\partial^{k+1}}{\partial x} g(x) \geq 0$. Again, by computation, we find that

$$\frac{\partial^{k+1}}{\partial x} g(x) = p(p-1)\ldots(p-k+1)(p-k)\left(c(p)^{k+1}(c(p)x+1)^{p-k-1} - x^{p-k-1}\right).$$

Therefore the desired result will follow if we show that

$$c(p)^{k+1}(c(p)x+1)^{p-k-1} - x^{p-k-1} \geq 0.$$

Since $p - k - 1 < 0$, this amounts to show

$$\frac{c(p)^{k+1}}{(c(p)x+1)^{k+1-p}} \geq \frac{1}{x^{k+1-p}} \Leftrightarrow \frac{c(p)^{\frac{k+1}{k+1-p}}}{c(p)x+1} \geq \frac{1}{x} \Leftrightarrow xc(p)\left(c(p)^{\frac{k+1}{k+1-p}} - 1\right) \geq 1.$$

But since

$$c(p)^{\frac{k+1}{k+1-p}} - 1 \geq 1 \Leftrightarrow c(p) \geq 2^{\frac{k+1}{p}-1},$$

the desired result follows. $\square$

4. Final Remarks

In this section we show the explicit connection of our rearrangement inequalities with the classical Sobolev inequalities, the Nash and Faber-Krahn inequalities and point out possible directions for future research. In particular, using Coulhon inequalities we will show a direct approach to some self-improving properties of Sobolev inequalities for $p > 1$.

4.1. **Nash Inequalities.** We start by giving a rearrangement characterization of the Nash type inequalities. It was shown in [2] (cf. also [9]), that the (S_ϕ^p) conditions are equivalent to Nash type inequalities. As a consequence, the results of this paper give a characterization of Nash inequalities in terms of rearrangements which we shall now describe.

We first observe that, with some trivial changes, one can adapt the proof of Proposition 2.4 in [9] (case $p = 2$) to obtain the following equivalence (for Nash inequalities for $p > 1$)

Proposition 1. *Let $p > 1$. The following are inequalities are equivalent up to multiplicative constants*
   (i) (S_ϕ^p) holds
   (ii) There exist positive constants c_1 and c_2 such that

$$\|f\|_p \le c_1 \phi \left(c_2 \left(\frac{\|f\|_1}{\|f\|_p} \right)^{\frac{p}{p-1}} \right) \||\nabla f|\|_p$$

for all $f \in Lip_0(X)$.

The case $\phi(t) = t^{1/n}$, $p = 2$, corresponds to the classical Nash inequality

$$\|f\|_2^{1+2/n} \le c \|f\|_1^{2/n} \||\nabla f|\|_2 .$$

Therefore, by Theorem 1, Nash's inequality is equivalent to

$$\left(\frac{f_{(2)}^{**}(t)}{t^{2/n}} \right)^{1/2} - \left(\frac{f_{(2)}^{*}(t)}{t^{2/n}} \right)^{1/2} \preceq \left(|\nabla f|_{(2)}^{**}(t) \right)^{1/2}, \ f \in Lip_0(\mathbb{R}^n).$$

4.2. **Classical Sobolev Inequalities.** We now consider a new approach, via rearrangement inequalities, of the known (cf. [7], [2], [9] and the references therein) equivalence between the classical Euclidean Sobolev inequalities and Coulhon inequalities. The case $p = 1$ of (1.5) gives us the inequality (1.3), whose connection to Sobolev inequalities was discussed extensively elsewhere (cf. [18]).

Let us consider the case $1 \le p < n$, $\frac{1}{\bar{p}} = \frac{1}{p} - \frac{1}{n}$. Let $\phi(t) = t^{1/n}$. We shall denote the corresponding (S_ϕ^p) condition by (S_n^p). Our aim is to prove that (S_n^p) implies the classical Sobolev inequality

$$\|f\|_{L(\bar{p},p)} \preceq \|\nabla f\|_{L^p}, f \in Lip_0(\mathbb{R}^n),$$

where for $1 \le r < \infty, 1 \le q \le \infty$,

$$\|f\|_{L(r,q)} = \left\{ \int_0^\infty \left(f^*(t) t^{\frac{1}{r}} \right)^q \frac{dt}{t} \right\}^{1/q} .$$

By a well known result, apparently originally due to Maz'ya, weak type Sobolev inequalities self-improve to strong type Sobolev inequalities (cf. [2], [22], and the references therein). We shall discuss this self-improvement in detail in the next subsection. Taking this fact for granted, it will be enough to show that (S_n^p) implies the weak type Sobolev inequality

(4.1) $$\|f\|_{L(\bar{p},\infty)} \preceq \|\nabla f\|_{L^p}, f \in Lip_0(\mathbb{R}^n),$$

where

$$\|f\|_{L(\bar{p},\infty)} = \sup_t \{f^*(t) t^{1/\bar{p}}\}.$$

To prove (4.1) let us first recall that since $\bar{p} > 1$, for $f \in Lip_0(\mathbb{R}^n)$ we have (cf. [5], [4]),

$$\|f\|_{L(\bar{p},\infty)} \simeq \sup_t \{(f^{**}(t) - f^*(t)) \, t^{1/\bar{p}}\}.$$

We have shown above that (S_n^p) implies (1.5); therefore it follows that

$$\left(f_p^{**}(t)\right)^{1/p} - f^*(t) \preceq t^{1/n} \left(|\nabla f|_p^{**}(t)\right)^{1/p}$$

$$= t^{1/n - 1/p} \left\{\int_0^t |\nabla f|^*(s)^p ds\right\}^{1/p}.$$

Combining the last inequality with Jensen's inequality we get

$$f^{**}(t) - f^*(t) \le \left(f_p^{**}(t)\right)^{1/p} - f^*(t)$$

$$(4.2) \qquad \preceq t^{1/n - 1/p} \left\{\int_0^t |\nabla f|^*(s)^p ds\right\}^{1/p}.$$

Summarizing, for $f \in Lip_0(\mathbb{R}^n)$,

$$\|f\|_{L(\bar{p},\infty)} \simeq \sup_t \{(f^{**}(t) - f^*(t)) \, t^{1/\bar{p}}\}$$

$$\preceq \sup_t \left\{\int_0^t |\nabla f|^*(s)^p ds\right\}^{1/p}$$

$$\le \||\nabla f|\|_p,$$

as we wished to show.

In this next section we shall discuss in detail the case $p = n$, and show the self-improvement of Sobolev-Coulhon inequalities.

4.3. Self-improvement.
There are several known mechanisms to show the self-improvement of Sobolev inequalities. Here we choose to adapt a variant the method apparently first developed by Maz'ya-Talenti (cf. [18] for a generalized version) using differential inequalities, focussing on the Euclidean case.

For a domain $\Omega \subset \mathbb{R}^n$, we have (cf. [22] for the classical Euclidean case or [18] for the general metric space case) the following formulation of the Polya-Szegö principle
(4.3)
$$\left(\int_0^{|\Omega|} \left(s^{1-\frac{1}{n}} \left(-f^*\right)'(s)\right)^p ds\right)^{1/p} \preceq \left(\int_0^{|\Omega|} \left(|\nabla f|^*(s)\right)^p ds\right)^{1/p}, \quad p \ge 1, \ f \in Lip_0(\Omega).$$

To use this powerful inequality we now reformulate (4.2) as an elementary differential inequality. For $f \in Lip_0(\Omega)$, let $F(t) := (f^{**}(t) - f^*(t))^p \, t^{1-\frac{p}{n}}, 1 \le p < n$. Then F is a positive, absolutely continuous function (cf. [17]), which by (4.2) satisfies

$$F(t) \preceq \int_0^t \left(|\nabla f|^*(s)\right)^p ds.$$

It follows that $F(0) = 0$, and therefore we can write $F(t) = \int_0^t F'(s)ds, t > 0$. We estimate F through this representation. By direct computation,

$$F'(t) = (1 - \frac{p}{n})t^{-\frac{p}{n}}[f^{**}(t) - f^*(t)]^p + t^{1-\frac{p}{n}}p[f^{**}(t) - f^*(t)]^{p-1}\left[(f^{**}(t))' - (f^*)'(t)\right]$$

$$= (1 - \frac{p}{n})t^{-\frac{p}{n}}[f^{**}(t) - f^*(t)]^p + t^{1-\frac{p}{n}}p[f^{**}(t) - f^*(t)]^{p-1}\left[(-1)\left(\frac{f^{**}(t) - f^*(t)}{t}\right) - (f^*)'(t)\right]$$

$$= (1 - \frac{p}{n} - p)t^{-\frac{p}{n}}[f^{**}(t) - f^*(t)]^p + t^{1-\frac{p}{n}}p[f^{**}(t) - f^*(t)]^{p-1}(-f^*)'(t).$$

The previous computation, combined with the fact that $F(t)$ is positive, yields

$$(-1)(1 - \frac{p}{n} - p)\int_0^{|\Omega|}[f^{**}(t) - f^*(t)]^p t^{-\frac{p}{n}}dt \leq p\int_0^{|\Omega|}t^{1-\frac{p}{n}}[f^{**}(t) - f^*(t)]^{p-1}(-f^*)'(t)dt.$$

Hölder's inequality and (4.3) yields

$$\int_0^{|\Omega|}\left([f^{**}(t) - f^*(t)]t^{\frac{1}{p}}\right)^p \frac{dt}{t}$$

$$= \int_0^{|\Omega|}t^{-\frac{p}{n}}[f^{**}(t) - f^*(t)]^p dt$$

$$\leq \frac{p}{(\frac{p}{n} + p - 1)}\int_0^{|\Omega|}t^{1-\frac{p}{n}}[f^{**}(t) - f^*(t)]^{p-1}(-f^*)'(t)dt$$

$$= \frac{p}{(p - 1 + \frac{p}{n})}\int_0^{|\Omega|}([f^{**}(t) - f^*(t)]^{p-1}t^{\frac{1-p}{n}})(t^{1-\frac{1}{n}}(f^*)'(t))dt$$

$$\preceq \left(\int_0^{|\Omega|}\left(t^{\frac{1-p}{n}}[f^{**}(t) - f^*(t)]^{p-1}\right)^{\frac{p}{p-1}}dt\right)^{(p-1)/p}\left(\int_0^{|\Omega|}\left(t^{1-\frac{1}{n}}(-f^*)'(t)\right)^p dt\right)^{1/p}$$

$$= c_{n,p}\left(\int_0^{|\Omega|}\left(t^{\frac{1}{p}}[f^{**}(t) - f^*(t)]\right)^p \frac{dt}{t}\right)^{1/p'}|||\nabla f|||_p.$$

Consequently, assuming *apriori* that $\int_0^{|\Omega|}\left(t^{\frac{1}{p}-\frac{1}{n}}[f^{**}(t) - f^*(t)]^p\right)\frac{dt}{t} < \infty$, we see that

$$\left\{\int_0^{|\Omega|}\left([f^{**}(t) - f^*(t)]t^{\frac{1}{p}}\right)^p \frac{dt}{t}\right\}^{1/p} \preceq |||\nabla f|||_p.$$

For $f \in Lip_0(\Omega)$ all the formal calculations above can be easily justified and we find the sharp Sobolev inequality

$$\|f\|_{L(\bar{p},p)} \simeq \left\{\int_0^{|\Omega|}\left([f^{**}(t) - f^*(t)]t^{\frac{1}{p}}\right)^p \frac{dt}{t}\right\}^{1/p} \preceq |||\nabla f|||_p.$$

Let us note that the previous calculation also works for $p = n$. In this case we should let $\frac{1}{\bar{p}} = 0$ and we obtain

$$\left\{\int_0^{|\Omega|}(f^{**}(t) - f^*(t))^n \frac{dt}{t}\right\}^{1/n} \preceq |||\nabla f|||_n.$$

In this case the left hand side should be re-interpreted as the *norm* of $L(\infty, n)$, the space defined by the condition (cf. [4])

$$\left\{ \int_0^{|\Omega|} (f^{**}(t) - f^*(t))^n \frac{dt}{t} \right\}^{1/n} < \infty.$$

It was shown in [4] that this condition implies the classical exponential integrability results of Trudinger and Brezis-Wainger.

Note that the self-improvement for general ϕ, which we have not discussed here, will involve the $p-$Lorentz Λ_ϕ spaces (for further related discussions we refer to [19]).

4.4. **The Morrey-Sobolev theorem.** The connection between rearrangement inequalities and the Morrey-Sobolev theorem (i.e. the case $p > n$ of the Sobolev embedding theorem) has been treated at great length in our recent article [21]. We consider here the corresponding Coulhon variant, but, once again for the sake of brevity, and to avoid technical complications, we shall only sketch the details for Sobolev spaces $W_0^1(Q)$ on the cube $Q = (0,1)^n$.

In this section we let $p > n$, then $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{n} < 0$. Using the fact that $(-f^{**}(t))' = \frac{f^{**}(t) - f^*(t)}{t}$ we can integrate the inequality (4.2) to obtain

$$\begin{aligned}
f^{**}(0) - f^{**}(1) &= \int_0^1 \frac{f^{**}(t) - f^*(t)}{t} dt \\
&\preceq \int_0^1 t^{-\frac{1}{\tilde{p}}} \left(\int_0^t |\nabla f|^*(s)^p ds \right)^{1/p} \frac{dt}{t} \\
&\leq \||\nabla f|\|_p \int_0^1 t^{-\frac{1}{\tilde{p}} - 1} dt \\
&= c_p \||\nabla f|\|_p.
\end{aligned}$$

Extending the inequalities we have obtained in this note through the use of signed rearrangements, and using an extension of a scaling argument that apparently goes back to [11] (we must refer to [21, pag. 3] for more details) we find that given $x, y \in Q$,

$$\begin{aligned}
|f(x) - f(y)| &\preceq \||\nabla f|\|_p |x - y|^{n(\frac{1}{n} - \frac{1}{p})} \\
&= \||\nabla f|\|_p |x - y|^{1 - \frac{n}{p}}.
\end{aligned}$$

4.5. **Further connections.** In this section we mention some problems and possible projects we find of some interest.

In the literature there are other definitions of the notion of gradient in the metric setting (e.g. [13] and the references therein) and it remains an open problem to fully explore the connections with our development here[7].

We hope to discuss the connection between isoperimetry, rearrangements and discrete Sobolev inequalities elsewhere.

For aficionados of interpolation theory we should note that, while there are obvious connections between the (S_ϕ^p) conditions and the $J-$method of interpolation or perhaps, even more appropriately, with the corresponding version of this method

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[7]For partial results (restricted to doubling measures) connecting different notions of the gradient with rearrangement inequalities we refer to [1], [15] and the references therein.

for the $E-$method of approximation (cf. [14]), we could not find a treatment in the literature. Such considerations are somehow implicit in the approach given in [2], and more explicitly in the unpublished manuscript [10]. Likewise, the ϕ inequalities that appear in the formulation of Nash's inequality above[8], appear directly related to the K/J inequalities of the extrapolation theory of [14].

Still another direction for future research is to develop in more detail the connection of the results in this paper and the work of Xiao [25] on the $p-$Faber-Krahn inequality.

Finally in this section we have discussed only the Euclidean case. It will be of interest to develop a detailed treatment of these applications in the general metric case.

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