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CONTACT GEOMETRY AND ISOSYSTOLIC INEQUALITIES

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ABSTRACT. A long-standing open problem asks whether a Riemannian metric on the real projective space with the same volume as the canonical metric carries a periodic geodesic whose length is at most π . A contact-geometric reformulation of systolic geometry and the use of canonical perturbation theory allow us to solve a parametric version of this problem: if g_s is a smooth, constant-volume deformation of the canonical metric that is not formally trivial, the length of the shortest periodic geodesic of the metric g_s attains π as a strict local maximum at $s = 0$. This result still holds for complex and quaternionic projective spaces as well as for the Cayley plane. Moreover, the same techniques can be applied to show that Zoll Finsler manifolds are the unique smooth critical points of the systolic volume.

Pour résoudre un problème nouveau, nous cherchons toujours à le simplifier par une série de transformations ; mais cette simplification a un terme, car il y a dans tout problème quelque chose d'essentiel, pour ainsi dire, que toute transformation est impuissante à modifier.

— Henri Poincaré

1. INTRODUCTION

The twofold purpose of this work is to introduce contact geometry as a natural setting for the study of systolic inequalities—inequalities involving the shortest length of periodic geodesics and the volume of a Riemannian manifold—and to exploit the large symmetry group that the theory inherits through the application of canonical perturbation theory.

The starting point of our investigations was the following concrete problem.

Local systolic minimality of the round sphere. Let $K \subset \mathbb{R}^3$ be a convex body with smooth boundary ∂K and let $\ell_1(\partial K)$ denote the length of the shortest closed geodesics on ∂K for the metric induced by the Euclidian metric. Is it true that

$$\text{Area}(\partial K) \geq \frac{1}{\pi} \ell_1(\partial K)^2$$

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whenever K is sufficiently close to the unit ball ?

To the authors's knowledge, this question appears in print for the first time in [3]. There, F. Balacheff shows that any infinitesimal deformation of the standard metric on the two-sphere can be integrated to a smooth deformation of Riemannian metrics satisfying the desired inequality. Moreover, it follows from Pu's theorem [22] that *if $K \subset \mathbb{R}^3$ is a centrally symmetric convex body with smooth boundary, then $\text{Area}(\partial K) \geq \ell_1(\partial K)^2/\pi$* . In this case the convex body need not be close to a ball.

One feature that makes the non-symmetric case interesting is that the round sphere does not minimize the ratio $\text{Area}(\partial K)/\ell_1(\partial K)^2$. Indeed, E. Calabi and C. Croke (see [8]) have remarked that if T denotes the singular convex surface consisting of two identical equilateral triangles glued along their boundary, then

$$\frac{\text{Area}(T)}{\ell_1(T)^2} = \frac{1}{2\sqrt{3}}.$$

While T is not the boundary of a convex body, it can be thickened to yield convex bodies with smooth boundary for which the ratio $\text{Area}(\partial K)/\ell_1(\partial K)^2$ is strictly smaller than $1/\pi$. In fact, the singular convex surface T is conjectured to realize the minimum over all Riemannian two-spheres M of the ratio $\text{Area}(M)/\ell_1(M)^2$. This minimum is positive by a theorem of C. Croke [8] and its best known lower bound is $1/32$, a result due to R. Rotman [26] who improved previous lower bounds due to C. Croke [8], herself in collaboration with A. Nabutovsky [19], and S. Sabourau [27].

Another feature that adds to the interest and difficulty of the problem is the existence of *Zoll surfaces* (i.e., surfaces all of whose geodesics are periodic of the same minimal period). By a theorem of V. Guillemin (see [15]) there are plenty of non-isometric Zoll surfaces in any neighborhood of the round metric on the sphere. Moreover, for any such surface the ratio of its area and the square of the length of its (prime) periodic geodesics is equal to $1/\pi$. As a result, the round sphere has no chance of being an isolated minimum, and any argument based on curvature flows—none of which is known to preserve the class of Zoll surfaces—is doomed to fail.

The local systolic minimality problem can be considered not just for the two-sphere, but for spheres and (real, complex, and quaternionic) projective spaces of all dimensions and, of course, for the Cayley plane. One would at least like to know whether the standard metrics on these spaces are critical points of the *systolic volume* functional $g \mapsto \text{vol}(M^n, g)/\ell_1(M^n, g)^n$. Here $\ell_1(M^n, g)$ denotes the *systole* defined as the infimum of the lengths of periodic geodesics. We shall show that this is indeed the case not only for compact rank-one symmetric spaces, but for all *Zoll Finsler manifolds* (i.e., Finsler manifolds all of whose geodesics are periodic with the same prime period).

Definition 1.1. A smooth one-parameter deformation F_s of a closed Finsler manifold (M, F_0) is said to be *isosystolic* if the quantity $\ell_1(M, F_s)$ remains constant along the deformation.

Theorem 1.2. *A closed Finsler manifold (M, F_0) is Zoll if and only if for every smooth isosystolic deformation F_s the derivative of the function $s \mapsto \text{vol}(M, F_s)$ vanishes at $s = 0$.*

When (M, g_0) is the canonical metric in a real projective space and the isosystolic deformation is required to be Riemannian, M. Berger [5] showed that $\text{vol}(M, g_s)$ has a critical point at $s = 0$. Theorem 1.2 improves upon Berger's result in that we allow deformations by (reversible and non-reversible) Finsler metrics and show that the standard metric on $\mathbb{R}P^n$ (and on any compact rank-one symmetric space) remains a critical point. Theorem 1.2 also implies that relatively few manifolds admit a *smooth* Finsler metric as a minimum of the systolic volume. Indeed, it seems to be an open question whether any manifold not diffeomorphic to a compact rank-one symmetric space admits a Zoll metric, Riemannian or Finsler. For example, it is easy to see that the torus does not admit any smooth Zoll Finsler metric.

In studying the functional $F \mapsto \text{vol}(M^n, F)/\ell_1(M^n, F)^n$ by perturbation techniques, we face the problem that it is not differentiable. In Theorem 1.2 we bypassed this difficulty by considering smooth isosystolic deformations. However, one of the key features of the present work is that we are able to tackle the problem head on and work with arbitrary smooth deformations, which we sometimes normalize to be volume-preserving. Roughly speaking, our main result states that if a one-parameter deformation F_s of a Zoll manifold (M, F_0) does not mischievously start tangent to all orders to deformations by Zoll metrics, the systolic volume $s \mapsto \text{vol}(M^n, F_s)/\ell_1(M^n, F_s)^n$ attains a *strict* minimum at $s = 0$.

Definition 1.3. A smooth Finsler deformation F_s of a Zoll manifold (M, F_0) is said to be *formally trivial* if for every $m \in \mathbb{N}$ there exists a deformation $F_s^{(m)}$ by Zoll Finsler metrics that has m -order contact with F_s at $s = 0$.

Theorem 1.4. *Let (M, F_s) be a smooth volume-preserving Finsler deformation of a Zoll manifold (M, F_0) . If the deformation is not formally trivial, then the function $s \mapsto \ell_1(M, F_s)$ attains a strict local maximum at $s = 0$. If, on the other hand, the deformation is formally trivial, then*

$$\ell_1(M, F_s) = \ell_1(M, F_0) + \mathcal{O}(|s|^k) \text{ for all } k > 0.$$

The notion of formally trivial deformation given in Definition 1.3 differs from the usual notion where for every $m \in \mathbb{N}$ the deformation F_s is required to have m -order contact at $s = 0$ with a deformation of the form $\phi_s^* F_0$ for some isotopy ϕ_s . This change is forced upon us by the wealth of non-isometric deformations of Zoll metrics in Finsler geometry. However, as in the abstract, we are able to revert to the more restrictive usual notion when

we consider Riemannian deformations of the canonical metrics on projective spaces.

Theorem 1.5. *Let g_s be a smooth volume-preserving deformation of the canonical metric in the projective space $\mathbb{K}P^n$. If at $s = 0$ the deformation g_s is not tangent to all orders to trivial deformations (i.e., to deformations of the form $\phi_s^*g_0$ for some isotopy ϕ_s), the length of the shortest periodic geodesic of the metric g_s attains π as a strict local maximum at $s = 0$.*

In other words, we are faced with the following non-exclusive alternatives: (1) either the length of the shortest periodic geodesic of the metric g_s attains π as a strict local maximum at $s = 0$, or (2) for every $k \geq 1$ the deformation g_s is tangent to order k at $s = 0$ to a trivial deformation. Spheres admit non-trivial Zoll deformations so the situation is more delicate. However, on the two-sphere the result takes a particularly simple form.

Theorem 1.6. *Let g_0 be the canonical metric on the two-sphere and let $\dot{\rho} : S^2 \rightarrow \mathbb{R}$ be a smooth function with zero average. If $\dot{\rho}$ is not odd and $e^{\rho_s}g_0$ is any smooth deformation satisfying $d\rho_s/ds|_{s=0} = \dot{\rho}$, then the length of the shortest periodic geodesic of $(S^2, e^{\rho_s}g_0)$ attains 2π as a strict local maximum at $s = 0$.*

This result is sharp: the main theorem of [15] states that if $\dot{\rho}$ is odd, then there exists a smooth deformation $e^{\rho_s}g_0$ by Zoll metrics satisfying $d\rho_s/ds|_{s=0} = \dot{\rho}$. The length of the shortest periodic geodesic is then constantly equal to 2π along the deformation.

Plan of the paper. Section 2 introduces contact geometry as a natural setting for the study of systolic inequalities. There the reader will find the statement of our main result: a systolic-geometric characterization of regular contact forms. The results on Zoll Finsler manifolds stated in this introduction follow easily from it. Section 3 is practically a short survey of what is known about regular contact manifolds, but also contains two new results: (1) the proof that regular contact forms are the only critical points of the systolic volume; (2) the proof that the systolic volume is a Lipschitz function in a C^2 neighborhood of any regular contact form. In Section 4 we use Lie transforms and the method of averaging to compute the normal form of deformations of regular contact forms. The proof of the main result will be found in Section 5 and a theorem that includes both Theorem 1.5 and Theorem 1.6 as particular cases will be found in Section 6.

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long, long ago. The idea to use the perturbation techniques of celestial mechanics to study the geodesic flows of metrics close to the round sphere was taken from Poincaré [21, §3] and Moser [18].

2. SYSTOLIC GEOMETRY FROM THE CONTACT VIEWPOINT

Although there is a wonderful reference in contact geometry ([12]), for the benefit of the reader we recall the basic notions and in so doing adapt the presentation to our needs.

2.1. Basic definitions. We follow Boothby and Wang's classic paper [7] and define a *contact manifold* as a pair (M, α) consisting of a $(2n + 1)$ -dimensional manifold together with a smooth 1-form α such that the top order form $\alpha \wedge d\alpha^n$ never vanishes.

Without exception, all our contact manifolds are closed and oriented in such a way that $\alpha \wedge d\alpha^n > 0$.

The kernel of α defines a field of hyperplanes in the tangent space of M (a vector sub-bundle of TM of co-dimension one) that is maximally non-integrable. Standard notation and terminology will have us denote this sub-bundle by ξ and call it the *contact structure* associated to the contact form α . Note that if (M, α) is a contact manifold and $\rho : M \rightarrow \mathbb{R}$ is a smooth function that never vanishes, the form $\rho\alpha$ is also a contact form which defines the same contact structure as α . If we are only interested in the contact structure associated to a contact form on M , we will write (M, ξ) .

Diffeomorphisms of (M, α) that preserve the contact structure are called *contactomorphisms* or *contact transformations*, while diffeomorphisms that preserve the contact form are *strict contactomorphisms*. Two contact manifolds (M_1, α_1) and (M_2, α_2) are *contactomorphic* if there exists a diffeomorphism $\phi : M_1 \rightarrow M_2$ and a nowhere-vanishing smooth function $\rho : M_1 \rightarrow \mathbb{R}$ such that $\phi^*\alpha_2 = \rho\alpha_1$. When $\rho \equiv 1$ (i.e., $\phi^*\alpha_2 = \alpha_1$), the contact manifolds are said to be *strictly contactomorphic*.

As we have defined them, contact manifolds come with a natural volume:

$$\text{vol}(M, \alpha) := \int_M \alpha \wedge d\alpha^n.$$

They also carry a natural vector field, the *Reeb vector field* R_α , defined by the equations $d\alpha(R_\alpha, \cdot) = 0$ and $\alpha(R_\alpha) = 1$. The flow of the vector field R_α (remember our contact manifolds are all closed) is called the *Reeb flow* and its orbits are the *Reeb orbits*. If we are not particularly interested in the parameterization of a Reeb orbit, we shall call it a *characteristic* (a closed characteristic if the orbit is periodic). Equivalently, characteristics are the 1-dimensional leaves of the *characteristic distribution* $\text{Ker } d\alpha$.

Our main interest is to find inequalities relating the volume of a contact manifold and the bottom of its *action spectrum*: the set of periods of its periodic Reeb orbits.

Definition 2.1. The *systole* of a contact manifold (M, α) , which we denote by $\ell_1(M, \alpha)$, is the smallest period of any of its periodic Reeb orbits. We define the *systolic volume* of a contact manifold (M, α) of dimension $2n + 1$ as the ratio

$$\mathfrak{S}(M, \alpha) = \frac{\text{vol}(M, \alpha)}{\ell_1(M, \alpha)^{n+1}}.$$

Seen as a functional on the space of contact forms inducing a given contact structure (M, ξ) , the systolic volume is upper semi-continuous. More precisely:

Proposition 2.2. *Let (M, α) be a contact manifold and let $C_+^\infty(M)$ be the set of (strictly) positive smooth functions on M . The functional*

$$\mathfrak{S} : C_+^\infty(M) \longrightarrow (0.. \infty)$$

defined by $\rho \mapsto \mathfrak{S}(M, \rho\alpha)$ is upper semi-continuous in the C^1 topology.

Proof. The proposition follows from the lower semi-continuity of the functional $\rho \mapsto \ell_1(M, \rho\alpha)$, which—once we pass from $\rho\alpha$ to its Reeb vector field at the cost of one derivative—follows in turn from a general principle: if X is a nowhere-zero C^1 vector field on a closed manifold and all its periodic orbits have periods strictly greater than T , then in the space of C^1 vector fields there is a C^0 neighborhood of X where every element shares this property (see Lemma 2.3 in [20]). \square

The reader may have already noted that in the definition of systolic volume and in the previous proposition we are implicitly assuming the existence of periodic Reeb orbits on closed contact manifolds. We can bypass this thorny issue by setting $\mathfrak{S}(M, \alpha) = 0$ if there are no periodic Reeb orbits, but the (Weinstein) conjecture is that they always exist. More importantly, their existence has been proved for all the contact manifolds that appear in our results. We will say more about this as we go along.

2.2. Symmetries in contact systolic geometry. The following simple result—whose proof is left to the reader—shows that the systolic volume is invariant under a very general class of transformations.

Proposition 2.3. *If (M_1, α_1) and (M_2, α_2) are contact manifolds for which there exists a diffeomorphism $\phi : M_1 \rightarrow M_2$ such that $\phi^*\alpha_2 = c\alpha_1 + df$, where c is a non-zero constant and f is a smooth function on M_1 , then*

$$\mathfrak{S}(M_1, \alpha_1) = \mathfrak{S}(M_2, \alpha_2).$$

From the viewpoint of perturbation theory, one advantage of working in the contact setting is that we may assume that every smooth deformation of a contact manifold (M, α_0) is of the form $\rho_s\alpha_0$, where ρ_s is a smooth function on M depending smoothly on the parameter. More precisely, *Gray's stability*

theorem (see Theorem 2.2.2 in [12]) states that given a smooth deformation α_s (s ranging over some compact interval), there exists an isotopy Φ_s such that $\Phi_s^* \alpha_s = \rho_s \alpha_0$. In other words, we may assume that the contact structure stays fixed along the deformation.

Note that if we apply a contact isotopy to a deformation of the form $\alpha_s = \rho_s \alpha_0$, the result will be another deformation of the same form. The measure in which we can simplify (i.e., bring to normal form) the deformation $\rho_s \alpha_0$ depends on our ability to construct large classes of contact isotopies. A convenient way to do this is by integrating the following class of vector fields:

Definition 2.4. A vector field X on a contact manifold (M, α) is said to be an *infinitesimal contact transformation* if $\mathcal{L}_X \alpha = \lambda \alpha$ for some smooth function $\lambda : M \rightarrow \mathbb{R}$.

The following standard result (cf., Theorem 2.3.1 in [12]) shows that infinitesimal contact transformations are easy to construct.

Proposition 2.5. *If h is a smooth function on a contact manifold (M, α) , there is a unique vector field X_h —the Hamiltonian vector field of h —that satisfies the equations $\alpha(X_h) = h$ and $\mathcal{L}_{X_h} \alpha = R_\alpha(h) \alpha$. In particular, X_h is an infinitesimal contact transformation.*

2.3. Main results.

Definition 2.6. A contact manifold (M, α) is said to be *regular* if its Reeb flow is periodic and all the Reeb orbits have prime period $\ell_1(M, \alpha)$.

Theorem 2.7. *A contact manifold (M, α) is regular if and only if for every smooth isosystolic deformation the derivative of the function $s \mapsto \text{vol}(M, \alpha_s)$ vanishes at $s = 0$.*

Definition 2.8. A smooth deformation α_s of a contact form α_0 is said to be *trivial* if there exist a smooth real-valued function $\lambda(s)$ depending only on the parameter s and an isotopy Φ_s such that $\alpha_s = \lambda(s) \Phi_s^* \alpha_0$. A smooth deformation α_s is said to be *formally trivial* if for every $m \in \mathbb{N}$ there exists a trivial deformation $\alpha_s^{(m)}$ that has m -order contact with α_s at $s = 0$.

Theorem 2.9. *Let (M, α_s) be a smooth deformation of a regular contact manifold (M, α_0) . If the deformation is not formally trivial, then the function $s \mapsto \mathfrak{S}(M, \alpha_s)$ attains a strict local minimum at $s = 0$. If, on the other hand, the deformation is formally trivial, then*

$$\mathfrak{S}(M, \alpha_s) = \mathfrak{S}(M, \alpha_0) + \mathcal{O}(|s|^k) \text{ for all } k > 0.$$

The proofs of these results will take up part of Section 3 and all of Sections 4 and 5.

First steps in contact systolic topology. It is natural to ask whether the *systolic constant* of a contact structure (M, ξ) , defined as

$$\sigma(M, \xi) := \inf \{ \mathfrak{S}(M, \alpha) : \alpha \text{ is a contact form with } \text{Ker } \alpha = \xi \},$$

is a non-trivial invariant in contact topology. Perhaps recent techniques such as embedded contact homology are sufficiently powerful to establish the strict positivity of the systolic constant for at least some classes of contact structures. Indeed, a very particular case of a conjecture of M. Hutchings (see Conjecture 8.5 in [16]) implies that the systolic constant of the standard contact structure on the 3-torus is greater than $1/2$ (in fact, we believe that it is equal to 3). At this time even the simplest questions about the systolic constant look impossibly hard. It seems we must follow D'Alembert's advice: *Avancez et la foi vous viendra.*

2.4. Applications of Theorems 2.7 and 2.9. We shall now show that most of the results stated in the introduction are easy consequences of the two preceding theorems. It is well known that geodesic flows of Riemannian and Finsler metrics are Reeb flows (see, for example, Theorem 1.5.2 in [12]). The precise setup is as follows: through the Legendre transform, a (not necessarily reversible) Finsler metric F on a manifold N gives rise to a Hamiltonian H defined in the slit cotangent bundle. The restriction α of the canonical one-form to the unit cotangent bundle S_H^*N (i.e., the set of covectors where $H = 1$) is a contact form and its Reeb flow is the geodesic flow of the metric. A periodic Reeb orbit in (S_H^*N, α) projects down to a closed geodesic on N whose length equals the period (and the action) of the orbit. In particular, $\ell_1(S_H^*N, \alpha)$ is the length of the shortest closed geodesic on N .

If the metric F is Riemannian and the manifold has dimension n , the Riemannian volume of (N, F) and the contact volume of (S_H^*N, α) are related by the equality

$$\text{vol}(S_H^*N, \alpha) = n! \varepsilon_n \text{vol}(N, F),$$

where ε_n is the volume of the n -dimensional Euclidean unit ball. When the metric is Finsler, we define the volume by the preceding equality. This is the *Holmes-Thompson* volume of a Finsler manifold (see [28] and [2] for a detailed discussion of this definition). It is important to underline that even when the metric is reversible this is not the Hausdorff measure of the Finsler manifold seen as a metric space. In fact, by a result of C. Durán (see [10]), the Holmes-Thompson volume of a reversible Finsler manifold is strictly smaller than its Hausdorff measure when the metric is not Riemannian. This has the agreeable consequence that any systolic inequality proved for the Holmes-Thompson volume is immediately true for the Hausdorff measure, and cases of equality hold only for Riemannian metrics.

The factor $n! \varepsilon_n$ that distinguishes the contact volume of the unit cotangent bundle from the Riemannian volume of the manifold may make the contact-geometric reformulation of classical isosystolic inequalities slightly unfamiliar to the initiated. For example, Pu's isosystolic inequality ([22]) reads as follows: *if H is the Hamiltonian of any Riemannian metric on the projective plane, then $\mathfrak{S}(S_H^*\mathbb{R}P^2, \alpha) \geq 4$.*

Proof of Theorems 1.2 and 1.4. Given that $\ell_1(S_H^*N, \alpha)$ is also the length of the shortest closed geodesic on (N, F) , an isosystolic deformation of the Finsler metric F corresponds to an isosystolic deformation of the canonical one-form restricted to S_H^*N . Conversely, a sufficiently small isosystolic deformation of the canonical 1-form restricted to S_H^*N corresponds to an isosystolic deformation of the Finsler metric. The key observation now is that a Finsler metric on a manifold N is Zoll if and only if the restriction of the canonical one-form to the cotangent bundle S_H^*N is a regular contact form. Theorem 1.2 follows at once from Theorem 2.7 and the equality $\text{vol}(S_H^*N, \alpha) = n! \varepsilon_n \text{vol}(N, F)$.

In order to deduce Theorem 1.4 from Theorem 2.9 we must show that formally trivial deformations of a Zoll Finsler manifold (N, F) —as defined in Definition 1.3—correspond precisely to formally trivial deformations of the regular contact manifold (S_H^*N, α) . One direction is immediate: a formally trivial deformation of (S_H^*N, α) yields a formally trivial deformation of (N, F) . The converse is not at all obvious, but follows at once from a result of A. Weinstein (see [32]):

Let α_0 be a regular contact form and let $(\alpha_s)_{s \in I}$ be a smooth deformation of α_0 for s ranging over some compact interval I . The deformation $(\alpha_s)_{s \in I}$ is trivial if and only if the forms α_s are regular contact forms for all values of the parameter s .

This means that a deformation of a Zoll metric F by Zoll metrics corresponds to a trivial deformation of (S_H^*N, α) and, therefore, formally trivial deformations of (N, F) correspond precisely to formally trivial deformations of (S_H^*N, α) . \square

To end this section, we underline that our methods work better for Finsler metrics than for Riemannian metrics because only the former are stable under small contact perturbations. Indeed, if (N, g) is a Riemannian manifold and S^*N is its unit cotangent bundle, small C^2 perturbations of the contact form α correspond to Finsler perturbations of the metric that are not necessarily Riemannian (nor reversible).

3. REGULAR CONTACT MANIFOLDS

Contact manifolds with periodic Reeb flows were introduced by G. Reeb in [23] under the cryptic names of S.D.F.I. (when all the Reeb orbits have the same prime period) and S.D.F'.I. (when the prime periods are not all equal). Reeb's S.D.F.I. (*systèmes dynamiques fibrés avec un invariant intégral*) were later studied by W.M. Boothby and H.C. Wang (see [7] and Section 7.2 in [12]) who coined the term *regular contact manifold*. Their main results are a structure theorem and a general construction:

Theorem 3.1 (Boothby-Wang [7]). *A regular contact manifold (M, α) with systole $T := \ell_1(M, \alpha)$ is a principal circle bundle over a base manifold B on which $(1/T)d\alpha$ induces an integral symplectic form ω (i.e., $[\omega] \in H^2(B, \mathbb{Z})$). Moreover, $[\omega]$ is the Euler class of the circle bundle.*

Theorem 3.2. *Let $\pi : S^{2n+1} \rightarrow \mathbb{C}P^n$ be the Hopf fibration and let α be the standard contact form on the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$. If B is a closed symplectic submanifold of $\mathbb{C}P^n$, the restriction of α to the submanifold $\pi^{-1}(B) \subset S^{2n+1}$ is a regular contact form. Moreover, every regular contact manifold (M, α) with $\ell_1(M, \alpha) = \pi$ is strictly contactomorphic to a manifold obtained by this construction.*

This last result is not precisely what Boothby and Wang proved in [7], but a small enhancement that uses the Gromov-Tischler characterization of those closed symplectic manifolds that admit a symplectic embedding in complex projective space (see [14] and [29]). In this form, Theorem 3.2 shows just how easy it is to construct regular contact manifolds.

We can apply Theorem 3.1 to complement our characterization of the critical points of the systolic volume by a characterization of its critical values:

Proposition 3.3. *The systolic volume of a regular contact manifold (M, α) is a positive integer. In particular, the systolic volume of a Zoll Finsler manifold is a positive integer.*

Proof. Let $\pi : M \rightarrow B$ be the Boothby-Wang fibration for (M, α) and assume without loss of generality that $\ell_1(M, \alpha) = 1$. Note that the fiber integration of the volume form $\alpha \wedge d\alpha^n$ yields precisely the form ω^n on B and, therefore,

$$\text{vol}(M, \alpha) = \int_M \alpha \wedge d\alpha^n = \int_B \omega^n = \langle [\omega^n], [B] \rangle.$$

Since $[\omega]$ and, therefore, $[\omega^n]$ are integral cohomology classes, it follows that $\text{vol}(M, \alpha)$ is a positive integer. \square

When (M, α) is the unit sphere bundle of a Zoll Riemannian manifold (N, g) , this result (and its proof) are due to A. Weinstein ([31]). In this case the systolic volume is what Weinstein calls $j(N, g)$, which is twice the *Weinstein integer* of the Zoll manifold.

The authors do not know of any criterion to determine whether a contact structure (M, ξ) admits a regular contact form. A result of D. Blair (see page 71 in [6]) states that the torus T^{2n+1} ($n > 0$), provided with any contact structure, does not. In particular, the torus does not admit any (smooth!) contact form that (globally or locally) minimizes the systolic volume.

Another natural question is whether two regular contact forms can define the same contact structure without being strictly contactomorphic. This is directly related to the old open problem that asks whether the geodesic flows of any two Zoll metrics on the same manifold are symplectically conjugate. Lastly, we mention the contact-geometric generalization of what C.T. Yang called the *Weak Blaschke Conjecture* in the theory of Zoll manifolds (see [33], [24], and [25]):

Question. If two regular contact manifolds are contactomorphic, do they have the same systolic volume ?

3.1. Systolic non-criticality of non-regular contact manifolds. The following result, together with Theorem 2.9, proves Theorem 2.7.

Theorem 3.4. *If (M, α_0) is not a regular contact manifold, then there exists a smooth isosystolic deformation α_s such that the derivative of the function $s \mapsto \text{vol}(M, \alpha_s)$ is negative at $s = 0$. Moreover, if α_0 is invariant under the action of a compact group G , the deformation can also be chosen to be G -invariant.*

Proof. Let us assume that α_0 is invariant under the action of a compact group G , which can be the trivial group $\{e\}$. Fix a G -invariant metric d that induces the standard (manifold) topology on M and for $\epsilon \geq 0$ define the set

$$M_T(\epsilon) = \{x \in M : d(x, \varphi_T(x)) \leq \epsilon\} ,$$

where $T = \ell_1(M, \alpha_0)$ and $\varphi_t : M \rightarrow M$ is the Reeb flow in (M, α_0) . Remark that the set $M_T(\epsilon)$ is nonempty, closed and G -invariant.

If α_0 is not a regular contact form, then for all sufficiently small positive values of ϵ , the set $M_T(\epsilon)$ is *properly* contained in M and its complement is nonempty, open, and G -invariant. We can then find a smooth function $\dot{\rho} : M \rightarrow \mathbb{R}$ that vanishes identically on $M_T(\epsilon)$ and such that

$$\int_M \dot{\rho} \alpha \wedge d\alpha^n < 0 .$$

Moreover, by averaging $\dot{\rho}$ over G if necessary, we may assume that $\dot{\rho}$ is G -invariant.

Claim. The deformation we seek is given by $\alpha_s = (1 + s\dot{\rho}) \alpha_0$.

Since M is closed, the function $1 + s\dot{\rho}$ is strictly positive for all sufficiently small values of s and, therefore, the α_s are G -invariant contact forms for small values of s . Furthermore,

$$\text{vol}(M, \alpha_s) = \int_M (1 + s\dot{\rho})^{n+1} \alpha_0 \wedge d\alpha_0^n$$

and the derivative of $s \mapsto \text{vol}(M, \alpha_s)$ evaluated at $s = 0$ equals

$$(n+1) \int_M \dot{\rho} \alpha \wedge d\alpha^n < 0 .$$

It remains to prove that α_s is an isosystolic deformation. For the rest of the proof we fix $\epsilon > 0$ such that $M_T(\epsilon)$ is *properly* contained in M and restrict the parameter s to a compact interval $-s_0 \leq s \leq s_0$ ($s_0 > 0$) chosen in such a way that

- (1) the α_s are contact forms;
- (2) if φ_t^s denotes the Reeb flow in (M, α_s) , then $d(\varphi_t^0(x), \varphi_t^s(x)) < \epsilon/4$ for all s in $[-s_0 \dots s_0]$, for all t in $[0 \dots T]$, and for all $x \in M$.

The identity $\alpha_s \equiv \alpha_0$ on $M_T(\epsilon)$ has two important consequences: (1) the systoles of (M, α) are also periodic Reeb orbits of (M, α_s) and we have the inequality

$$T = \ell_1(M, \alpha_0) \geq \ell_1(M, \alpha_s);$$

(2) any periodic orbit of φ^s with period strictly between 0 and T must pass through the complement of $M_T(\epsilon)$.

To complete the proof let us choose a number b ($0 < b < T$) sufficiently close to T so that $d(x, \varphi_t^0(x)) > \epsilon/2$ for all x in the complement of $M_T(\epsilon)$ and all $t \in [b, T]$. By the lower semi-continuity of the systole (Proposition 2.2), $\ell_1(M, \alpha_s) > b$ for all sufficiently small values of s . Moreover, if $s \in [-s_0, s_0]$, then for all x in the complement of $M_T(\epsilon)$ and all $t \in [b, T]$, we have that $d(x, \varphi_t^s(x)) > \epsilon/4$. We conclude that if the parameter s is sufficiently small, the flow φ^s has no periodic orbit with period strictly between 0 and T . \square

3.2. Lipschitz continuity of the systolic volume. The statement of Theorem 2.9 takes for granted the existence of periodic Reeb orbits for every contact form in a smooth deformation (M, α_s) of a regular contact manifold (M, α_0) , at least for all sufficiently small values of the parameter. As A. Banyaga remarks in [4], this follows from a result of V. Ginzburg (Theorem 2 in [13]). Specializing Ginzburg's proof to the contact setting, we obtain a very simple construction that is the key to understanding the regularity of the systolic volume functional in a neighborhood of a regular contact form.

Theorem 3.5 (V. Ginzburg [13]). *Given a regular contact manifold (M, α) , there exists an open neighborhood of zero $U \subset C^2(M)$ such that if $f \in U$, the Reeb flow of the contact form $(1 + f)\alpha$ has a periodic orbit.*

Sketch of the proof. The key idea is to construct a map $A : U \rightarrow C^2(M)$ such that critical points of $A_f : M \rightarrow \mathbb{R}$ lie on periodic Reeb orbits of the contact form $(1 + f)\alpha$.

Let $\pi : M \rightarrow B$ be the Boothby-Wang fibration of the regular contact form (M, α) . On M we can easily construct a Riemannian metric such that (1) for every $x \in M$ the Reeb vector $R_\alpha(x)$ is a unit vector orthogonal to the contact hyperplane ξ_x ; (2) the Reeb flow of (M, α) acts by isometries. Indeed, all that needs to be done is to prescribe that the Reeb vector field and the contact hyperplane, which are always transversal, shall be orthogonal and if the chosen metric is not invariant by the Reeb flow, we simply average by the circle action it to make it so. We call this metric g and fix it once and for all. Note that g induces a submersive metric on B via the projection $\pi : M \rightarrow B$ and that the contact hyperplanes are precisely the horizontal subspaces of the Riemannian submersion.

Let $\epsilon > 0$ be smaller than the injectivity radius of (M, g) and, for every $x \in M$, let $D_x = D_x(\epsilon)$ denote the image under the exponential map of the Euclidean disc

$$\{v_x \in \xi_x \subset T_x M : \|v_x\| < \epsilon\}.$$

In a Riemannian fibration, geodesics that start horizontal stay horizontal. Therefore, if y is any point in D_x , there exists a unique horizontal geodesic in D_x joining y to x . We shall denote this geodesic by $\sigma(x, y)$ and remark that σ depends smoothly on the choice of x and y .

Let $f > -1$ and denote the Reeb flow of the contact form $(1 + f)\alpha$ by φ_t^f . For f small enough in the C^2 norm, the system of hypersurfaces D_x ($x \in M$) are Poincaré sections. We denote the first return time of x to D_x along the flow φ_t^f by $\tau_f(x)$, and let

$$\gamma_f(x) := \{\varphi_t^f(x) : 0 \leq t \leq \tau_f(x)\}.$$

Think of $\gamma_f(x)$ as an oriented, unparameterized curve joining x and $x_f := \varphi_{\tau_f(x)}(x)$ and complete it to a closed curve $\tilde{\gamma}_f(x)$ by adding the geodesic segment $\sigma(x, x_f)$.

The function $A_f : M \rightarrow \mathbb{R}$ is defined by the equation

$$A_f(x) = \int_{\tilde{\gamma}_f(x)} (1 + f)\alpha = \int_{\gamma_f(x)} (1 + f)\alpha,$$

where the last equation follows from the fact that the geodesic $\sigma(x, x_f)$ is horizontal.

The assignment $x \mapsto \tilde{\gamma}_f(x)$ defines an embedding of M into its space of unparameterized oriented loops, and the function A_f is the composition of the action functional with this embedding. Ginzburg shows in [13] that the critical points of A_f correspond to the fixed points of the map $x \mapsto x_f$ and, therefore, to periodic Reeb orbits. \square

Note that the map $f \mapsto \ell_1(M, (1 + f)\alpha)$ can now be seen as a composition $f \mapsto A_f \mapsto \min A_f$. This is the key to the proof of the following result:

Theorem 3.6. *Let (M, α) be regular contact manifold. There exists a small open neighborhood U of zero in the Banach space $C^2(M)$ such that the function $f \mapsto \ell_1(M, (1 + f)\alpha)$ is Lipschitz in U . In particular, the systolic volume functional $f \mapsto \mathfrak{S}(M, (1 + f)\alpha)$ is Lipschitz in a small neighborhood of zero in $C^2(M)$.*

Proof. If in the proof of Theorem 3.5 we consider the Reeb vector field of $(1 + f)\alpha$ as a vector field depending in a continuously differentiable way on the parameter f , we clearly see that the map $f \mapsto A_f$ is a C^1 map between Banach manifolds and, by the intermediate value theorem, it is Lipschitz in a small convex neighborhood of $0 \in C^2(M)$.

Since the function $\min : C^0(M) \rightarrow \mathbb{R}$ is Lipschitz with constant one, it is a fortiori a Lipschitz function on $C^2(M)$. It follows that the composition $\ell_1(M, (1 + f)\alpha) = \min A_f$ is Lipschitz in a small convex neighborhood of $0 \in C^2(M)$. \square

4. LIE TRANSFORMS AND THE METHOD OF AVERAGING

The origin of many papers is a simple observation on which everything hinges, and so it is with this one.

Theorem 4.1. *Let (M, α) be a regular contact manifold. If a smooth, positive function $\rho : M \rightarrow (0, \infty)$ is invariant under the Reeb flow of α , then $\ell_1(M, \rho\alpha) \leq \min \rho \ell_1(M, \alpha)$. In particular, $\mathfrak{S}(M, \rho\alpha) \geq \mathfrak{S}(M, \alpha)$ and equality holds if and only if ρ is constant.*

Proof. The key observation is that if $u \in M$ is a point where this minimum is attained, then the closed characteristic of (M, α) passing through u is also a closed characteristic for $(M, \rho\alpha)$. Indeed, if $\gamma(t)$ is any parameterization of this characteristic,

$$d(\rho\alpha)(\dot{\gamma}(t), \cdot) = d\rho \wedge \alpha(\dot{\gamma}(t), \cdot) + \rho(\gamma(t))d\alpha(\dot{\gamma}(t), \cdot)$$

vanishes identically because $d\rho(\gamma(t))$ and $d\alpha(\dot{\gamma}(t), \cdot)$ vanish identically. Moreover, since $\rho(\gamma(t))$ is constantly equal to $\min \rho$, we have that the action of γ in $(M, \rho\alpha)$ equals

$$\int_{\gamma} \rho\alpha = \min \rho \int_{\gamma} \alpha = (\min \rho) \ell_1(M, \alpha).$$

It follows that $\ell_1(M, \rho\alpha) \leq (\min \rho) \ell_1(M, \alpha)$.

To prove the second part of the theorem, note that

$$\text{vol}(M, \rho\alpha) = \int_M \rho^{n+1} \alpha \wedge d\alpha^n \geq (\min \rho)^{n+1} \text{vol}(M, \alpha)$$

with equality if and only if ρ is constant. \square

Constructing integrals of motion (i.e., smooth functions that are invariant under the Reeb flow) in a regular contact manifold (M, α) is an easy matter: if $\pi : M \rightarrow B$ is the Boothby-Wang fibration of (M, α) and f is any smooth function on B , the function $f \circ \pi$ is an integral of motion. Alternatively, we can take any function ρ on M and construct its *averaged function*

$$\bar{\rho}(x) := \frac{1}{T} \int_0^T \rho(\varphi_t(x)) dt,$$

where φ_t is the Reeb flow of (M, α) and $T := \ell_1(M, \alpha)$. Clearly, $\bar{\rho}$ is an integral of motion.

Theorem 4.1 implies that if a deformation of a regular contact form α_0 is of the form $\alpha_s = \Phi_s^* \rho_s \alpha_0$, where Φ_s is an isotopy and ρ_s is a smooth one-parameter family of integrals of motion for the Reeb flow of α_0 , then the systolic volume of α_s attains a minimum at $s = 0$. Given the large number of isotopies and integrals of motion for a periodic Reeb flow, we could hope that every deformation of α_0 is of this form and thus prove the local systolic minimality of regular contact manifolds. Of course, this idea does not work. However, the theory of normal forms—which we adapt to contact geometry—tells us that it *almost* works.

Definition 4.2. Let $\alpha_s = \rho_s \alpha_0$ be a smooth deformation of the regular contact form α_0 and let k be a non-negative integer. We shall say that α_s is in *normal form to order k* if

$$\alpha_s = (1 + s\mu^{(1)} + \cdots + s^k \mu^{(k)} + s^{k+1} r_s) \alpha_0$$

where the functions $\mu^{(i)}$ ($1 \leq i \leq k$) are integrals of motion for the Reeb flow of α_0 and the function r_s is a smooth function on M depending smoothly on the parameter s .

Theorem 4.3. *Let $\alpha_s = \rho_s \alpha_0$ be a smooth deformation of the regular contact form α_0 . Given a non-negative integer k , there exists a contact isotopy such that $\Phi_s^{(k)*} \alpha_s$ is in normal form to order k .*

There are several inductive procedures to construct the contact isotopy appearing in this theorem. We shall follow a technique known as the *method of Dragt and Finn* (see [9] and [11]). The idea is to construct the isotopy as a composition $\phi_{s^k}^{(k)} \circ \dots \circ \phi_s^{(1)}$, where $\phi_s^{(i)}$ denotes the flow of a vector field X_i on M .

The term *Lie transform* that is usually associated to this and other equivalent methods of computing normal forms has its origin in a simple remark: if ϕ_τ is the flow of a vector field X and β is any smooth function or differential form on M , the Taylor expansion of $\phi_\tau^* \beta$ around $\tau = 0$ can be written in terms of the exponential series of the Lie derivative operator \mathcal{L}_X :

$$\exp(\tau \mathcal{L}_X) \beta = \beta + \tau \mathcal{L}_X \beta + \frac{\tau^2}{2} \mathcal{L}_X (\mathcal{L}_X \beta) + \dots.$$

In what concerns us, this boils down to remembering that, up to terms of order $k+1$ and higher, $\phi_{s^k}^* \beta = \beta + s^k \mathcal{L}_X \beta$.

In order to construct the vector fields X_1, \dots, X_k that the method requires, we shall need the following

Lemma 4.4. *If (M, α) is a regular contact manifold, the space of smooth functions on M decomposes as a direct sum of the kernel and the image of the operator $f \mapsto R_\alpha(f)$.*

Proof. If φ_t denotes the Reeb flow of (M, α) and $T := \ell_1(M, \alpha)$, the projection onto $\text{Ker } R_\alpha$ associated to this decomposition is the operator that sends a function smooth function f on M to its averaged function

$$\bar{f}(x) := \frac{1}{T} \int_0^T f(\varphi_t(x)) dt.$$

If the averaged function is identically zero, it is easily checked that $f = R_\alpha(h)$, where

$$h(x) = \frac{1}{T} \int_0^T t f(\varphi_t(x)) dt.$$

Therefore, we can uniquely decompose every smooth function f on M as $\bar{f} + (f - \bar{f}) \in \text{Ker } R_\alpha \oplus \text{Im } R_\alpha$. \square

Proof of Theorem 4.3. This is a simple proof by induction. The case $k = 0$ follows immediately from the Taylor expansion $\rho_s = (1 + sr_s)$ around $s = 0$. Let us now assume that the deformation α_s is already in normal form to order $k-1$ and show that there is a contact isotopy Φ_s such that $\Phi_s^* \alpha_s$ is in contact form to order k .

By hypothesis $\alpha_s = (1 + s\mu^{(1)} + \dots + s^{k-1}\mu^{(k-1)} + s^k r_s)\alpha_0$. Expanding r_s around $s = 0$ we have

$$\alpha_s = (1 + s\mu^{(1)} + \dots + s^{k-1}\mu^{(k-1)} + s^k \nu + s^{k+1} r'_s)\alpha_0.$$

As in Lemma 4.4, write ν as $\bar{\nu} + (\nu - \bar{\nu})$ and let h be a function such that $R_{\alpha_0}(h) = -(\nu - \bar{\nu})$.

Claim. Let X_h be the Hamiltonian vector field of h and let ϕ_t be its flow. If Φ_s is the contact isotopy defined by $s \mapsto \phi_{s^k}$, then $\Phi_s^* \alpha_s$ is in normal form to order k .

Indeed, up to terms of order $k + 1$ and higher

$$\Phi_s^* \alpha_s = (1 + s\mu^{(1)} + \dots + s^{k-1}\mu^{(k-1)} + s^k \nu)\alpha_0 + s^k \mathcal{L}_{X_h} \alpha_0.$$

By Proposition 2.5, $\mathcal{L}_{X_h} \alpha_0 = R_{\alpha_0}(h) = -(\nu - \bar{\nu})$ and, therefore,

$$\Phi_s^* \alpha_s = (1 + s\mu^{(1)} + \dots + s^{k-1}\mu^{(k-1)} + s^k \bar{\nu} + s^{k+1} r''_s)\alpha_0.$$

This gives us the desired normal form with $\mu^{(k)} = \bar{\nu}$. \square

5. PROOF OF THE MAIN THEOREM

We recall Theorem 2.9 assuming, without loss of generality, that our deformations are volume-preserving.

Theorem 5.1. *Let (M, α_s) be a smooth volume-preserving deformation of a regular contact manifold (M, α_0) . If the deformation is not formally trivial, then the function $s \mapsto \ell_1(M, \alpha_s)$ attains a strict local maximum at $s = 0$. If, on the other hand, the deformation is formally trivial, then*

$$\ell_1(M, \alpha_s) = \ell_1(M, \alpha_0) + \mathcal{O}(|s|^k) \text{ for all } k > 0.$$

Proof. The key elements in the proof are Theorem 3.6 on the Lipschitz continuity of the systole in a neighborhood of a regular contact form, Theorem 4.1, and the induction step in the proof of Theorem 4.3. The proof is broken down into five steps, where the first four are technical statements followed by their proofs.

1. *If $\alpha_s = (1 + s\nu_s + s^k r_s)\alpha_0$, where ν_s and r_s are smooth functions on M depending smoothly on the parameter s and $k > 1$, then*

$$\ell_1(M, \alpha_s) = \ell_1(M, (1 + s\nu_s)\alpha_0) + \mathcal{O}(|s|^k).$$

By Theorem 3.6, there is a C^2 neighborhood of zero in $C^\infty(M)$ for which the function $f \mapsto \ell_1(M, (1 + f)\alpha_0)$ is Lipschitz. It follows that for all sufficiently small values of s ,

$$|\ell_1(M, (1 + s\nu_s + s^k r_s)\alpha_0) - \ell_1(M, (1 + s\nu_s)\alpha_0)| \leq L \|s^k r_s\|_2 = \mathcal{O}(|s|^k).$$

2. *If μ is an integral of motion for the Reeb flow of α_0 and k is a positive integer, then*

$$\ell_1(M, (1 + s^k \mu)\alpha_0) \leq \begin{cases} (1 + s^k \min \mu) \ell_1(M, \alpha_0) & \text{if } s \geq 0 \text{ or } k \text{ is even} \\ (1 + s^k \max \mu) \ell_1(M, \alpha_0) & \text{if } s < 0 \text{ and } k \text{ is odd} \end{cases}$$

In particular, if $(1 + s^k \mu) \alpha_0$ is a volume-preserving deformation and μ is not identically zero, then $\ell_1(M, (1 + s^k \mu) \alpha_0) < \ell_1(M, \alpha_0)$ for all values of s different from zero.

By Theorem 4.1, $\ell_1(M, (1 + s^k \mu) \alpha_0)$ is less than $\ell_1(M, \alpha_0)$ times the minimum over M of the function $(1 + s^k \mu)$. We take the minimum or the maximum of μ depending on whether s^k is positive or negative.

If the deformation is volume-preserving, the equality

$$\begin{aligned} \text{vol}(M, \alpha_s) &= \int_M (1 + s^k \mu)^{n+1} \alpha_0 \wedge d\alpha_0^n \\ &= \text{vol}(M, \alpha_0) + (n+1) s^k \int_M \mu \alpha_0 \wedge d\alpha_0^n + \mathcal{O}(|s|^{k+1}) \end{aligned}$$

implies that the integral of μ over M is zero. Therefore, if μ is not identically zero, its extrema must have opposites signs. We conclude that the function $\ell_1(M, (1 + s^k \mu) \alpha_0)$ attains a strict maximum at $s = 0$.

3. If $\alpha_s = (1 + s^k \rho + s^{k+1} r_s) \alpha_0$ and $\bar{\rho}$ denotes the function obtained by averaging ρ along the orbits of the Reeb vector field of α_0 , then there exists a contact isotopy $\Phi_s : M \rightarrow M$ such that

$$\Phi_s^* \alpha_s = (1 + s^k \bar{\rho} + s^{k+1} r'_s) \alpha_0,$$

where r'_s is some smooth function on M depending smoothly on the parameter.

This is precisely the induction step in Theorem 4.3.

4. Let $\alpha_s = (1 + s^k \rho + s^{k+1} r_s) \alpha_0$ be a smooth volume-preserving deformation.

- (1) If $\bar{\rho}$ is not identically zero, then $\ell_1(M, \alpha_s) < \ell_1(M, \alpha_0)$ for all sufficiently small values of s different from zero.
- (2) If $\bar{\rho}$ is identically zero, then $\ell_1(M, \alpha_s) = \ell_1(M, \alpha_0) + \mathcal{O}(|s|^{k+1})$.

Using **3** and **1**, we have that

$$\begin{aligned} \ell_1(M, \alpha_s) &= \ell_1(M, (1 + s^k \bar{\rho} + s^{k+1} r'_s) \alpha_0) \\ &= \ell_1(M, (1 + s^k \bar{\rho}) \alpha_0) + \mathcal{O}(|s|^{k+1}). \end{aligned}$$

It follows immediately that if $\bar{\rho} \equiv 0$, then $\ell_1(M, \alpha_s) = \ell_1(M, \alpha_0) + \mathcal{O}(|s|^{k+1})$. Assume now that $\bar{\rho}$ is not identically zero. By **2**,

$$\ell_1(M, (1 + s^k \bar{\rho}) \alpha_0) = \begin{cases} (1 + s^k \min \bar{\rho}) \ell_1(M, \alpha_0) & \text{if } s \geq 0 \text{ or } k \text{ is even} \\ (1 + s^k \max \bar{\rho}) \ell_1(M, \alpha_0) & \text{if } s < 0 \text{ and } k \text{ is odd} \end{cases}$$

and, therefore, in order to prove that $\ell_1(M, \alpha_s) < \ell_1(M, \alpha_0)$ for all sufficiently small values of s different from zero, we just need prove that $\max \bar{\rho} > 0$ and $\min \bar{\rho} < 0$. This follows from the volume-preserving character of the deformation exactly as in **2**.

5. *The upshot:*

Let $\alpha_s = \rho_s \alpha_0$ be a smooth volume-preserving deformation of the regular contact form α_0 . Developing ρ_s around $s = 0$, we write

$$\alpha_s = (1 + s\rho_{(1)} + s^2 r_s) \alpha_0, \text{ where } \rho_{(1)} = d\rho_s/ds|_{s=0}.$$

By **4**, if the average $\bar{\rho}_{(1)}$ is not identically zero, then $s \mapsto \ell_1(M, \alpha_s)$ attains a strict maximum at $s = 0$ and we're done. If, on the other hand, $\bar{\rho}_{(1)} \equiv 0$, then, also by **4**, $\ell_1(M, \alpha_s) = \ell_1(M, \alpha_0) + \mathcal{O}(|s|^2)$.

More importantly, **3** tells us that if $\bar{\rho}_{(1)} \equiv 0$, there is a contact isotopy Φ_s such that $\Phi_s^* \alpha_s = (1 + s^2 r'_s) \alpha_0$. Since this new deformation is also a smooth volume-preserving deformation of α_0 and $\ell_1(M, \Phi_s^* \alpha_s) = \ell_1(M, \alpha_s)$, we can rewrite $\alpha_s = (1 + s^2 r'_s) \alpha_0$ and start anew. If we repeat this process an arbitrary number of times, we see that either $s \mapsto \ell_1(M, \alpha_s)$ attains a strict maximum at $s = 0$ or that $\ell_1(M, \alpha_s) = \ell_1(M, \alpha_0) + \mathcal{O}(|s|^k)$ for all $k > 0$.

It also follows from the proof that if $\ell_1(M, \alpha_s)$ does not attain a strict maximum at $s = 0$, then for any positive integer k , there exists a contact isotopy $\Phi_s^{(k)}$ and a smooth function $\nu_s^{(k)}$ on M depending smoothly of the parameter s , such that $\Phi_s^{(k)*} \alpha_s = (1 + s^{k+1} \nu_s^{(k)}) \alpha_0$. In other words, either $\ell_1(M, \alpha_s)$ attains a strict maximum at $s = 0$ or the deformation α_s is formally trivial. \square

6. APPLICATIONS TO RIEMANNIAN GEOMETRY

In the results that follow we are solely concerned with Zoll Riemannian metrics and their Riemannian deformations.

Definition 6.1. Let (N, g_0) be a Zoll manifold whose geodesic flow φ_t has prime period T . A symmetric 2-tensor h on M is said to satisfy the *zero-energy condition* (with respect to g_0) if the X-ray transform

$$\hat{h}(v) = \int_0^T h(\varphi_t(v), \varphi_t(v)) dt$$

is identically zero as a function on the unit tangent bundle of N .

Theorem 6.2. *Let (N, g_s) be a smooth volume-preserving deformation of a Zoll manifold (N, g_0) . If the deformation tensor $\dot{g} := dg_s/ds|_{s=0}$ does not satisfy the zero-energy condition, then the infimum of the lengths of periodic geodesics of (N, g_s) attains a strict local maximum at $s = 0$.*

Before going into the proof of this result, we remark that Theorem 1.6 follows easily from it.

Proof of Theorem 1.6. If the deformation of the standard metric on the two-sphere is given as $e^{\rho_s} g_0$, the zero-energy condition is precisely the condition that the integral of $\dot{\rho} = d\rho_s/ds|_{s=0}$ over any great circle be equal to zero. By a classic result of Funk this condition is verified if and only if $\dot{\rho}$ is an odd function on the two-sphere. We now apply Theorem 6.2 to conclude the proof. \square

As the reader may have already guessed, the gist of the proof of Theorem 6.2 is the (obvious once you see it) relation between the zero-energy condition and the method of averaging in canonical perturbation theory.

Proof of Theorem 6.2. In order to apply the techniques of this and the previous section, we work on the unit cotangent bundle of the Zoll manifold (N, g_0) , which we denote by S^*N . Note that if α_0 is the restriction of the canonical 1-form to the unit cotangent bundle, (S^*N, α_0) is a regular contact manifold.

For each $x \in N$, let $g_s^*(x) : T_x^*N \times T_x^*N \rightarrow \mathbb{R}$ be the inner product dual to $g_s(x)$. We define $g_s^* : T^*N \rightarrow \mathbb{R}$ as the function $p_x \mapsto g_s^*(x)(p_x, p_x)$ and ρ_s as the restriction of $(g_s^*)^{-1/2}$ to S^*N . The deformation of Riemannian metrics g_s can be now seen as the deformation of contact forms $\alpha_s = \rho_s \alpha_0$.

Note that

$$\dot{\rho} := d\rho_s/ds|_{s=0} = d(g_s^*)^{-1/2}/ds|_{s=0} = -(1/2)dg_s^*/ds|_{s=0}.$$

Using the isomorphism $\# : T^*N \rightarrow TN$ defined by g_0 , we see that the function $\dot{g}^* := dg_s^*/ds|_{s=0}$ is related to the deformation tensor of the deformation g_s by the identity

$$\dot{g}^*(p_x) = \dot{g}(\#p_x, \#p_x).$$

It follows that the zero-energy condition for \dot{g} is satisfied if and only if the average of $\dot{\rho}$ along the Reeb orbits of (S^*N, α_0) is identically zero.

If the deformation tensor \dot{g} does not satisfy the zero-energy condition, Point 4 in the proof of Theorem 5.1 tells us that $\ell_1(M, \alpha_s)$ attains a strict local maximum at $s = 0$ and this concludes the proof. \square

We are now ready for the *proof of Theorem 1.5*.

By the Michel-Tsukamoto solution of the infinitesimal Blaschke conjecture (see [17] and [30]), a symmetric 2-tensor h on a projective space satisfies the zero-energy condition with respect to the canonical metric g_0 if and only if there exists a vector field X such that $\mathcal{L}_X g_0 = h$. In other words, the zero energy condition characterizes the deformation tensor of (trivial) deformations of the form $\phi_s^* g_0$.

Suppose first that the smooth deformation g_s of the canonical metric g_0 on $\mathbb{K}P^n$ does not agree to first order with a trivial deformation. The deformation tensor $\dot{g} := dg_s/ds|_{s=0}$ does not satisfy the zero-energy condition and the application of Theorem 6.2 concludes the proof in that case.

When the deformation agrees precisely to order $k - 1$ ($1 < k < \infty$) with a trivial deformation, we owe to Larry Guth the idea of reparametrizing it so that the new deformation does not agree *to first order* with a trivial deformation.

Composing with an isotopy if necessary, we may assume that the deformation is of the form $g_s = g_0 + s^k h + \mathcal{O}(s^{k+1})$, where h is a symmetric two-tensor that does not satisfy the zero-energy condition. The deformation

$$\tilde{g}_s := g_{\sqrt[k]{s}}$$

is no longer smooth at $s = 0$, but nevertheless the corresponding contact form on $S^*\mathbb{K}P^n$ can be written as $\tilde{\rho}_s\alpha_0$ with

$$\tilde{\rho}_s = 1 + s\dot{\rho} + \sum_{l=1}^k s^{\frac{k+l}{k}} r_{l,s},$$

where the functions $\dot{\rho}$ and $r_{l,s}$ depend smoothly on all of their variables. Using Points **3** and **1** in the proof of Theorem 5.1, we have that

$$\begin{aligned} \ell_1(S^*\mathbb{K}P^n, \tilde{\rho}_s\alpha_0) &= \ell_1(S^*\mathbb{K}P^n, (1 + s\dot{\rho})) + \mathcal{O}(|s|^{\frac{k+1}{k}}) \\ &= \ell_1(S^*\mathbb{K}P^n, (1 + s\bar{\rho})) + \mathcal{O}(|s|^{\frac{k+1}{k}}). \end{aligned}$$

Since h does not satisfy the zero-energy condition, the averaged function $\bar{\rho}$ is not identically zero, so Point **4** in the proof of Theorem 5.1 implies that $\ell_1(S^*\mathbb{K}P^n, \tilde{\rho}_s\alpha_0)$ attains a strict local maximum at $s = 0$. It follows at once that the length of the shortest periodic geodesic of the metric g_s also attains a strict local maximum at $s = 0$. \square

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